CHAPTER 05

Virtual Work

5.1 Concept of Virtual Work

The net work done by applied forces on a body is zero if the body is in static equilibrium, if we assume that the body in equilibrium undergoes an infinitely small imaginary displacement (virtual displacement) some work will be imagined to be done by the applied forces and inertial forces of the body. Such an imaginary work is called virtual work.

If some unknown forces which are acting on the body and keep the body in equilibrium condition, are to be calculated, then the method of virtual work can be used.

Concept of virtual work is useful in finding some unknown forces which are acting on the body and keep the body in static equilibrium condition.

5.2 Principle of Virtual Work

If a system of forces acting on a body or a system of bodies be in equilibrium and the system be imagined to undergo a small displacement consistent with the geometrical conditions, then the algebraic sum of the virtual work done by the forces of the system is zero.

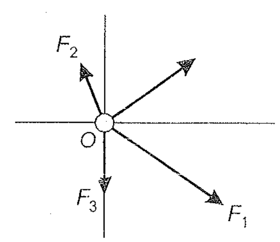


Fig. 5.1

Consider a plane concurrent system of forces F_1, F_2, F_3 ...F_n acting on a particle at O as shown in fig. 5.1. Let the resultant of forces be R. If system is in equilibrium then

$$R = F_1 + F_2 + F_3 .....F_N = 0$$

Now we impart an infinitely small virtual displacement δr to the particle at O in any arbitrary direction. The work done by the forces F_i through the virtual displacement is

$$\delta U = \delta r \cdot F_1 + \delta r \cdot F_2 + \delta r \cdot F_3 ...... \delta r \cdot F_n$$

$$\delta U = \delta r \cdot (F_1 + F_2 + F_3 ...... F_n)$$

$$= \delta r \cdot R$$

Now if $R = 0$ then $\delta U = 0$. Thus virtual work done by the forces acting on the particle through any virtual displacement is zero.

Example 5.1 A simple supported beam at the ends, 5 m span carries a load of 15 kN at a distance of 2 m from one end. Determine the end reaction using the principle of virtual work.

Solution:

As per problem statement the configuration is shown in fig.(a). Assume the virtual displacement given in vertical upper direction at point B is y, then from geometry of fig.(b) displacement y_c at C is

$$y_C = \frac{AC}{AB}y = \frac{2}{5}y.$$

and displacement y_A at A is

$$y_A = 0$$

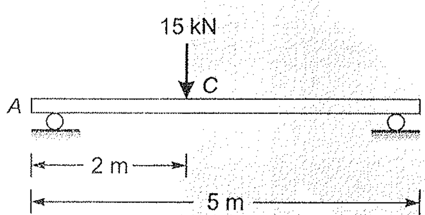


Fig. (a)

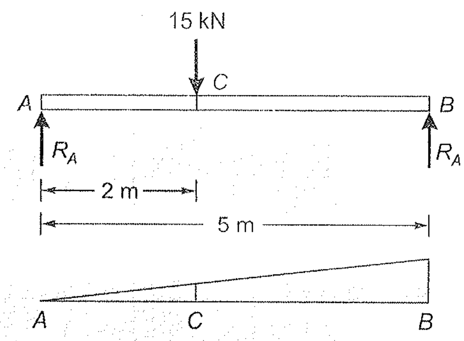


Fig. (b)

Here beam is in equilibrium under following forces as shown in fig.(b)

(1) 15 kN acting at C, (2) Reaction force R_A acting at A, (3) Reaction force R_B acting at B.

Total work done by these force due to virtual work must be zero.

