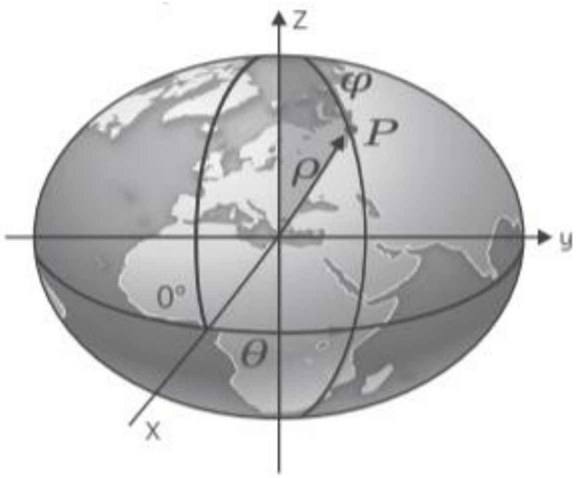
Gravitation

Case Study Based Questions

Read the following passages and answer the questions that follow:

1. The earth is not a perfect sphere. It is an ellipsoid. The radius in the equatorial plane is 21 km larger than the radius along the poles. Due to this, the gravity is more at poles and less at the equator.



Both, due to rotation and the bulging at the equator, the value of the g is smaller at the equator than the poles. The surface of the planet plays an important role in the value of gravity. The sun does not revolve around hence it is nearly spherical. The mass of the sun causes it to have the greatest value of gravity in our solar system.

(A) At what altitude above the earth's surface does the value of 'g' equal that of a 100-kilometer-deep mine?

(B) At which height from the earth's surface does the acceleration due to gravity decrease by 75% of its value at the earth's surface?

(C) The commodity weights as per its real weight at equators, whereas it differs at poles, making a product more cost- effective to purchase at the equator than at the poles. Why there is a difference in weighing at both places?

Ans. (A) Let the acceleration due to gravity at height h be the same as that at a depth d, deep into the earth i.e., d= 100 km, Therefore,

$$g' = g\left(1 - \frac{2h}{R}\right) = g\left(1 - \frac{d}{R}\right)$$

$$\frac{2h}{R} = \frac{d}{R}$$

$$2h = d$$

$$h = \frac{d}{2} = 50 \text{ km}$$

(B) Since, the acceleration due to gravity reduces by 75%, then the value of acceleration due to gravity there is g'= 100-75 = 25%.

It means, $g' = \frac{25}{100} g.$

If h is the height of location above the surface of the earth, then

$$g' = \frac{g}{\left(1 + \frac{h}{R_e}\right)^2}$$

$$\frac{g'}{g} = \frac{1}{\left(1 + \frac{h}{R}\right)^2} = \frac{25}{100}$$

$$= \frac{1}{\left(1 + \frac{h}{R}\right)^2}$$

$$1 + \frac{h}{R} = 2$$

$$R + h = 2R$$

$$H = R$$

Therefore, at R = 6400 km from the earth surface acceleration due to gravity is decreased by 75% of its value at the earth surface.

h = R = 6400 km

(C) We know that the value of acceleration due to gravity, g is deduced to be:

$g = \frac{GM}{R^2}$ by the Law of Gravitational attraction. According to this, the value of, g depends on the mass and the radius of the Earth (G being a constant). However, as we know that the Earth is not perfectly spherical but bulged out (R is slightly more at the equator) at the equator on account of its rotation. Thus, it is found that the value of g is 0.5% more at the poles than it is at the equator. As a result, the weight (W = mg) of the same amount of sugar must also be greater at the poles.

Weight of 1 kg sugar at equator measured by the weighing machine = 1 x 9.8 = 9.8 N

Mass measured (as per the calibrated weighing machine) will be

$$= \frac{9.8}{9.8} = 1\text{ kg}$$

weight of 1kg sugar at poles

= 1 x 9.85

= 9.85 N.

