Standard X

MATHEMATICS

Part-2

Government of Kerala
Department of Education

State Council of Educational Research and Training (SCERT)
2016

THE NATIONAL ANTHEM

Jana-gana-mana adhinayaka, jaya he
Bharatha-bhagya-vidhata.
Punjab-Sindh-Gujarat-Maratha
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchala-Jaladhi-taranga
Tava subha name jage,
Tava subha asisa mage,
Gahe tava jaya gatha.
Jana-gana-mangala-dayaka jaya he
Bharatha-bhagya-vidhata.
Jaya he, jaya he, jaya he,
Jaya jaya jaya, jaya he!

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give respect to my parents, teachers and all elders and treat everyone with courtesy.

I pledge my devotion to my country and my people. In their well-being and prosperity alone lies my happiness.

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Dear children,

Mathematics starts in counting and measuring. In the age of agriculture, it becomes the second degree equations of areas; rises to astronomy for weather prediction. Grows into the branch of mathematics called trigonometry. In Renaissance Europe, trigonometry forms the foundation of navigation. It becomes the basis of locating places using satellites in today's world. The mathematical principles which seventeenth century mathematicians developed as purely mathematical operations of numbers are now used to make security systems in e-transactions. I wish all of you would recognize the innumerable applications of mathematics and revel in its theoretical rhythms.

With love and regards

Dr. P. A. Fathima
Director, SCERT

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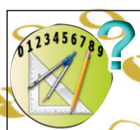
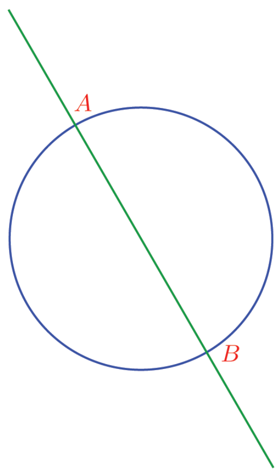


For Discussion

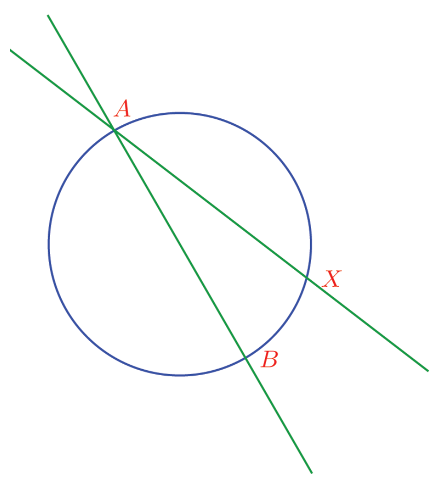
Line and circle

See this picture :

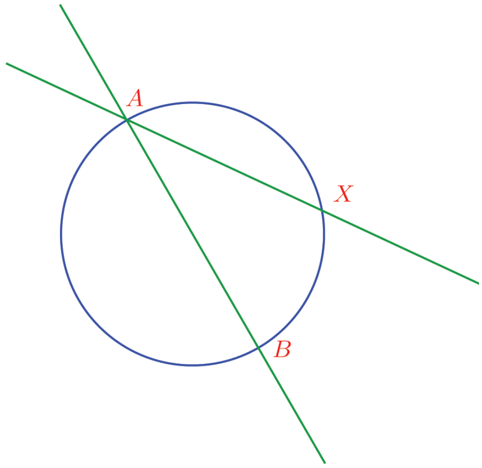


AB is the diameter through the point A on the circle; and it is extended a bit on either side.

This picture shows another chord through A, instead of a diameter, which is also extended.

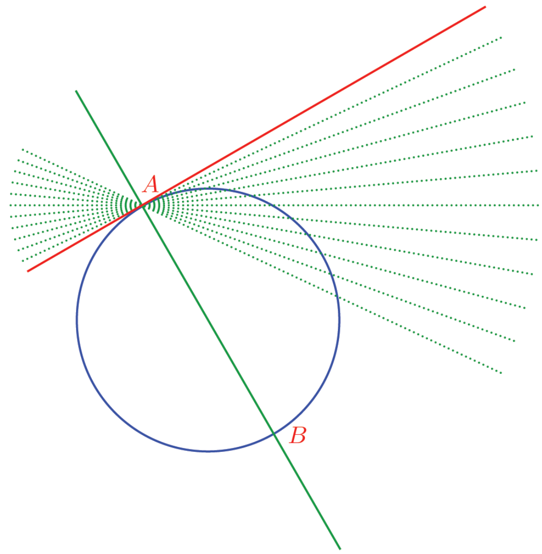


Without altering the position of A, suppose we make X closer to A.



What if we make X closer and closer to A along the circle?

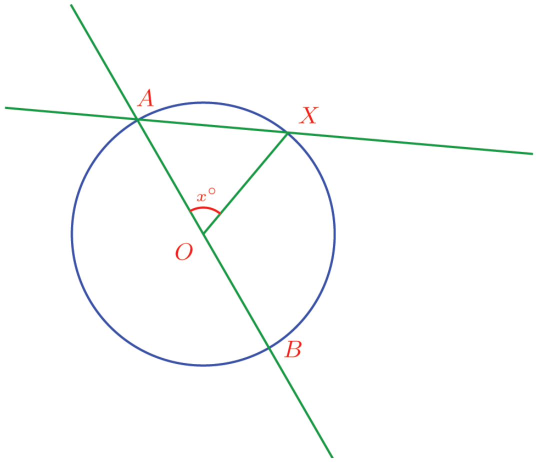
Draw a circle centered at a point O in GeoGebra and mark points A, X on it. Draw lines joining O, A and A, X. What happens to the line AX, when X is moved closer to A? When X coincides with A? Join OX. What happens to the angles OAX and OXA as X is moved closer and closer to A?



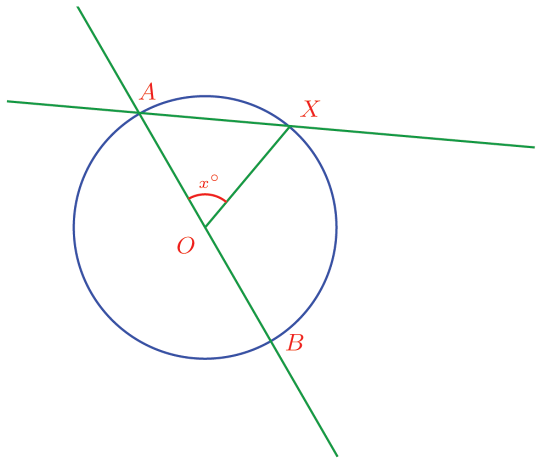
The red line in the picture just touches the circle at A, right?

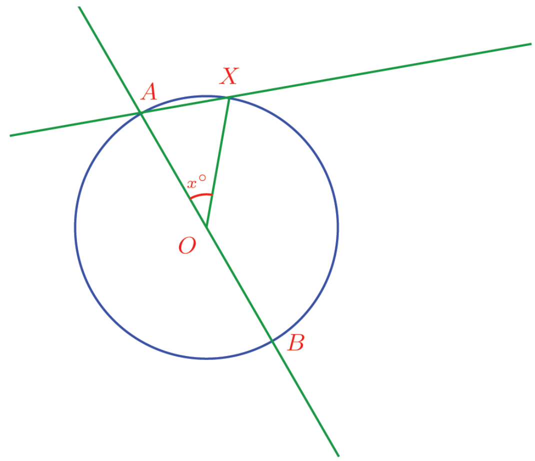
This line is called the *tangent* to the circle at A. Look at the picture again; see any relation between the tangent and the diameter?

To make this clear, let's take the central angle of chord AX as x°.



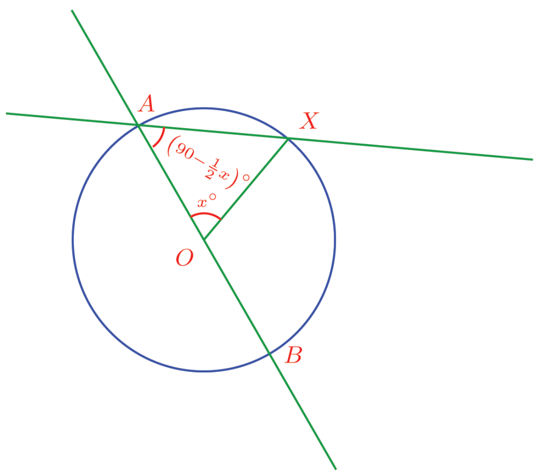
As X gets closer to A, the length of the chord AX and its central angle becomes smaller; that is, the number x gets closer to zero.





What about the angle between the chord and the diameter? Since ΔAOX is isosceles, this angle is

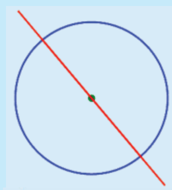
$$\frac{1}{2}(180 - x)^{\circ} = \left(90 - \frac{1}{2}x\right)^{\circ}$$



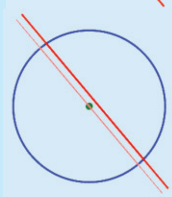
As X gets closer to A, this angle gets closer to 90°. And when the extended chord becomes a tangent, the angle becomes exactly 90°.

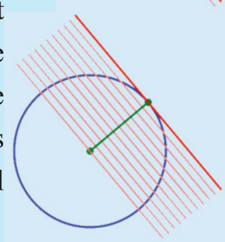
Sliding line

See this picture – a circle and a line through the centre.

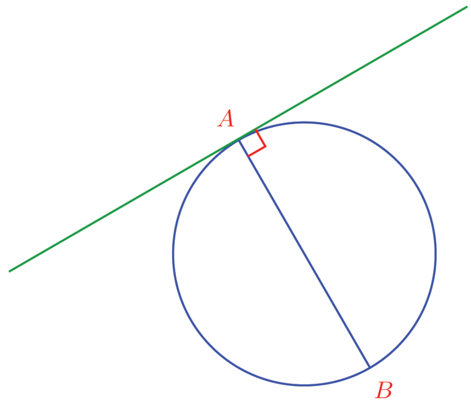


Suppose we slide the line a little higher. And sliding it more and more, we finally get a line just touching the circle at a single point. And the line joining the centre and this point is perpendicular to all these parallel lines.





Draw a circle and a radius in GeoGebra. Choose a point on the radius and draw the perpendicular through it. Now change the position of the point. What happens when it is on the circle?



We state this as a general principle:

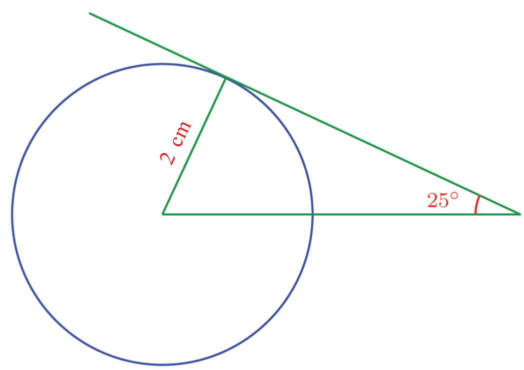
The tangent at a point on a circle is perpendicular to the diameter through that point.

To draw a tangent to a circle in GeoGebra, choose **Tangents.** Click on the circle and a point on the circle. What if we click on a point outside the circle?

Draw the tangent at a point on a circle and enable **Trace On** for it and **Animation** for the point. What do we get?

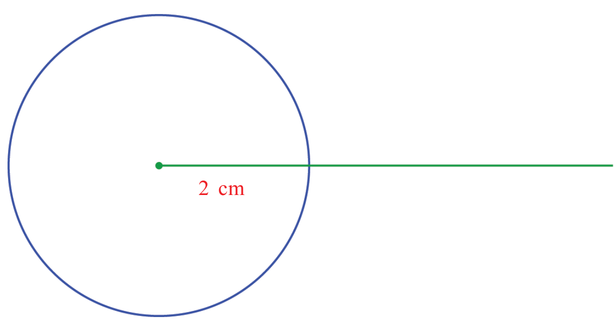
Let's look at some problems based on this.

In the picture below, the top line is a tangent to the circle:



Can you draw this picture in your notebook?

Draw a circle of radius 2 centimetres and a horizontal line through its centre.

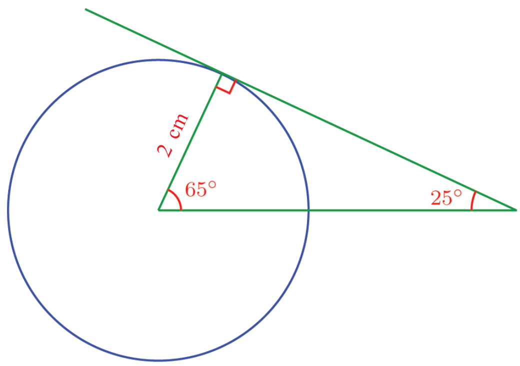


Now at what point on the circle do we draw the required tangent?

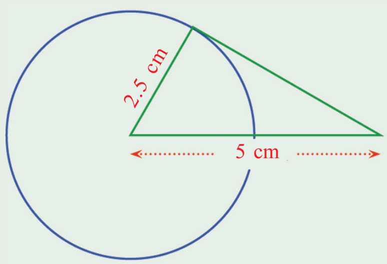
Look at the first picture again; the top angle of the triangle is 90°; and another angle is 25°.

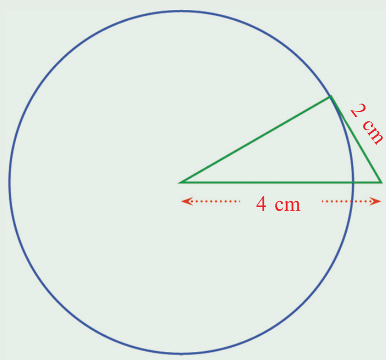
So, the third angle is 65°.

Can't you complete the picture now?



(1) In each of the two pictures below, a triangle is formed by a tangent to a circle, the radius though the point of contact and a line through the centre:

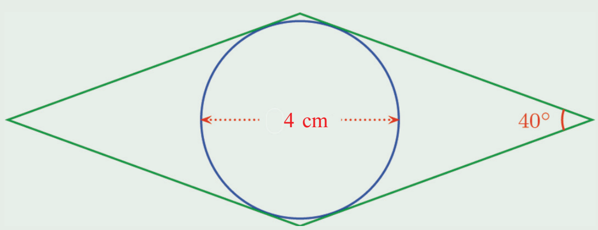




Draw these in your note book.

(2) In the picture, all sides of a rhombus are tangents to a circle.

Draw this picture in your notebook.

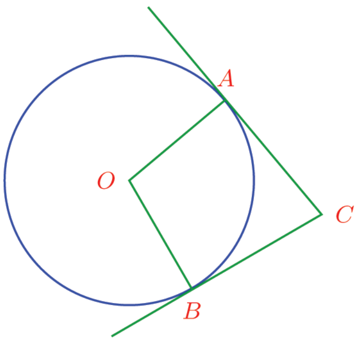


(3) What sort of a quadrilateral is formed by the tangents at the ends of two diameters of a circle?

(4) Prove that the tangents drawn to a circle at the two ends of a diameter are parallel.