Thus

$$0 \times R_A - y_C \times 15 + y \times R_B = 0$$

$$0 \times R_A - \frac{2}{5}y \times 15 + y \times R_B = 0$$

or

$$-6 + R_B = 0$$

or

$$R_B = 6 \text{ kN}$$

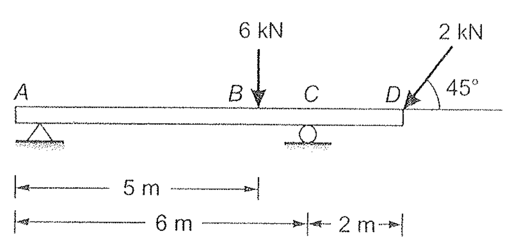
The work done by R_A is zero because there is no displacement.

Now resolving forces vertically we get

$$R_A + R_B = 15 \text{ kN}$$

$$R_A = 15 - R_B = 15 - 6 = 9 \text{ kN}$$

Example 5.2 A beam AD of 8 m is hinged at A and simply supported at C. Beam is loaded as shown in fig. Determine the reaction at A and C.



Solution:

Here beam is hinged at A and simple supported at C. Thus at hinged point A reaction force acts in both directions i.e. in horizontal and vertical direction. At C only vertical force will act as shown in fig. (a). By giving virtual displacement at D we can determine R_C. To determine R_{Ay} we have to give virtual displacement at A also and to determine R_{Az} we have to give horizontally virtual displacement also.

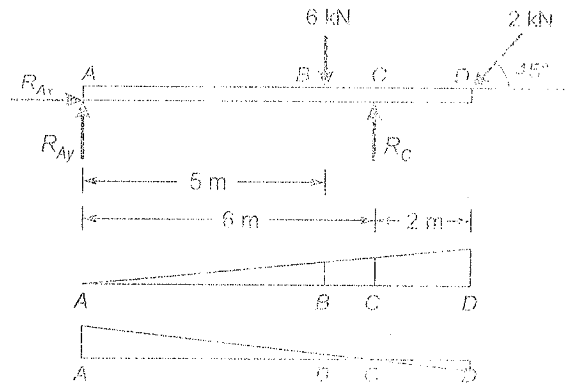


Fig. (a)

Assume beam is given a virtual displacement y in vertical upward direction at point D as shown in fig. (a). From geometry of fig. (a) the virtual displacement in vertical direction at point B and C are

$$y_B = \frac{AB}{AD}y = \frac{5}{8}y$$

$$y_C = \frac{AC}{AD}y = \frac{6}{8}y$$

Since algebraic sum of virtual work done by all forces must be zero

Therefore

$$-6 \times y_B + R_C \times y_C - 2 \times \sin 45° \, y = 0$$

or $$-6 \times \frac{5}{8}y + R_C \times \frac{6}{8}y - \frac{2}{\sqrt{2}}y = 0$$

or $$-30 + 6R_C - 8\sqrt{2} = 0$$

or $$R_C = \frac{30 + 8\sqrt{2}}{6} = 6.89 \text{ kN}$$

The work done by other forces is zero because of no displacement.

Now virtual displacement y is given in vertical direction at point A as shown in fig (a). Virtual displacement at point B, C and D are

$$y_B = \frac{BC}{AC}y = \frac{1}{6}y$$

$$y_D = \frac{CD}{AC}y = \frac{2}{6}y$$

Displacement at C is zero. Algebraic sum of virtual work must be zero. Therefore

or $$R_{Ay} \times y - 6 \times y_B - 2 \times \sin 45° \times y_D = 0$$

or $$R_{Ay} \times y - 6 \times \frac{1}{6}y + 2 \cdot \frac{1}{\sqrt{2}} \times \frac{2}{6}y = 0$$

or $$R_{Ay} - 1 + \frac{\sqrt{2}}{3} = 0$$

or $$R_{Ay} = 1 - \frac{\sqrt{2}}{3} = 0.53 \text{ kN}$$

It may be easily seen that horizontal component R_{Ax} =2 cos 45° but may be calculated from principal of virtual work also. We have to give horizontal displacement x at A. There will be virtual displacement at D also. At both point some work is done by virtual work. Therefore

$$R_{Ax} \times x - 2 \cos 45° \times x = 0$$

or $$R_{Ax} = 2\cos 45° = \frac{2}{\sqrt{2}} = 1.414 \text{ kN}$$

The work done by other forces is zero because of no force in the direction of displacement.