Mass measured (as per the calibrated weighing machine) will be

$$= \frac{9.85}{9.8}$$
$$= 1.005\text{ kg.}$$

So at the poles, even 1 kg sugar will be read as 1.005 kg by the vendor and you will get lesser sugar from him. Thus, it is more profitable to buy sugar at the equator than on the poles. Mathematically it is,

$$g' = g - \omega^2 R_e \cos^2 \lambda$$
$$\text{At poles, } \lambda = 90^o, \cos 90^o = 0$$
$$\text{At Equator, } \lambda = 0^o, \cos 0^o = 1$$
$$\therefore \quad g_{poles} > g_{equator}$$

2. To test the validity of the Copernicus model (Heliocentric model), the great Danish astronomer Tycho Braho made extraordinary observations by studying the motions of planets and stars without the aid of a telescope. This data was critically analyzed by Johannes Kepler. From these complicated data; Kepler deduced simple relations that governed planetary motion. These are three famous laws of Kepler which strongly supported the Copernicus model of the solar system and played a major role in the discovery of Newton's Law of Gravitation. The laws are

(i) Law of orbits (ii) Law of areas (iii) Law of periods.

(A) Kepler's second law is the consequence of the law of conservation of:

(a) linear momentum

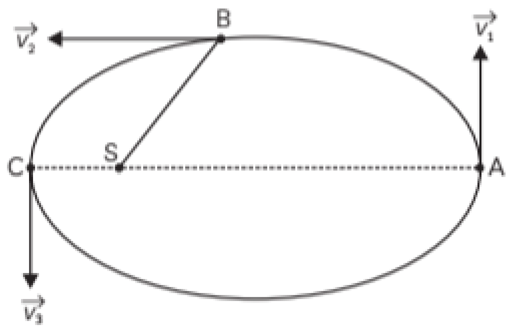
(b) energy

(c) angular momentum

(d) mass

(B)The distance of two planets from the sun is 1013 m and 1012 m respectively. The ratio of the periods is:

(a) $10\sqrt{10}:1$
(b) $1:10\sqrt{9}$
(c) 1:1
(d) 103:1
(C) The different positions of a planet around the sun in an elliptical orbit are shown by A, B and C. If V_1, V_2 and v3 be the tangential speeds of the planet at A, B and C respectively, then:



(a) $V_1 = V_2 = V_3$
(c) $V_1 < V_2 < V_3$
(b) $v_1 > V_2 > V_3$
(d) $v_1 = V_2 > V_3$
(D) If the distance between the earth and the sun were one-third of its present value, the number of days in a year would have been:
(a) increased
(b) decreased
(c) remains same
(d) cannot say
(E) A planet moves around the sun in an elliptical orbit with the sun at one of its foci. The physical quantity associated with the motion of the planet that remains constant with time is:
(a) velocity
(b) centripetal force
(c) linear momentum
(d) angular momentum

Ans. (A) (c) angular momentum
Explanation: When a planet revolves around its orbit, it covers equal areas in

similar amounts of time, according to Kepler's second rule of planetary motion,

which states that $\frac{dA}{dt}$ = constant.

But, $\frac{dA}{dt} = \frac{L}{2m}$

Thus, L is constant.

(B) (a) 10√10:1

Explanation: By Kepler's law $T_2 \propto r_3$

$$\text{or} \quad \frac{T_1}{T_2} = \left(\frac{r_1}{r_2}\right)^{3/2}$$

$$\frac{T_1}{T_2} = \left(\frac{10^{13}}{10^{12}}\right)^{3/2}$$

$$\frac{T_1}{T_2} = 10\sqrt{10}$$

(C) (c) $V_1 < V_2 < V_3$

Explanation: Kepler's Law

$$\frac{dA}{dt} = \text{constant and}$$

$$T^2 \propto r^3$$

$$\frac{dA}{dt} = \text{constant}$$

$$v_1 < v_2 < v_3$$

(D) (b) decreased

Explanation: According to Kepler's law

$$T^2 \propto r^3$$

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{r_1}{r_2}\right)^3$$

$$\left(\frac{365}{T_2}\right)^2 = \left(\frac{3r_1}{r_1}\right)^3$$

$$\left(\frac{365}{T_2}\right)^2 = 27$$

$$\left(\frac{365}{T_2}\right) = 5.19$$

$$T_2 = \frac{365}{5.19}$$

$$T_2 = 70 \text{ days}$$

There will be 70 days in an year. Hence, number of days will be decreased to approximately 70 days.