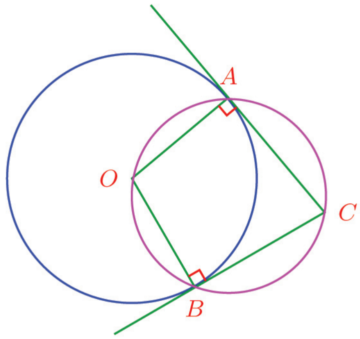
Tangents and angles

See this picture:



The tangents at the points A, B on a circle centered at O meet at C.

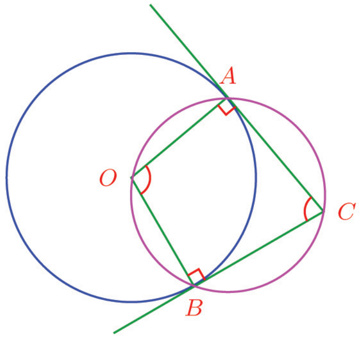
In the quadrilateral $OACB$, the angles at the opposite corners A, B are right; so their sum is 180°. Thus the quadrilateral is cyclic.



That is,

The quadrilateral with vertices at the centre of a circle, two points on it and the point where the tangents at these points meet, is cyclic.

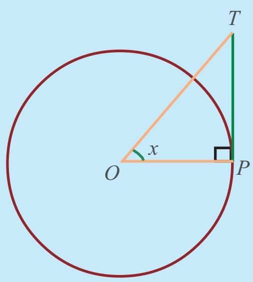
In such a quadrilateral the sum of the other two angles is also 180°.



The name

The name tangent is derived from the Latin word *tangere*, meaning 'to touch'.

The full name of the tan measure is also tangent. What is its connection with a line touching a circle?

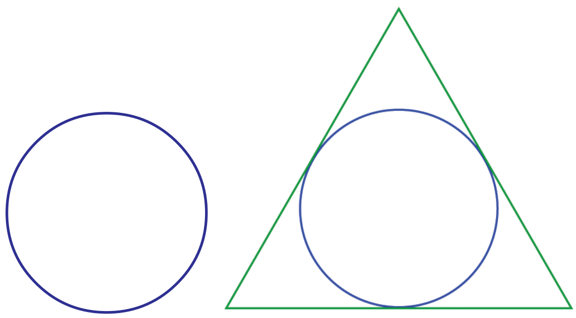


If we take the radius of the circle as the unit of length, then the length of the tangent PT is $\tan x$, right?

This also is a general idea worth noting:

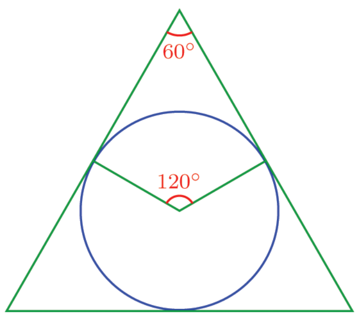
In a circle, the angle between the radii through two points and the angle between the tangents at these points are supplementary.

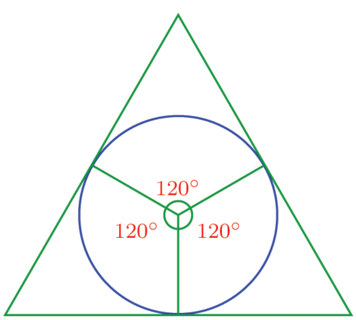
Let's look at some figures which can be drawn using these ideas. First we draw an equilateral triangle, exactly covering a circle.



The sides of the triangle must be tangents to the circle. And since the triangle is to be equilateral, the angle between them must be 60°.

What about the angle between the radii through the points of contact of the tangents?

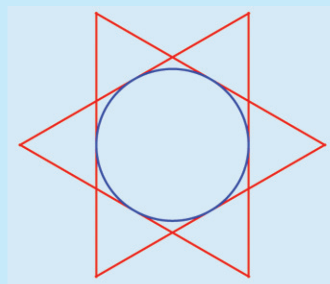




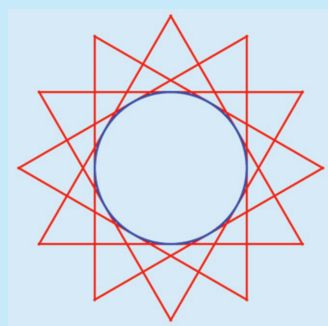
Thus we can see that all the angles between the radii through the points of contact are 120°.

Circle with lines

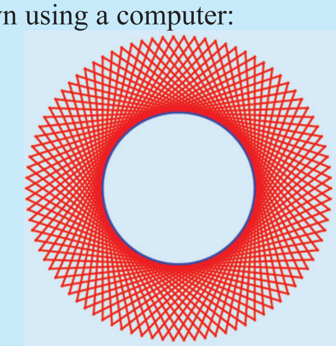
The picture below shows a star made by six tangents to a circle.



How about 12 tangents?

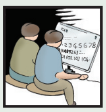


The picture below shows 90 tangents drawn using a computer:



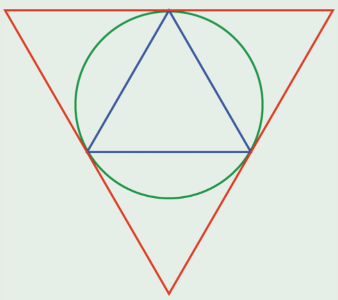
Thus we need only draw three radii of the circle 120° apart and draw the tangents at their ends to get our triangle.

Draw a circle of radius 3 centimetres and draw an equilateral triangle like this.



(1) Draw a circle of radius 2.5 centimetres. Draw a triangle of angles 40°, 60°, 80° with all its sides touching the circle.

(2) In the picture, the small (blue) triangle is equilateral. The sides of the large (red) triangle are tangents to the circumcircle of the small triangle at its vertices.

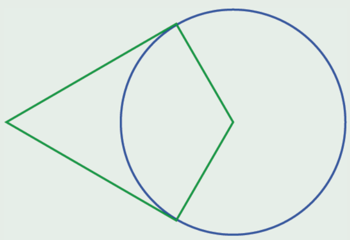


i) Prove that the large triangle is also equilateral and its sides are double that of the small triangle.

ii) Draw this picture, with sides of the smaller triangle 3 centimetres.

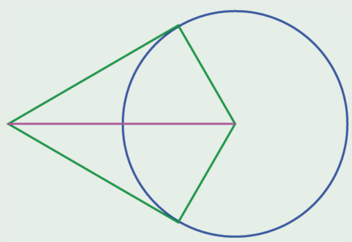
iii) Instead of an equilateral triangle, if we draw the tangents to the circumcircle of any other triangle at its vertices, do we get a similar triangle with double the sides? Justify.

(3) The picture shows the tangents at two points on a circle and the radii through the points of contact.

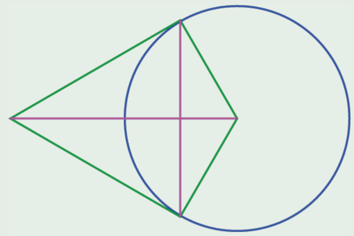


i) Prove that the tangents have the same length.

ii) Prove that the line joining the centre and the points where the tangents meet bisects the angle between the radii.

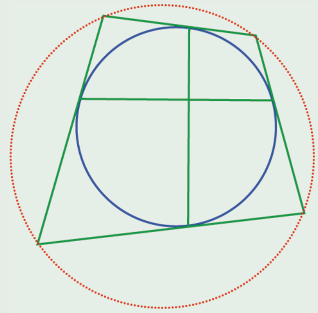


iii) Prove that this line is the perpendicular bisector of the chords joining the points of contact.



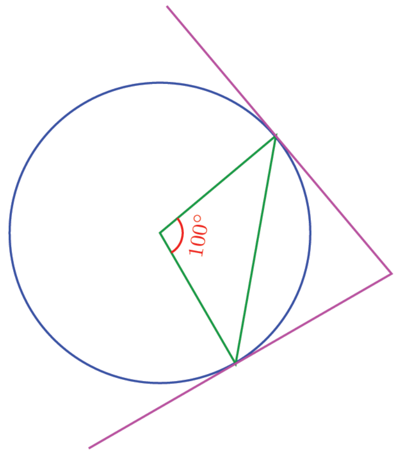
(4) Prove that the quadrilateral with sides as the tangents at the ends of a pair of perpendicular chords of a circle is cyclic.

What sort of a quadrilateral do we get if one chord is a diameter? And if both chords are diameters?

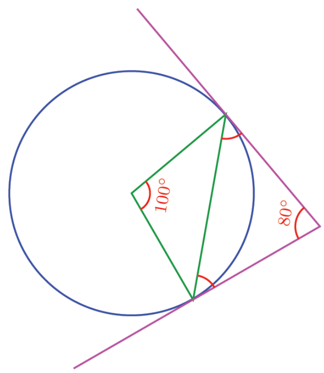


Chord and tangent

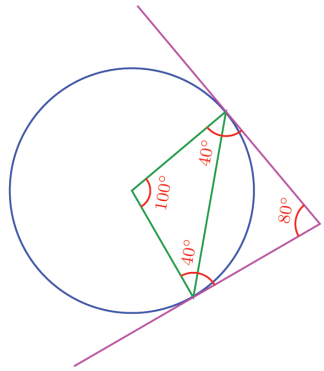
The picture shows the tangents at the two ends of a chord of a circle:



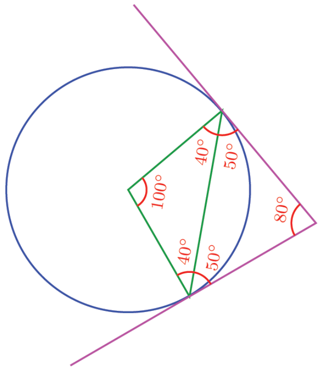
We know that angle between the tangents is 80°. What about the angles between the tangents and the chord?



The two sides of the green triangle in the picture are equal and so are the angles opposite them. Since the sum of these angles is $180^\circ - 100^\circ = 80^\circ$, each measure 40°.



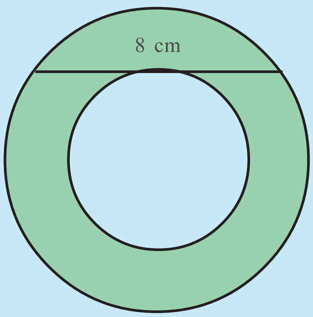
The angle between the radius and the tangent is 90°. So the angle between the chord and the tangent is $90^\circ - 40^\circ = 50^\circ$.



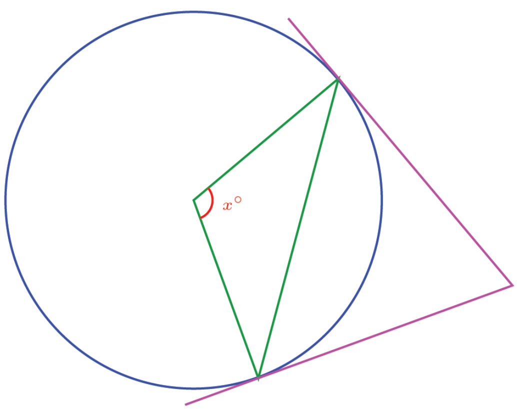
This is half the central angle of the chord, isn't it?

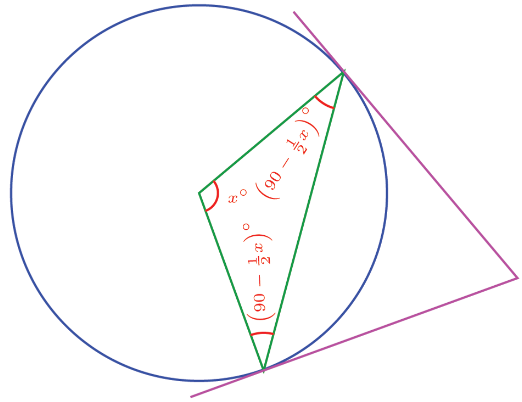
Area problem

What is the area of the green ring?



Is it true for any chord? To check this, let's take the central angle as x°.

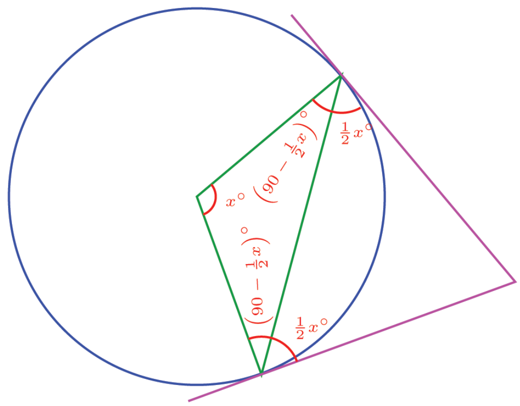




So the other two angles of the green triangle are

$$\frac{1}{2}(180 - x)^\circ = \left(90 - \frac{1}{2}x\right)^\circ$$

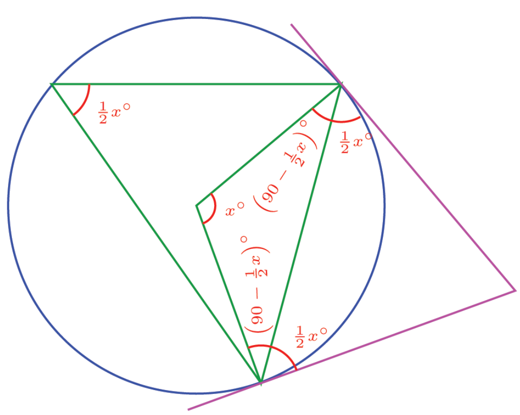
Since the angle between the tangent and the radius is 90°, we can see that the angle between the tangent and chord is $\frac{1}{2}x^\circ$.



In GeoGebra, draw a circle and a chord. Draw the tangents at the ends of this chord. Mark the central angle of the chord and the angles between the chord and the tangents. What is the relation between these angles? Draw several chords and see.

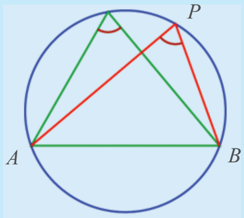
In a circle, the angle between a chord and tangent at either end is half the central angle of the chord.

The angle made by the chord on the larger part of the circle is also half the central angle, isn't it?

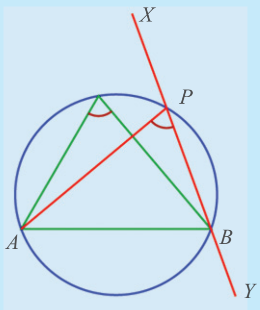


Unchanging angle

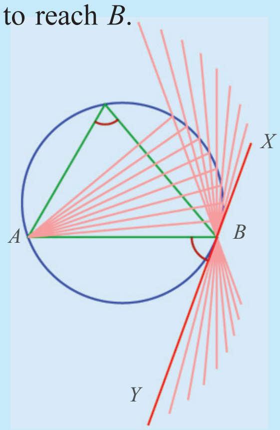
We have seen that the angles on the same part of a circle are equal.



Let's extend PB a bit.

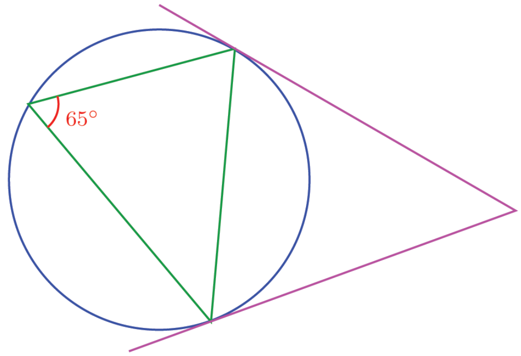


Now suppose P moves along the circle to reach B.