Now $$R_A = \sqrt{R_{Ax}^2 + R_{Ay}^2} = \sqrt{1.414^2 + 0.53^2} = 1.51$$

$$\theta = \tan^{-1}\frac{R_{Ax}}{R_{Ay}} = \tan^{-1}\frac{0.53}{1.414} = 20.55°$$

Example 5.3 A uniform ladder of 500 N weight rests on the smooth vertical wall and rough horizontal ground making an angle 45° with the horizontal ground. Find the frictional force of the ground using principle of virtual work.

Solution:

As per problem statement configuration is shown in fig. (a) and all force acting on ladder is shown in fig. (b). Here *AB* is ladder and its weight *W* is acting at the center of ladder. R_A and R_B are reaction at *A* and *B* and R_F is the fictional force acting at point at *A*.

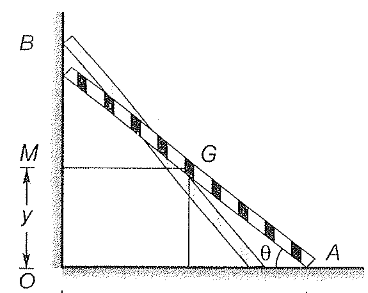


Fig. (a)

Let *L* be length of ladder. Then from geometry of fig. (a).

$$AB = L$$

$$OM = y = \frac{OB}{2} = \frac{L\sin\theta}{2} \quad ...(1)$$

$$AG = \frac{L}{2}$$

$$OA = x = L\cos\theta \quad ...(2)$$

Differentiating equation (1) and (2) we get

$$\frac{dy}{d\theta} = \frac{L}{2}\cos\theta \text{ or } dy = \frac{L}{2}\cos\theta d\theta$$

$$\frac{dx}{d\theta} = -L\sin\theta \text{ or } dx = -L\sin\theta\, d\theta$$

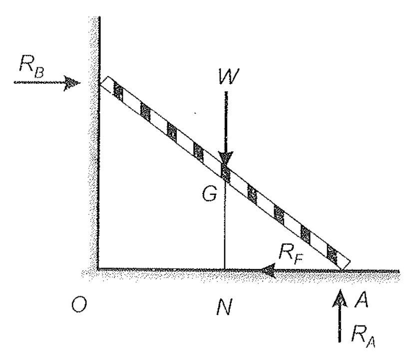


Fig. (b)

From figure it may be easily seen that if point *A* move towards −*x* axis, the point *B* and *G* moves towards +*y* axis and the point G moves towards both −*x* axis and +*y* axis. When point *A* and *B* moves then θ change from θ to θ + *d*θ. Assume the point *A* moves along −*x* axis direction through a distance of *dx*, then *G* also moves by *dx* and *dy* where

$$dx = -L\sin\theta\, d\theta$$

$$dy = \frac{1}{2}\cos\theta d\theta$$

The virtual work is done by weight of ladder $W \times dy$ and by friction force is $R_F \times dx$. As per principal of virtual work, algebraic sum of total work must be zero. Therefore

$$W \times dy + R_F \times dx = 0$$

or $$W \times \left(\frac{1}{2}\cos\theta d\theta\right) + R_F(-\sin\theta d\theta) = 0$$

or $$\frac{W}{2} \times \cos\theta - R_F\sin\theta = 0$$

or $$R_F = \frac{W}{2\tan\theta}$$

Here $$\theta = 45° \text{ and } W = 500 \text{ N}$$

Thus $$R_F = \frac{500}{2\tan 45°} = 250 \text{ N}$$

The work done by other forces is zero because of no displacement.

Example 5.4 A ladder weighing 200 N rests on a smooth floor A and against a smooth wall at B as shown in fig. A horizontal rope MG prevent the ladder from slipping. Determine the tension in the rope using method of virtual work.

