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Limits and Derivatives

Short Answer Type Questions

Q. 1 Evaluate $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$.

Sol. Given,

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{x^2 - (3)^2}{x - 3}$$

$$= \lim_{x \to 3} \frac{(x + 3)(x - 3)}{(x - 3)} = \lim_{x \to 3} (x + 3)$$

$$= 3 + 3 = 6$$

Q. 2 Evaluate $\lim_{x \to 1/2} \frac{4x^2 - 1}{2x - 1}$.

Sol. Given,

$$\lim_{x \to 1/2} \frac{4x^2 - 1}{2x - 1} = \lim_{x \to 1/2} \frac{(2x)^2 - (1)^2}{2x - 1}$$

$$= \lim_{x \to 1/2} \frac{(2x + 1)(2x - 1)}{(2x - 1)} = \lim_{x \to 1/2} (2x + 1)$$

$$= 2 \times \frac{1}{2} + 1 = 1 + 1 = 2$$

Q. 3 Evaluate $\lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h}$.

Sol. Given,

$$\lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{(x + h)^{1/2} - (x)^{1/2}}{x + h - x}$$

$$= \lim_{h \to 0} \frac{(x + h)^{1/2} - (x)^{1/2}}{(x + h) - x} \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n - 1}\right]$$

$$= \frac{1}{2} x^{\frac{1}{2} - 1} = \frac{1}{2} x^{-1/2} \qquad [\because h \to 0 \Rightarrow x + h \to x]$$

$$= \frac{1}{2\sqrt{x}}$$

Q. 4 Evaluate $\lim_{x \to 0} \dfrac{(x+2)^{1/3} - 2^{1/3}}{x}$.

Sol. Given, $\lim_{x \to 0} \dfrac{(x+2)^{1/3} - 2^{1/3}}{x} = \lim_{x \to 0} \dfrac{(x+2)^{1/3} - 2^{1/3}}{(x+2) - 2}$

$$= \frac{1}{3} \times 2^{\frac{1}{3} - 1}$$

$$= \frac{1}{3} \times (2)^{-2/3} \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$= \frac{1}{3(2)^{2/3}} \qquad [\because x \to 0 \Rightarrow x + 2 \to 2]$$

Q. 5 Evaluate $\lim_{x \to 0} \dfrac{(1+x)^6 - 1}{(1+x)^2 - 1}$.

Sol. Given, $\lim_{x \to 0} \dfrac{(1+x)^6 - 1}{(1+x)^2 - 1} = \lim_{x \to 0} \dfrac{\dfrac{(1+x)^6 - 1}{x}}{\dfrac{(1+x)^2 - 1}{x}}$ [dividing numerator and denominator by x]

$$= \lim_{x \to 0} \frac{\dfrac{(1+x)^6 - 1}{(1+x) - 1}}{\dfrac{(1+x)^2 - 1}{(1+x) - 1}} \qquad [\because x \to 0 \Rightarrow 1 + x \to 1]$$

$$= \frac{\lim_{x \to 0} \dfrac{(1+x)^6 - (1)^6}{(1+x) - 1}}{\lim_{x \to 0} \dfrac{(1+x)^2 - (1)^2}{(1+x) - 1}} \qquad \left[\because \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}\right]$$

$$= \frac{6(1)^{6-1}}{2(1)^{2-1}} \qquad \left[\therefore \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$= \frac{6 \times 1}{2 \times 1} = \frac{6}{2} = 3$$

Q. 6 Evaluate $\lim_{x \to a} \dfrac{(2+x)^{5/2} - (a+2)^{5/2}}{x - a}$.

Sol. Given, $\lim_{x \to a} \dfrac{(2+x)^{5/2} - (a+2)^{5/2}}{x - a} = \lim_{x \to a} \dfrac{(2+x)^{5/2} - (a+2)^{5/2}}{(2+x) - (a+2)}$

$$= \frac{5}{2}(a+2)^{\frac{5}{2} - 1} \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$= \frac{5}{2}(a+2)^{3/2} \qquad [\because x \to a \Rightarrow x + 2 \to a + 2]$$

Q. 7 Evaluate $\lim_{x \to 1} \frac{x^4 - \sqrt{x}}{\sqrt{x} - 1}$.

Sol. Given, $\lim_{x \to 1} \frac{x^4 - \sqrt{x}}{\sqrt{x} - 1} = \lim_{x \to 1} \frac{\sqrt{x}[(x)^{7/2} - 1]}{\sqrt{x} - 1}$

$$= \lim_{x \to 1} \frac{(x)^{7/2} - 1}{\sqrt{x} - 1} \cdot \lim_{x \to 1} \sqrt{x} \qquad \left[\because \lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)\right]$$

$$= \lim_{x \to 1} \frac{\frac{x^{7/2} - 1}{x - 1}}{\frac{(x)^{1/2} - 1}{x - 1}} \cdot 1$$

$$= \frac{\lim_{x \to 1} \frac{x^{7/2} - 1}{x - 1}}{\lim_{x \to 1} \frac{(x)^{1/2} - 1}{x - 1}} \qquad \left[\because \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}\right]$$

$$= \frac{\frac{7}{2}(1)^{\frac{7}{2} - 1}}{\frac{1}{2}(1)^{\frac{1}{2} - 1}} = \frac{\frac{7}{2}}{\frac{1}{2}} = 7$$

Q. 8 Evaluate $\lim_{x \to 2} \frac{x^2 - 4}{\sqrt{3x - 2} - \sqrt{x + 2}}$.

Sol. Given, $\lim_{x \to 2} \frac{x^2 - 4}{\sqrt{3x - 2} - \sqrt{x + 2}} = \lim_{x \to 2} \frac{(x^2 - 4)\sqrt{3x - 2} + \sqrt{x + 2})}{(\sqrt{3x - 2} - \sqrt{x + 2})(\sqrt{3x - 2} - \sqrt{x + 2})}$

$$= \lim_{x \to 2} \frac{(x^2 - 4)(\sqrt{3x - 2} + \sqrt{x + 2})}{(\sqrt{3x - 2})^2 - (\sqrt{x + 2})^2}$$

$[\because (a + b)(a - b) = a^2 - b^2]$

$$= \lim_{x \to 2} \frac{(x^2 - 4)(\sqrt{3x - 2} + \sqrt{x + 2})}{(3x - 2) - (x + 2)}$$

$$= \lim_{x \to 2} \frac{(x^2 - 4)(\sqrt{3x - 2} + \sqrt{x + 2})}{3x - 2 - x - 2}$$

$$= \lim_{x \to 2} \frac{(x^2 - 4)(\sqrt{3x - 2} + \sqrt{x + 2})}{2x - 4}$$

$$= \lim_{x \to 2} \frac{(x + 2)(x - 2)(\sqrt{3x - 2} + \sqrt{x + 2})}{2(x - 2)}$$

$$= \lim_{x \to 2} \frac{(x + 2)(\sqrt{3x - 2} + \sqrt{x + 2})}{2}$$

$$= \frac{(2 + 2)(\sqrt{6 - 2} + \sqrt{2 + 2})}{2}$$

$$= \frac{4(2 + 2)}{2} = 8$$

Q. 9 Evaluate $\lim_{x \to \sqrt{2}} \dfrac{x^4 - 4}{x^2 + 3\sqrt{2}x - 8}$.

Sol. Given, $\lim_{x \to \sqrt{2}} \dfrac{x^4 - 4}{x^2 + 3\sqrt{2}x - 8} = \lim_{x \to \sqrt{2}} \dfrac{(x^2)^2 - (2)^2}{x^2 + 3\sqrt{2}x - 8}$

$$= \lim_{x \to \sqrt{2}} \frac{(x^2 - 2)(x^2 + 2)}{x^2 + 4\sqrt{2}x - \sqrt{2}x - 8}$$

$$= \lim_{x \to \sqrt{2}} \frac{(x - \sqrt{2})(x + \sqrt{2})(x^2 + 2)}{x(x + 4\sqrt{2}) - \sqrt{2}(x + 4\sqrt{2})}$$

$$= \lim_{x \to \sqrt{2}} \frac{(x - \sqrt{2})(x + \sqrt{2})(x^2 + 2)}{(x - \sqrt{2})(x + 4\sqrt{2})}$$

$$= \lim_{x \to \sqrt{2}} \frac{(x + \sqrt{2})(x^2 + 2)}{(x + 4\sqrt{2})}$$

$$= \frac{(\sqrt{2} + \sqrt{2})[(\sqrt{2})^2 + 2]}{(\sqrt{2} + 4\sqrt{2})}$$

$$= \frac{2\sqrt{2}(2 + 2)}{5\sqrt{2}} = \frac{8}{5}$$

Q. 10 Evaluate $\lim_{x \to 1} \dfrac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$.

Sol. Given, $\lim_{x \to 1} \dfrac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$ $\left[\dfrac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 1} \frac{x^7 - x^5 - x^5 + 1}{x^3 - x^2 - 2x^2 + 2}$$

$$= \lim_{x \to 1} \frac{x^5(x^2 - 1) - 1(x^5 - 1)}{x^2(x - 1) - 2(x^2 - 1)}$$

On dividing numerator and denominator by $(x - 1)$, then

$$= \lim_{x \to 1} \frac{\dfrac{x^5(x^2 - 1)}{(x - 1)} - \dfrac{1(x^2 - 1)}{(x - 1)}}{\dfrac{x^2(x - 1)}{(x - 1)} - \dfrac{2(x^2 - 1)}{(x - 1)}}$$

$$= \frac{\lim_{x \to 1} x^5(x + 1) - \lim_{x \to 1} \left(\dfrac{x^5 - 1}{x - 1}\right)}{\lim_{x \to 1} x^2 - \lim_{x \to 1} (x + 1)}$$

$$= \frac{1 \times 2 - 5 \times (1)^4}{1 - 2 \times 2} = \frac{2 - 5}{1 - 4}$$

$$= \frac{-3}{-3} = 1$$

Q. 11 Evaluate $\lim_{x \to 0} \dfrac{\sqrt{1+x^3} - \sqrt{1-x^3}}{x^2}$.

Sol. Given, $\lim_{x \to 0} \dfrac{\sqrt{1+x^3} - \sqrt{1-x^3}}{x^2} = \lim_{x \to 0} \dfrac{\sqrt{1+x^3} - \sqrt{1-x^3}}{x^2} \cdot \dfrac{\sqrt{1+x^3} + \sqrt{1-x^3}}{\sqrt{1+x^3} + \sqrt{1-x^3}}$