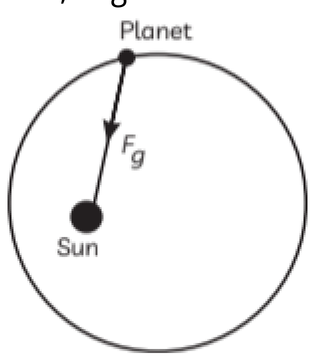
(E) (d) angular momentum

Explanation: A planet revolves around the sun, in an elliptical orbit under the effect of gravitational pull on the planet

So, torque, T =|rx F = rF sin 180° = 0

$$\text{As } T = \frac{dL}{dt}; \text{ so } L = a \text{ constant}$$

Hence, angular momentum is constant.



3. The satellite stays in orbit because it still has momentum energy it picked up from the rocket pulling it in one direction. Earth's gravity pulls it in another direction. This balance between gravity and momentum keeps the satellite orbiting around Earth. Satellites that orbit close to Earth feel a stronger tug of Earth's gravity. To stay in orbit, they must travel faster than a satellite orbiting farther away. The International Space Station orbits about 250 miles above the Earth and travels at a speed of about 17,150 miles per hour.

(A) A rocket is launched into a circular orbit around the Earth. What more speed should be given to the spaceship now in order for it to overcome the earth's gravitational pull?
(B) An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the earth. (i) Determine the height of the circular orbit around the earth's surface. (ii) If the satellite is stopped suddenly in its orbit and allowed to fall freely on the earth, find the speed with which it hits the surface of the earth. Given M = mass of the earth and R = radius of earth.
(C) An astronaut inside a small spaceship orbiting around the earth cannot detect gravity. If the space station orbiting the earth has a large size, can he hope to detect gravity?

Ans. (A) Let ΔK be the additional kinetic energy imparted to the spaceship to overcome the gravitational pull then,

$$\Delta K = -\text{(total energy of spaceship)} = \frac{GMm}{2R}$$

Total kinetic energy

$$= \frac{GMm}{2R} + \Delta K$$

$$= \frac{GMm}{2R} + \frac{GMm}{2R}$$

$$= \frac{GMm}{R}$$

then $\frac{1}{2}mv_2^2 = \frac{GMm}{R}$

$\Rightarrow$ $v_2 = \sqrt{\frac{2GM}{R}}$

But $v_1 = \sqrt{\frac{GM}{R}}$.

so additional velocity required

$$= v_2 - v_1$$
$$= \sqrt{\frac{2GM}{R}} - \sqrt{\frac{GM}{R}}$$
$$= (\sqrt{2} - 1)\sqrt{\frac{GM}{R}}$$

Alternate Solution:

Additional velocity

= escape velocity

− orbital velocity

$$= v_{esc} - v_0.$$
$$= \sqrt{\frac{2GM}{R}} - \sqrt{\frac{GM}{R}}$$
$$= (\sqrt{2} - 1)\sqrt{\frac{GM}{R}}$$

(B) (1) Let height above the earth's surface = h then

$$v_{orbital} = \sqrt{\frac{GM}{R+h}}$$
$$= \frac{1}{2}v_e = \frac{1}{2}\sqrt{\frac{2GM}{R}}$$
$$\Rightarrow \quad R + h = 2R$$
$$\Rightarrow \quad h = R$$

(ii) If the satellite is stopped suddenly, then its total energy

$$E_1 = -\frac{GMm}{2R}$$

Let its speed be v when it hits the earth's surface than its total energy on earth surface

$$E_2 = -\frac{GMm}{R} + \frac{1}{2}mv^2$$

Conservation law for mechanical energy yield v

$$E_1 = E_2$$
$$\Rightarrow \quad \frac{-GMm}{2R} = -\frac{GMm}{R} + \frac{1}{2}mv^2$$
$$\Rightarrow \quad v = \sqrt{\frac{GM}{R}}$$

(C) An Astronaut inside a small spaceship experiences a very small negligible constant acceleration and hence the astronaut feels weightlessness. If the space station has too much mass and size, then it can experience acceleration due to gravity, e.g., on the moon.