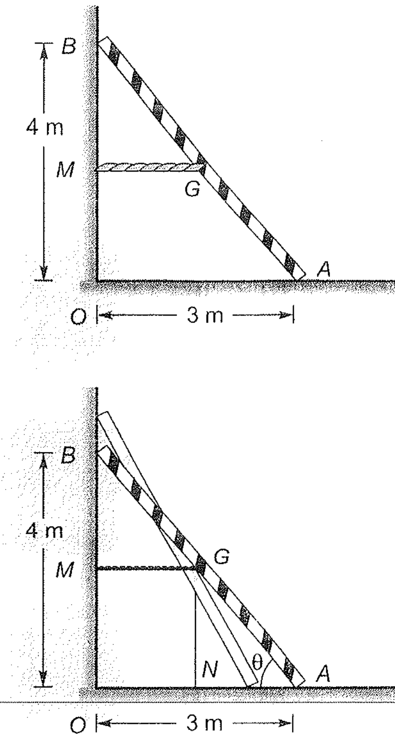


Fig. (a)

Solution:

Let L be the length of ladder. From geometry of fig. (a) we get

$$MG = x = \frac{L}{2}\cos\theta$$

$$GN = y = \frac{L}{2}\sin\theta$$

or we have $$dx = -\frac{L}{2}\sin\theta d\theta$$

$$dy = \frac{L}{2}\cos\theta d\theta$$

The ladder is equilibrium under following forces (fig.(b))

(1) Reaction R_A at point A, (2) Weight W of ladder at G, (3) Tension T in rope in MG direction, (4) Reaction R_B at point B.

When point A moves towards $-x$, the point M moves towards $-x$ and towards $+y$ also. Point B moves towards $+y$. Assume a virtual displacement dx is given in $-x$ direction at point A.

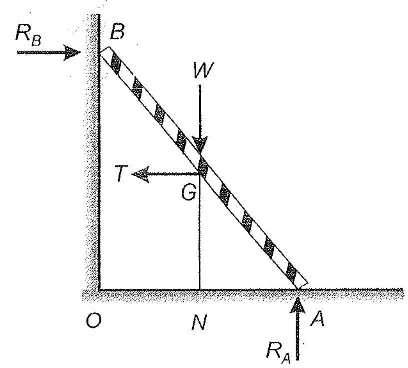


Fig. (b)

$$\text{Virtual work done by } W \text{ is} = W \times dy$$

$$\text{Virtual work done by } T \text{ is} = T \times dx$$

Summation of virtual work done must be zero. Then

$$Wdy + Tdx = 0$$

Now, putting the value of dy and dx,

$$W\left(\frac{L}{2}\cos\theta d\theta\right) + T\left(-\frac{L}{2}\sin\theta d\theta\right) = 0$$

or $$W\cos\theta - T\sin\theta = 0$$

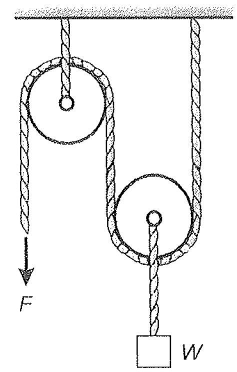
$$T = \frac{W}{\tan\theta}$$

In problem statement $$W = 250 \text{ N}$$

$$\tan\theta = \frac{OB}{OA} = \frac{4}{3}$$

$$T = \frac{200}{4/3} = 150 \text{ N}$$

Example 5.5 A weight of 10 kN is raised by two pulley system as shown in fig. Determine the force F required to hold the weight in equilibrium.



Solution:

Assume that F goes down through a distance y. From the geometry of fig. it may be easily seen that weight moves upward by 1/2 y distance.

