Chapter

3

Statics

CONTENTS

Varignon

Historically, Mechanics was the earliest branch of Physics to be developed as an exact science. The Laws of levers and of fluids *were known to the Greeks in third century B.C. The fundamental theorem of statics, or rather another form of its, viz., the Triangle of Forces was first enunciated by Stevinus of Bruges in the year 1586. It was, however, left to Galileo (1564-1642) and Newton (1642-1727) to formulate the* laws of mechanics *and to place mechanics on a sound footing as an exact science. Newton was also the first to formulate correctly the* law of universal gravitation. *Following Newton's time, important contributions to mechanics were made by* Euler, D' Alembert, Lagrange, Laplace, Poinsot *and* Coriolis. *All these contributions were however within framework of Newton's laws of motion made.*

Statics

3.1 Introduction

Statics is that branch of mechanics which deals with the study of the system of forces in equilibrium.

Matter : Matter is anything which can be perceived by our senses of which can exert, or be acted on, by forces.

Force : Force is anything which changes, or tends to change, the state of rest, or uniform motion, of a body. To specify a force completely four things are necessary they are magnitude, direction, sense and point of application. Force is a vector quantity.

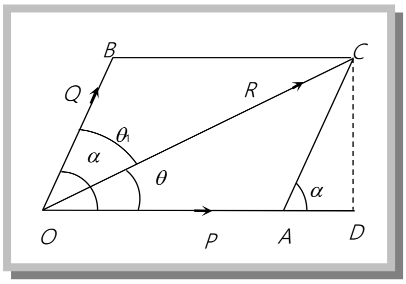
3.2 Parallelogram law of Forces

If two forces, acting at a point, be represented in magnitude and direction by the two sides of a parallelogram drawn from one of its angular points, their resultant is represented both in magnitude and direction of the parallelogram drawn through that point.

If *OA* and *OB* represent the forces *P* and *Q* acting at a point *O* and inclined to each other at an angle α. If *R* is the resultant of these forces represented by the diagonal *OC* of the parallelogram *OACB* and *R* makes an angle θ with *P* *i.e.* $\angle COA = \theta$, then $R^2 = P^2 + Q^2 + 2PQ\cos\alpha$ and $\tan\theta = \dfrac{Q\sin\alpha}{P + Q\cos\alpha}$

The angle θ_1 which the resultant *R* makes with the direction of the force *Q* is given by

$$\theta_1 = \tan^{-1}\left(\frac{P\sin\alpha}{Q + P\cos\alpha}\right)$$



Case (i) : If $P = Q$

$\therefore R = 2P\cos(\alpha/2)$ and $\tan\theta = \tan(\alpha/2)$ or $\theta = \alpha/2$

Case (ii) : If $\alpha = 90°$, *i.e.* forces are perpendicular

$\therefore R = \sqrt{P^2 + Q^2}$ and $\tan\theta = \frac{Q}{P}$

Case (iii) : If $\alpha = 0°$, *i.e.* forces act in the same direction

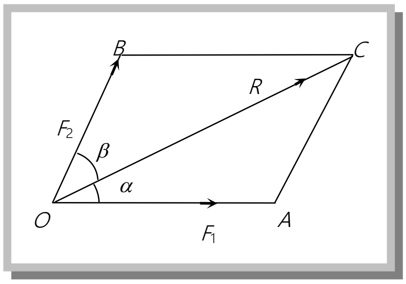
$\therefore R_{max} = P + Q$

Case (iv) : If $\alpha = 180°$, *i.e.* forces act in opposite direction

$\therefore R_{min} = P - Q$

Note : ❑ The resultant of two forces is closer to the larger force.

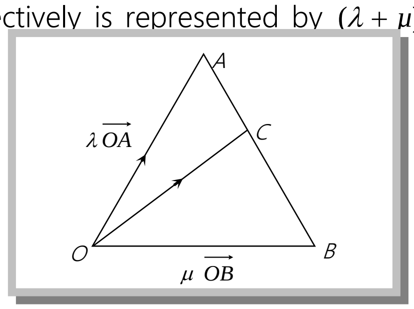
- ❑ The resultant of two equal forces of magnitude *P* acting at an angle α is $2P\cos\frac{\alpha}{2}$ and it bisects the angle between the forces.
- ❑ If the resultant *R* of two forces *P* and *Q* acting at an angle α makes an angle θ with the direction of *P*, then $\sin\theta = \frac{Q\sin\alpha}{R}$ and $\cos\theta = \frac{P + Q\cos\alpha}{R}$
- ❑ If the resultant *R* of the forces *P* and *Q* acting at an angle α makes an angle θ with the direction of the force *Q*, then $\sin\theta = \frac{P\sin\alpha}{R}$ and $\cos\theta = \frac{Q + P\sin\alpha}{R}$



- ❑ **Component of a force in two directions :** The component of a force *R* in two directions making angles α and β with the line of action of *R* on and opposite sides of it are

$$F_1 = \frac{OC.\sin\beta}{\sin(\alpha+\beta)} = \frac{R\sin\beta}{\sin(\alpha+\beta)} \text{ and } F_2 = \frac{OC.\sin\alpha}{\sin(\alpha+\beta)} = \frac{R.\sin\alpha}{\sin(\alpha+\beta)}$$

λ-μ theorem : The resultant of two forces acting at a point *O* in directions *OA* and *OB* represented in magnitudes by $\lambda.OA$ and $\mu.OB$ respectively is represented by $(\lambda + \mu)OC$, where *C* is a point in *AB* such that $\lambda.CA = \mu.CB$



Important Tips

☞ *The forces P, Q, R act along the sides BC, CA, AB of ΔABC.*

Their resultant passes through.