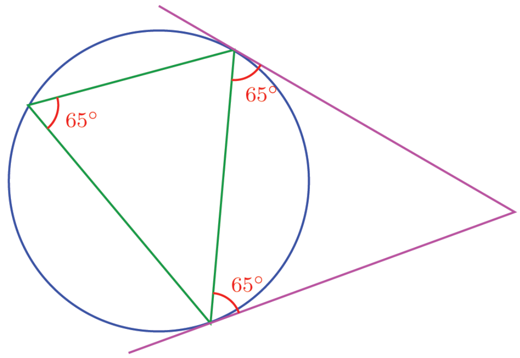


The line XY becomes the tangent at B. And the angle does not change.

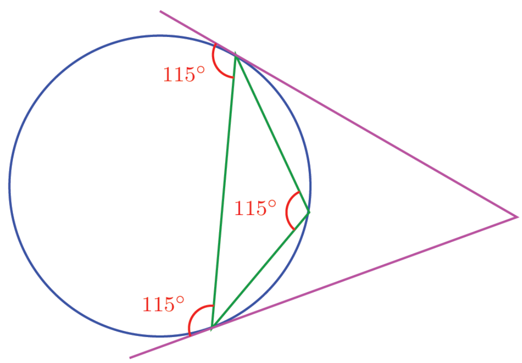
So in this picture, what is the angle which the tangents make with the chord?



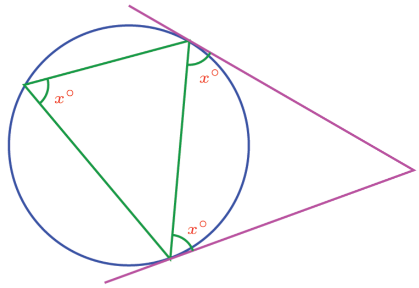
The angles on the right are 65°.

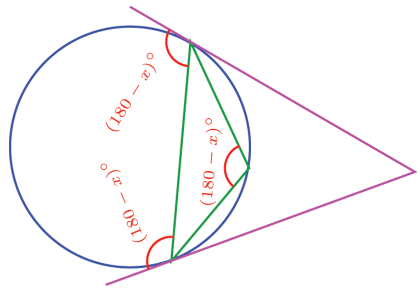


The angles which the tangents make on the left of the chord are $180° - 65° = 115°$. This is the angle which the chord makes on the smaller part of the circle, isn't it?



So the relation between the angles which the chord makes with the tangents at its ends and the angles which it makes on the circle can be shown like this:

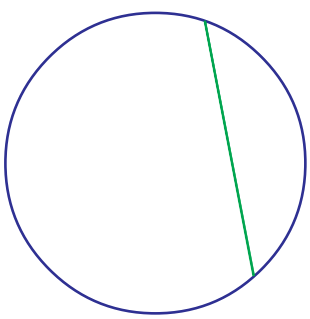


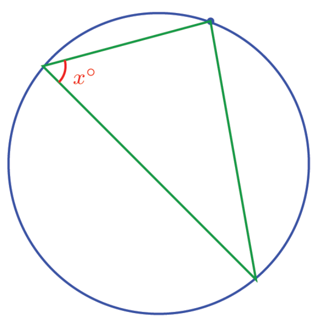


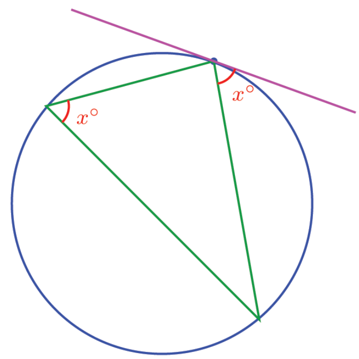
And we can write it like this:

In a circle, the angles which a chord makes with the tangents at its ends on any side are equal to the angle which it makes on the part of the circle on the other side.

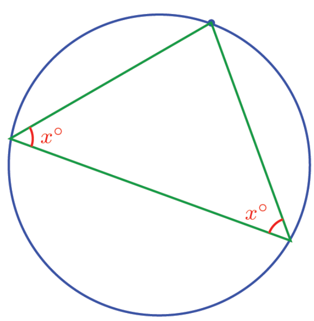
We draw a tangent to a circle at a point, by drawing the perpendicular to the diameter through this point, right? The above idea can be used to draw tangents, even if the centre is not known. We need only draw a chord through this point and draw the angle which it makes on one part of a circle on the opposite side of the chord:

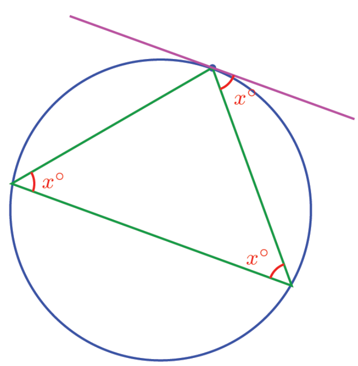






If the triangle drawn is isosceles, then we need draw only the line through the point, parallel to the bottom side.

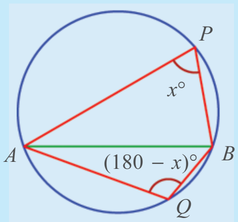




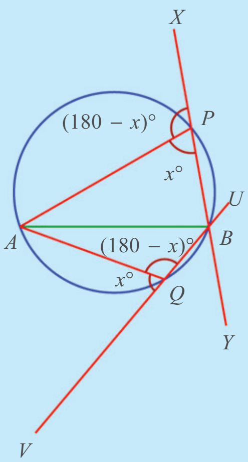
So, to draw the tangent to a circle at a specified point, we first draw an arc of a circle centred at this point and join the points where it cuts the first circle:

Flip – flop

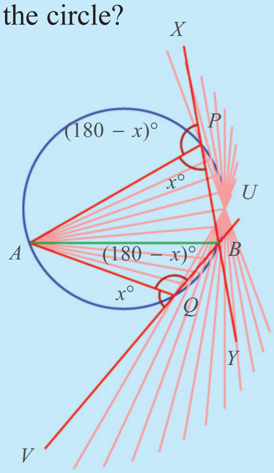
Angles on the two parts of a circle are supplementary:



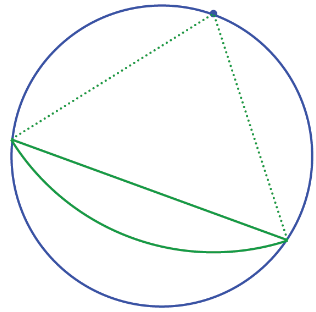
Let's extend the lines as before.



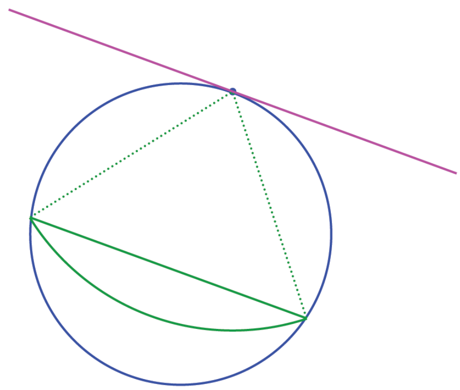
What happens as P moves towards along the circle?



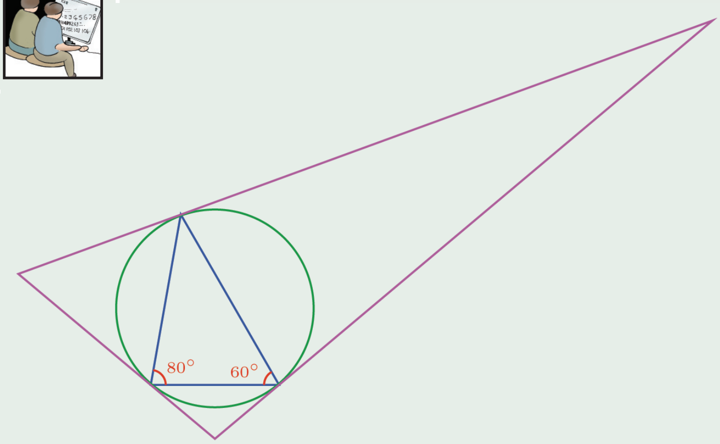
Throughout the motion, $x°$ is below AP and $(180 - x)°$ above AP, isn't it?



Now we need only draw a line parallel to this line through the point at which we want the tangent:



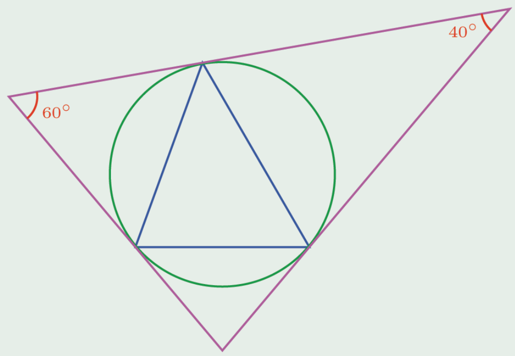
(1) In the picture, the sides of the large triangle are tangents to the circumcircle of the small triangle, through its vertices.



Calculate the angles of the large triangle.

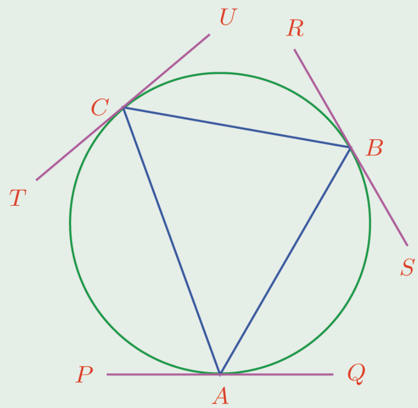
(2) In the picture, the sides of the large triangle are tangents of the circumcircle of the smaller triangle, through its vertices.

Calculate the angles of the smaller triangle.



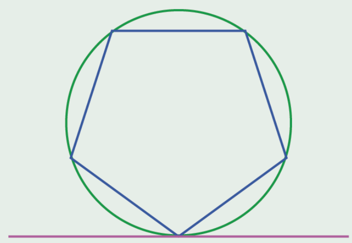
(3) In the picture, *PQ*, *RS*, *TU* are tangents to the circumcircle of ΔABC.

Sort out the equal angles in the picture.



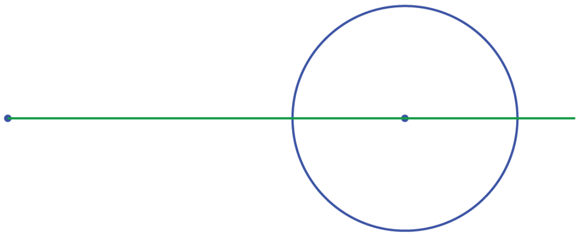
(4) In the picture, the tangent to the circumcircle of a regular pentagon through a vertex is shown.

Calculate the angle when the tangent makes with the two sides of the pentagon through the point of contact.



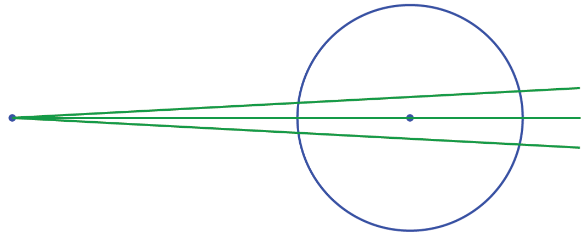
A tangent from outside

See this picture:

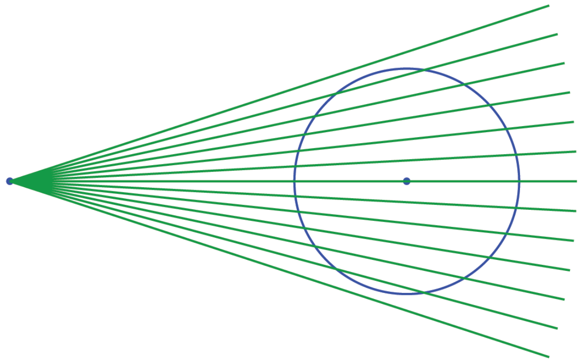


A point outside a circle is joined to the centre and extended. It cuts the circle at two points; and these points are the ends of a diameter.

Suppose we join the same point outside the circle to a point a little above or below the centre?

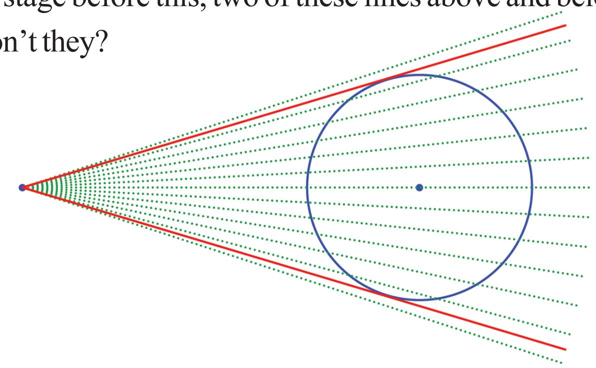


The points where the line cuts the circle gets a little closer. Let's continue this:



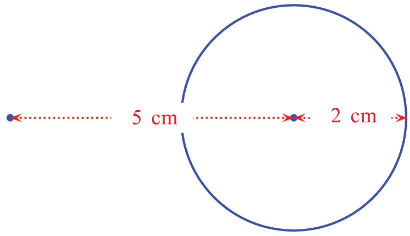
The lines which cut the circle at closer and closer points, leave the circle entirely after a stage.

But at some stage before this, two of these lines above and below just touch the circle, don't they?



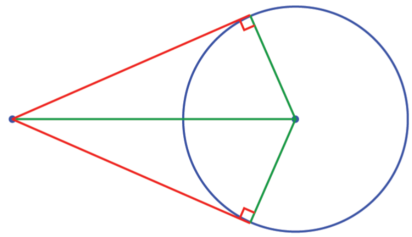
From a point outside a circle, two tangents can be drawn.

But we haven't discussed say how we can actually draw such a pair of tangents. See this picture:



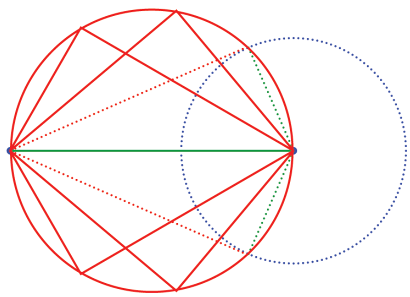
A point is marked 5 centimetres away from the centre of a circle of radius 2 centimetres

How do we draw the pair of tangents to the circle from this point?



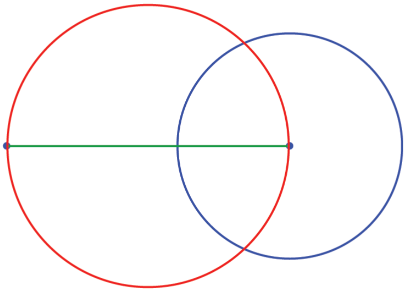
Perhaps it would be clear if we imagine how the picture would be after they are drawn;

We need two pairs of mutually perpendicular lines from the centre of the circle and the point outside.