Using the principal of virtual work

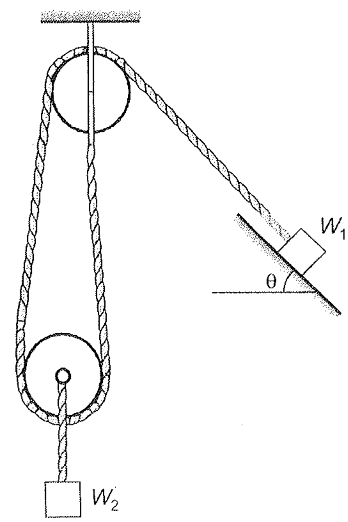
$$F \times y - W \times \frac{y}{2} = 0$$

or $$F = \frac{W}{2}$$

Here $$W = 10 \text{ kN}$$

Thus $$F = \frac{10}{2} = 5 \text{ kN}$$

Example 5.6 A block of W_1 = 6 kN rests on the smooth surface inclined at θ = 30° with the horizontal. The block is supported by an weight W hung from a pulley as shown in fig. Using the principle of virtual work determine the required W_2 for equilibrium condition.



Solution:

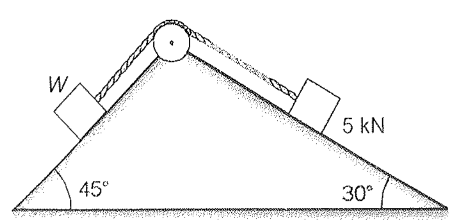
Assume that a virtual displacement y is given to block W_1 in the direction of inclined plane. From figure it may be easily seen that displacement of weight W_2 is y/2 in upward direction. The displacement of block W_1, in vertical direction is y sin θ in downward. Thus by principle of virtual work.

$$-W_2 \times \frac{y}{2} + W_1 \times y \sin\theta = 0$$

or $$W_2 = 2\, W_1 \sin\theta$$

or $$W_2 = 2 \times 6 \times \sin 30° = 6 \text{ kN}$$

Example 5.7 A weight of 5 kN resting on a smooth surface inclined 30° to the horizontal is supported by a load W, resting on the another smooth surface inclined to horizontal by 45° as shown in fig. Both weight are connected with a string carried over a smooth pulley. Using the principle of virtual work determine the value of W.



Solution:

Consider the fig.(a). Assume that W_1 is pulled by y along the inclined plane AC, then W_2 moves by y along BC.

Displacement of W_1 in vertical direction = $y \sin \alpha$ upward

Displacement of W_2 in vertical direction = $y \sin \beta$ downward

Thus by principle of virtual work

$$-W_1 \times y \sin \alpha + W_2 \times y \sin \alpha = 0$$

or $$\frac{W_1}{W_2} = \frac{\sin\beta}{\sin\alpha}$$

Here $$W_2 = 5 \text{ kN} \quad \text{and} \quad W_1 = W$$

$$\alpha = 45° \quad \text{and} \quad \beta = 30°$$

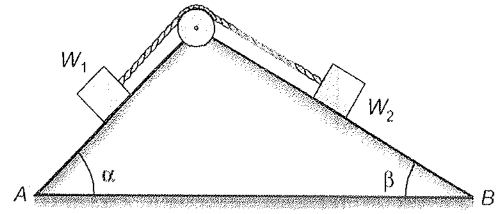
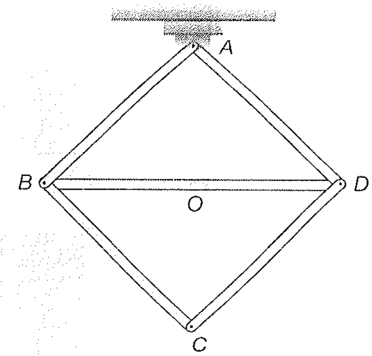


Fig. (a)

Thus $$\frac{W}{5} = \frac{\sin 30°}{\sin 45°}$$

or $$W = 5 \times \frac{1}{2} \times \frac{\sqrt{2}}{1} = \frac{5}{\sqrt{2}} = 3.53 \text{ kN}$$

Example 5.8 A square frame *ABCDA* is made up of 4 equal bar of weight 50 N/m. The frame is stiffened by fixing a rod *BD* of same metal with same diameter. The frame is suspended at point *A* as shown in fig. If length of each of square rod is 50 cm, find the tension *T* in the rod *BD* using principal of virtual work.