(a) Incentre, if $P + Q + R = 0$

(b) Circumcentre, if $P\cos A + Q\cos B + R\cos C = 0$

(c) Orthocentre, if $P\sec A + Q\sec B + R\sec C = 0$

(d) Centroid, if $P\text{ cosec } A + Q\text{ cosec } B + R\text{ cosec } C = 0$

or $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$

Example: 1 Forces *M* and *N* acting at a point *O* make an angle 150°. Their resultant acts at *O* has magnitude 2 units and is perpendicular to *M*. Then, in the same unit, the magnitudes of *M* and *N* are [BIT Ranchi 1993]

(a) $2\sqrt{3}, 4$ (b) $\sqrt{\frac{3}{2}}, 2$

(c) 3, 4 (d) 4. 5

Solution: (a) We have, $2^2 = M^2 + N^2 + 2MN\cos 150° \Rightarrow 4 = M^2 + N^2 - \sqrt{3}MN$ .....(i)

and, $\tan\frac{\pi}{2} = \frac{M\sin 150°}{M + N\cos 150°} \Rightarrow M + N\cos 150° = 0$

$\Rightarrow M - N\frac{\sqrt{3}}{2} = 0 \Rightarrow M = \frac{N\sqrt{3}}{2}$ .....(ii)

Solving (i) and (ii), we get $M = 2\sqrt{3}$ and $N = 4$.

Example: 2 If the resultant of two forces of magnitude *P* and 2*P* is perpendicular to *P*, then the angle between the forces is [Roorkee 1997]

(a) 2π/3 (b) 3π/4 (c) 4π/5 (d) 5π/6

Solution: (a) Let the angle between the forces *P* and 2*P* be α. Since the resultant of *P* and 2*P* is perpendicular to *P*. Therefore,

$\tan\pi/2 = \frac{2P\sin\alpha}{P + 2P\cos\alpha} \Rightarrow P + 2P\cos\alpha = 0 \Rightarrow \cos\alpha = \frac{-1}{2} \Rightarrow \alpha = \frac{2\pi}{3}$

Example: 3 If the line of action of the resultant of two forces *P* and *Q* divides the angle between them in the ratio 1 : 2, then the magnitude of the resultant is [Roorkee 1993]

(a) $\frac{P^2 + Q^2}{P}$ (b) $\frac{P^2 + Q^2}{Q}$ (c) $\frac{P^2 - Q^2}{P}$ (d) $\frac{P^2 - Q^2}{Q}$

Solution: (d) Let 3θ be the angle between the forces *P* and *Q*. It is given that the resultant *R* of *P* and *Q* divides the angle between them in the ratio 1 : 2. This means that the resultant makes an angle θ with the direction of *P* and angle 2θ with the direction of *Q*.

Therefore, $P = \dfrac{R\sin 2\theta}{\sin 3\theta}$ and $\theta = \dfrac{R\sin\theta}{\sin 3\theta}$

$\Rightarrow \dfrac{P}{Q} = \dfrac{\sin 2\theta}{\sin\theta} = 2\cos\theta$ .....(i)

Also $Q = \dfrac{R\sin\theta}{\sin 3\theta} \Rightarrow Q = \dfrac{R}{3-4\sin^2\theta}$

$\Rightarrow \dfrac{R}{Q} = 3-4\sin^2\theta \Rightarrow \dfrac{R}{Q} = -1+4\cos^2\theta \;\Rightarrow \dfrac{R}{Q}+1 = (2\cos\theta)^2$ .....(ii)

From (i) and (ii), we get, $\left(\dfrac{P}{Q}\right)^2 = \dfrac{R}{Q}+1 \Rightarrow \dfrac{R}{Q} = \dfrac{P^2-Q^2}{Q^2} \;\Rightarrow R = \dfrac{P^2-Q^2}{Q}$

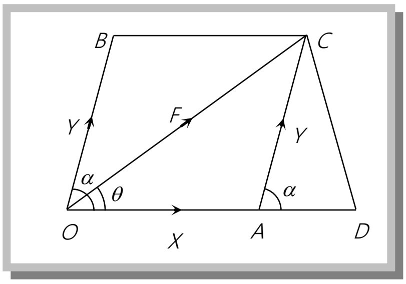
Example: 4 Two forces *X* and *Y* have a resultant *F* and the resolved part of *F* in the direction of *X* is of magnitude *Y*. Then the angle between the forces is

(a) $\sin^{-1}\sqrt{\dfrac{X}{2Y}}$ (b) $2\sin^{-1}\sqrt{\dfrac{X}{2Y}}$ (c) $4\sin^{-1}\sqrt{\dfrac{X}{2Y}}$ (d) None of these

Solution: (b) Let *OA* and *OB* represent two forces *X* and *Y* respectively. Let α be the angle between them and θ, the angle which the resultant *F* (represented by *OC*) makes with *OA*.

Now, resolved part of *F* along *OA*.

$F\cos\theta = OC \times \dfrac{OD}{OC} = OD \; = OA + AD = OA + AC\cos\alpha \; = \; X + Y\cos\alpha$



But resolved part of *F* along *OA* is given to by *Y*.

$\therefore Y = X + Y\cos\alpha$ or $Y(1-\cos\alpha) = X \;\Rightarrow Y.2\sin^2\dfrac{\alpha}{2} = X, \;\therefore \sin^2\alpha/2 = \dfrac{X}{2Y}$

i.e., $\sin\dfrac{\alpha}{2} = \sqrt{\dfrac{X}{2Y}}$ or $\dfrac{\alpha}{2} = \sin^{-1}\sqrt{\dfrac{X}{2Y}}$

Thus, $\alpha = 2\sin^{-1}\sqrt{\dfrac{X}{2Y}}$

Example: 5 The greatest and least magnitude of the resultant of two forces of constant magnitude are *F* and *G*. When the forces act an angle 2α, the resultant in magnitudes is equal to [UPSEAT 2001]

(a) $\sqrt{F^2\cos^2\alpha + G^2\sin^2\alpha}$ (b) $\sqrt{F^2\sin\alpha + G^2\cos^2\alpha}$ (c) $\sqrt{F^2+G^2}$ (d) $\sqrt{F^2-G^2}$

Solution: (a) Greatest resultant $= F = A + B$

Least resultant $= G = A - B$

On solving, we get $A = \dfrac{F+G}{2}$, $B = \dfrac{(F-G)}{2}$

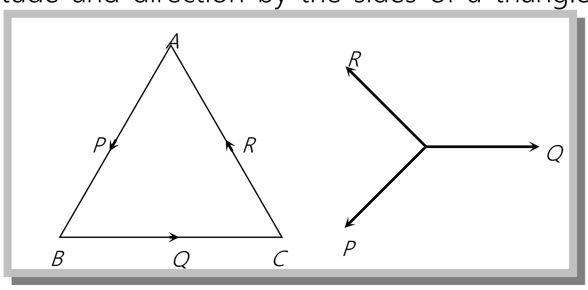
where *A* and *B* act an angle 2α, the resultant

$R = \sqrt{A^2+B^2+2AB\cos 2\alpha} \;\Rightarrow\; R = \sqrt{F^2\cos^2\alpha + G^2\sin^2\alpha}$

3.3 Triangle law of Forces

If three forces, acting at a point, be represented in magnitude and direction by the sides of a triangle, taken in order, they will be in equilibrium.