$$= \lim_{x \to 0} \frac{(1+x^3) - (1-x^3)}{x^2(\sqrt{1+x^3} + \sqrt{1-x^3})}$$

$$= \lim_{x \to 0} \frac{1+x^3 - 1 + x^3}{x^2(\sqrt{1+x^3} + \sqrt{1-x^3})}$$

$$= \lim_{x \to 0} \frac{2x^3}{x^2(\sqrt{1+x^3} + \sqrt{1-x^3})}$$

$$= \lim_{x \to 0} \frac{2x}{(\sqrt{1+x^3} + \sqrt{1-x^3})}$$

$$= 0$$

Q. 12 Evaluate $\lim_{x \to -3} \dfrac{x^3 + 27}{x^5 + 243}$.

Sol. Given, $\lim_{x \to -3} \dfrac{x^3 + 27}{x^5 + 243} = \lim_{x \to -3} \dfrac{\dfrac{x^3 + 27}{x + 3}}{\dfrac{x^5 + 243}{x + 3}}$

$$= \lim_{x \to -3} \frac{\dfrac{x^3 - (-3)^3}{x - (-3)}}{\dfrac{x^5 - (-3)^5}{x - (-3)}} = \frac{\lim_{x \to -3} \dfrac{x^3 - (-3)^3}{x - (-3)}}{\lim_{x \to -3} \dfrac{x^5 - (-3)^5}{x - (-3)}} \qquad \left[\because \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}\right]$$

$$= \frac{3(-3)^{3-1}}{5(-3)^{5-1}} = \frac{3(-3)^2}{5(-3)^4} \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$= \frac{3}{5(-3)^2} = \frac{3}{45} = \frac{1}{15}$$

Q. 13 Evaluate $\lim_{x \to 1/2} \left(\dfrac{8x - 3}{2x - 1} - \dfrac{4x^2 + 1}{4x^2 - 1}\right)$.

Sol. Given, $\lim_{x \to 1/2} \left(\dfrac{8x - 3}{2x - 1} - \dfrac{4x^2 + 1}{4x^2 - 1}\right) = \lim_{x \to 1/2} \left[\dfrac{(8x-3)(2x+1) - (4x^2+1)}{(4x^2 - 1)}\right]$

$$= \lim_{x \to 1/2} \left[\frac{16x^2 + 8x - 6x - 3 - 4x^2 - 1}{4x^2 - 1}\right]$$

$$= \lim_{x \to 1/2} \left[\frac{12x^2 + 2x - 4}{4x^2 - 1}\right]$$

$$= \lim_{x \to 1/2} \frac{2(6x^2 + x - 2)}{4x^2 - 1}$$

$$= \lim_{x \to 1/2} \frac{2(6x^2 + 4x - 3x - 2)}{4x^2 - 1}$$

$$= \lim_{x \to 1/2} \frac{2[2x(3x+2) - 1(3x+2)]}{4x^2 - 1}$$

$$= \lim_{x \to 1/2} \frac{2[(3x+2)(2x-1)]}{(2x)^2 - (1)^2}$$

$$= \lim_{x \to 1/2} \frac{2(3x+2)(2x-1)}{(2x-1)(2x+1)}$$

$$= \lim_{x \to 1/2} \frac{2(3x+2)}{2x-1} = \frac{2\left(3 \times \frac{1}{2} + 2\right)}{2 \times \frac{1}{2} + 1}$$

$$= \frac{3}{2} + 2 = \frac{7}{2}$$

Q. 14 Find the value of n, if $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 80$, $n \in N$.

Sol. Given, $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 80$

$\Rightarrow \quad n(2)^{n-1} = 80 \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$

$\Rightarrow \quad n(2)^{n-1} = 5 \times 16$

$\Rightarrow \quad n \times 2^{n-1} = 5 \times (2)^4$

$\Rightarrow \quad n \times 2^{n-1} = 5 \times (2)^{5-1}$

$\therefore \quad n = 5$

Q. 15 Evaluate $\lim_{x \to 0} \frac{\sin 3x}{\sin 7x}$.

Sol. Given, $\lim_{x \to 0} \frac{\frac{\sin 3x}{3x} \cdot 3x}{\frac{\sin 7x}{7x} \cdot 7x} = \frac{\lim_{x \to 0} \frac{\sin 3x}{3x}}{\lim_{x \to 0} \frac{\sin 7x}{7x}} \cdot \frac{3x}{7x}$

$$= \frac{3}{7} \cdot \frac{\lim_{x \to 0} \frac{\sin 3x}{3x}}{\lim_{x \to 0} \frac{\sin 7x}{7x}} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= \frac{3}{7} \qquad [\because x \to 0 \Rightarrow (kx \to 0), \text{ here } k \text{ is real number}]$$

Q. 16 Eavaluate $\lim_{x \to 0} \frac{\sin^2 2x}{\sin^2 4x}$.

Sol. Given,

$$\lim_{x \to 0} \frac{\sin^2 2x}{\sin^2 4x} = \lim_{x \to 0} \frac{\sin^2 2x}{[\sin 2\,(2x)]^2}$$

$$= \lim_{x \to 0} \frac{\sin^2 2x}{(2\sin 2x \cos 2x)^2}$$

$$= \lim_{x \to 0} \frac{\sin^2 2x}{4\sin^2 2x \cos^2 2x} \qquad [\because \sin 2\theta = 2\sin\theta\cos\theta]$$

$$= \lim_{x \to 0} \frac{1}{4\cos^2 2x} = \frac{1}{4} \qquad [\because \cos 0 = 1]$$

Q. 17 Evaluate $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$.

Sol. Given,

$$\lim_{x \to 0} \frac{1 - \cos 2x}{x^2} = \lim_{x \to 0} \frac{1 - 1 + 2\sin^2 x}{x^2} \qquad [\because \cos 2x = 1 - 2\sin^2 x]$$

$$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2} = 2 \lim_{x \to 0} \frac{\sin^2 x}{x^2}$$

$$= 2 \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= 2 \times 1 = 2$$

Q. 18 Evaluate $\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3}$.

Sol. Given,

$$\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3} = \lim_{x \to 0} \frac{2\sin x - 2\sin x \cos x}{x^3} \qquad [\because \sin 2x = 2\sin x \cos x]$$

$$= \lim_{x \to 0} \frac{2\sin x(1 - \cos x)}{x^3}$$

$$= 2 \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \left(\frac{1 - \cos x}{x^2}\right)$$

$$= 2 \cdot 1 \lim_{x \to 0} \frac{1 - \cos x}{x^2} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= 2 \lim_{x \to 0} \frac{1 - 1 + 2\sin^2 \frac{x}{2}}{x^2} = 2 \lim_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{4 \times \frac{x^2}{4}}$$

$$= \frac{2 \cdot 2}{4} \lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = \lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = 1$$

Q. 19 Evaluate $\lim_{x \to 0} \dfrac{1 - \cos mx}{1 - \cos nx}$.

Sol. Given, $\lim_{x \to 0} \dfrac{1-\cos mx}{1-\cos nx} = \lim_{x \to 0} \dfrac{1-1+2\sin^2 \frac{mx}{2}}{1-1+2\sin^2 \frac{nx}{2}}$ $\left[\because \cos mx = 1-2\sin^2 \frac{mx}{2} \text{ and } \sin nx = 1-2\sin^2 \frac{nx}{2}\right]$

$$= \lim_{x \to 0} \frac{\sin^2 \frac{mx}{2}}{\sin^2 \frac{nx}{2}} = \lim_{x \to 0} \frac{\dfrac{\sin^2 \frac{mx}{2}}{\left(\frac{mx}{2}\right)^2} \cdot \left(\frac{mx}{2}\right)^2}{\dfrac{\sin^2 \frac{nx}{2}}{\left(\frac{nx}{2}\right)^2} \cdot \left(\frac{nx}{2}\right)^2} = \frac{\lim_{x \to 0}\left(\dfrac{\sin \frac{mx}{2}}{\frac{mx}{2}}\right)^2}{\lim_{x \to 0}\left(\dfrac{\sin \frac{nx}{2}}{\frac{nx}{2}}\right)^2} \cdot \frac{m^2 \frac{x^2}{4}}{n^2 \frac{x^2}{4}}$$

$$= \frac{m^2}{n^2} \cdot \frac{\lim_{x \to 0}\left(\dfrac{\sin \frac{mx}{2}}{\frac{mx}{2}}\right)^2}{\lim_{x \to 0}\left(\dfrac{\sin \frac{nx}{2}}{\frac{nx}{2}}\right)^2} = \frac{m^2}{n^2}$$ $\left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$

$[\because x \to 0 \Rightarrow k\,x \to 0]$

Q. 20 Evaluate $\lim_{x \to \pi/3} \dfrac{\sqrt{1 - \cos 6x}}{\sqrt{2}\left(\frac{\pi}{3} - x\right)}$.

Sol. Given, $\lim_{x \to \pi/3} \dfrac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)} = \lim_{x \to \pi/3} \dfrac{\sqrt{1-1+2\sin^2 3x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)}$ $[\because \cos 2x = 1-2\sin^2 x]$

$$= \lim_{x \to \pi/3} \frac{\sqrt{2}\sin 3x}{\sqrt{2}\left(\frac{\pi}{3}-x\right)} = \lim_{x \to \pi/3} \frac{\sin 3x}{\frac{\pi}{3}-x}$$

$$= \lim_{x \to \pi/3} \frac{\sin(\pi - 3x)}{\frac{\pi - 3x}{3}}$$ $[\because \sin(\pi - \theta) = \sin\theta]$

$$= 3 \lim_{x \to \pi/3} \frac{\sin(\pi - 3x)}{(\pi - 3x)} = 3 \times 1$$ $\left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$

$$= 3$$ $\left[\because x \to \frac{\pi}{3} \Rightarrow \left(x - \frac{\pi}{3}\right) \to 0\right]$