Solution:

Let *W* be the weight of each bar of square frame. Then $\sqrt{2}W$ is weight of rod *BD* because $BD = \sqrt{2}AB$.

Let *T* be thrust in rod *BD*. The effect of rod *BD* can be replaced by introducing *T* horizontally at *B* and *D* and a load of $\frac{\sqrt{2}W}{2}$ vertically at *B* and *D* as shown in fig.(a). The weight of each rod is acting at *CG*. Now assume a virtual displacement *dx* and *dy* to C.G of *AB* due to its weight. From geometry of figure it may be easily seen that

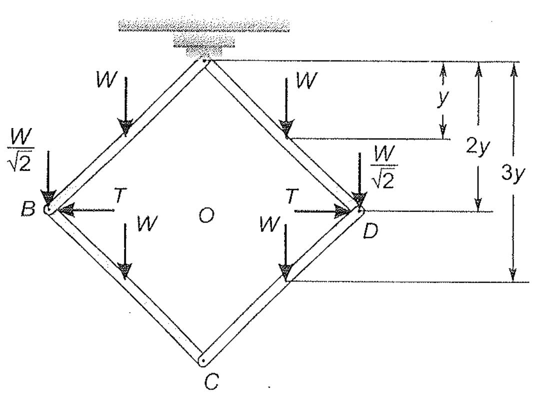


Fig. (a)

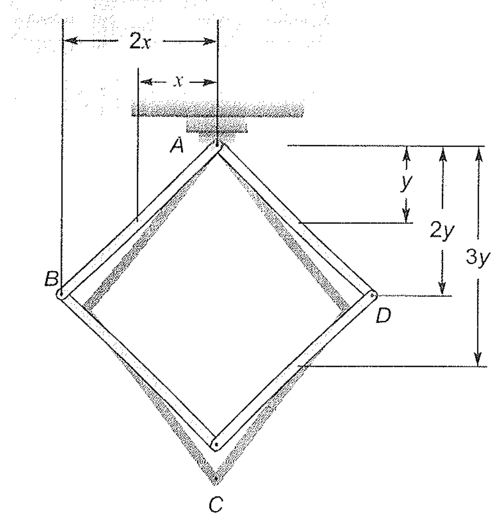


Fig. (b)

Then $2dy$ = Vertical displacement of weight $\frac{W}{\sqrt{2}}$ at point *B* and *D*.

$3dy$ = Vertical displacement of C.G of rod BC and DC.

$2dx$ = Horizontal displacement of point B and D with respect to point O.

From fig.(a) it may be noted that

$$OA = OB \text{ or } 2x = 2y \text{ or } 2dx = 2dy$$

Thus by principle of virtual work

$$2\times W\times dy + 2\times \frac{W}{\sqrt{2}}\times 2dy + 2\times W\times 3dy - 2T\times 2dx = 0$$

or
$$W + \sqrt{2}W + 3W - 2T = 0$$

or
$$T = W\left(\frac{4+\sqrt{2}}{2}\right) = W\left(2+\frac{1}{\sqrt{2}}\right)$$

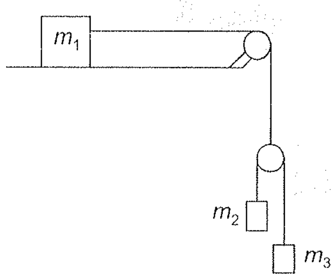
Here the length of each rod is 50 cm or 0.5 m and density is 50 N/m.