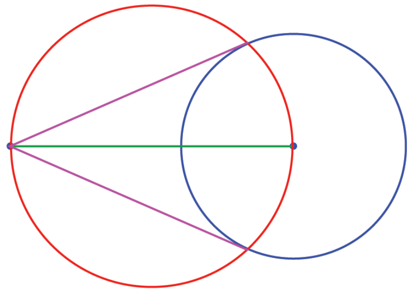
We have seen in the lesson **Circles**, that all such pairs of mutually perpendicular lines meet on the circle with the line joining these points as diameter:

In those pairs we want, one line should be a radius of our original circle; that is, the lines should meet on this circle. For that, we

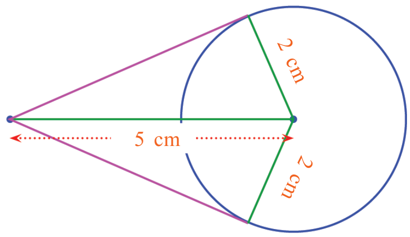


Draw a circle centred at a point O in GeoGebra and mark points A, B on it. Draw the tangents at these points and mark their point of intersection as C. Draw the quadrilateral OACB. Is it cyclic? We can check by drawing the circle through O, A, B using. **Circle Through Three Points**. Move A, B and see what happens when they get closer and farther apart. What happens when they are the ends of a diameter?

Joining the points of intersection of these circles to the point outside, we get the tangents from it:



In our problem, the radius of the original circle is 2 centimetres and the distance from the centre to the point outside is 5 centimetres.



So, we can calculate the lengths of the tangents using Pythagoras Theorem:

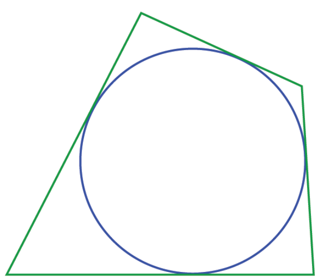
$$\sqrt{5^2 - 2^2} = \sqrt{21} \text{ centimetres}$$

We have already seen that if tangents are drawn from two points on a circle, then their lengths from the point of contact to the point of intersection are equal. We can now state it like this:

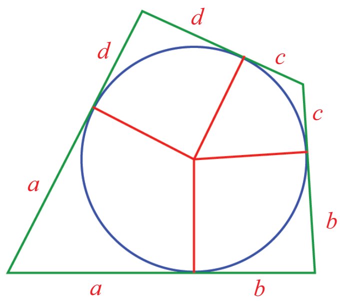
The tangents to a circle from a point are of the same length.

Let's look at a problem based on this. The picture shows the quadrilateral formed by the tangents at four points on a circle.

Let's join the centre to these points.



Taking the lengths of the tangents from the corners as a, b, c, d, we can mark these lengths as below:



So, the sum of the lengths of the bottom and top sides of the quadrilateral is $(a + b) + (c + d)$.

What about the sum of the left and right sides?
$(a + d) + (b + c)$

Both sums are $a + b + c + d$. Thus we have the following:

Draw a circle in GeoGebra and mark four points on it. Draw tangents at these points and mark their points of intersection. Draw the quadrilateral with these as vertices. Then we can hide the tangents. Mark the lengths of the sides of the quadrilateral and note the relation between them, as we change the positions of the points on the circle.

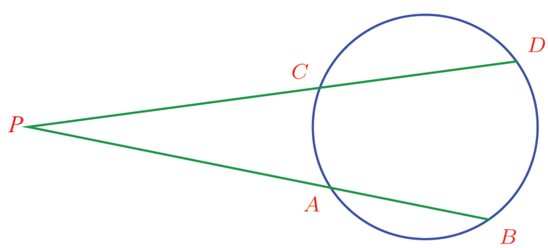
In a quadrilateral formed by the tangents at four points on a circle, the sum of the opposite sides are equal.

If the sum of the opposite sides of a quadrilateral are equal, can we draw a circle with the four sides as tangents?

Recall the fact we have seen earlier: in a quadrilateral formed by joining four points on a circle, the sum of the opposite angles are equal.

We have seen that among the lines drawn to a circle from a point, those which touch the circle at a single point are equal. We have also seen in the lesson, **Circles**, that for all lines cutting the circle at two points, the product of the whole line and the part outside the circle are equal.

Remember this picture and its equation?

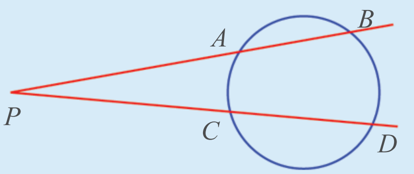


$$PA \times PB = PC \times PD$$

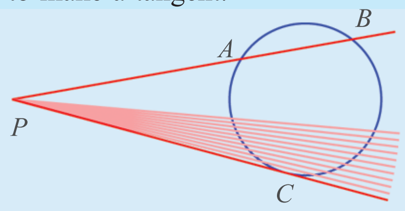
Now suppose, we draw a line which touches a circle and another line intersecting the circle.

Same relation

See this picture:



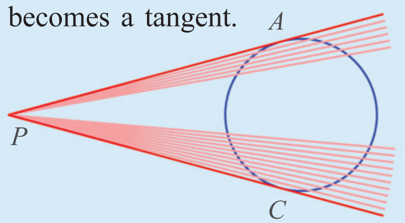
We have $PA \times PB = PC \times PD$. Suppose the lower line is rotated to make a tangent.



Then PD is the same as PC and the relation above becomes

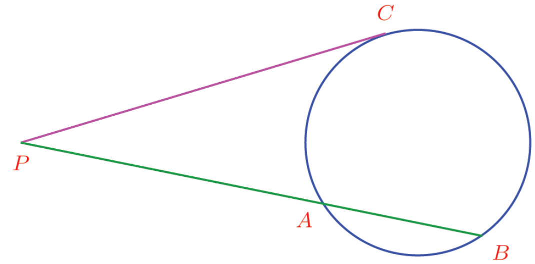
$$PA \times PB = PC^2$$

Suppose the upper line also becomes a tangent.

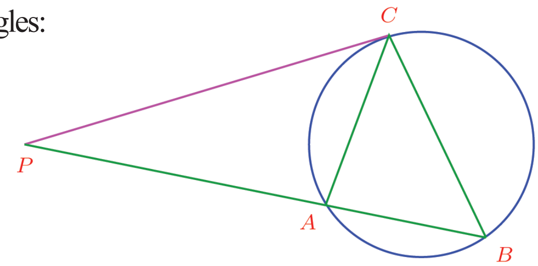


The relation becomes $PA^2 = PC^2$ or $PA = PC$.

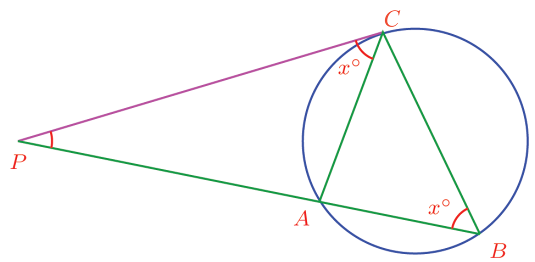
We have already seen that the length of the tangents from a point to a circle are equal.



To find out the relation between these, join AC, BC to make triangles:



The chord AC makes angle PCA at C; it is equal to the angle ABC, which AC makes on the other side of the circle, isn't it?



That is, the angle at C in ΔABC is equal to the angle at B in ΔPBC. And in both triangles the angle at P is the same.

Thus these triangles have the same angles and so pairs of sides opposite equal angles have the same ratio.

In ΔPAC, the side opposite the angle of x° is PA and in ΔPBC, the side opposite the angle of x° is PC. In ΔPAC, the longest side is PC and in ΔPBC, the longest side is PB. So,

$$\frac{PA}{PC} = \frac{PC}{PB}$$

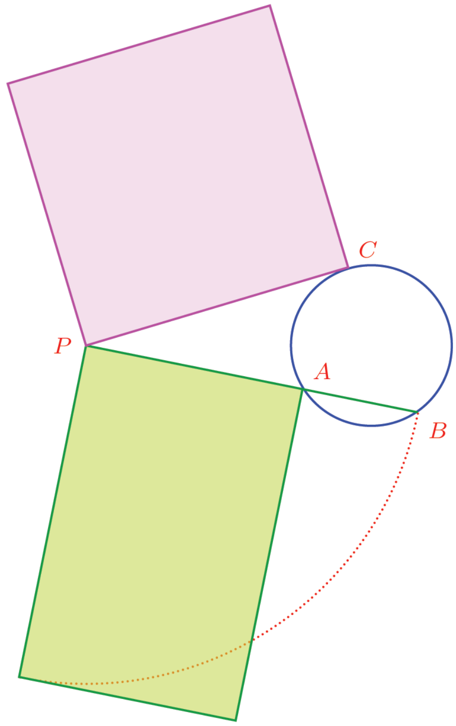
We can write this as,

$$PA \times PB = PC^2$$

The product of an intersecting line and the part of it outside the circle is equal to the square of the tangent.

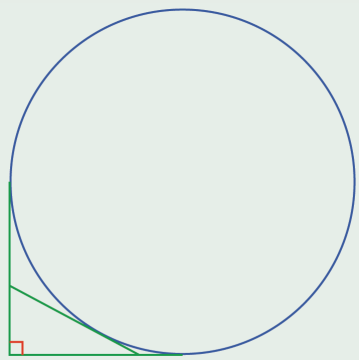
As in the case of intersecting chords, this can be stated in terms of areas.

The rectangle with the intersecting line and its part outside the circle as sides and the square with sides equal to the tangent have the same area.

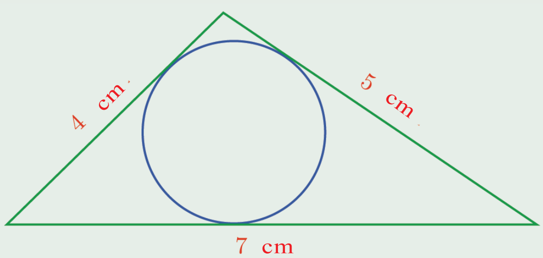


(1) In the picture, a triangle is formed by two mutually perpendicular tangents to a circle and a third tangent.

Prove that the perimeter of the triangle is equal to the diameter of the circle.

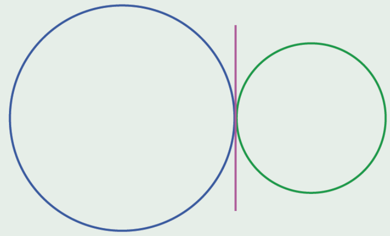


(2) The picture shows a triangle formed by three tangents to a circle.

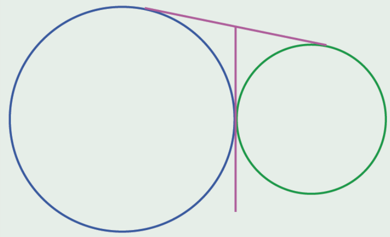


Calculate the length of each tangent from the corner of the triangle to the point of contact.

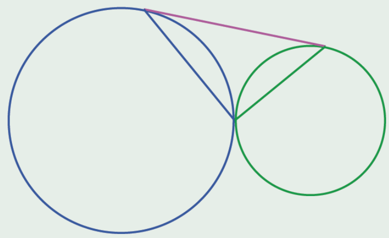
(3) In the picture, two circles touch at a point and the common tangent at this point is drawn.



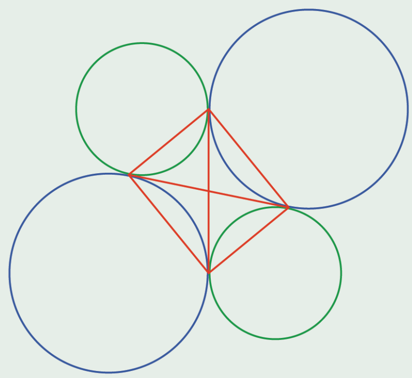
i) Prove that this tangent bisects another common tangent of these circles.



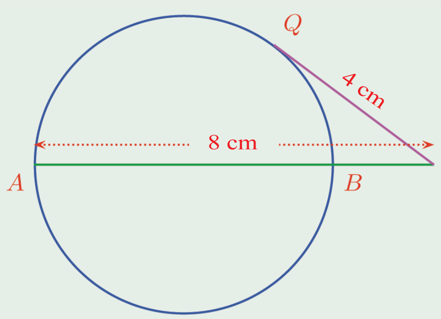
ii) Prove that the points of contact of these two tangents form the vertices of a right triangle.



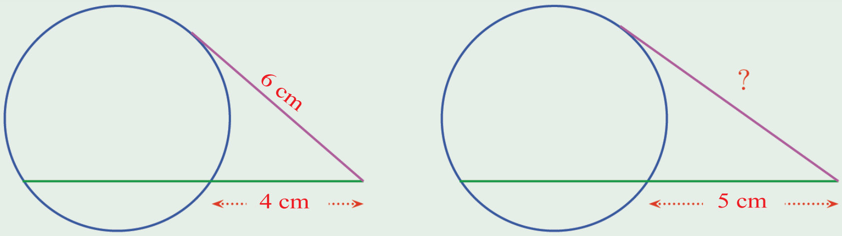
iii) Draw the picture on the right in your notebook, using convenient lengths.



(4) In the picture below, AB is a diameter and P is a point on AB extended. A tangent from P touches the circle at Q. What is the radius of the circle?



(5) In the first picture below, the line joining two points on a circle is extended by 4 centimetres and a tangent is drawn from this point. Its length is 6 centimetres, as shown:



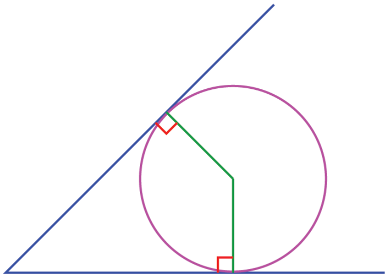
The second picture shows the same line extended by 1 centimetre more and a tangent drawn from this point. What is the length of this tangent?

(6) Draw a rectangle of one side 6 centimetres and area equal to that of a square of side 5 centimetres.

Circle touching a line

We have seen that from a point, two lines touching a circle can be drawn and also how we can draw these lines.

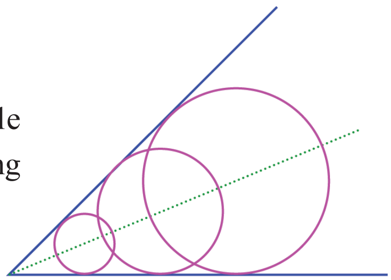
Now let's ask in reverse: can we draw a circle touching two lines meeting at a point? See this picture.



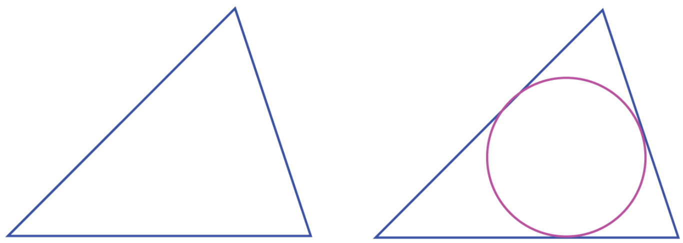
The radii are perpendicular to these lines. In other words, the centre of the circle must be at the same distance from these lines. So it must be on the bisector of this angle.

The centre of a circle touching two lines meeting at a point is on the bisector of the angle formed by the lines.

We can take any point on the angle bisector as centre to draw a circle touching the lines.

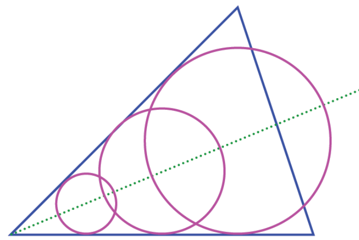


So, the next question is whether we can draw a circle touching all three sides of a triangle.