Q. 21 Evaluate $\lim_{x \to \frac{\pi}{4}} \dfrac{\sin x - \cos x}{x - \dfrac{\pi}{4}}$.

Sol. Given, $\lim_{x \to \pi/4} \dfrac{\sqrt{2}\left(\sin x \cdot \dfrac{1}{\sqrt{2}} - \cos x \cdot \dfrac{1}{\sqrt{2}}\right)}{\left(x - \dfrac{\pi}{4}\right)} = \lim_{x \to \pi/4} \dfrac{\sqrt{2}\left(\sin x \cos \dfrac{\pi}{4} - \cos x \cdot \sin \dfrac{\pi}{4}\right)}{\left(x - \dfrac{\pi}{4}\right)}$

$$= \lim_{x \to \pi/4} \frac{\sqrt{2}\left\{\sin\left(x - \dfrac{\pi}{4}\right)\right\}}{\left(x - \dfrac{\pi}{4}\right)}$$

$$= \sqrt{2} \lim_{x \to \pi/4} \frac{\sin\left(x - \dfrac{\pi}{4}\right)}{\left(x - \dfrac{\pi}{4}\right)} = \sqrt{2} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$\left[\because x \to \frac{\pi}{4} \Rightarrow \left(x - \frac{\pi}{4}\right) \to 0\right]$$

Q. 22 Evaluate $\lim_{x \to \pi/6} \dfrac{\sqrt{3}\sin x - \cos x}{x - \dfrac{\pi}{6}}$.

Sol. Given, $\lim_{x \to \pi/6} \dfrac{\sqrt{3}\sin x - \cos x}{x - \dfrac{\pi}{6}} = \lim_{x \to \pi/6} \dfrac{2\left(\dfrac{\sqrt{3}}{2}\sin x - \dfrac{1}{2}\cos x\right)}{\left(x - \dfrac{\pi}{6}\right)}$

$$= \lim_{x \to \pi/6} \frac{2\left(\sin x \cos \dfrac{\pi}{6} - \cos x \sin \dfrac{\pi}{6}\right)}{\left(x - \dfrac{\pi}{6}\right)} = 2 \lim_{x \to \pi/6} \frac{\sin\left(x - \dfrac{\pi}{6}\right)}{\left(x - \dfrac{\pi}{6}\right)}$$

$$[\because \sin A \cos B - \cos A \sin B = \sin (A - B)]$$

$$= 2 \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$\left[\because x \to \frac{\pi}{6} \Rightarrow \left(x - \frac{\pi}{6}\right) \to 0\right]$$

Q. 23 Evaluate $\lim_{x \to 0} \dfrac{\sin 2x + 3x}{2x + \tan 3x}$.

Sol. Given, $\lim_{x \to 0} \dfrac{\sin 2x + 3x}{2x + \tan 3x} = \lim_{x \to 0} \dfrac{\dfrac{\sin 2x + 3x}{2x} \cdot 2x}{\dfrac{2x + \tan 3x}{3x} \cdot 3x}$

$$= \lim_{x \to 0} \frac{\left(\dfrac{\sin 2x}{2x} + \dfrac{3x}{2x}\right) 2x}{\left(\dfrac{2x}{3x} + \dfrac{\tan 3x}{3x}\right) 3x} = \lim_{x \to 0} \frac{\dfrac{\sin 2x}{2x} + \dfrac{3}{2}}{\dfrac{2}{3} + \dfrac{\tan 3x}{3x}} \cdot \frac{2}{3}$$

$$= \frac{2}{3} \lim_{x \to 0} \frac{\frac{\sin 2x}{2x} + \frac{3}{2}}{\frac{2}{3} + \lim_{x \to 0} \frac{\tan 3x}{3x}}$$

$$= \frac{2}{3}\left(\frac{1 + \frac{3}{2}}{\frac{2}{3} + 1}\right) \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \to 0} \frac{\tan x}{x} = 1\right]$$

$$= \frac{2}{3} \times \frac{\frac{5}{2}}{\frac{5}{3}} = \frac{2}{3} \times \frac{5}{2} \times \frac{3}{5} = 1$$

Q. 24 Evaluate $\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$.

Sol. Given, $\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}} = \lim_{x \to 0} \frac{2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{\sqrt{x} - \sqrt{a}}$

$$\left[\because \sin C - \sin D = 2\cos\frac{C+D}{2} . \sin\frac{C-D}{2}\right]$$

$$= \lim_{x \to a} \frac{2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)(\sqrt{x} + \sqrt{a})}{(\sqrt{x} - \sqrt{a})\ (\sqrt{x} + \sqrt{a})}$$

$$= \lim_{x \to 0} \frac{2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)\ (\sqrt{x} + \sqrt{a})}{x - a}$$

$$= 2 \lim_{x \to a} \cos\left(\frac{x+a}{2}\right)\ (\sqrt{x} + \sqrt{a}) \lim_{x \to 0} \frac{\sin\left(\frac{x-a}{2}\right)}{2\left(\frac{x-a}{2}\right)}$$

$$= 2 \lim_{x \to 0} \cos\left(\frac{x+a}{2}\right)(\sqrt{x} + \sqrt{a}) \cdot \frac{1}{2} \lim_{x \to 0} \frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)}$$

$$= 2 \cdot \cos\frac{a}{2} \cdot \sqrt{a} \cdot \frac{1}{2} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= \sqrt{a}\cos\frac{a}{2}$$

Q. 25 Evaluate $\lim_{x \to \pi/6} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$.

Sol. Given $\lim_{x \to \pi/6} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2} = \lim_{x \to \pi/6} \frac{\operatorname{cosec}^2 x - 1 - 3}{\operatorname{cosec} x - 2}$ $\qquad [\because \operatorname{cosec}^2 x = 1 + \cot^2 x]$

$$= \lim_{x \to \pi/6} \frac{\operatorname{cosec}^2 x - 4}{\operatorname{cosec} x - 2} = \lim_{x \to \pi/6} \frac{(\operatorname{cosec} x)^2 - (2)^2}{\operatorname{cosec} x - 2}$$

$$= \lim_{x \to \pi/6} \frac{(\operatorname{cosec} x + 2)(\operatorname{cosec} x - 2)}{(\operatorname{cosec} x - 2)} = \lim_{x \to \pi/6} (\operatorname{cosec} x + 2)$$

$$= \operatorname{cosec}\frac{\pi}{6} + 2 = 2 + 2 = 4$$

Q. 26 Evaluate $\lim_{x \to 0} \dfrac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$.

Sol. Given, $\lim_{x \to 0} \dfrac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x} = \lim_{x \to 0} \dfrac{\sqrt{2} - \sqrt{1 + 2\cos^2 \frac{x}{2} - 1}}{\sin^2 x}$ $\left[\because \cos x = 2\cos^2 \frac{x}{2} - 1\right]$

$$= \lim_{x \to 0} \frac{\sqrt{2} - \sqrt{2\cos^2 \frac{x}{2}}}{\sin^2 x} \qquad \left[\because \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right]$$

$$= \lim_{x \to 0} \frac{\sqrt{2}\left(1 - \cos\frac{x}{2}\right)}{\sin^2 x} = \lim_{x \to 0} \frac{\sqrt{2}\left(1 - 1 + 2\sin^2\frac{x}{4}\right)}{\sin^2 x}$$

$$= \lim_{x \to 0} \frac{\sqrt{2}\left(2\sin^2\frac{x}{4}\right)}{\sin^2 x} = \lim_{x \to 0} 2\sqrt{2} \frac{\sin^2\frac{x}{4}}{\left(\frac{x}{4}\right)^2} \cdot \frac{\left(\frac{x}{4}\right)^2}{\sin^2 x}$$

$$= 2\sqrt{2} \lim_{x \to 0} \left(\frac{\sin\frac{x}{4}}{\frac{x}{4}}\right)^2 \cdot \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^2 \cdot \frac{1}{16}$$

$$= 2\sqrt{2} \cdot 1 \cdot 1 \cdot \frac{1}{16} = \frac{1}{4\sqrt{2}}$$

Q. 27 Evaluate $\lim_{x \to 0} \dfrac{\sin x - 2\sin 3x + \sin 5x}{x}$.

Sol. Given,

$$\lim_{x \to 0} \frac{\sin x - 2\sin 3x + \sin 5x}{x} = \lim_{x \to 0} \frac{\sin 5x + \sin x - 2\sin 3x}{x}$$

$$= \lim_{x \to 0} \frac{2\sin 3x \cos 2x - 2\sin 3x}{x} = \lim_{x \to 0} \frac{2\sin 3x(\cos 2x - 1)}{x}$$

$$= \lim_{x \to 0} \frac{2\sin 3x}{\frac{1}{3} \times 3x}(\cos 2x - 1) = 6 \lim_{x \to 0} \frac{\sin 3x}{3x}(\cos 2x - 1)$$

$$= 6 \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \lim_{x \to 0} (\cos 2x - 1) = 6 \times 1 \times 0 = 0$$

Q. 28 If $\lim_{x \to 1} \dfrac{x^4 - 1}{x - 1} = \lim_{x \to k} \dfrac{x^3 - k^3}{x^2 - k^2}$, then find the value of k.

Sol. Given,

$$\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim_{x \to k} \frac{x^3 - k^3}{x^2 - k^2}$$

$$\Rightarrow \quad 4(1)^{4-1} = \lim_{x \to k} \frac{\frac{x^3 - k^3}{x - k}}{\frac{x^2 - k^2}{x - k}} \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$\Rightarrow \quad 4 = \frac{\lim_{x \to k} \frac{x^3 - k^3}{x - k}}{\lim_{x \to k} \frac{x^2 - k^2}{x - k}} 1 \qquad \left[\because \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \right]$$

$$\Rightarrow \quad 4 = \frac{3k^2}{2k} \quad \Rightarrow \quad 4 = \frac{3}{2}k$$

$$\therefore \quad k = \frac{4 \times 2}{3} = \frac{8}{3}$$

Differentiate each of the functions w.r.t. x in following questions

Q. 29 $\dfrac{x^4 + x^3 + x^2 + 1}{x}$

Sol. $$\frac{d}{dx}\left(\frac{x^4 + x^3 + x^2 + 1}{x}\right) = \frac{d}{dx}\left(x^3 + x^2 + x + \frac{1}{x}\right)$$

$$= \frac{d}{dx}x^3 + \frac{d}{dx}x^2 + \frac{d}{dx}x + \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= 3x^2 + 2x + 1 + \left(-\frac{1}{x^2}\right)$$

$$= 3x^2 + 2x + 1 - \frac{1}{x^2}$$

$$= \frac{3x^4 + 2x^3 + x^2 - 1}{x^2}$$

Q. 30 $\left(x + \dfrac{1}{x}\right)^3$

Sol. Let $$y = \left(x + \frac{1}{x}\right)^3$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(x + \frac{1}{x}\right)^3 = 3\left(x + \frac{1}{x}\right)^{3-1} \frac{d}{dx}\left(x + \frac{1}{x}\right) \qquad \text{[by chain rule]}$$