Thus
$$W = 0.5\times 50 = 25 \text{ N}$$

$$T = 25\times\left(2+\frac{1}{\sqrt{2}}\right) = 67.68 \text{ N}$$

Objective Brain Teasers

Q.1 Three blocks of masses m_1, m_2 and m_3 are connected as shown in the figure. All surfaces are frictionless and the string and the pulleys are light. Then the acceleration of m_1 is



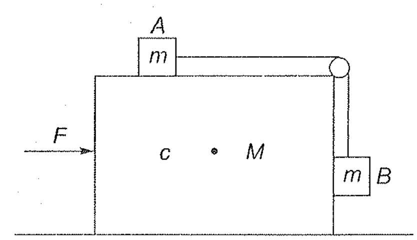
(a) $\dfrac{g}{1+\dfrac{m_1}{4}\left(\dfrac{1}{m_2}+\dfrac{1}{m_3}\right)}$

(b) $\dfrac{g}{1+\dfrac{m_1}{2(m_2+m_3)}}$

(c) $\dfrac{g}{1+\dfrac{m_1}{m_3-m_2}}$

(d) $\dfrac{g}{m_1}(m_2+m_3)$

Q.2 Consider the situation as shown in the figure. The horizontal surface below the bigger block is smooth. The coefficient of friction between the blocks is μ. Find the minimum force F that can be applied in order to keep the smaller blocks at rest with respect to bigger blocks.



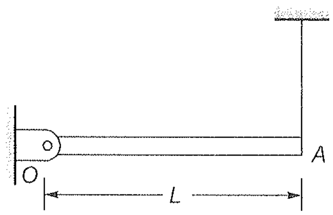
(a) $(M+2m)\left(\dfrac{1-\mu^2}{1+\mu^2}\right)g$

(b) $(M+2m)\left(\dfrac{1+\mu}{1-\mu}\right)g$

(c) $(M+2m)\left(\dfrac{1-\mu}{1+\mu}\right)g$

(d) $(M+2m)\left(\dfrac{1+\mu^2}{1+\mu^2}\right)g$

Q.3 A uniform bar of mass m, length L, hinged at O and supported at A by a string as shown in figure, suddenly the string breaks and bar starts rotating about O. The angular acceleration of the bar is $K\times(g/L)$, where g is gravitational acceleration, then the value of K should be



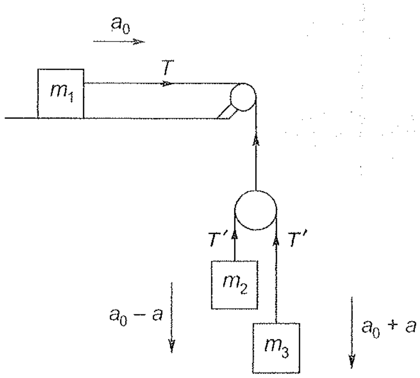
(a) 2 (b) 1.5
(c) 1 (d) 2.5

ANSWERS

1. (a) 2. (c) 3. (b)

Hints & Explanation

1. (a)



$$T' = \frac{T}{2}$$

Force on M_1,

$\Rightarrow$ $$T = m_1 a_0 \qquad ...(1)$$

Force on M_2,

$\Rightarrow$ $$m_2 g - \frac{T}{2} = m_2(a_0 - a) \qquad ...(2)$$

Force on M_3,

$\Rightarrow$ $$m_3 g - \frac{T}{2} = m_3(a_0 + a) \qquad ...(3)$$

From eq. (1), (2) and (3) we get,

$\Rightarrow$ $$a_0 - a = g - \frac{m_1 a_0}{2m_2}$$

$\Rightarrow$ $$a_0 + a = g - \frac{m_1 a_0}{2m_3}$$

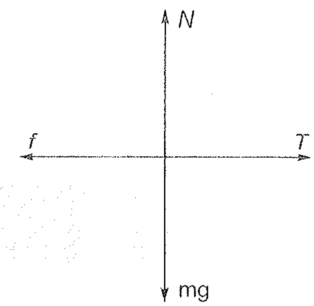
Our adding we get,

$$2a_0 = 2g - \frac{m_1 a_0}{2}\left(\frac{1}{m_2} + \frac{1}{m_3}\right)$$

On solving, $$a_0 = \frac{g}{1 + \frac{m_1}{4}\left(\frac{1}{m_2} + \frac{1}{m_3}\right)}$$