Here $\overrightarrow{AB} = P, \;\; \overrightarrow{BC} = Q, \;\overrightarrow{CA} = R$



In triangle *ABC*, we have $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0$

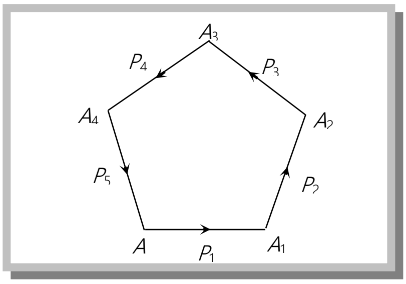
$\Rightarrow P+Q+R=0$

Hence the forces *P,Q,R* are in equilibrium.

Converse : If three forces acting at a point are in equilibrium, then they can be represented in magnitude and direction by the sides of a triangle, taken in order.

3.4 Polygon law of Forces

If any number of forces acting on a particle be represented in magnitude and direction by the sides of a polygon taken in order, the forces shall be in equilibrium.



Example: 6 *D* and *E* are the mid-points of the sides *AB* and *AC* respectively of a ΔABC. The resultant of the forces is represented by $\overrightarrow{BE}$ and $\overrightarrow{DC}$ is

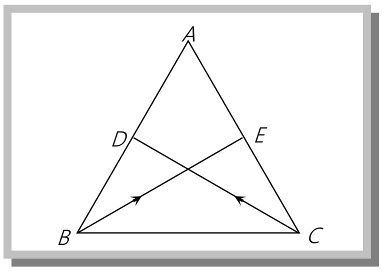
(a) $\frac{3}{2}\overrightarrow{AC}$ (b) $\frac{3}{2}\overrightarrow{CA}$ (c) $\frac{3}{2}\overrightarrow{AB}$ (d) $\frac{3}{2}\overrightarrow{BC}$

Solution: (d) We have,

$\overrightarrow{BE}+\overrightarrow{DC}=\left(\overrightarrow{BC}+\overrightarrow{CE}\right)+\left(\overrightarrow{DB}+\overrightarrow{BC}\right)$

$=2\overrightarrow{BC}+\frac{1}{2}\overrightarrow{CA}+\frac{1}{2}\overrightarrow{AB}=2\overrightarrow{BC}+\frac{1}{2}\left(\overrightarrow{CA}+\overrightarrow{AB}\right)$

$=2\overrightarrow{BC}+\frac{1}{2}\overrightarrow{CB}=2\overrightarrow{BC}-\frac{1}{2}\overrightarrow{BC}=\frac{3}{2}\overrightarrow{BC}$



Example: 7 *ABCDE* is pentagon. Forces acting on a particle are represented in magnitude and direction by $\overrightarrow{AB},\overrightarrow{BC},\overrightarrow{CD},2\overrightarrow{DE},\overrightarrow{AD}$, and $\overrightarrow{AE}$. Their resultant is given by

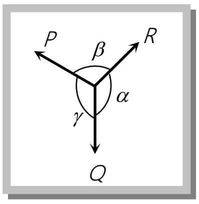
(a) $\overrightarrow{AE}$ (b) $2\overrightarrow{AB}$ (c) $3\overrightarrow{AE}$ (d) $4\overrightarrow{AE}$

Solution: (c) We have, $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+2\overrightarrow{DE}+\overrightarrow{AD}+\overrightarrow{AE}=\left(\overrightarrow{AB}+\overrightarrow{BC}\right)+\left(\overrightarrow{CD}+\overrightarrow{DE}\right)+\left(\overrightarrow{AD}+\overrightarrow{DE}\right)+\overrightarrow{AE}$

$=\left(\overrightarrow{AC}+\overrightarrow{CE}\right)+\overrightarrow{AE}+\overrightarrow{AE}=\overrightarrow{AE}+\overrightarrow{AE}+\overrightarrow{AE}=3\overrightarrow{AE}$.

3.5 Lami's Theorem

If three forces acting at a point be in equilibrium, each force is proportional to the sine of the angle between the other two. Thus if the forces are *P*, *Q* and *R*, α, β, γ be the angles between *Q* and *R*, *R* and *P*, *P* and *Q* respectively. If the forces are in equilibrium, we have, $\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$.



The converse of this theorem is also true.

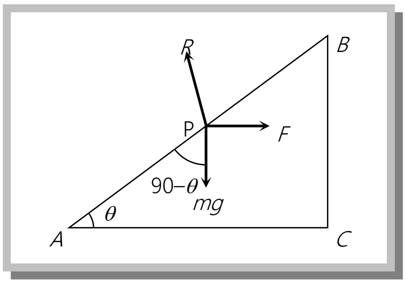
Example: 8 A horizontal force *F* is applied to a small object *P* of mass *m* on a smooth plane inclined to the horizon at an angle θ. If *F* is just enough to keep *P* in equilibrium, then *F* = [BIT Ranchi 1993]

(a) $mg\cos^2\theta$ (b) $mg\sin^2\theta$ (c) $mg\cos\theta$ (d) $mg\tan\theta$

Solution: (d) By applying Lami's theorem at *P*, we have

$$\frac{R}{\sin 90^\circ} = \frac{F}{\sin(180^\circ - \theta)} = \frac{mg}{\sin(90^\circ + \theta)}$$

$$\Rightarrow \frac{R}{1} = \frac{F}{\sin\theta} = \frac{mg}{\cos\theta} \Rightarrow F = mg\tan\theta$$



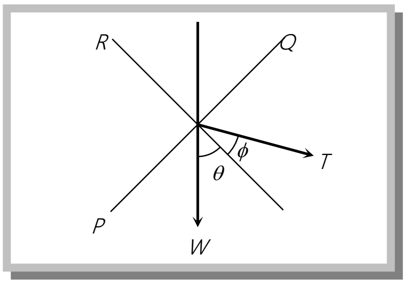
Example: 9 A kite of weight *W* is flying with its string along a straight line. If the ratios of the resultant air pressure *R* to the tension *T* in the string and to the weight of the kite are $\sqrt{2}$ and $(\sqrt{3}+1)$ respectively, then [Roorkee 1990]