$$= 3\left(x + \frac{1}{x}\right)^2\left(1 - \frac{1}{x^2}\right)$$

$$= 3x^2 - \frac{3}{x^2} - \frac{3}{x^4} + 3$$

Q. 31 $(3x + 5)(1 + \tan x)$

Sol. Let $$y = (3x + 5)(1 + \tan x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[(3x + 5)(1 + \tan x)]$$

$$= (3x + 5)\frac{d}{dx}(1 + \tan x) + (1 + \tan x)\frac{d}{dx}(3x + 5) \qquad \text{[by product rule]}$$

$$= (3x + 5)(\sec^2 x) + (1 + \tan x) \cdot 3$$

$$= (3x + 5)\sec^2 x + 3(1 + \tan x)$$

$$= 3x\sec^2 x + 5\sec^2 x + 3\tan x + 3$$

Q. 32 $(\sec x - 1)(\sec x + 1)$

Sol. Let

$$y = (\sec x - 1)(\sec x + 1)$$

$$y = (\sec^2 - 1) \qquad [\because (a+b)(a-b) = a^2 - b^2]$$

$$= \tan^2 x$$

$$\therefore \quad \frac{dy}{dx} = 2\tan x \cdot \frac{d}{dx}\tan x$$

$$= 2\tan x \cdot \sec^2 x \qquad \text{[by chain rule]}$$

Q. 33 $\dfrac{3x+4}{5x^2 - 7x + 9}$

Sol. Let

$$y = \frac{3x+4}{5x^2 - 7x + 9}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(5x^2 - 7x + 9)\frac{d}{dx}(3x+4) - (3x+4)\frac{d}{dx}(5x^2 - 7x + 9)}{(5x^2 - 7x + 9)^2} \qquad \text{[by quotient rule]}$$

$$= \frac{(5x^2 - 7x + 9)\cdot 3 - (3x+4)(10x - 7)}{(5x^2 - 7x + 9)^2}$$

$$= \frac{15x^2 - 21x + 27 - 30x^2 + 21x - 40x + 28}{(5x^2 - 7x + 9)^2}$$

$$= \frac{-15x^2 - 40x + 55}{(5x^2 - 7x + 9)^2}$$

$$= \frac{55 - 15x^2 - 40x}{(5x^2 - 7x + 9)^2}$$

Q. 34 $\dfrac{x^5 - \cos x}{\sin x}$

Sol. Let

$$y = \frac{x^5 - \cos x}{\sin x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sin x \frac{d}{dx}(x^5 - \cos x) - (x^5 - \cos x)\frac{d}{dx}\sin x}{(\sin x)^2} \qquad \text{[by quotient rule]}$$

$$= \frac{\sin x(5x^4 + \sin x) - (x^5 - \cos x)\cos x}{\sin^2 x}$$

$$= \frac{5x^4 \sin x + \sin^2 x - x^5 \cos x + \cos^2 x}{\sin^2 x}$$

$$= \frac{5x^4 \sin x - x^5 \cos x + \sin^2 x + \cos^2 x}{\sin^2 x}$$

$$= \frac{5x^4 \sin x - x^5 \cos x + 1}{\sin^2 x}$$

Q. 35 $\dfrac{x^2 \cos \dfrac{\pi}{4}}{\sin x}$

Sol. Let
$$y = \frac{x^2 \cos \frac{\pi}{4}}{\sin x} = \frac{\frac{x^2}{\sqrt{2}}}{\sin x}$$
$$y = \frac{1}{\sqrt{2}} \cdot \frac{x^2}{\sin x}$$
$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{2}}\left[\frac{\sin x \cdot \frac{d}{dx} x^2 - x^2 \frac{d}{dx} \sin x}{\sin^2 x}\right] \quad \text{[by quotient rule]}$$
$$= \frac{1}{\sqrt{2}}\left[\frac{\sin x \cdot 2x - x^2 \cdot \cos x}{\sin^2 x}\right]$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{2x \sin x - x^2 \cos x}{\sin^2 x}$$
$$= \frac{x}{\sqrt{2}}[2 \operatorname{cosec} x - x \cot x \operatorname{cosec} x]$$
$$= \frac{x}{\sqrt{2}} \operatorname{cosec} [2 - x \cot x]$$

Q. 36 $(ax^2 + \cot x)(p + q \cos x)$

Sol. Let $y = (ax^2 + \cot x)(p + q \cos x)$
$$\therefore \quad \frac{dy}{dx} = (ax^2 + \cot x)\frac{d}{dx}(p + q \cos x) + (p + q \cos x)\frac{d}{dx}(ax^2 + \cot x) \quad \text{[by product rule]}$$
$$= (ax^2 + \cot x)(-q \sin x) + (p + q \cos x)(2ax - \operatorname{cosec}^2 x)$$
$$= -q \sin x(ax^2 + \cot x) + (p + q \cos x)(2ax - \operatorname{cosec}^2 x)$$

Q. 37 $\dfrac{a + b \sin x}{c + d \cos x}$

Sol. Let
$$y = \frac{a + b \sin x}{c + d \cos x}$$
$$\therefore \quad \frac{dy}{dx} = \frac{(c + d \cos x)\frac{d}{dx}(a + b \sin x) - (a + b \sin x)\frac{d}{dx}(c + d \cos x)}{(c + d \cos x)^2} \quad \text{[by quotinet rule]}$$
$$= \frac{(c + d \cos x)(b \cos x) - (a + b \sin x)(-d \sin x)}{(c + d \cos x)^2}$$
$$= \frac{bc \cos x + bd \cos^2 x + ad \sin x + bd \sin^2 x}{(c + d \cos x)^2}$$
$$= \frac{bc \cos x + ad \sin x + bd(\cos^2 x + \sin^2 x)}{(c + d \cos x)^2}$$
$$= \frac{bc \cos x + ad \sin x + bd}{(c + d \cos x)^2}$$

Q. 38 $(\sin x + \cos x)^2$

Sol. Let $y = (\sin x + \cos x)^2$

$\therefore$ $\frac{dy}{dx} = 2(\sin x + \cos x)(\cos x - \sin x)$

$= 2(\cos^2 x - \sin^2 x)$ [by chain rule]

$= 2\cos 2x$ [$\because \cos 2x = \cos^2 x - \sin^2 x$]

Q. 39 $(2x - 7)^2 (3x + 5)^3$

Sol. Let $y = (2x - 7)^2 (3x + 5)^3$

$\frac{dy}{dx} = (2x - 7)^2 \frac{d}{dx}(3x + 5)^3 + (3x + 5)^3 \frac{d}{dx}(2x - 7)^2$ [by product rule]

$= (2x - 7)^2 (3)(3x + 5)^2 (3) + (3x + 5)^3 2(2x - 7)(2)$ [by chain rule]

$= 9(2x - 7)^2 (3x + 5)^2 + 4(3x + 5)^3 (2x - 7)$

$= (2x - 7)(3x + 5)^2 [9(2x - 7) + 4(3x + 5)]$

$= (2x - 7)(3x + 5)^2 (18x - 63 + 12x + 20)$

$= (2x - 7)(3x + 5)^2 (30x - 43)$

Q. 40 $x^2 \sin x + \cos 2x$

Sol. Let $y = x^2 \sin x + \cos 2x$

$\frac{dy}{dx} = \frac{d}{dx}(x^2 \sin x) + \frac{d}{dx}\cos 2x$

$= x^2 \cdot \cos x + \sin x \, 2x + (-\sin 2x) \cdot 2$ [by product rule]

$= x^2 \cos x + 2x \sin x - 2\sin 2x$ [by chain urle]

Q. 41 $\sin^3 x \cos^3 x$

Sol. Let $y = \sin^3 x \cos^3 x$

$\therefore$ $\frac{dy}{dx} = \sin^3 x \cdot \frac{d}{dx}\cos^3 x + \cos^3 x \frac{d}{dx}\sin^3 x$ [by product rule]

$= \sin^3 x \cdot 3\cos^2 x(-\sin x) + \cos^3 x \cdot 3\sin^2 x \cos x$ [by chain rule]

$= -3\cos^2 x \sin^4 x + 3\sin^2 x \cos^4 x$

$= 3\sin^2 x \cos^2 x(\cos^2 x - \sin^2 x)$

$= 3\sin^2 x \cos^2 x \cos 2x$

$= \frac{3}{4}(2\sin x \cos x)^2 \cos 2x$

$= \frac{3}{4}\sin^2 2x \cos 2x$

Q. 42 $\dfrac{1}{ax^2+bx+c}$

Sol. Let $$y = \frac{1}{ax^2+bx+c} = (ax^2+bx+c)^{-1}$$

$\therefore$ $$\frac{dy}{dx} = -(ax^2+bx+c)^{-2}(2ax+b) \qquad \text{[by chain rule]}$$
$$= \frac{-(2ax+b)}{(ax^2+bx+c)^2}$$

Long Answer Type Questions

Differentiate each of the functions with respect to x in following questions using first principle.