2. (c)

If no force is applied the block A will slip on C towards right and the block B will move downward. Let the minimum force required to prevent slipping be F. (Taking block A as system)



For vertical, $N = mg$

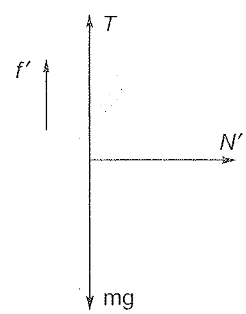
Minimum force in order to prevent slipping, $f = \mu N = \mu mg$ ...(1)

As block moves towards right with acceleration a.

$\Rightarrow$ $$T - f = ma$$

$\Rightarrow$ $$T - \mu mg = ma \qquad ...(2)$$

Taking block B as a system



As block moves towards right with acceleration a

$\Rightarrow$ $$N' = ma$$

$\therefore$ $$f' = \mu ma$$

For vertical equilibrium $T + f' = mg$

$$T + \mu ma = mg \qquad ...(3)$$

Solving eq. (2) and (3)

$\Rightarrow$ $$a_{min} = \left(\frac{1-\mu}{1+\mu}\right)g$$

$\therefore$ $F_{min} = (M + 2m) \times a_{min} = (M + 2m)\left(\frac{1-\mu}{1+\mu}\right)$

3. (b)

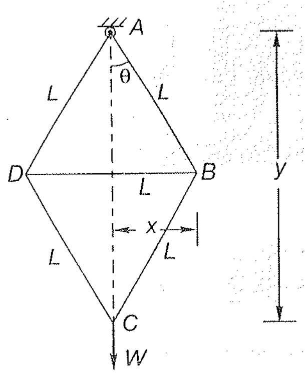
$$T = mg \times \frac{L}{2}$$

$$I_0 = \frac{mL^2}{3}$$

$$\alpha = \frac{T}{I_0} = \frac{mgL}{2} \times \frac{3}{mL^2} = \frac{1.5g}{L}$$

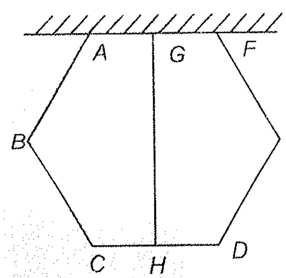
Student's Assignments

Q.1 Five bars *AB*, *BC*, *CD*, *DA* and *BD* each of equal length *L* and equal cross sectional area '*a*' are pin jointed so as to form a plane frame *ABCD* with a diagonal member *BD*. The frame is suspended from the joint *A* and a weight *W* is attached at the lower joint *C* as shown in figure. Neglecting self weight of the bars, determine magnitude of thrust in bar *BD*, using the principle of virtual work.



Q.2 The potential energy of a mechanical system is given by $V = 8x^4 - 4x^2 + 8$, where x is the position co-ordinate defining the configuration of a single degree of freedom system. Determine the values of x and stability condition of each.

Q.3 A hexagon frame is made up of six bars of equal weight and equal length as shown in figure. Rod *AF* is fixed in a horizontal plane. A rod *GH* is fixed at the mid-points of rods *AF* and *CD*. If weight of each rod is *W*, then by using the principle of virtual work show that tension in member *GH* is equal to 3*W*.



Q.4 Determine the horizontal reaction *Q* for the frame consisting of two rhombuses of sides 750 mm and 500 mm each as shown in figure. The frame carries two loads *P* as shown. If $\theta = 45°$, then what is the reaction *Q*?