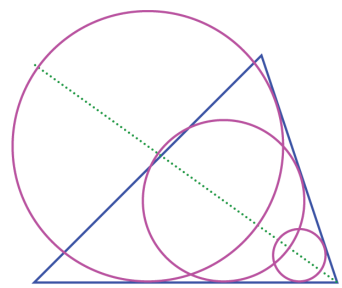


Draw an angle and its bisector in GeoGebra. Mark a point on this bisector and draw the perpendicular from this point to one of the sides of the angle. Mark their point of intersection. Draw the circle with centre at the point on the bisector and passing through the point on the side. Doesn't it touch both sides? Move the centre along the bisector and see.

We can take any point on the bisector of the angle made by the bottom and left side to draw a circle touching these two sides.



Taking any point on the bisector of the angle made by the bottom and right side as centre, we can draw a circle touching these two sides.

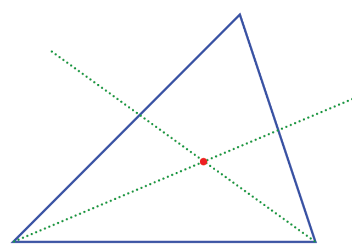


So, what if we take the point of both these bisectors, that is their point of intersection?

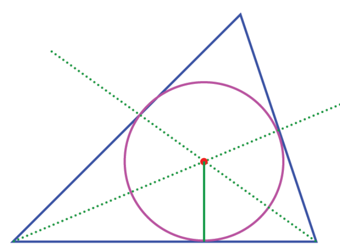
The lengths of the perpendiculars from this point to all three sides are equal, right?

What about the circle of radius this length, centred at this point?

This circle is called the *incircle* of the triangle.



We note another thing here. Since the centre of the incircle is at the same distance from the left and right sides, it is also on the bisector of the angle joining these sides.

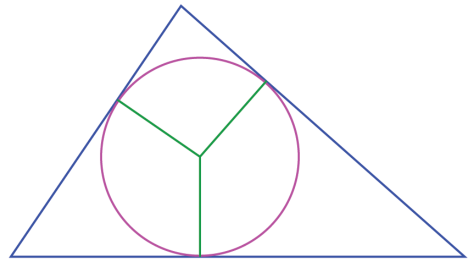


Draw a triangle and its incircle in GeoGebra.

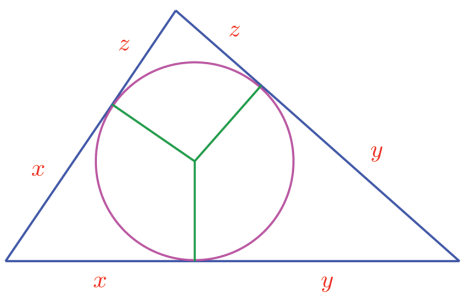
The bisectors of all three angles of a triangle meet at a point.

There are some relations between the points where the incircle touches the triangle and the sides of the triangle.

To see it, we draw the incircle of a triangle and join its centre to the points of contact with the sides.

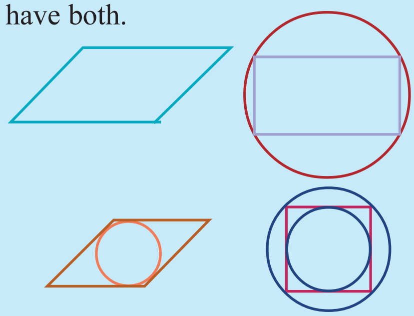


The sides of the triangle are formed by the tangents to the incircle from the vertices. And the lengths of the tangents from each corner are equal. So taking the lengths of the tangents as x, y, z, we can mark them as below:



Circumcircle and incircle

Every triangle has a circumcircle and an incircle. But in the case of quadrilaterals, some have neither of these, some have only one and some have both.



The sum of all these lengths is the perimeter of the triangle. That is, the perimeter of the triangle is $2(x+y+z)$. In other words, $x+y+z$ is half the perimeter of the triangle. Taking it as s,

$$x + y + z = s$$

Next if we take the lengths of the sides of the triangle as a, b, c, the picture gives

$$x + y = a$$

$$y + z = b$$

$$z + x = c$$

Now to get x, we need only subtract $y + z$ from $x + y + z$:

$$x = (x + y + z) - (y + z) = s - b$$

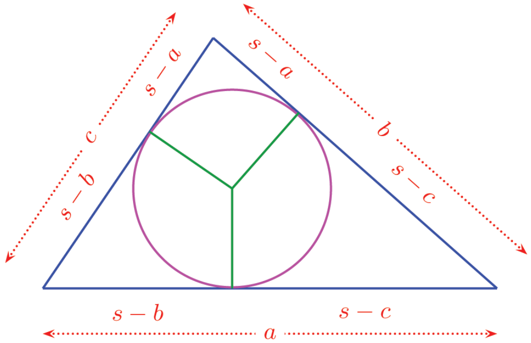
Similarly, we see that

$$y = (x + y + z) - (z + x) = s - c$$

and

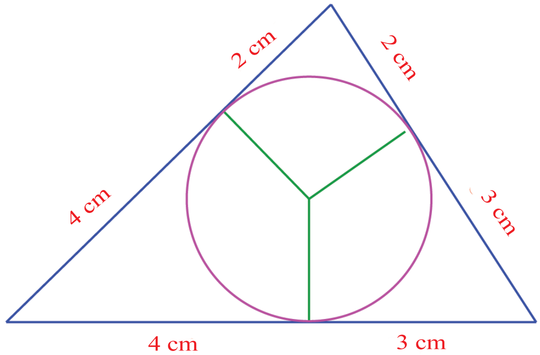
$$z = (x + y + z) - (x + y) = s - a$$

So the lengths of tangents are like this:

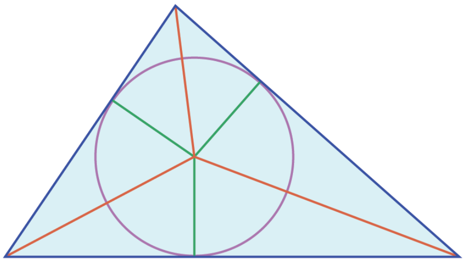


For example, consider a triangle of sides 5 centimetres, 6 centimetres and 7 centimetres. Half the perimeter is 9 centimetres.

So the points of contact with the incircle divides the sides like this: $9 - 5 = 4$, $9 - 6 = 3$, $9 - 7 = 2$



The radius of the incircle has a relation with the area of the triangle. The lines joining the centre of the incircle to the vertices divide the triangle into three.



One side of each of these small triangles is a side of the original large triangle and the height from it is equal to the radius of the incircle. So, if we take the sides of the triangle as a, b, c and the radius of the incircle as r, then the areas of these small triangles are $\frac{1}{2}ar$, $\frac{1}{2}br$, $\frac{1}{2}cr$.

Their sum is the whole area of the large triangle. Taking it as A, we have

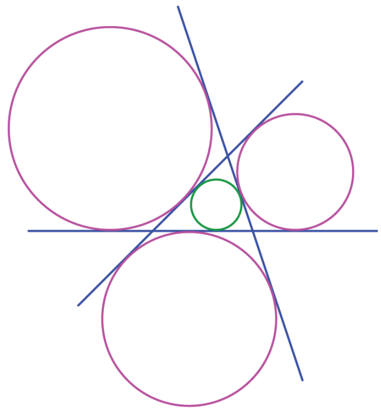
$$A = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = \frac{1}{2}(a+b+c)\,r = sr$$

This equation can written as

$$r = \frac{A}{s}$$

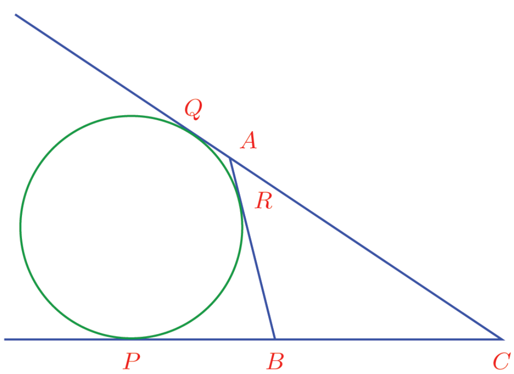
The radius of the incircle of a triangle is its area divided by half the perimeter.

The incircle of a triangle is the circle inside the triangle, which touches all three sides. If the requirement is only touching all three sides, then we can draw three more such circles.



These are called *excircles* of the triangle. They are drawn using the bisectors of the external angles.

Consider a triangle and one of its excircles:



P, Q, R are the points where the circle touches the sides of the triangle.

Let's find the lengths of the tangents CP and CQ. Taking the length BC, CA, AB of the triangle as a, b, c, we get

$$CP = CB + BP = a + BP \qquad CQ = CA + AQ = b + AQ$$

Now $BP = BR$, being tangents from B and $AQ = AR$, being tangents from A. So,

$$CP = a + BR \qquad CQ = b + AR$$

We also see that $AR + RB = AB$. This together with the two equations above gives

$$CP + CQ = a + b + BR + AR = a + b + c$$

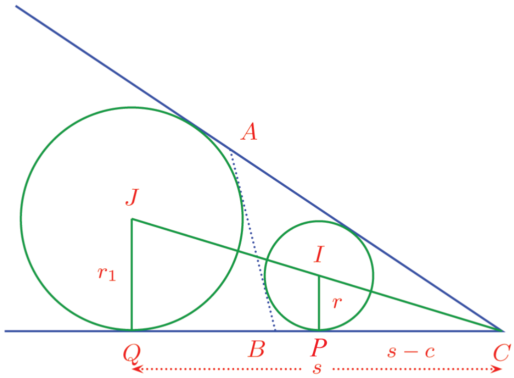
But this is the perimeter of the triangle. Also, CP and CQ have the same length. So, what do we get?

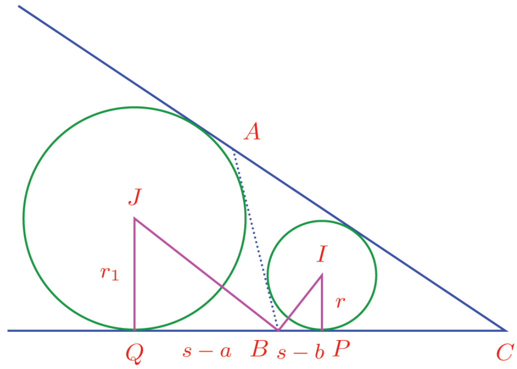
$$CP = CQ = s$$

Thus we have the following:

> The lengths of the tangents from a vertex of the triangle to the excircle opposite it are equal to the perimeter of the triangle.

Now let's draw the incircle also. Let r be the radius of the incircle and r_1, the radius of the excircle.





ΔCIP and ΔCJQ of the picture on the left have the same angles, and so ratio of sides opposite equal angles are also the same.

(1) $$\frac{r}{r_1} = \frac{s-c}{s}$$

Now in the picture on the right, look at the angles of ΔBIP and ΔBJQ at B. Since BI, BJ are the bisectors of the internal and external angles at B of ΔABC,

$$\angle QBJ = \frac{1}{2}\angle QBA = \frac{1}{2}(180^\circ - \angle CBA) = 90^\circ - \angle PBI$$

Thus ΔPBI and ΔQBJ have the same angles, which gives

$$\frac{r}{s-a} = \frac{s-b}{r_1}$$

Using cross multiplication, we get

(2) $$rr_1 = (s-a)(s-b)$$

Now from equations (1) and (2), we have

$$\frac{r}{r_1} \times rr_1 = \frac{s-c}{s} \times (s-a)(s-c)$$

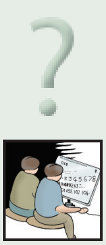
That is,

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s}$$

We have already noted that the area of ΔABC is rs. Using the above equation the area is

$$\sqrt{s(s-a)(s-b)(s-c)}$$

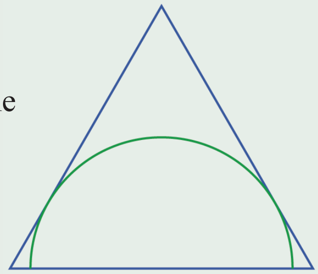
This method of computing the area of a triangle using the lengths of its side is known as Heron's Formula.



(1) Draw a triangle of sides 4 centimetres, 5 centimetres, 6 centimetres and draw its incircle. Calculate its radius.

(2) Draw a rhombus of sides 5 centimetres and one angle 50° and draw its incircle.

(3) Draw an equilateral triangle and a semicircle touching its two sides, as in the picture.



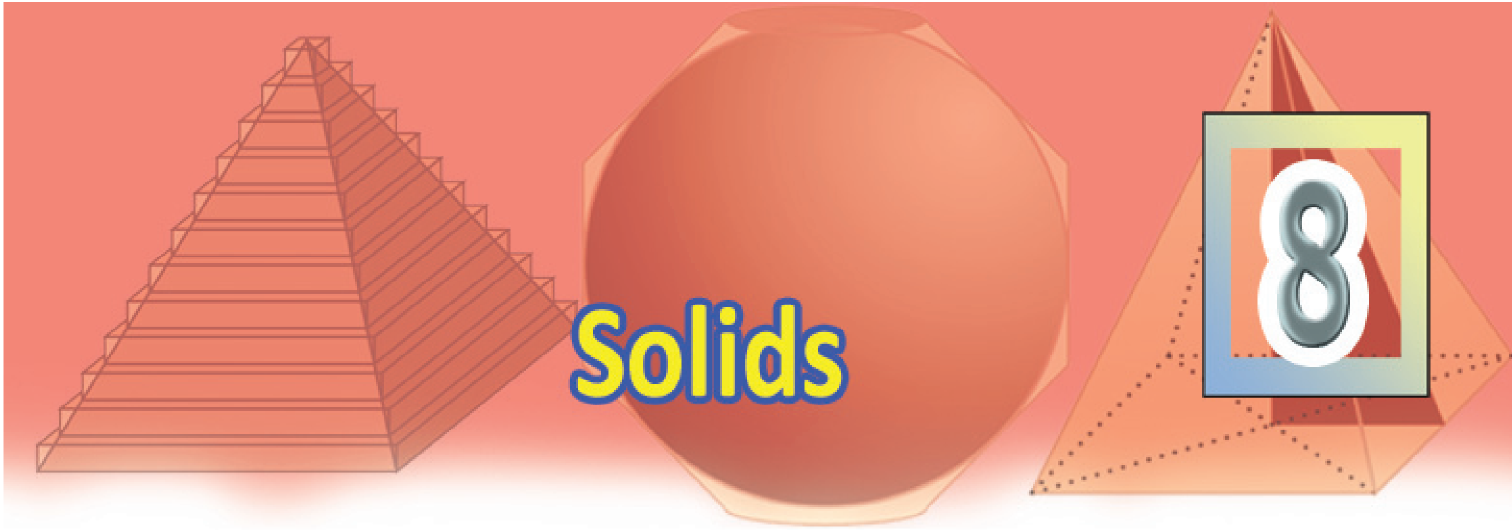
(4) Prove the radius of the incircle of an equilateral triangle is half the radius of its circumcircle.

(5) Prove that if the hypotenuse of a right triangle is h and the radius of its incircle is r, then its area is $r(h+r)$.