Q. 43 $\cos(x^2+1)$

Sol. Let $f(x) = \cos(x^2+1)$ and $f(x+h) = \cos\{(x+h)^2+1\}$

$\therefore$ $$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{\cos\{(x+h)^2+1\}-\cos(x^2+1)}{h}$$
$$= \lim_{h\to 0}\frac{-2\sin\left\{\dfrac{(x+h)^2+1+x^2+1}{2}\right\}\sin\left\{\dfrac{(x+h)^2+1-x^2-1}{2}\right\}}{h}$$
$$\left[\because \cos C-\cos D = -2\sin\frac{C+D}{2}\cdot\sin\frac{C-D}{2}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[-2\sin\left\{\frac{(x+h)^2+x^2+2}{2}\right\}\sin\left\{\frac{(x+h)^2-x^2}{2}\right\}\right]$$
$$= \lim_{h\to 0}\frac{1}{h}\left[-2\sin\left\{\frac{(x+h)^2+x^2+2}{2}\right\}\sin\left\{\frac{x^2+h^2+2xh-x^2}{2}\right\}\right]$$
$$= \lim_{h\to 0}\frac{1}{h}\left[-2\sin\left\{\frac{(x+h)^2+x^2+2}{2}\right\}\sin\left\{\frac{h^2+2hx}{2}\right\}\right]$$
$$= -2\lim_{h\to 0}\sin\left\{\frac{(x+h)^2+x^2+2}{2}\right\}\lim_{h\to 0}\left\{\frac{\sin h\left(\dfrac{h+2x}{2}\right)}{h\left(\dfrac{h+2x}{2}\right)}\right\}\times\left(\frac{h+2x}{2}\right)$$
$$= -2\lim_{h\to 0}\sin\left\{\frac{(x+h)^2+x^2+2}{2}\right\}\lim_{h\to 0}\left(\frac{h+2x}{2}\right) \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$
$$= -2x\sin(x^2+1) \qquad [\because x\to 0 \Rightarrow kx\to 0]$$

Q. 44 $\dfrac{ax+b}{cx+d}$

Sol. Let

$$f(x) = \frac{ax + b}{cx + d}$$

$$f(x + h) = \frac{a\,(x + h) + b}{c\,(x + h) + d}$$

$\therefore$

$$\frac{d}{dx}f(x) = \lim_{h \to 0}\frac{1}{h}[f(x + h) - f(x)]$$

$$= \lim_{h \to 0}\frac{1}{h}\left[\frac{a\,(x + h) + b}{c\,(x + h) + d} - \frac{ax + b}{cx + d}\right]$$

$$= \lim_{h \to 0}\frac{1}{h}\left[\frac{ax + b + ah}{c(x + h) + d} - \frac{ax + b}{cx + d}\right]$$

$$= \lim_{h \to 0}\frac{1}{h}\left[\frac{(ax + ah + b)(cx + d) - (ax + b\{c(x + h) + d\}}{\{c(x + h) + d\}(cx + d)}\right]$$

$$= \lim_{h \to 0}\frac{1}{h}\left[\frac{(ax + ah + b)(cx + d) - (ax + b)(cx + ch + d)}{\{c(x + h) + d)\}(cx + d)}\right]$$

$$= \lim_{h \to 0}\frac{1}{h}\left[\frac{acx^2 + achx + bcx + adx + adh + bd - \{acx^2 + achx + adx + bcx + bch + bd\}}{\{c(x + h) + d\}(cx + d)}\right]$$

$$= \lim_{h \to 0}\frac{1}{h}\left[\frac{acx^2 + achx + bcx + adx + adh + bd - acx^2 - achx - adx - bcx - bch - bd\}}{\{c(x + h) + d\}(cx + d)}\right]$$

$$= \lim_{h \to 0}\frac{1}{h}\left[\frac{adh - bch}{\{c(x + h) + d\}(cx + d)}\right]$$

$$= \lim_{h \to 0}\frac{ac - bd}{\{c(x + h) + d\}(cx + d)}$$

$$= \frac{ac - bd}{(cx + d)^2}$$

Q. 45 $x^{2/3}$

Sol. Let

$$f(x) = x^{2/3}$$

$$f(x + h) = (x + h)^{2/3}$$

Now,

$$\frac{d}{dx}f(x) = \lim_{h \to 0}\frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0}\frac{1}{h}[(x + h)^{2/3} - x^{2/3}]$$

$$= \lim_{h \to 0}\frac{1}{h}\left[x^{2/3}\left(1 + \frac{h}{x}\right)^{2/3} - x^{2/3}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[x^{2/3}\left(1 + \frac{h}{x}\cdot\frac{2}{3} + \frac{2}{3}\left(\frac{2}{3} - 1\right)\frac{h^2}{x^2} + \cdots\right) - 1\right]$$

$$\left[\because (1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[x^{2/3}\left(\frac{2}{3}\cdot\frac{h}{x} - \frac{2}{9}\cdot\frac{h^2}{x^2} + \cdots\right)\right]$$

$$= \lim_{h \to 0} \frac{x^{2/3}}{h}\cdot\frac{2}{3}\frac{h}{x}\left(1 - \frac{1}{3}\cdot\frac{h}{x} + \cdots\right)$$

$$= \frac{2}{3}x^{2/3 - 1} = \frac{2}{3}x^{-1/3}$$

Alternate Method

Let $f(x) = x^{2/3}$

$f(x + h) = (x + h)^{2/3}$

$\therefore$ $\frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$$= \lim_{(h \to 0)}\left[\frac{(x + h)2/3 - x^{2/3}}{h}\right] = \lim_{(x+h) \to x}\left[\frac{(x + h)\,2/3 - x^{2/3}}{(x + h) - x}\right]$$

$$= \frac{2}{3}(x)^{2/3-1} \qquad \left[\because \lim_{x \to a}\frac{x^x - a^n}{x - a} = na^{n-1}\right]$$

$$= \frac{2}{3}x^{-1/3}$$

Q. 46 $x\cos x$

Sol. Let $f(x) = x\cos x$

$\therefore$ $f(x + h) = (x + h)\cos(x + h)$

$\therefore$ $\frac{d}{dx}f(x) = \lim_{h \to 0}\frac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0}\frac{1}{h}[(x + h)\cos(x + h) - x\cos x]$$

$$= \lim_{h \to 0}\frac{1}{h}[x\cos(x + h) + h\cos(x + h) - x\cos x]$$

$$= \lim_{h \to 0}\frac{1}{h}[x\{\cos(x + h) - \cos x\} + h\cos(x + h)]$$

$$= \lim_{h \to 0}\frac{1}{h}\left[x\left\{-2\sin\left(\frac{2x + h}{2}\right)\sin\frac{h}{2}\right\} + h\cos(x + h)\right]$$

$$= \lim_{h \to 0}\left[-2x\sin\left(x + \frac{h}{2}\right)\frac{\sin\frac{h}{2}}{h} + \cos(x + h)\right]$$

$$\left[\because \cos C - \cos D = -2\sin\frac{C + D}{2}\cdot\sin\frac{C - D}{2}\right]$$

$$= -2\lim_{h \to 0}x\sin\left(x + \frac{h}{2}\right)\lim_{h \to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\cdot\frac{1}{2} + \lim_{h \to 0}\cos(x + h)$$

$$= -2\cdot\frac{1}{2}x\sin x + \cos x$$

$$= \cos x - x\sin x$$

Evaluate each of the following limits in following questions

Q. 47 $\lim_{y \to 0} \frac{(x+y)\sec(x+y) - x\sec x}{y}$

Sol. Given , $\lim_{y \to 0} \frac{(x+y)\sec(x+y) - x\sec x}{y}$

$$= \lim_{y \to 0} \frac{\frac{x+y}{\cos(x+y)} - \frac{x}{\cos x}}{y}$$

$$= \lim_{y \to 0} \frac{(x+y)\cos x - x\cos(x+y)}{y\cos x\cos(x+y)}$$

$$= \lim_{y \to 0} \left[\frac{x\cos x + y\cos x - x\cos(x+y)}{y\cos x\cos(x+y)}\right]$$

$$= \lim_{y \to 0} \left[\frac{x\cos x - x\cos(x+y) + y\cos x}{y\cos x\cos(x+y)}\right]$$

$$= \lim_{y \to 0} \frac{x\{\cos x - \cos(x+y)\} + y\cos x}{y\cos x\cos(x+y)}$$

$$= \lim_{y \to 0} \frac{x\left[-2\sin\left(x + \frac{y}{2}\right)\sin\left(\frac{-y}{2}\right)\right] + y\cos x}{y\cos x\cos(x+y)}$$

$$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2}\cdot\sin\frac{C-D}{2}\right]$$

$$= \lim_{y \to 0} \left[\frac{x\left\{2\sin\left(x + \frac{y}{2}\right)\sin\frac{y}{2}\right\} + y\cos x}{y\cos x\cos(x+y)}\right]$$

$$= \lim_{y \to 0} \frac{2x\sin\left(x + \frac{y}{2}\right)}{\cos x\cos(x+y)} \cdot \lim_{y \to 0} \frac{\sin\frac{y}{2}}{\frac{y}{2}} \cdot \frac{1}{2} + \lim_{y \to 0} \sec(x+y)$$

$$\left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ and } x \to 0 \Rightarrow kx \to 0\right]$$

$$= \lim_{y \to 0} \frac{2x\sin\left(x + \frac{y}{2}\right)}{\cos x\cos(x+y)} \cdot \frac{1}{2} + \lim_{y \to 0} \sec(x+y)$$

$$= \frac{2x\sin x}{\cos x\cos x} \cdot \frac{1}{2} + \sec x$$

$$= x\tan x\sec x + \sec x$$

$$= \sec x\,(x\tan x + 1)$$

Q. 48 $\lim_{x \to 0} \dfrac{\sin(\alpha+\beta)x + \sin(\alpha-\beta)x + \sin 2\alpha x}{\cos 2\beta x - \cos 2\alpha x} \cdot x$

Sol. Given, $\lim_{x \to 0} \dfrac{[\sin(\alpha+\beta)x + \sin(\alpha-\beta)x + \sin 2\alpha x]}{\cos 2\beta x - \cos 2\alpha x} \cdot x$

$$= \lim_{x \to 0} \frac{[2\sin\alpha x \cdot \cos\beta x + \sin 2\alpha x]}{\cos 2\beta x - \cos 2\alpha x} \cdot x \quad \left[\because \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$$= \lim_{x \to 0} \frac{[2\sin\alpha x \cos\beta x + \sin 2\alpha x]x}{2\sin(\alpha+\beta)x \sin(\alpha-\beta)x} \quad \left[\because \cos C - \cos D = 2\sin\frac{C+D}{2} \cdot \sin\frac{D-C}{2}\right]$$

$$= \lim_{x \to 0} \frac{[2\sin\alpha x \cos\beta x + 2\sin\alpha x \cos\alpha x]x}{2\sin(\alpha+\beta)x \sin(\alpha-\beta)x}$$

$$= \lim_{x \to 0} \frac{2\sin\alpha x[\cos\beta x + \cos\alpha x]x}{2\sin(\alpha+\beta)x \sin(\alpha-\beta)x}$$

$$= \lim_{x \to 0} \frac{\sin\alpha x\left[2\cos\left(\frac{\alpha+\beta}{2}\right)x \cos\left(\frac{\alpha-\beta}{2}\right)x\right]x}{2\sin\left(\frac{\alpha+\beta}{2}\right)x \cos\left(\frac{\alpha+\beta}{2}\right)x \cdot 2\sin\left(\frac{\alpha-\beta}{2}\right)x \cos\left(\frac{\alpha-\beta}{2}\right)x}$$

$$\left[\because \cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2} \text{ and } \sin 2\theta = 2\sin\theta\cos\theta\right]$$

$$= \lim_{x \to 0} \frac{\sin\alpha x \cdot x}{2\sin\left(\frac{\alpha+\beta}{2}\right)x \sin\left(\frac{\alpha-\beta}{2}\right)x}$$