(a) $T = (\sqrt{6}+\sqrt{2})W$ (b) $R = (\sqrt{3}+1)W$ (c) $T = \frac{1}{2}(\sqrt{6}-\sqrt{2})W$ (d) $R = (\sqrt{3}-1)W$

Solution: (b) From Lami's theorem,

$$\frac{R}{\sin(\theta+\phi)} = \frac{T}{\sin(180^o - \theta)} = \frac{W}{\sin(180^o - \phi)}$$

$$\Rightarrow \frac{R}{\sin(\theta+\phi)} = \frac{T}{\sin\theta} = \frac{W}{\sin\phi} \quad .....(i)$$



Given, $\frac{R}{T} = \sqrt{2}$ .....(ii) and $\frac{R}{W} = \sqrt{3}+1$ .....(iii)

Dividing (iii) by (ii), we get $\frac{\frac{R}{W}}{\frac{R}{T}} = \frac{\sqrt{3}+1}{\sqrt{2}}$

$$\Rightarrow \frac{T}{W} = \frac{\sqrt{3}+1}{\sqrt{2}} \Rightarrow T = \frac{\sqrt{3}+1}{\sqrt{2}}W = \frac{1}{2}(\sqrt{6}+\sqrt{2})W \Rightarrow R = T\sqrt{2} = \frac{\sqrt{2}}{\sqrt{2}}(\sqrt{3}+1)W = (\sqrt{3}+1)W$$

Example: 10 Three forces $\vec{P}, \vec{Q}\ \vec{R}$ are acting at a point in a plane. The angles between $\vec{P}$ and $\vec{Q}$ and $\vec{Q}$ and $\vec{R}$ are 150° and 120° respectively, then for equilibrium, forces *P*, *Q*, *R* are in the ratio [MNR 1991; UPSEAT 2000]

(a) $1:2:\sqrt{3}$ (b) $1:2:3$ (c) 3:2:1 (d) $\sqrt{3}:2:1$

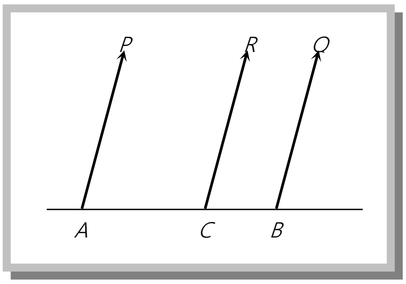
Solution: (d) Clearly, the angle between *P* and *R* is $360^\circ - (150^\circ + 120^\circ) = 90^\circ$. By Lami's theorem,

$$\frac{P}{\sin 120°} = \frac{Q}{\sin 90°} = \frac{R}{\sin 150°} \Rightarrow \frac{P}{\sqrt{3}/2} = \frac{Q}{1} = \frac{R}{1/2} \Rightarrow \frac{P}{\sqrt{3}} = \frac{Q}{2} = \frac{R}{1}$$

3.6 Parallel Forces

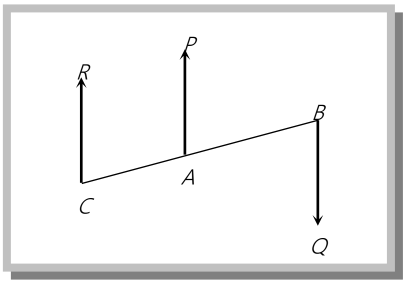
(1) **Like parallel forces :** Two parallel forces are said to be like parallel forces when they act in the same direction.

The resultant R of two like parallel forces P and Q is equal in magnitude of the sum of the magnitude of forces and R acts in the same direction as the forces P and Q and at the point on the line segment joining the point of action P and Q, which divides it in the ratio $Q : P$ internally.



(2) **Two unlike parallel forces :** Two parallel forces are said to be unlike if they act in opposite directions.

If P and Q be two unlike parallel force acting at A and B and P is greater in magnitude than Q. Then their resultant R acts in the same direction as P and acts at a point C on BA produced. Such that $R = P - Q$ and $P.CA = Q.CB$



Then in this case C divides BA externally in the inverse ratio of the forces,

$$\frac{P}{CB} = \frac{Q}{CA} = \frac{P-Q}{CB-CA} = \frac{R}{AB}$$

Important Tips

☞ *If three like parallel forces P, Q, R act at the vertices A, B, C repectively of a triangle ABC, then their resultant act at the*

(i) Incentre of ΔABC, if $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$

(ii) Circumcentre of ΔABC, if $\frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$

(iii) Orthocentre of ΔABC, if $\frac{P}{\tan A} = \frac{Q}{\tan B} = \frac{R}{\tan C}$

(iv) Centroid of ΔABC, if P = Q = R.

Example: 11 Three like parallel forces *P, Q, R* act at the corner points of a triangle *ABC*. Their resultant passes through the circumcentre,if [Rookee 1995]

(a) $\frac{P}{a}=\frac{Q}{b}=\frac{R}{c}$ (b) $P=Q=R$ (c) $P+Q+R=0$ (d) None of these

Solution: (c) Since the resultant passes through the circumcentre of ΔABC, therefore, the algebraic sum of the moments about it, is zero.

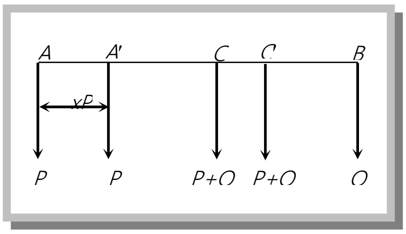
Hence, $P+Q+R=0$.

Example: 12 *P* and *Q* are like parallel forces. If *P* is moved parallel to itself through a distance *x*, then the resultant of *P* and *Q* moves through a distance. [Rookee 1995]

(a) $\frac{Px}{P+Q}$ (b) $\frac{Px}{P-Q}$ (c) $\frac{Px}{P+2Q}$ (d) None of these

Solution: (a) Let the parallel forces *P* and *Q* act at *A* and *B* respectively. Suppose the resultant *P* + *Q* acts at *C*.