(6) Calculate the area of a triangle of sides 13 centimetres, 14 centimetres, 15 centimetres.

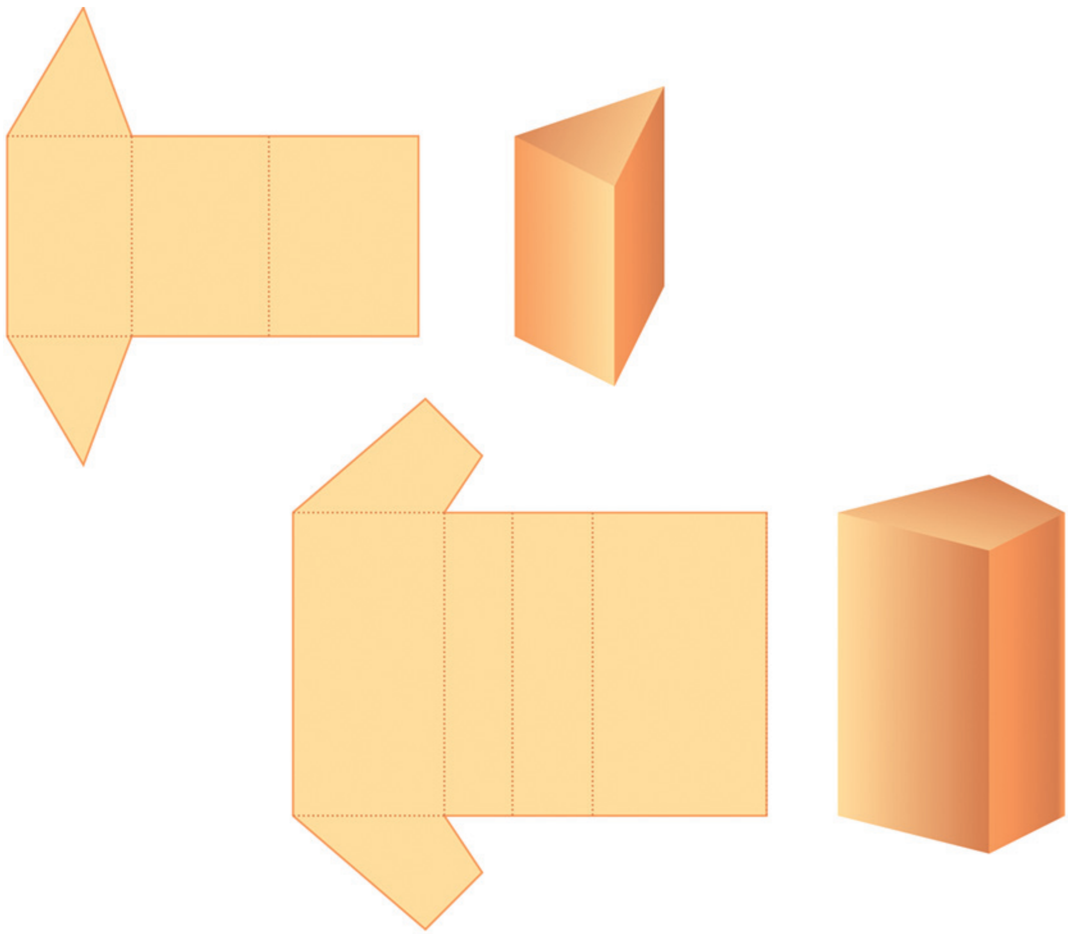
Looking back

Learning outcomes	On my own	With teacher's help	Must improve
• Understanding what happens to a line joining two points on a circle when they are closer and closer, and hence the idea of a tangent.			
• Realising that the tangent at a point on a circle and the radius through the point are perpendicular to each other.			
• Understanding the relations between the angle between the tangents to a circle at the ends of a chord, the central angle of the chord and the angle which the chord makes at a point on the circle.			
• Realising that two tangents can be drawn to a circle from a point outside and actually drawing them.			
• Drawing a circle touching all three sides of a triangle.			



Pyramids

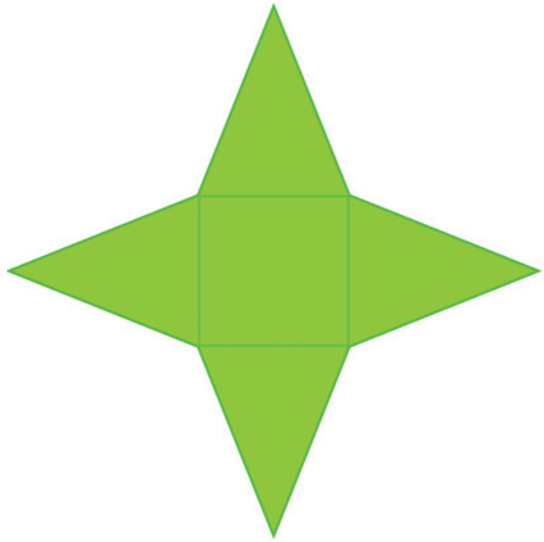
We can make prisms by cutting thick paper in various ways and pasting the edges.



And we have learnt much about them.

Let's make another kind of solid:

First, cut out a figure like this in paper.

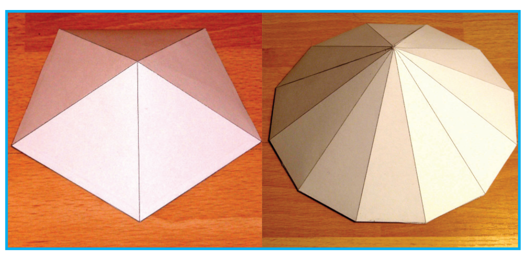


A square in the middle and four triangles around it; all four of them are isosceles triangles and they are equal. Now fold and paste as shown below.

Pyramids in GeoGebra

We have seen in Class 9 how prisms can be drawn in GeoGebra. Now let's see how pyramids can be drawn. Open 3D Graphics and make the necessary preparations (See the section, **Solids in GeoGebra** of the lesson, **Prisms** in the Class 9 textbook) Draw a square in Graphics. Choose **Extrude to Pyramid or Cone** in 3D Graphics and click on the square. In the window opening up, type the height of the pyramid. (we can also make a slider and give its name as the height).

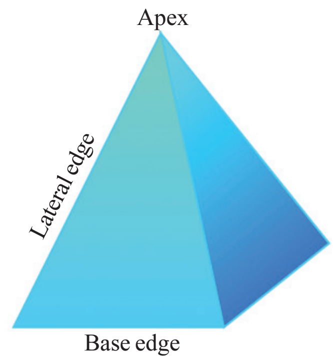
What shape is this? Can't be called a prism; prisms have two equal bases and rectangles on the sides. In the shape we have made now, we have a square at the bottom, a point on top and triangles all around.



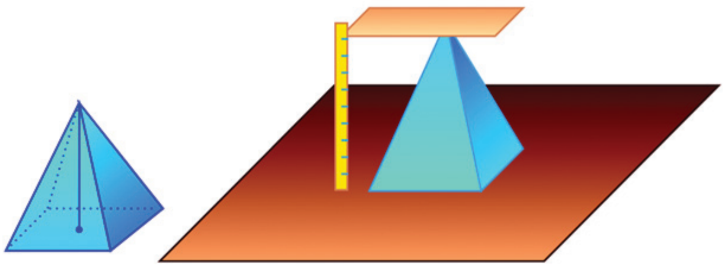
Instead of square, the base can be some other rectangle, a triangle or some other polygon. Try! (It is better looking when the base is a regular polygon.)

Such a solid is generally called a *pyramid*.

The sides of the polygon forming the base of a pyramid are called *base edges* and the other sides of the triangles are called *lateral edges*. The topmost point of a pyramid is called its *apex*.

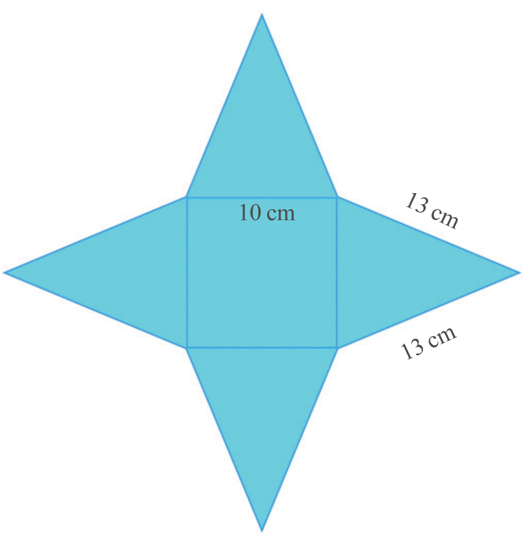


The height of a prism is the distance between its bases, isnt it? The height of a pyramid is the perpendicular distance from the apex to the base.



Area

What is the surface area of a square pyramid of base edges 10 centimetres and lateral edges 13 centimetres? Surface area is the area of paper needed to make it. How will it look, if we cut this pyramid open and lay it flat?



Let's see how GeoGebra helps us to see the 'cut and spread' shape of a pyramid. Make a pyramid in 3D graphics as described earlier. Choose **Net** and click on the pyramid. We get the shape of the paper used to make it (It is called the net of the solid). We also get a slider in **Graphics**. By moving the slider, we can see how the pyramid is made from the net. We can also hide the original pyramid by clicking against the pyramid in the **Algebra** window.

The area of the square is easily seen to be 100 square centimetres. What about the triangles?

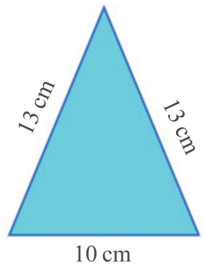
The sides of the triangle are 10, 13, 13 centimetres. We have Heron's help to compute the areas; subtracting each side from half the perimeter, we can calculate the area as

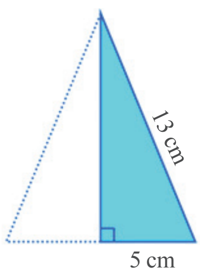
$$\sqrt{18\times8\times5\times5} = \sqrt{9\times16\times5\times5} = 60$$

Thus the area of each triangular face is 60 square centimetres. So the surface area of the pyramid is,

$$100 + (4\times60) = 340 \text{ square centimetres}$$

The area of the triangle can also be computed as half the product of its base and height.





For that, we need the height of the triangle. Since the triangle is isosceles, this altitude bisects the base.

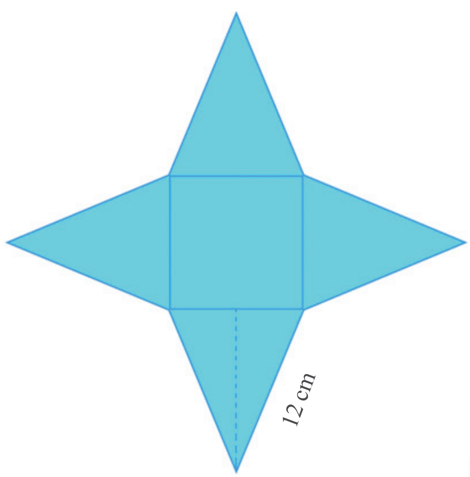
So using Pythagoras Theorem, the height of the triangle is

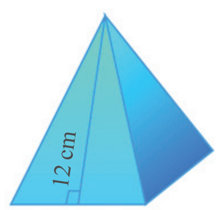
$$\sqrt{13^2-5^2} = 12 \text{ centimetres}$$

Height and slant height

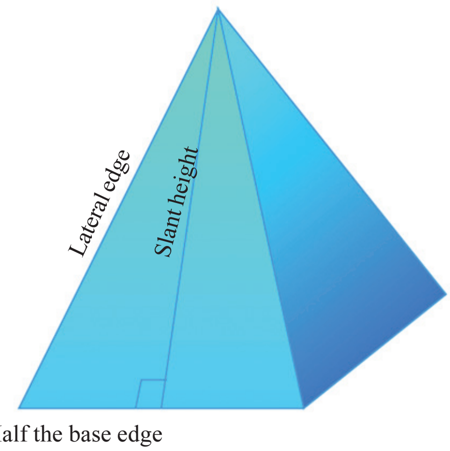
Draw a pyramid in GeoGebra. Click **Midpoint or Centre** to mark the mid points of a base edge and the midpoint of a diagonal of the base. Use **Segment** to mark the height and slant height of the pyramid. Use **Polygon** to make the right triangle with height, slant height and half the base edge as sides. Using **Net**, the pyramid can be cut and spread. We can also hide the pyramid.

Thus the area of the triangle is 60 square centimetres. What will be the height of the triangle, when the paper is turned into a pyramid?



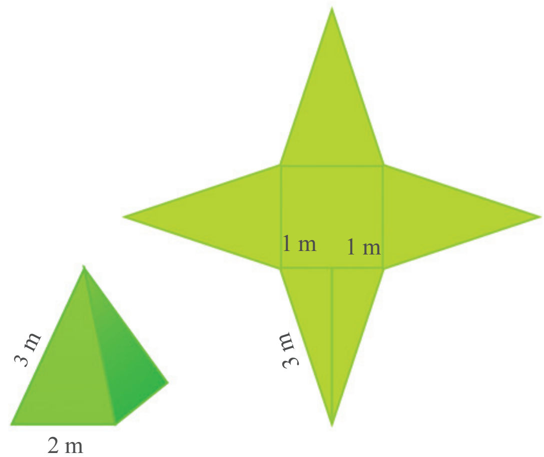


This length is called the *slant height* of the pyramid.



We have seen the relation between the base edge, lateral edge and slant height of a pyramid in the problem we did just now. As shown in the picture on the right, there is a right triangle on each side of the pyramid - its perpendicular sides are the slant height and half the base edge, the hypotenuse is a lateral edge.

Now do this problem: what is the surface area of a pyramid with base edges 2 metres and lateral edges 3 metres?



The base area is 4 square metres. To compute the areas of lateral faces, we need the slant height. In the right triangle mentioned above, one side is half the base edge, that is, 1 metre and the hypotenuse is the lateral edge of 3 metres. So, slant height is

$$\sqrt{3^2 - 1^2} = 2\sqrt{2} \text{ metres}$$

Using this, the area of each triangular face is

$$\frac{1}{2} \times 2 \times 2\sqrt{2} = 2\sqrt{2} \text{ square metres.}$$

So, the surface area of the pyramid is,

$$4 + \left(4 \times 2\sqrt{2}\right) = 4 + 8\sqrt{2} \text{ square metres.}$$

If not satisfied with this, a calculator can be used, (or an approximate value of $\sqrt{2}$ recalled) to compute this as 15.1 square metres.

(1) A square of side 5 centimetres, and four isosceles triangles of base 5 centimetres and height 8 centimetres, are to be put together to make a square pyramid. How many square centimetres of paper is needed?

(2) A toy is in the shape of a square pyramid of base edge 16 centimetres and slant height 10 centimetres. What is the total cost of painting 500 such toys, at 80 rupees per square metre?

(3) The lateral faces of a square pyramid are equilateral triangles and the length of a base edge is 30 centimetres. What is its surface area?

(4) The perimeter of the base of square pyramid is 40 centimetres and the total length of all its edges is 92 centimetres. Calculate its surface area.

(5) Can we make a square pyramid with the lateral surface area equal to the base area?