$$= \frac{1}{2}\lim_{x \to 0} \frac{\frac{\sin\alpha x}{\alpha x} \cdot x \cdot (\alpha x)}{2\dfrac{\sin\left(\frac{\alpha+\beta}{2}\right)x}{\left(\frac{\alpha+\beta}{2}\right)x} \cdot \dfrac{\sin\left(\frac{\alpha-\beta}{2}\right)x}{\left(\frac{\alpha-\beta}{2}\right)x} \cdot \left(\frac{\alpha+\beta}{2}\right)x \cdot \left(\frac{\alpha-\beta}{2}\right)x}$$

$$= \frac{\frac{1}{2}\lim_{x \to 0} \frac{\sin\alpha x}{\alpha x} \cdot \alpha x^2}{\lim_{x \to 0} \dfrac{\sin\left(\frac{\alpha+\beta}{2}\right)x}{\left(\frac{\alpha+\beta}{2}\right)x} \lim_{x \to 0} \dfrac{\sin\left(\frac{\alpha-\beta}{2}\right)x}{\left(\frac{\alpha-\beta}{2}\right)x} \cdot \left(\frac{\alpha^2-\beta^2}{4}\right)x^2}$$

$$\left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ and } x \to 0 \Rightarrow kx \to 0\right]$$

$$= \frac{1}{2} \cdot \frac{\alpha \cdot 4}{\alpha^2 - \beta^2}\left[\frac{\lim_{x \to 0} \frac{\sin\alpha x}{\alpha x}}{\lim_{x \to 0} \sin\dfrac{\left(\frac{\alpha+\beta}{2}\right)x}{\left(\frac{\alpha+\beta}{2}\right)x} \lim_{x \to 0} \sin\dfrac{\left(\frac{\alpha-\beta}{2}\right)x}{\left(\frac{\alpha-\beta}{2}\right)x}}\right]$$

$$= \frac{1}{2} \cdot \frac{4\alpha}{\alpha^2 - \beta^2} = \frac{2\alpha}{\alpha^2 - \beta^2}$$

Q. 49 $\lim_{x \to \pi/4} \dfrac{\tan^3 x - \tan x}{\cos\left(x + \dfrac{\pi}{4}\right)}$

Sol. Given, $\lim_{x \to \pi/4} \dfrac{\tan^3 x - \tan x}{\cos\left(x + \dfrac{\pi}{4}\right)}$ $\left[\dfrac{0}{0} \text{ form}\right]$

$$= \lim_{x \to \pi/4} \frac{\tan x\,(\tan^2 x - 1)}{\cos\left(x + \frac{\pi}{4}\right)} = \lim_{x \to \pi/4} \tan x \cdot \lim_{x \to \pi/4}\left(\frac{1 - \tan^2 x}{\cos\left(x + \frac{g\pi}{4}\right)}\right)$$

$$= -1 \times \lim_{x \to \pi/4} \frac{(1 + \tan x)(1 - \tan x)}{\cos\left(x + \frac{\pi}{4}\right)} \qquad [\because a^2 - b^2 = (a+b)(a-b)]$$

$$= -\lim_{x \to \pi/4}(1 + \tan x) \lim_{x \to \pi/4}\left[\frac{\cos x - \sin x}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)}\right]$$

$$= -(1+1) \times \lim_{x \to \pi/4} \frac{\sqrt{2}\left[\frac{1}{\sqrt{2}} \cdot \cos x - \frac{1}{\sqrt{2}} \cdot \sin x\right]}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)} = -2\sqrt{2} \lim_{x \to \pi/4}\left[\frac{\cos\frac{\pi}{4} \cdot \cos x - \sin\frac{\pi}{4} \cdot \sin x}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)}\right]$$

$$[\because \cos A \cdot \cos B - \sin A \sin B = \cos(A + B)]$$

$$= -2\sqrt{2} \lim_{x \to \pi/4} \frac{\cos\left(x + \frac{\pi}{4}\right)}{\cos x \cdot \cos\left(x + \frac{\pi}{4}\right)} = -2\sqrt{2} \times \frac{1}{\frac{1}{\sqrt{2}}} = -2\sqrt{2} \times \sqrt{2} = -4$$

Q. 50 $\lim_{x \to \pi} \dfrac{1 - \sin\dfrac{x}{2}}{\cos\dfrac{x}{2}\left(\cos\dfrac{x}{4} - \sin\dfrac{x}{4}\right)}$

Sol. Given, $\lim_{x \to \pi} \dfrac{1 - \sin\frac{x}{2}}{\cos\frac{x}{2}\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)}$

$$= \lim_{x \to \pi} \frac{\cos^2\frac{x}{4} + \sin^2\frac{x}{4} - 2 \cdot \sin\frac{x}{4} \cdot \cos\frac{x}{4}}{\cos\frac{x}{2}.(\cos\frac{x}{4} - \sin\frac{x}{4})} \qquad [\because \sin^2\theta + \cos^2\theta = 1 \sin 2\theta = 2\sin\theta\cos\theta]$$

$$= \lim_{x \to \pi} \frac{\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)^2}{\left(\cos^2\frac{x}{4} - \sin^2\frac{x}{4}\right)\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)} \qquad [\because \cos^2 2\theta = \cos^2\theta - \sin^2\theta]$$

$$= \lim_{x\to\pi} \frac{\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)}{\left(\cos\frac{x}{4} + \sin\frac{x}{4}\right)\left(\cos\frac{x}{4} - \sin\frac{x}{4}\right)} \qquad [\because a^2 - b^2 = (a+b)(a-b)]$$

$$\lim_{x\to\pi} \frac{1}{\cos\frac{x}{4} + \sin\frac{x}{4}} = \frac{1}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

Q. 51 Show that $\lim_{x \to \pi/4} \frac{|x-4|}{x-4}$ does not exist,

Sol. Given,

$$\lim_{x \to \pi/4} \frac{|x-4|}{x-4}$$

$$\text{LHL} = \lim_{x \to \frac{\pi^-}{4}} \frac{-(x-4)}{x-4} \qquad [\because |x-4| = -(x-4),\ x < 4]$$

$$= -1$$

$$\text{RHL} = \lim_{x \to \frac{\pi^+}{4}} \frac{(x-4)}{x-4} = 1 \qquad [\because |x-4| = (x-4),\ x > 4]$$

$\therefore$ $\qquad \text{LHL} \neq \text{RHL}$

So, limit does not exist.

Q. 52 If $f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & \text{when } x \neq \frac{\pi}{2} \\ 3, & \text{when } x = \frac{\pi}{2} \end{cases}$ and $\lim_{x \to \pi/2} f(x) = f\left(\frac{\pi}{2}\right)$, then find the value of k.

Sol. Given, $\qquad f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & x \neq \frac{\pi}{2} \\ 3, & x = \frac{\pi}{2} \end{cases}$

$$\therefore \qquad \text{LHL} = \lim_{x \to \frac{\pi^-}{2}} \frac{k\cos x}{\pi - 2x} = \lim_{h\to 0} \frac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)}$$

$$= \lim_{h\to 0} \frac{k\sin h}{\pi - \pi + 2h} = \lim_{h\to 0} \frac{k\sin h}{2h}$$

$$= \frac{k}{2} \lim_{h\to 0} \frac{\sin h}{h} = \frac{k}{2} \cdot 1 = \frac{k}{2} \qquad \left[\because \lim_{h\to 0} \frac{\sin x}{x} = 1\right]$$

$$\text{RHL} = \lim_{x \to \frac{\pi^+}{2}} \frac{k\cos x}{\pi - 2x} = \lim_{x \to \frac{\pi}{2}} + \frac{k\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)}$$

$$= \lim_{h\to 0} \frac{-k\sin h}{\pi - \pi - 2h} = \lim_{h\to 0} \frac{k\sin h}{2h} = \frac{k}{2} \lim_{h\to 0} \frac{\sin h}{2h} = \frac{k}{2} \text{ and } f\left(\frac{\pi}{2}\right) = 3$$

It is given that, $\qquad \lim_{x \to \pi/2} f(x) = f\left(\frac{\pi}{2}\right) \Rightarrow \frac{k}{2} = 3$

$\therefore$ $\qquad k = 6$

Q. 53 If $f(x) = \begin{cases} x+2, & x \le -1 \\ cx^2, & x > -1 \end{cases}$, then find c when $\lim_{x \to -1} f(x)$ exists.

Sol. Given,

$$f(x) = \begin{cases} x+2, & x \le -1 \\ cx^2, & x > -1 \end{cases}$$

$$\text{LHL} = \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (x+2)$$

$$= \lim_{h \to 0} (-1-h+2) = \lim_{h \to 0} (1-h) = 1$$

$$\text{RHL} = \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} cx^2 = \lim_{h \to 0} c(-1+h)^2$$

$\therefore$ $$= c$$

If $\lim_{x \to -1} f(x)$ exist, then LHL = RHL

$\therefore$ $$c = 1$$

Objective Type Questions

Q. 54 $\lim_{x \to \pi} \dfrac{\sin x}{x - \pi}$ is equal to

(a) 1 (b) 2

(c) –1 (d) –2

Sol. *(c)* Given, $\lim_{x \to \pi} \dfrac{\sin x}{x-\pi} = \lim_{x \to \pi} \dfrac{\sin(\pi - x)}{-(\pi - x)}$

$[\because \sin\theta = \sin(\pi - \theta)]$

$$= -\lim_{x \to \pi} \frac{\sin(\pi - x)}{(\pi - x)} = -1 \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ and } \pi - x \to 0 \Rightarrow x \to \pi\right]$$

Q. 55 $\lim_{x \to 0} \dfrac{x^2 \cos x}{1 - \cos x}$ is equal to

(a) 2 (b) $\dfrac{3}{2}$

(c) $\dfrac{-3}{2}$ (d) 1

Sol. *(a)* Given, $\lim_{x \to 0} \dfrac{x^2 \cos x}{1 - \cos x} = \lim_{x \to 0} \dfrac{x^2 \cos x}{2\sin^2 \dfrac{x}{2}}$ $\left[\because 1 - \cos x = 2\sin^2 \dfrac{x}{2}\right]$

$$= 2 \lim_{x \to 0} \frac{\left(\frac{x}{2}\right)^2}{\sin^2 \frac{x}{2}} \cdot \lim_{x \to 0} \cos x = 2 \cdot 1 = 2$$