Then, $AC=\left(\frac{AB}{P+Q}\right)Q$ .....(i)



If *P* is moved parallel to itself through a distance *x i.e.* at *A*'.

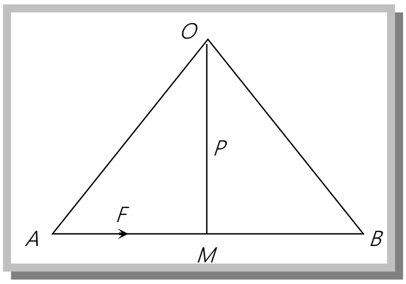
Suppose the resultant now acts at *C*'. Then ,

$A'C'=\left(\frac{A'B}{P+Q}\right)Q \Rightarrow A'C'=\left(\frac{AB-x}{P+Q}\right)Q$ .....(ii)

Now $CC'=AC'-AC=AA'+A'C'-AC$

$\Rightarrow CC'=x+\left(\frac{AB-x}{P+Q}\right)Q-\left(\frac{AB}{P+Q}\right)Q \Rightarrow CC'=x-\frac{Qx}{P+Q} \Rightarrow CC'=\frac{Px}{P+Q}$

3.7 Moment

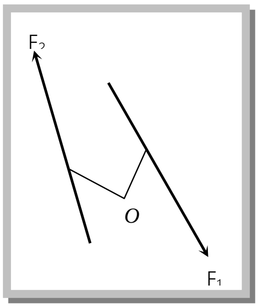


The moment of a force about a point *O* is given in magnitude by the product of the forces and the perpendicular distance of *O* from the line of action of the force.

If *F* be a force acting a point *A* of a rigid body along the line *AB* and *OM* (= *p*) be the perpendicular distance of the fixed point *O* from *AB*, then the moment of force about $O=F.p=AB\times OM=2\left[\frac{1}{2}(AB\times OM)\right]=2(\text{area of }\Delta AOB)$

The S.I. unit of moment is *Newton-meter* (*N-m*).

(1) Sign of the moment : The moment of a force about a point measures the tendency of the force to cause rotation about that point. The tendency of the force F_1 is to turn the lamina in the clockwise direction and of the force F_2 is in the anticlockwise direction.



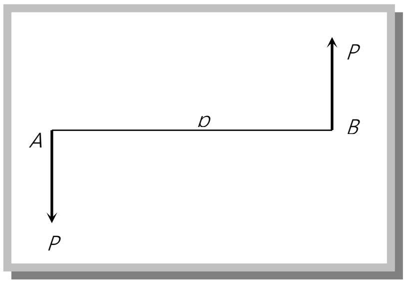
The usual convention is to regard the moment which is anticlockwise direction as positive and that in the clockwise direction as negative.

(2) **Varignon's theorem :** The algebraic sum of the moments of any two coplanar forces about any point in their plane is equal to the moment of their resultant about the same point.

Note :
- Thy algebraic sum of the moments of any two forces about any point on the line of action of their resultant is zero.
- Conversely, if the algebraic sum of the moments of any two coplanar forces, which are not in equilibrium, about any point in their plane is zero, their resultant passes through the point.
- If a body, having one point fixed, is acted upon by two forces and is at rest. Then the moments of the two forces about the fixed point are equal and opposite.

3.8 Couples

Two equal unlike parallel forces which do not have the same line of action, are said to form a couple.



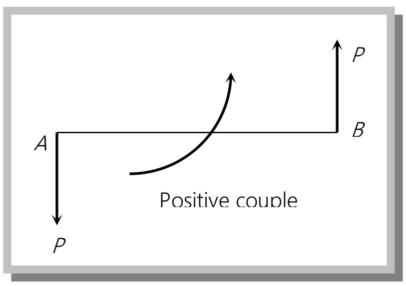
Example : Couples have to be applied in order to wind a watch, to drive a gimlet, to push a cork screw in a cork or to draw circles by means of pair of compasses.

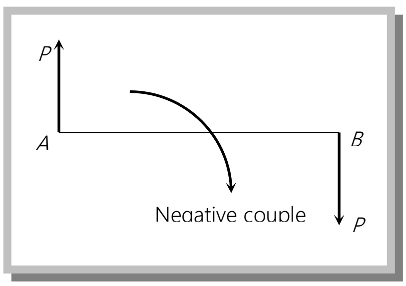
(1) **Arm of the couple :** The perpendicular distance between the lines of action of the forces forming the couple is known as the arm of the couple.

(2) **Moment of couple :** The moment of a couple is obtained in magnitude by multiplying the magnitude of one of the forces forming the couple and perpendicular distance between the lines of action of the force. The perpendicular distance between the forces is called the arm of the couple. The moment of the couple is regarded as positive or negative according as it has a tendency to turn the body in the anticlockwise or clockwise direction.

Moment of a couple = Force × Arm of the couple = $P.p$

(3) **Sign of the moment of a couple :** The moment of a couple is taken with positive or negative sign according as it has a tendency to turn the body in the anticlockwise or clockwise direction.



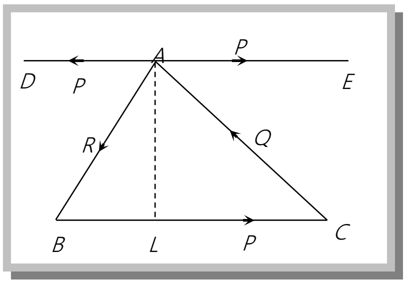


Note : ❑ A couple can not be balanced by a single force, but can be balanced by a couple of opposite sign.

3.9 Triangle theorem of Couples

If three forces acting on a body be represented in magnitude, direction and line of action by the sides of triangle taken in order, then they are equivalent to a couple whose moment is represented by twice the area of triangle.

Consider the force *P* along *AE*, *Q* along *CA* and *R* along *AB*. These forces are three concurrent forces acting at *A* and represented in magnitude and direction by the sides *BC, CA* and *AB* of ΔABC. So, by the triangle law of forces, they are in equilibrium.