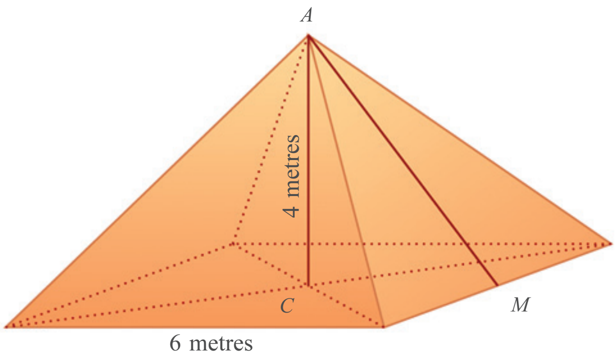
Height and slant height

The height of a pyramid is often an important measure. See this problem:

> A tent is to be made in the shape of a square pyramid of base edges 6 metres and height 4 metres. How many square metres of canvas is needed to make it?

To calculate the area of the triangular faces of the tent, we need the slant height. How do we compute it using the given specifications?

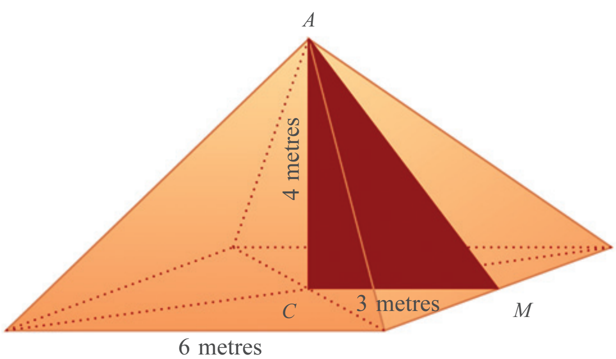
See this picture:



The slant height we need is AM. Joining CM, we get a right triangle with AM as hypotenuse. What is the length of CM in it?

Pyramids of Egypt

The very word pyramid brings to our mind the great pyramids of Egypt. 138 such pyramids are found from various parts of Egypt. Many of them were built around 2000 BC.

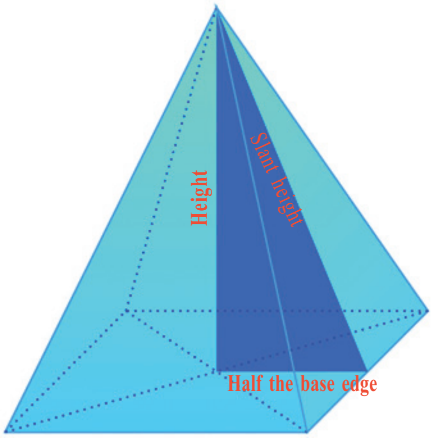


From the picture, $AM = \sqrt{3^2 + 4^2} = 5$ metres.

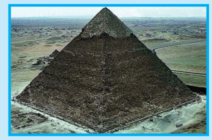
So to make the tent, four isosceles triangles of base 6 metres and height 6 metres are needed. Their total area is $4 \times \frac{1}{2} \times 6 \times 5 = 60$ square metres.

So this much canvas is needed to make the tent.

In this problem, we have found something which is true in the case of all square pyramids. Within every square pyramid, we can imagine a right triangle with perpendicular sides as the height of the pyramid and half the base edge and hypotenuse as the slant height.



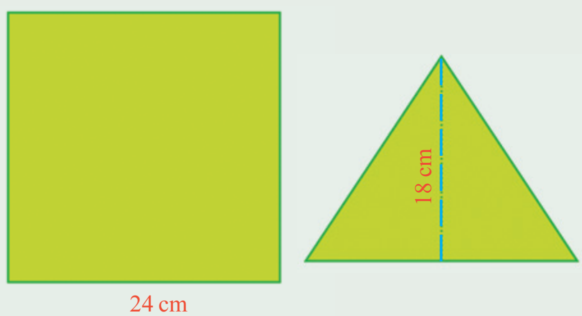
Great Pyramid



The largest pyramid in Egypt is the Great Pyramid of Giza.

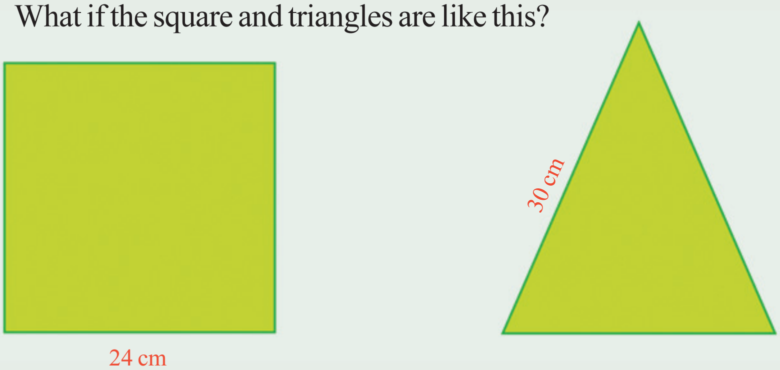
Its base is a square of almost half a lakh square metres and its height is about 140 metres. It is estimated that about 20 years would have been needed to complete it. These royal tombs built with huge blocks of stones stacked with precision to end in a point are living symbols of human labour, engineering skill and mathematical knowledge.

(1) Using a square and four triangles with dimensions as specified in the picture, a pyramid is made.



What is the height of this pyramid?

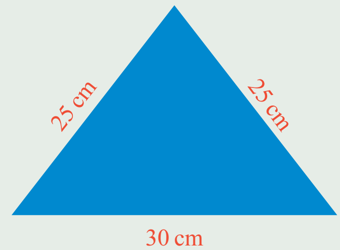
What if the square and triangles are like this?



(2) A square pyramid of base edge 10 centimetres and height 12 centimetres is to be made of paper. What should be the dimensions of the triangles?

(3) Prove that in any square pyramid, the squares of the height, slant height and lateral edge are in arithmetic sequence.

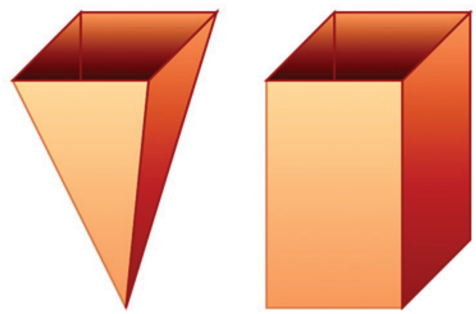
(4) A square pyramid is to be made with the triangle shown here as a lateral face. What would be its height? What if the base edge is 40 centimetres instead of 30 centimetres?



Can we make a square pyramid with any four equal isosceles triangles?

Volume of a pyramid

We have seen that the volume of any prism is equal to the product of the base area and the height. What about the volume of a pyramid?



Let's take the case of a square pyramid. Make a hollow square pyramid with thick paper and also a square prism of the same base and height.

Draw a square pyramid and square prism of the same base and with the same height in GeoGebra. To distinguish between them, change the colour of the pyramid and make Opacity 100. (**Object properties** → **Colour**). Find their volumes using **Volume.** What is the relation between them? Change the base and height and see.

Fill the pyramid with sand and transfer it to the prism. Measure the height of the sand in the prism and see what fraction of the height of the prism it is. A third isn't it? So to fill the prism, how many times should we fill the pyramid?

Thus we see that the volume of the prism is three times the volume of the pyramid. (A mathematical explanation of this is given at the end of this lesson).

We have seen in Class 9 that the volume of a prism is equal to the product of the base area and the height.

So what can we say about the volume of a square pyramid?

The volume of a square pyramid is equal to a third of the product of the base area and the height.

For example, the volume of a square pyramid of base edge 10 centimetres and height 8 centimetres is $\frac{1}{3} \times 10^2 \times 8 = 266\frac{2}{3}$ cubic centimetres.

A metal cube of edges 15 centimetres is melted and recast into a square pyramid of base edge 25 centimetres. What is its height?

The volume of the cube is 15^3 cubic centimetres. The volume of the square pyramid is also this. And the volume of a pyramid is a third of the product of the base area and height. Since the base area of our pyramid is 25^2 square centimetres, a third of the height is $\frac{15^3}{25^2}$ and so the height is,

$$3 \times \frac{15^3}{25^2} = 16.2 \text{ centimetres.}$$

?

(1) What is the volume of a square pyramid of base edge 10 centimetres and slant height 15 centimetres?

(2) Two square pyramids have the same volume. The base edge of one is half that of the other. How many times the height of the second pyramid is the height of the first?

(3) The base edges of two square pyramids are in the ratio 1 : 2 and their heights in the ratio 1 : 3. The volume of the first is 180 cubic centimetres. What is the volume of the second?

(4) All edges of a square pyramid are 18 centimetres. What is its volume?

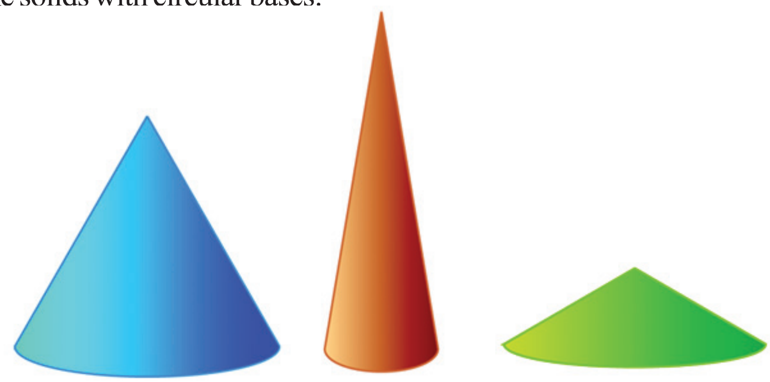
(5) The slant height of a square pyramid is 25 centimetres and its surface area is 896 square centimetres. What is its volume?

(6) All edges of a square pyramid are of the same length and its height is 12 centimetres. What is its volume?

(7) What is the surface area of a square pyramid of base perimeter 64 centimetres and volume 1280 cubic centimetres?

Cone

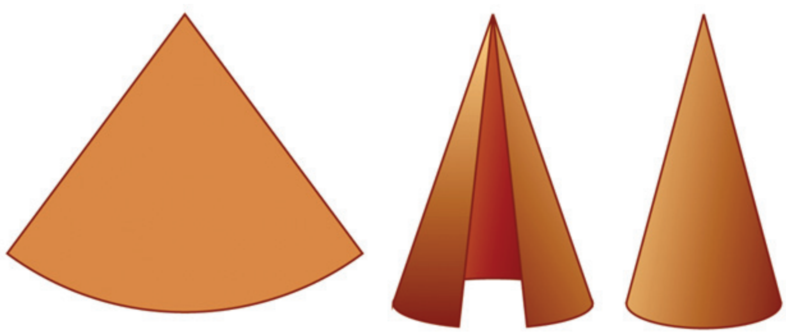
Cylinders are prism-like solids with circular bases. Similarly, we have pyramid-like solids with circular bases:



They are called *cones*.

We can make a cylinder by rolling up a rectangle. Likewise, we can make a cone by rolling up a sector of a circle.

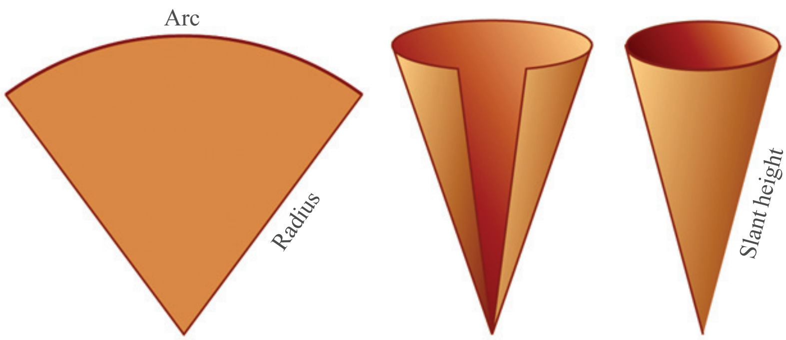
What is the relation between the dimensions of the sector we start with and the cone we end up with?



Cone

We can draw cones in GeoGebra, just as we drew pyramids. Draw a circle in **Graphic** and in **3D Graphics**, use **Extrude to Pyramid or Cone**. Base radius and height can be changed using sliders.

The radius of the sector becomes the slant height of the cone. The arc length of the sector becomes the circumference of the base of the cone.



We often specify the size of a sector in terms of the central angle. See this problem:

> From a circle of radius 12 centimetres, a sector of central angle 45° is cut out and made into a cone. What are the slant height and base radius of this cone?

Slant height of the cone is the radius of the circle itself: 12 centimetres. What about its base radius?

45° is $\frac{1}{8}$ of 360°. And the arc length of a sector is proportional to the central angle. So this arc length is $\frac{1}{8}$ of the circumference of the full circle.

This arc becomes the base circle of the cone. Thus the circumference of the base circle of the cone is $\frac{1}{8}$ of the circumference of the larger circle from which the sector was cut out. Since radii of circles are proportional to their circumferences, the radius of the smaller circle is $\frac{1}{8}$ of the radius of the large circle. Thus the radius of the base of the cone is $\frac{1}{8} \times 12 = 1.5$ centimetres.

How about a question in the reverse direction?

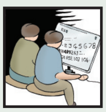
> How do we make a cone of base radius 5 centimetres and slant height 15 centimetres?

To make a cone, we need a sector. Since the slant height is to be 15 centimetres, the sector must be cut out from a circle of radius 15 centimetres.

What should be its central angle?

The radius of the small circle forming the base of the cone is $\frac{5}{15} = \frac{1}{3}$ of the radius of the large circle from which the sector is to be cut out. (How do we get this?). So, the circumference of the small circle is also $\frac{1}{3}$ of the circumference of the large circle.

The circumference of the small circle is the arc length of the sector. Thus the arc of the sector is $\frac{1}{3}$ of the circle from which it is cut out. So its central angle must be $360 \times \frac{1}{3} = 120°$.



(1) What are the radius of the base and slant height of a cone made by rolling up a sector of central angle 60° cut out from a circle of radius 10 centimetres?

(2) What is the central angle of the sector to be used to make a cone of base radius 10 centimetres and slant height 25 centimetres?

(3) What is the ratio of the base-radius and slant height of a cone made by rolling up a semicircle?

Curved surface area

As in the case of a cylinder, a cone also has a curved surface - the part which rises up at a slant. The area of this curved surface is the area of the sector used to make the cone. (For a cylinder also, the area of the curved surface is the area of the rectangle rolled up to make it, isn't it?)

See this problem:

> To make a conical hat of base radius 8 centimetres and slant height 30 centimetres, how much square centimetres of paper do we need?

What we need here is the area of the circular sector we roll up to make this hat. Since the slant height is to be 30 centimetres, we must cut out the sector from a circle of this radius. Also the radius of the small circle forming the base of the cone must be 8 centimetres, that is $\frac{8}{30} = \frac{4}{15}$ of the radius of the large circle from which the sector is cut out. So the circumference of small circle is also the same fraction of the circumference of the large circle. The arc length of the sector is the circumference of the small circle. Thus the sector to be cut out is $\frac{4}{15}$ of the full circle. So, its area is the fraction of the area of the circle; that is,

$$\pi \times 30^2 \times \frac{4}{15} = \pi \times 2 \times 30 \times 4 = 240\pi$$

Thus we need 240π square centimetres of paper to make the hat (It can be computed as approximately 754 square centimetres).