Q. 56 $\lim_{x \to 0} \dfrac{(1+x)^n - 1}{x}$ is equal to

(a) n
(b) 1
(c) $-n$
(d) 0

Sol. ***(a)*** Given, $\lim_{x \to 0} \dfrac{(1+x)^n - 1}{x} = \lim_{x \to 0} \dfrac{(1+x)^n - 1}{(1+x)-1} = \lim_{x \to 0} \dfrac{(1+x)^n - 1}{(1+x)-1}$

$$= \lim_{x \to 0} \frac{(1+x)^n - 1^n}{(1+x)-1} = \lim_{(1+x) \to 1} \frac{(1+x)^n - 1^n}{(1+x)-1}$$

$$= n \cdot (1)^{n-1} = n \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

Q. 57 $\lim_{x \to 1} \dfrac{x^m - 1}{x^n - 1}$ is equal to

(a) 1
(b) $\dfrac{m}{n}$
(c) $-\dfrac{m}{n}$
(d) $\dfrac{m^2}{n^2}$

Sol. ***(b)*** Given,
$$\lim_{x \to 1} \frac{x^m - 1}{x^n - 1} = \lim_{x \to 1} \frac{\dfrac{x^m - 1}{x-1}}{\dfrac{x^n - 1}{x-1}} = \frac{\lim_{x \to 1} \dfrac{x^m - 1^m}{x-1}}{\lim_{x \to 1} \dfrac{x^n - 1^n}{x-1}}$$

$$= \frac{m\,(1)^{m-1}}{n\,(1)^{n-1}} = \frac{m}{n} \qquad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

Q. 58 $\lim_{\theta \to 0} \dfrac{1 - \cos 4\theta}{1 - \cos 6\theta}$ is equal to

(a) $\dfrac{4}{9}$
(b) $\dfrac{1}{2}$
(c) $\dfrac{-1}{2}$
(d) -1

Sol. ***(a)*** Given, $\lim_{\theta \to 0} \dfrac{1 - \cos 4\theta}{1 - \cos 6\theta} = \lim_{\theta \to 0} \dfrac{2\sin^2 2\theta}{2\sin^2 3\theta}$ $\qquad [\because 1 - \cos 2\theta = 2\sin^2 \theta]$

$$= \frac{\lim_{\theta \to 0} \dfrac{\sin^2 2\theta}{(2\theta)^2} \cdot (2\theta)^2}{\lim_{\theta \to 0} \dfrac{\sin^2 3\theta}{(3\theta)^2} \cdot (3\theta)^2} = \frac{4}{9} \cdot \frac{\lim_{\theta \to 0} \left(\dfrac{\sin 2\theta}{2\theta}\right)^2}{\lim_{\theta \to 0} \left(\dfrac{\sin 3\theta}{3\theta}\right)^2} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ and } x \to 0 \Rightarrow kx \to 0\right]$$

$$= \frac{4}{9}$$

Q. 59 $\lim_{x \to 0} \dfrac{\operatorname{cosec} x - \cot x}{x}$ is equal to

(a) $\dfrac{-1}{2}$ (b) 1 (c) $\dfrac{1}{2}$ (d) 1

Sol. ***(c)*** Given,

$$\lim_{x \to 0} \frac{\operatorname{cosec} x - \cot x}{x}$$

$$= \lim_{x \to 0} \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x} = \lim_{x \to 0} \frac{1 - \cos x}{x \cdot \sin x}$$

$$= \lim_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{x \cdot 2\sin\frac{x}{2}\cos\frac{x}{2}} = \lim_{x \to 0} \frac{\tan\frac{x}{2}}{x}$$

$$= \lim_{x \to 0} \frac{\tan\frac{x}{2}}{\frac{x}{2}} \cdot \frac{1}{2} = \frac{1}{2} \qquad \left[\because \lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1\right]$$

Q. 60 $\lim_{x \to 0} \dfrac{\sin x}{\sqrt{x+1} - \sqrt{1-x}}$ is equal to

(a) 2 (b) 0 (c) 1 (d) –1

Sol. ***(c)*** Given,

$$\lim_{x \to 0} \frac{\sin x}{\sqrt{x+1} - \sqrt{1-x}}$$

$$= \lim_{x \to 0} \frac{\sin x}{\sqrt{x+1} - \sqrt{1-x}} \cdot \frac{\sqrt{x+1} + \sqrt{1-x}}{\sqrt{x+1} + \sqrt{1-x}}$$

$$= \lim_{x \to 0} \frac{\sin x(\sqrt{x+1} + \sqrt{1-x})}{(x+1) - (1-x)}$$

$$= \lim_{x \to 0} \frac{\sin x(\sqrt{x+1} + \sqrt{1-x})}{x + 1 - 1 + x}$$

$$= \frac{1}{2} \lim_{x \to 0} \frac{\sin x}{x} \lim_{x \to 0} (\sqrt{x+1} + \sqrt{1-x})$$

$$= \frac{1}{2} \cdot 1 \cdot 2 = 1 \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

Q. 61 $\lim_{x \to \pi/4} \dfrac{\sec^2 x - 2}{\tan x - 1}$ is

(a) 3 (b) 1 (c) 0 (d) 2

Sol. ***(d)*** Given,

$$\lim_{x \to \pi/4} \frac{\sec^2 x - 2}{\tan x - 1}$$

$$= \lim_{x \to \pi/4} \frac{1 + \tan^2 x - 2}{\tan x - 1} = \lim_{x \to \pi/4} \frac{\tan^2 x - 1}{\tan x - 1}$$

$$= \lim_{x \to \pi/4} \frac{(\tan x + 1)(\tan x - 1)}{(\tan x - 1)} = \lim_{x \to \pi/4} (\tan x + 1)$$

$$= 2$$

Q. 62 $\lim_{x \to 1} \frac{(\sqrt{x}-1)(2x-3)}{2x^2+x-3}$ is equal to

(a) $\frac{1}{10}$ (b) $\frac{-1}{10}$ (c) 1 (d) None of these

Sol. ***(b)*** Given, $\lim_{x \to 1} \frac{(\sqrt{x}-1)(2x-3)}{2x^2+x-3} = \lim_{x \to 1} \frac{(\sqrt{x}-1)(2x-3)}{(2x+3)(x-1)}$

$$= \lim_{x \to 1} \frac{(\sqrt{x}-1)(2x-3)}{(2x+3)(\sqrt{x}-1)(\sqrt{x}+1)}$$

$$= \lim_{x \to 1} \frac{2x-3}{(2x+3)(\sqrt{x}+1)} = \frac{-1}{5 \times 2} = \frac{-1}{10}$$

Q. 63 If $f(x) = \begin{cases} \frac{\sin[x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$, where $[\cdot]$ denotes the greatest integer function, then $\lim_{x \to 0} f(x)$ is equal to

(a) 1 (b) 0 (c) -1 (d) Does not exist

Sol. ***(d)*** Given, $f(x) = \begin{cases} \frac{\sin[x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$

$\therefore$ $\text{LHL} = \lim_{x \to 0^-} f(x)$

$$= \lim_{x \to 0^-} \frac{\sin[x]}{[x]} = \lim_{h \to 0} \frac{\sin[0-h]}{[0-h]}$$

$$= \lim_{h \to 0} \frac{-\sin[-h]}{[-h]} = -1$$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\sin[x]}{[x]}$$

$$= \lim_{x \to 0^+} \frac{\sin[0+h]}{[0+h]} = \lim_{h \to 0} \frac{\sin[h]}{[h]} = 1$$

$\because$ LHL $\neq$ RHL

So, limit does not exist.

Q. 64 $\lim_{x \to 0} \frac{|\sin x|}{x}$ is equal to

(a) 1 (b) $= -1$

(c) Does not exist (d) None of these

Sol. ***(c)*** Given, $\text{limit} = \lim_{x \to 0} \frac{|\sin x|}{x}$

$\therefore$ $\text{LHL} = \lim_{x \to 0^-} \left(\frac{-\sin x}{x}\right) = -\lim_{x \to 0^-} \frac{\sin x}{x} = -1$

$$\text{RHL} = \lim_{x \to 0^+} \frac{\sin x}{x} = 1$$

$\because$ LHL $\neq$ RHL

So , limit does not exist.

Q. 65 If $f(x)=\begin{cases} x^2-1, & 0<x<2 \\ 2x+3, & 2\le x<3 \end{cases}$, then the quadratic equation whose roots are $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$ is

(a) $x^2-6x+9=0$
(b) $x^2-7x+8=0$
(c) $x^2-14x+49=0$
(d) $x^2-10x+21=0$

Sol. ***(d)*** Given, $f(x)=\begin{cases} x^2-1, & 0<x<2 \\ 2x+3, & 2\le x<3 \end{cases}$

$$\therefore \quad \lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(x^2-1)$$

$$=\lim_{h\to 0}[(2-h)^2-1]=\lim_{h\to 0}(4+h^2-4h-1)$$

$$=\lim_{h\to 0}(h^2-4h+3)=3$$

and
$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(2x+3)$$

$$=\lim_{h\to 0}[2(2+h)+3]=\lim_{h\to 0}(4+2h+3)=7$$

So, the quadratic equation whose roots are 3 and 7 is $x^2-(3+7)x+3\times 7=0$ *i.e.*, $x^2-10x+21=0$.