4. If we throw a ball vertically upwards from the surface of the earth; it rises to a certain height and falls back. If we throw it at a greater velocity, it rises to a greater height. If we throw it with sufficient velocity, it may never come
back. It will escape from the gravitational pull of the earth. This minimum velocity is called escape velocity. [Delhi Gov. QB 2022]

(A) Escape speed of a body of mass m depends upon its mass as:

(a) mo

(b) m

(c) m^2

(d) m^3

(B) The escape velocity for an object projected vertically upward from the earth's surface is approx. 11 km/s. If the body is projected at an angle of 45° with the vertical, then the escape velocity will be:

(a) $\frac{11}{\sqrt{2}}$ km/s

(b) 11 km/s

(c) $11\sqrt{2}$ km/s

(d) 22 km/s

(C) The value of escape velocity on a certain planet is 2 km/s. Then the value of orbital speed of a satellite orbiting close to its surface is:

(a) 12 km/s

(b) 1 km/s

(c) $\sqrt{2}$ km/s

(d) $2\sqrt{2}$km/s

(D) The escape speed of the planet is v. If the

radius of the planet contracts to $\frac{1}{4}$ th of

the present value, without any change in mass, the escape speed would become:

(a) halved

(b) doubled

(c) quadrupled

(d) one fourth

(E) The moon has no atmosphere as:

(a) The escape speed on the moon is very large as the thermal speed of the molecules of gases on the moon.

(b) The escape speed on the moon is equal to the thermal speed of the gaseous molecules on the moon.

(c) The escape speed on the moon is very small as compared to the thermal speed of the molecules of gases on moon.

(d) Size of the moon as compared to the earth is very less and hence escape speed of the moon is large.

Ans. (A) (a) mo

Explanation: Escape speed of a body from Earth's surface is given by: Vmm = √2gR As we can see from the above equation, there is no 'mass' term. Implies it can be written as m° So, the escape speed of a body is independent of its mass.

(B) (b) 11 km/s

Explanation: Escape speed of a body from Earth's surface is given by: Vmm = √2gR This expression is obtained by conservation of energy and doesn't involve in which direction the body is thrown/projected. So, irrespective of the angle of projection, escape speed of the body from Earth's surface remains constant i.e.≈ 11 km/s

(C) (c) √2 km/s

Explanation:

v_2 = 2 kms-^{1}

Vo=?

The orbital speed can be written in terms of escape velocity.

$$v_0 = \frac{v_o}{\sqrt{2}}$$

Vo is the orbital speed for a satellite orbital close to its surface. v_e is the escape velocity.

$$v_0 = \frac{v_e}{\sqrt{2}}$$

$$v_0 = \frac{2}{\sqrt{2}} = \sqrt{2}\,kms^{-1}$$

(D) (b) doubled

Explanation: Escape velocity on a planet is

$$v = \sqrt{\frac{2GM}{R}}$$

Now the radius of plant contracts to one fourth of present value without change in its mass.

So $R = \frac{R}{4}$

So new escaplee velocity

$$v = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{4 \times 2GM}{R}} = 2V$$

(E) (c) The escape speed on the moon is very small as compared to the thermal speed of the molecules of gases on the moon

Explanation: The escape velocity (the minimum velocity with which a body is to be projected so as to escape from the gravitational pulls on the surface of moon is very much less than the rms velocity of the molecules of gas at the surface temperature of moon. Therefore, the molecules will escape and therefore moon cannot hold an atmosphere.