Curved surface

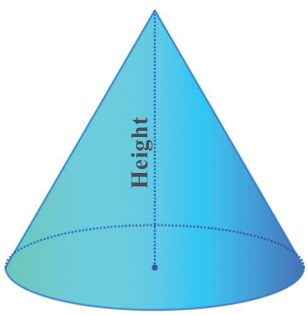
The area of the curved surface of a cone is the area of the sector used to make it. If we take the base radius of the cone as r and its slant height as l, then the radius of the sector is l and its central angle is $\frac{r}{l} \times 360^\circ$. So its area is

$$\frac{1}{360} \times \left(\frac{r}{l} \times 360\right) \times \pi l^2 = \pi r l$$

(Recall the computation of the area of a sector in Class 9).

Thus the area of the curved surface of a cone is half the product of the base circumference and the slant height.

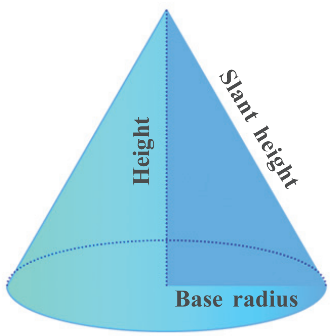
As in a pyramid, the height of a cone is the perpendicular distance from the apex to the base, and it is the distance between the apex and the centre of the base circle.



Again as in the case of a square pyramid, the height is related to the slant height via a right triangle.

For example, in a cone of base radius 5 centimetres and height 10 centimetres the slant height is,

$$\sqrt{5^2 + 10^2} = \sqrt{125} = 5\sqrt{5} \text{ centimetres}$$



(1) What is the area of the curved surface of a cone of base radius 12 centimetres and slant height 25 centimetres?

(2) What is the surface area of a cone of base diameter 30 centimetres and height 40 centimetres?

(3) A conical fire work is of base diameter 10 centimetres and height 12 centimetres. 10000 such fireworks are to be wrapped in colour paper. The price of the colour paper is 2 rupees per square metre. What is the total cost?

(4) Prove that for a cone made by rolling up a semicircle, the area of the curved surface is twice the base area.

Volume of a cone

To find the volume of a cone, we can do an experiment similar to the one we did to find the volume of a square pyramid. Make a cone and a cylinder of the same base and height. Fill the cone with sand and transfer it to the cylinder. Here also, we can see that the volume of the cone is a third of the volume of the cylinder. Thus we have the following:

The volume of a cone is equal to a third of the product of the base area and height.

As in the case of square pyramids, draw a cylinder and a cone of same base and height in GeoGebra. Compare their volumes.

(A mathematical explanation of this also is given at the end of this lesson)

For example, the volume of a cone of base radius 4 centimetres and height 6 centimetres is

$$\frac{1}{3} \times \pi \times 4^2 \times 6 = 32\pi \text{ cubic centimetres.}$$

(1) The base radius and height of a cylindrical block of wood are 15 centimetres and 40 centimetres. What is the volume of the largest cone that can be carved out of this?

(2) The base radius and height of a solid metal cylinder are 12 centimetres and 20 centimetres. By melting it and recasting, how many cones of base radius 4 centimetres and height 5 centimetres can be made?

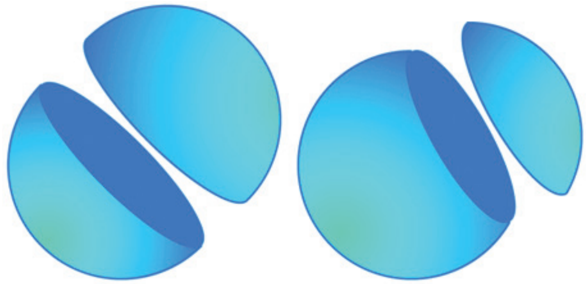
(3) A sector of central angle 216° is cut out from a circle of radius 25 centimetres and is rolled up into a cone. What are the base radius and height of the cone? What is its volume?

(4) The base radii of two cones are in the ratio 3 : 5 and their heights are in the ratio 2 : 3. What is the ratio of their volumes?

(5) Two cones have the same volume and their base radii are in the ratio 4 : 5. What is the ratio of their heights?

Sphere

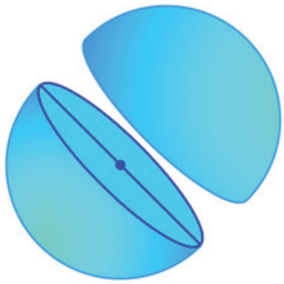
Round solids enter our lives in various ways - as the thrill of ball games and as the sweetness of laddus. Now let's look at the mathematics of such solids called *spheres*.



It we slice cylinder or cone parallel to a base, we get a circle. In whatever way we slice a sphere, we get a circle.

The distance of any point on a circle from the centre is the same. A sphere also has a *centre*, from which the distance to any point on its surface is the same. This distance is called the *radius* of the sphere and double this is called the *diameter*.

If we slice a sphere into exact halves, we get a circle whose centre, radius and diameter are those of the sphere itself.



We cannot cut open a sphere and spread it flat, as we did with other solids. The fact is that we cannot make the surface of a sphere flat without some folding or stretching.

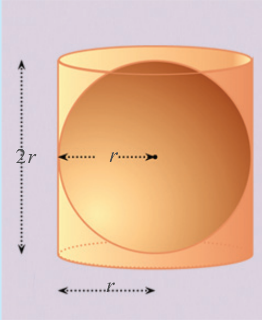
But we can prove that the surface area of a sphere of radius r is $4\pi r^2$ (An explanation is given at the end of the lesson).

The surface area of a sphere is equal to the square of its radius multiplied by 4π.

Also, we can prove that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ (An explanation of this also is given at the end of the lesson.)

Sphere and cylinder

Consider a cylinder which can cover a sphere precisely. Its base radius is the radius of the sphere and its height is double its radius.



So if we take the radius of the sphere as r, the base radius and height of the cylinder are r and $2r$.

So its surface area is

$$(2\pi r \times 2r) + (2 \times \pi r^2) = 6\pi r^2$$

The surface area of the sphere is $4\pi r^2$. Thus the ratio of these surface areas is 3 : 2.

Again, the volume of the cylinder is

$$\pi r^2 \times 2r = 2\pi r^3$$

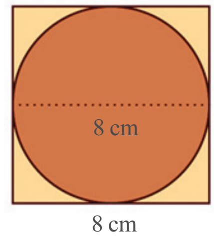
and the volume of the sphere is $\frac{4}{3}\pi r^3$, so that the ratio of the volumes is also 3 : 2

See this problem:

What is the surface area of the largest sphere that can be carved from a cube of edges 8 centimetres?

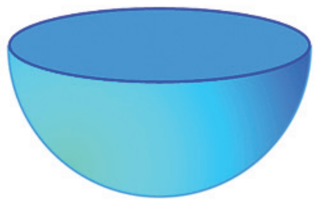
We can see from the picture that the diameter of the sphere is the length of an edge of the cube. So, the surface area of the sphere is

$4\pi \times 4^2 = 64\pi$ square centimetres



Another problem:

A solid sphere of radius 12 centimetres is cut into two equal halves. What is the surface area of each hemisphere?



The surface of the hemisphere consists of half the surface of the sphere and a circle.

Since the radius of the sphere is 12 centimetres, its area is,

$$4\pi \times 12^2 = 576\pi \text{ square centimetres}$$

Since the radius of the circle is 12 centimetres, its area is

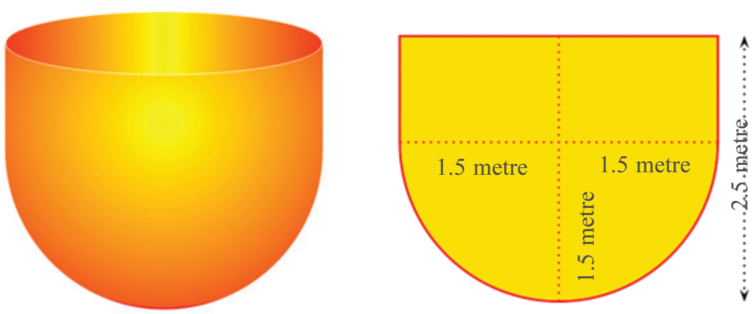
$$\pi \times 12^2 = 144\pi \text{ square centimetres}$$

So the surface area of the hemisphere is,

$$\frac{1}{2} \times 576\pi + 144\pi = 432\pi \text{ square centimetres}$$

One more example:

A water tank is in the shape of a hemisphere attached to a cylinder. Its radius is 1.5 metres and the total height is 2.5 metres. How many litres of water can it hold?



The volume of the hemispherical part of the tank is,

$$\frac{2}{3}\pi\times(1.5)^3 = 2.25\pi \text{ cubic metres.}$$

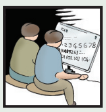
And the volume of the cylindrical part is

$$\pi\times(1.5)^2(2.5 - 1.5) = 2.25\pi \text{ cubic metres.}$$

So, the total volume is

$$2.25\pi + 2.25\pi = 4.5\pi \approx 14.13 \text{ cubic metres.}$$

Since one cubic metre is 1000 litres, the tank can hold about 14130 litres of water.



(1) The surface area of a solid sphere is 120 square centimetres. If it is cut into two halves, what would be the surface area of each hemisphere?

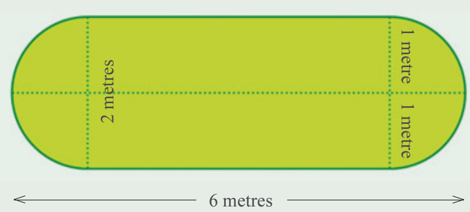
(2) The volume of two spheres are in the ratio 27 : 64. What is the ratio of their radii? And the ratio of their surface areas?

(3) The base radius and length of a metal cylinder are 4 centimetres and 10 centimetres. If it is melted and recast into spheres of radius 2 centimetres, how many spheres can be made?

(4) A metal sphere of radius 12 centimetres is melted and recast into 27 small spheres. What is the radius of each sphere?

(5) From a solid sphere of radius 10 centimetres, a cone of height 16 centimetres is carved out. What fraction of the volume of the sphere is the volume of the cone?

(6) The picture shows the dimensions of a petrol tank.



How many litres of petrol can it hold?

(7) A solid sphere is cut into two hemispheres. From one, a square pyramid and from the other a cone, each of maximum possible size are carved out. What is the ratio of their volumes?

What is the speciality of the square pyramid of maximum volume that can be cut out from a solid hemisphere?

Looking back

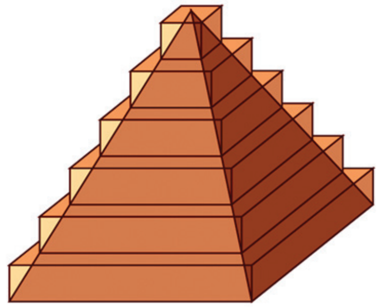
Learning outcomes	On my own	With teacher's help	Must improve
• Finding the relation between the base edge, lateral edge, height, slant height and base diagonal of a square pyramid.			
• Making square pyramids of specified dimensions by cutting out squares and isosceles triangles.			
• Computing the surface area and volume of a square pyramid using given measurements.			
• Computing the dimensions of sectors needed to make cones of specified dimensions.			
• Explaining the relation between the base radius, height and slant height of a cone.			
• Explaining the methods to compute surface areas and volumes of cones.			
• Explaining the methods to compute the surface areas and volumes of spheres and hemispheres.			

Appendix

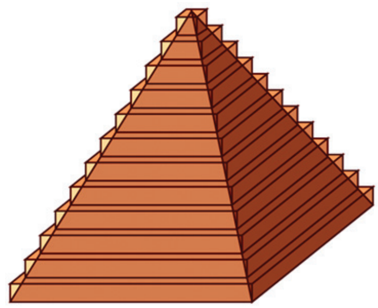
We have seen only the techniques of calculating volumes of pyramids and cones, and also the surface area and volume of a sphere. For those who may be interested in knowing how they are actually got, we give some explanations below.

Volume of a pyramid

We can think of a stack of square plates, of decreasing size as an approximation to a square pyramid.



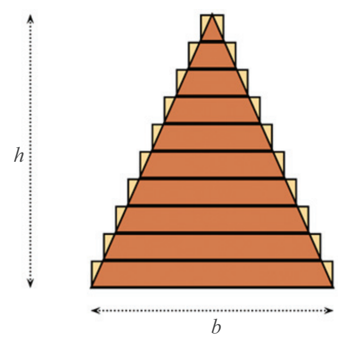
As we decrease the thickness of the plates and increase their number, we get better approximations.



And the sum of the volumes of these plates get nearer to the volume of the pyramid.

Suppose we use 10 plates, to start with. Each plate is a square prism of small height. Let's use plates of same height. So, if we take the height of the pyramid as h ,each plate is of height $\frac{1}{10}h$. How do we compute the base of each plate?

If we imagine the pyramid and the stack of plates sliced vertically down from the vertex, we get a picture like this.



Starting from the top, we have isosceles triangles of increasing size. Their heights increase at the rate of $\frac{1}{10}h$ for each plate.

Since these triangles are all similar (why?) their bases also increase at the same rate. So, if we take the base edge of the bases of the pyramid to be b, the bases of the triangles starting from the top are $\frac{1}{10}b, \frac{2}{10}b, ..., b$.

So, the volumes of the plates are

$$\left(\frac{1}{10}b\right)^2 \times \frac{1}{10}h, \left(\frac{2}{10}b\right)^2 \times \frac{1}{10}h, ..., b^2 \times \frac{1}{10}h$$

And their sum?

$$\frac{1}{10}b^2h\left(\frac{1}{10^2} + \frac{2^2}{10^2} + ... + \frac{9^2}{10^2} + \frac{10^2}{10^2}\right) = \frac{1}{1000}b^2h\left(1^2 + 2^2 + 3^2 + ... + 10^2\right)$$

We have seen how such sums can be computed in the section, **Sum of squares** of the lesson, **Arithmetic Sequences**.

$$1^2 + 2^2 + 3^2 + ... + 10^2 = \frac{1}{6} \times 10 \times (10 + 1) \times (2 \times 10 + 1)$$

Thus the sum of the volumes

$$\frac{1}{1000}b^2h \times \frac{1}{6} \times 10 \times 11 \times 21 = \frac{1}{6}b^2h \times \frac{10}{10} \times \frac{11}{10} \times \frac{21}{10} = \frac{1}{6}b^2h \times 1.1 \times 2.1$$

Now imagine 100 such plates (we cannot draw it anyway.)

The thickness of a plate becomes $\frac{1}{100}h$ and the base edges would be $\frac{1}{100}b, \frac{2}{100}b, \frac{3}{100}b, ..., b$. So the sum of the volumes would be

$$\frac{1}{100^3}b^2h\left(1^2 + 2^2 + 3^2 + ... + 100^2\right) = \frac{1}{100^3}b^2h \times \frac{1}{6} \times 100 \times 101 \times 201$$

$$= \frac{1}{6}b^2h \times \frac{100}{100} \times \frac{101}{100} \times \frac{210}{100}$$

$$= \frac{1}{6}b^2h \times 1.01 \times 2.01$$

What if we increase the number of plates to 1000? Without going through detailed computations, we can see that the sum of volumes would be

$$\frac{1}{6}b^2h \times 1.001 \times 2.001$$

What is the number to which these sums get closer and closer to?

It is the volume of the pyramid; and it is

$$\frac{1}{6}b^2h \times 1 \times 2 = \frac{1}{3}b^2h$$