The remaining two forces *P* along *AD* and *P* along *BC* form a couple, whose moment is $m = P.AL = BC.AL$

Since $\frac{1}{2}(BC.AL) = 2\left(\frac{1}{2}\text{area of the }\Delta ABC\right)$

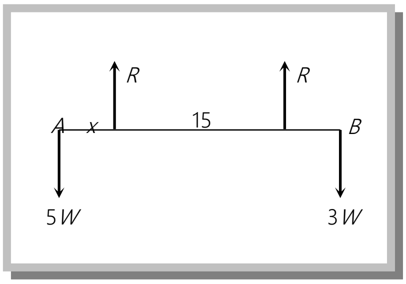
$\therefore$ Moment = $BC.AL$ = 2 (Area of ΔABC)

Example: 13 A light rod *AB* of length 30 *cm.* rests on two pegs 15 *cm.* apart. At what distance from the end *A* the pegs should be placed so that the reaction of pegs may be equal when weight 5*W* and 3*W* are suspended from *A* and *B* respectively

[Roorkee 1995, UPSEAT 2001]

(a) 1.75 *cm.*, 15.75 *cm.* (b) 2.75 *cm.*, 17.75 *cm.* (c) 3.75 *cm.*, 18.75 *cm.* (d) None of these

Solution: (c) Let *R, R* be the reactions at the pegs *P* and *Q* such that $AP = x$



Resolving all forces vertically, we get

$R + R = 8W \Rightarrow R = 4W$

Take moment of forces about *A*, we get

$R.AP + R.AQ = 3W.AB$

$\Rightarrow 4W.x + 4W.(x+15) = 3W.30$

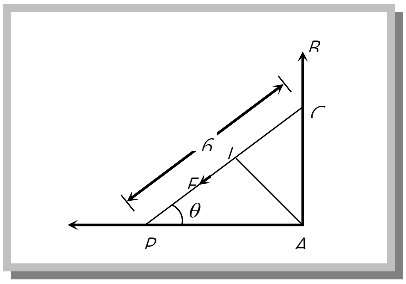
$\Rightarrow x = 3.75 cm$

$\therefore AP = x = 3.75 cm$ and $AQ = 18.75 cm$

Example: 14 At what height from the base of a vertical pillar, a string of length 6 *metres* be tied, so that a man sitting on the ground and pulling the other end of the string has to apply minimum force to overturn the pillar

(a) 1.5 *metres* (b) $3\sqrt{2}$ *metres* (c) $3\sqrt{3}$ *metres* (d) $4\sqrt{2}$ *metres*

Solution: (b) Let the string be tied at the point *C* of the vertical pillar, so that *AC* = *x*



Now moment of *F* about *A* = *F*. *AL*

$= F.\ AP \sin\theta$

$= F.6 \cos\theta\ \sin\theta$

$= 3\ F \sin 2\theta$

To overturn the pillar with maximum (fixed) force *F*, moment is maximum if

$\sin 2\theta = 1$ (max.)

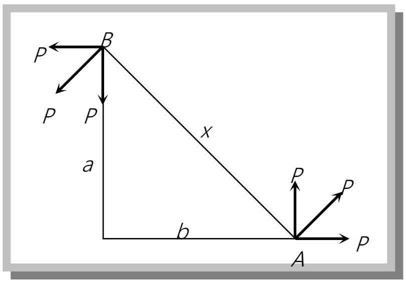
$\Rightarrow 2\theta = 90°$, *i.e.* $\theta = 45°$

$\therefore AC = PC \sin 45° = 6.\frac{1}{\sqrt{2}} = 3\sqrt{2}$

Example: 15 Two unlike parallel forces acting at points *A* and *B* form a couple of moment *G*. If their lines of action are turned through a right angle, they form a couple of moment *H*. Show that when both act at right angles to *AB*, they form a couple of moment.

(a) *GH* (b) $G^2 + H^2$ (c) $\sqrt{G^2 + H^2}$ (d) None of these

Solution: (c) We have, $Pa = G$ and $Pb = H$ .....(i)



Clearly, $a^2 + b^2 = x^2$

$\Rightarrow x = \sqrt{\frac{G^2}{P^2} + \frac{H^2}{P^2}}$ [from (i)]

$\Rightarrow Px = \sqrt{G^2 + H^2}$

Hence, required moment = $\sqrt{G^2 + H^2}$

Example: 16 The resultant of three forces represented in magnitude and direction by the sides of a triangle *ABC* taken in order with *BC* = 5 *cm*, *CA* = 5 *cm*, and *AB* = 8 *cm*, is a couple of moment

(a) 12 units (b) 24 units (c) 36 units (d) 16 units

Solution: (b) Resultant of three forces represented in magnitude and direction by the sides of a triangle taken in order is a couple of moment equal to twice the area of triangle.

$\therefore$ the resultant is a couple of moment = 2 × (area of ΔABC)

Here, *a* = 5 *cm*, *b* = 5 *cm* and c = 8 *cm*

$\therefore\ 2S = 5 + 5 + 8 \Rightarrow S = 9$.

Area = $\sqrt{S(S-a)(S-b)(S-c)} = \sqrt{9(9-5)(9-5)(9-8)} = 12$

$\therefore$ Required moment = 2 (12) = 24 units.

3.10 Equilibrium of Coplanar Forces

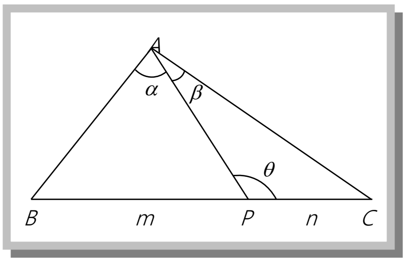
(1) If three forces keep a body in equilibrium, they must be coplanar.

(2) If three forces acting in one plane upon a rigid body keep it in equilibrium, they must either meet in a point or be parallel.

(3) When more than three forces acting on a rigid body, keep it in equilibrium, then it is not necessary that they meet at a point. The system of forces will be in equilibrium if there is neither translatory motion nor rotatory motion.

i.e. $X = 0$, $Y = 0$, $G = 0$ or $R = 0$, $G = 0$.

(4) A system of coplanar forces acting upon a rigid body will be in equilibrium if the algebraic sum of their resolved parts in any two mutually perpendicular directions vanish separately, and if the algebraic sum of their moments about any point in their plane is zero.



(5) A system of coplanar forces acting upon a rigid body will be in equilibrium if the algebraic sum of the moments of the forces about each of three non-collinear points is zero.

(6) Trigonometrical theorem : If P is any point on the base BC of ΔABC such that $BP : CP = m : n$.