Q. 66 $\lim_{x\to 0}\dfrac{\tan 2x-x}{3x-\sin x}$ is equal to

(a) 2
(b) $\frac{1}{2}$
(c) $\frac{-1}{2}$
(d) $\frac{1}{4}$

Sol. ***(b)*** Given,

$$\lim_{x\to 0}\frac{\tan 2x-x}{3x-\sin x}=\lim_{x\to 0}\frac{x\left[\dfrac{\tan 2x}{x}-1\right]}{x\left[3-\dfrac{\sin x}{x}\right]}$$

$$=\frac{\lim_{x\to 0} 2\times\dfrac{\tan 2x}{2x}-1}{3-\lim_{x\to 0}\dfrac{\sin x}{x}}=\frac{2-1}{3-1}=\frac{1}{2}$$

Q. 67 If $f(x)=x-[x]$, $\in R$, then $f'\left(\frac{1}{2}\right)$ is equal to

(a) $\frac{3}{2}$
(b) 1
(c) 0
(d) -1

Sol. ***(b)*** Given, $f(x)=x-[x]$

Now, first we have to check the differentiability of $f(x)$ at $x=\frac{1}{2}$.

$$\therefore \quad Lf'\left(\frac{1}{2}\right)=\text{LHD}=\lim_{h\to 0}\frac{f\left(\frac{1}{2}-h\right)-f\left(\frac{1}{2}\right)}{-h}$$

$$= \lim_{n \to 0} \frac{\left(\frac{1}{2} - h\right) - \left(\frac{1}{2} - h\right) - \frac{1}{2} + \left(\frac{1}{2}\right)}{h} = \lim_{h \to 0} \frac{\frac{1}{21} - h - 0 - \frac{1}{2} + 0 = 1}{h}$$

and $$Rf'\left(\frac{1}{2}\right) = \text{RHD } \lim_{h \to 0} \frac{f\left(\frac{1}{2} + h\right) - f\left(\frac{1}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{1}{2} + h\right) - \left(\frac{1}{2} + h\right) - \frac{1}{2} + \left(\frac{1}{2}\right)}{h} = \lim_{h \to 0} \frac{\frac{1}{2} + h - 0 - \frac{1}{2} + 0}{h} = 1$$

$\because$ LHL = RHD

$\therefore$ $f'\left(\frac{1}{2}\right) = 1$

Q. 68 If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, then $\frac{dy}{dx}$ at $x = 1$ is equal to

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) 0

Sol. *(d)* Given, $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

Now, $$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$$

$\therefore$ $$\left(\frac{dy}{dx}\right)_{x=1} = \frac{1}{2} - \frac{1}{2} = 0$$

Q. 69 If $f(x) = \frac{x - 4}{2\sqrt{x}}$, then $f'(1)$ is equal to

(a) $\frac{5}{4}$ (b) $\frac{4}{5}$ (c) 1 (d) 0

Sol. *(a)* Given, $f(x) = \frac{x - 4}{2\sqrt{x}}$

Now, $$f'(x) = \frac{2\sqrt{x} - (x - 4) \cdot 2 \cdot \frac{1}{2\sqrt{x}}}{4x}$$

$$= \frac{2x - (x - 4)}{4x^{3/2}} = \frac{2x - x + 4}{4x^{3/2}}$$

$$= \frac{x + 4}{4x^{3/2}}$$

$\therefore$ $$f'(1) = \frac{1 + 4}{4 \times (1)^{3/2}} = \frac{5}{4}$$

Q. 70 If $y = \frac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{-4x}{(x^2 - 1)^2}$ (b) $\frac{-4x}{x^2 - 1}$

(c) $\frac{1 - x^2}{4x}$ (d) $\frac{4x}{x^2 - 1}$

Sol. ***(a)*** Given,

$$y = \frac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}} \Rightarrow y = \frac{x^2 + 1}{x^2 - 1}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x^2 - 1)\,2x - (x^2 + 1)(2x)}{(x^2 - 1)^2} \qquad \text{[by quotient rule]}$$

$$\frac{dy}{dx} = \frac{2x(x^2 - 1 - x^2 - 1)}{(x^2 - 1)^2}$$

$$= \frac{2x\,(-2)}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

Q. 71 If $y = \dfrac{\sin x + \cos x}{\sin x - \cos x}$, then $\dfrac{dy}{dx}$ at $x = 0$ is equal to

(a) -2 (b) 0 (c) $\frac{1}{2}$ (d) Does not exist

Sol. ***(a)*** Given, $y = \dfrac{\sin x + \cos x}{\sin x - \cos x}$

$$\therefore \quad \frac{dy}{dx} = \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2} \qquad \text{[by quotient rule]}$$

$$= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2}$$

$$= \frac{-[(\sin x - \cos x)^2 + (\sin x + \cos x)^2]}{(\sin x - \cos x)^2}$$

$$= \frac{-[\sin^2 x + \cos^2 x - 2\sin x\cos x + \sin^2 x + \cos^2 x + 2\sin x\cos x]}{(\sin x - \cos x)^2}$$

$$= \frac{-2}{(\sin x - \cos x)^2}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=0} = -2$$

Q. 72 If $y = \dfrac{\sin(x + 9)}{\cos x}$, then $\dfrac{dy}{dx}$ at $x = 0$ is equal to

(a) $\cos 9$ (b) $\sin 9$ (c) 0 (d) 1

Sol. ***(a)*** Given,

$$y = \frac{\sin(x + 9)}{\cos x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos x\cos(x + 9) - \sin(x + 9)\,(-\sin x)}{(\cos x)^2} \qquad \text{[by quotient rule]}$$

$$= \frac{\cos x\cos(x + 9) + \sin x\,\sin(x + 9)}{\cos^2 x}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=0} = \frac{\cos 9}{1}$$

$$= \cos 9$$

Q. 73 If $f(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^{100}}{100}$, then $f'(1)$ is equal to

(a) $\frac{1}{100}$ (b) 100

(c) 0 (d) Does not exist

Sol. ***(b)*** Given,
$$f(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^{100}}{100}$$

$\therefore$
$$f'(x) = 0 + 1 + 2 \times \frac{x}{2} + \ldots + 100\frac{x^{99}}{100}$$
$$f'(x) = 1 + x + x^2 + \ldots + x^{99}$$

Now,
$$f'(1) = 1 + 1 + 1 + \ldots + 1 \text{ (100 times)}$$
$$= 100$$

Q. 74 If $f(x) = \frac{x^n - a^n}{x - a}$ for some constant a, then $f'(a)$ is equal to

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) Does not exist

Sol. ***(d)*** Given,
$$f(x) = \frac{x^n - a^n}{x - a}$$

$\therefore$
$$f'(x) = \frac{(x - a)nx^{n-1} - (x^n - a^n)(1)}{(x - a)^2}$$ [by quotient rule]

$\Rightarrow$
$$f'(x) = \frac{nx^{n-1}(x - a) - x^n + a^n}{(x - a)^2}$$

Now,
$$f'(a) = \frac{na^{n-1}(0) - a^n + a^n}{(x - a)^2}$$

$\Rightarrow$
$$f'(a) = \frac{0}{0}$$

So, $f'(a)$ does not exist,
Since, $f(x)$ is not defined at $x = a$.
Hence, $f'(x)$ at $x = a$ does not exist.

Q. 75 If $f(x) = x^{100} + x^{99} + \ldots + x + 1$, then $f'(1)$ is equal to

(a) 5050 (b) 5049

(c) 5051 (d) 50051

Sol. ***(a)*** Given,
$$f(x) = x^{100} + x^{99} + \ldots + x + 1$$

$\therefore$
$$f'(x) = 100x^{99} + 99x^{98} + \ldots + 1 + 0$$
$$= 100x^{99} + 99x^{98} + \ldots + 1$$

Now,
$$f'(1) = 100 + 99 + \ldots + 1$$
$$= \frac{100}{2}[2 \times 100 + (100 - 1)(-1)]$$
$$= 50[200 - 99] \qquad \left[\because S_n = \frac{n}{2}\{2a + (n-1)d\}\right]$$
$$= 50 \times 101$$
$$= 5050$$

Q. 76 If $f(x) = 1 - x + x^2 - x^3 + \ldots - x^{99} + x^{100}$, then $f'(1)$ is equal to

(a) 150 (b) −50 (c) −150 (d) 50

Sol. *(d)* Given, $f(x) = 1 - x + x^2 - x^3 + \ldots - x^{99} + x^{100}$

$$f'(x) = 0 - 1 + 2x - 3x^2 + \ldots - 99x^{98} + 100x^{99}$$
$$= -1 + 2x - 3x^2 + \ldots - 99x^{98} + 100x^{99}$$

$\therefore$ $f'(1) = -1 + 2 - 3 + \ldots - 99 + 100$

$$= (-1 - 3 - 5 - \ldots - 99) + (2 + 4 + \ldots + 100) \quad \left[\because S_n = \frac{n}{2}\{2a + (n-1)d\}\right]$$
$$= -\frac{50}{2}[2 \times 1 + (50 - 1)2] + \frac{50}{2}[2 \times 2 + (50 - 1)2]$$
$$= -25[2 + 49 \times 2] + 25[4 + 49 \times 2]$$
$$= -25(2 + 98) + 25(4 + 98)$$
$$= -25 \times 100 + 25 \times 102$$
$$= -2500 + 2550$$
$$= 50$$

Fillers

Q. 77 If $f(x) = \dfrac{\tan x}{x - \pi}$, then $\lim\limits_{x \to \pi} f(x) =$ ....

Sol. Given, $f(x) = \dfrac{\tan x}{x - \pi} = \lim\limits_{x \to \pi} \dfrac{\tan x}{x - \pi} = \lim\limits_{\pi - x \to 0} \dfrac{-\tan(\pi - x)}{-(\pi - x)}$ $\left[\because \lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1\right]$

$$= 1$$

Q. 78 $\lim\limits_{x \to 0} \left(\sin mx \cot \dfrac{x}{\sqrt{3}}\right) = 2$, then $m =$ ......

Sol. Given, $\lim\limits_{x \to 0} \left(\sin mx \cot \dfrac{x}{\sqrt{3}}\right) = 2$

$\Rightarrow$ $$\lim_{x \to 0} \frac{\sin mx}{mx} \cdot mx \cdot \frac{1}{\tan \frac{x}{\sqrt{3}}} = 2$$

$\Rightarrow$ $$\lim_{x \to 0} \frac{\sin mx}{mx} \cdot mx \cdot \frac{\frac{x}{\sqrt{3}}}{\tan \frac{x}{\sqrt{3}}} \cdot \frac{1}{\frac{x}{\sqrt{3}}} = 2$$

$\Rightarrow$ $$\lim_{x \to 0} \frac{\sin mx}{mx} \cdot \lim_{x \to 0} \frac{\frac{x}{\sqrt{3}}}{\tan \frac{x}{\sqrt{3}}} \cdot \lim_{x \to 0} \frac{mx}{\frac{x}{\sqrt{3}}} = 2$$

$\Rightarrow$ $$\sqrt{3}x = 2$$

$\therefore$ $$m = \frac{2\sqrt{3}}{3}$$

Q. 79 If $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$, then $\frac{dy}{dx} = \dots\dots$

Sol. Given,

$$y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\therefore \quad \frac{dy}{dx} = 0 + 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{4!}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= y$$

Q. 80 $\lim_{x \to 3^+} \frac{x}{[x]} = \dots$

Sol. Given,

$$\lim_{x \to 3^+} \frac{x}{[x]} = \lim_{h \to 0} \frac{(3+h)}{[3+h]}$$

$$= \lim_{h \to 0} \frac{(3+h)}{3} = 1$$