Then, (i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$ where $\angle BAP = \alpha, \angle CAP = \beta$

(ii) $(n+n)\cot\theta = n\cot B - m\cot C$

Example: 17 Two smooth beads A and B, free to move on a vertical smooth circular wire, are connected by a string. Weights W_1, W_2 and W are suspended from A, B and a point C of the string respectively.

In equilibrium, A and B are in a horizontal line. If $\angle BAC = \alpha$ and $\angle ABC = \beta$, then the ratio $\tan\alpha : \tan\beta$ is

[Roorkee 1996, UPSEAT 2001]

(a) $\frac{\tan\alpha}{\tan\beta} = \frac{W - W_1 + W_2}{W + W_1 - W_2}$ (b) $\frac{\tan\alpha}{\tan\beta} = \frac{W + W_1 - W_2}{W - W_1 + W_2}$ (c) $\frac{\tan\alpha}{\tan\beta} = \frac{W + W_1 + W_2}{W + W_1 - W_2}$ (d) None of these

Solution: (a) Resolving forces horizontally and vertically at the points A, B and C respectively, we get

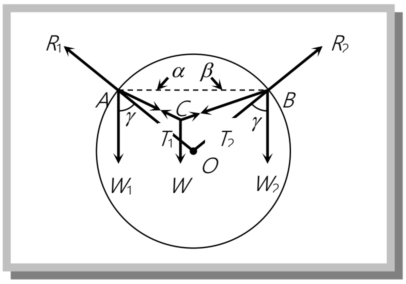
$T\cos\alpha = R_1\sin\gamma$ .....(i)

$T_1\sin\alpha + W_1 = R_1\cos\gamma$ .....(ii)

$T_1\cos\beta = R_2\sin\gamma$ .....(iii)

$T_2\sin\beta + W_2 = R_2\cos\gamma$ .....(iv)

$T_1\cos\alpha = T_2\cos\beta$ .....(v)



and $T_1 \sin\alpha + T_2 \sin\beta = W$ .....(vi)

Using (v), from (i)and (ii), we get, $R_1 = R_2$

$\therefore$ From (ii) and (vi), we have

$T_1 \sin\alpha + W_1 = T_2 \sin\beta + W_2$

or $T_1 \sin\alpha - T_2 \sin\beta = W_2 - W_1$ .....(vii)

Adding and subtracting (vi) and (vii), we get

$2T_1 \sin\alpha = W + W_2 - W_1$ ......(viii)

$2T_2 \sin\beta = W - W_2 + W_1$ ......(ix)

Dividing (viii) by (ix), we get

$\frac{T_1}{T_2} \cdot \frac{\sin\alpha}{\sin\beta} = \frac{W - W_1 + W_2}{W + W_1 - W_2}$ or $\frac{\cos\beta}{\cos\alpha} \cdot \frac{\sin\alpha}{\sin\beta} = \frac{W - W_1 + W_2}{W + W_1 - W_2}$ (from (v)) or $\frac{\tan\alpha}{\tan\beta} = \frac{W - W_1 + W_2}{W + W_1 - W_2}$

Example: 18 A uniform beam of length $2a$ rests in equilibrium against a smooth vertical plane and over a smooth peg at a distance h from the plane. If θ be the inclination of the beam to the vertical, then $\sin^3\theta$ is **[MNR 1996]**

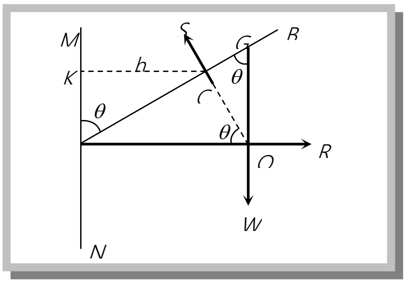
(a) $\frac{h}{a}$ (b) $\frac{h^2}{a^2}$ (c) $\frac{a}{h}$ (d) $\frac{a^2}{h^2}$

Solution: (a) Let AB be a rod of length $2a$ and weight W. It rests against a smooth vertical wall at A and over peg C, at a distance h from the wall. The rod is in equilibrium under the following forces :

(i) The weight W at G

(ii) The reaction R at A

(iii) The reaction S at C perpendicular to AB.



Since the rod is in equilibrium. So, the three force are concurrent at O.

In ΔACK, we have, $\sin\theta = \frac{h}{AC}$

In ΔACO, we have, $\sin\theta = \frac{AO}{a}$

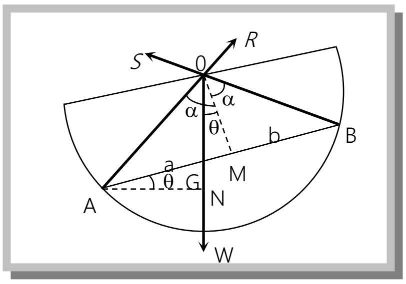
In ΔAGO, we have $\sin\theta = \frac{AO}{a}$; $\therefore \sin^3\theta = \frac{h}{AC} \cdot \frac{AC}{AO} \cdot \frac{AO}{a} = \frac{h}{a}$

Example: 19 A beam whose centre of gravity divides it into two portions a and b, is placed inside a smooth horizontal sphere. If θ be its inclination to the horizon in the position of equilibrium and 2α be the angle subtended by the beam at the centre of the sphere, then **[Roorkee 1994]**

(a) $\tan\theta = (b-a)(b+a)\tan\alpha$ (b) $\tan\theta = \frac{(b-a)}{(b+a)}\tan\alpha$ (c) $\tan\theta = \frac{(b+a)}{(b-a)}\tan\alpha$ (d) $\tan\theta = \frac{1}{(b-a)(b+a)}\tan\alpha$

Solution: (b) Applying $m-n$ theorem in ΔABC, we get

$(AG + GB)\cot\angle OGB = GB\cot\angle OAB - AG\cot\angle OBG$

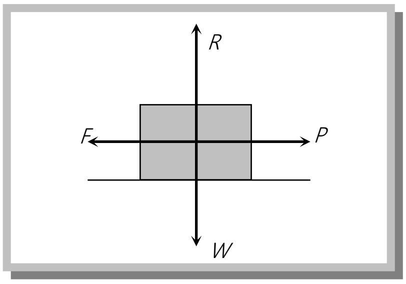


$$\Rightarrow (a+b)\cot(90^\circ - \theta) = b\cot\left(\frac{\pi}{2} - \alpha\right) - a\cot\left(\frac{\pi}{2} - \alpha\right)$$

$$\Rightarrow (a+b)\tan\theta = b\tan\alpha - a\tan\alpha \quad \Rightarrow \tan\theta = \left(\frac{b-a}{a+b}\right)\tan\alpha$$

3.11 Friction

Friction is a retarding force which prevent one body from sliding on another.



It is, therefore a reaction.

When two bodies are in contact with each other, then the property of roughness of the bodies by virtue of which a force is exerted between them to resist the motion of one body upon the other is called friction and the force exerted is called force of friction.

(1) **Friction is a self adjusting force :** Let a horizontal force *P* pull a heavy body of weight *W* resting on a smooth horizontal table. It will be noticed that up to a certain value of *P*, the body does not move. The reaction *R* of the table and the weight *W* of the body do not have any effect on the horizontal pull as they are vertical. It is the force of friction *F*, acting in the horizontal direction, which balances *P* and prevents the body from moving.

As *P* is increased, *F* also increases so as to balance *P*. Thus *F* increases with *P.* A stage comes when *P* just begins to move the body. At this stage *F* reaches its maximum value and is equal to the value of *P* at that instant. After that, if *P* is increased further, *F* does not increase any more and body begins to move.

This shows that friction is self adjusting, *i.e.* amount of friction exerted is not constant, but increases gradually from zero to a certain maximum limit.

(2) **Statical friction :** When one body tends to slide over the surface of another body and is not on the verge of motion then the friction called into play is called statical friction.

(3) **Limiting friction :** When one body is on the verge of sliding over the surface of another body then the friction called into play is called limiting friction.

(4) **Dynamical friction :** When one body is actually sliding over the surface of another body the friction called into play is called dynamical friction.

(5) **Laws of limiting friction/statical friction/Dynamical friction :**

(i) Limiting friction acts in the direction opposite to that in which the body is about to move.

(ii) The magnitude of the limiting friction between two bodies bears a constant ratio depends only on the nature of the materials of which these bodies are made.

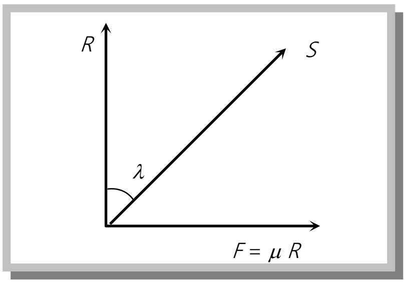
(iii) Limiting friction is independent of the shape and the area of the surfaces in contact, so long as the normal reaction between them is same, if the normal reaction is constant.

(iv) Limiting friction f_s is directly proportional to the normal reaction R, *i.e.* $f_s \propto R$

$f_s = \mu_s.R;\ \mu_s = f_s / R$, where μ_s is a constant which is called coefficient of statical friction.

In case of dynamic friction, $\mu_k = f_k/R$, where μ_k is the coefficient of dynamic friction.

(6) **Angle of friction :** The angle which the resultant force makes with the direction of the normal reaction is called the angle of friction and it is generally denoted by λ.



Thus λ is the limiting value of α, when the force of friction F attains its maximum value.

$$\therefore \tan\lambda = \frac{\text{Maximum force of friction}}{\text{Normal reaction}}$$

Since R and μ R are the components of S, we have, $S\cos\lambda = R$, $S\sin\lambda = \mu R$.

Hence by squaring and adding, we get $S = R\sqrt{1+\mu^2}$ and on dividing them, we get $\tan\lambda = \mu$. Hence we see that the coefficient of friction is equal to the tangent of the angle of friction.

3.12 Coefficient of Friction

When one body is in limiting equilibrium in contact with another body, the constant ratio which the limiting force of friction bears to normal reaction at their point of contact, is called the coefficient of friction and it is generally denoted by μ.

Thus, μ is the ratio of the limiting friction and normal reaction.

Hence, $\mu = \tan\lambda = \dfrac{\text{Maximum force of friction}}{\text{Normal reaction}}$

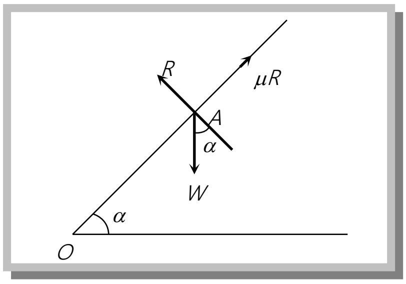
$\Rightarrow \mu = \dfrac{F}{R} \Rightarrow F = \mu R$, where F is the limiting friction and R is the normal reaction.

Note : ❑ The value of μ depends on the substance of which the bodies are made and so it differs from one body to the other. Also, the value of μ always lies between 0 and 1. Its value is zero for a perfectly smooth body.

- **Cone of friction :** A cone whose vertex is at the point of contact of two rough bodies and whose axis lies along the common normal and whose semi-vertical angle is equal to the angle of friction is called cone of friction.

3.13 Limiting equilibrium on an Inclined Plane

Let a body of weight W be on the point of sliding down a plane which is inclined at an angle α to the horizon. Let R be the normal reaction and μR be the limiting friction acting up the plane.



Thus, the body is in limiting equilibrium under the action of three forces : R, μR and W.

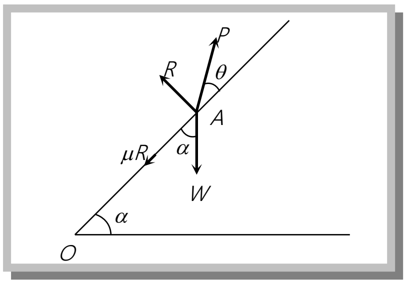
Resolving the forces along and perpendicular to the plane, we have

$\mu R = W \sin\alpha$ and $R = W\cos\alpha$

$$\Rightarrow \frac{\mu R}{R} = \frac{W\sin\alpha}{\cos\alpha} \Rightarrow \mu = \tan\alpha \Rightarrow \tan\lambda = \tan\alpha \Rightarrow \alpha = \lambda$$

Thus, if a body be on the point of sliding down an inclined plane under its own weight, the inclination of the plane is equal to the angle of the friction.

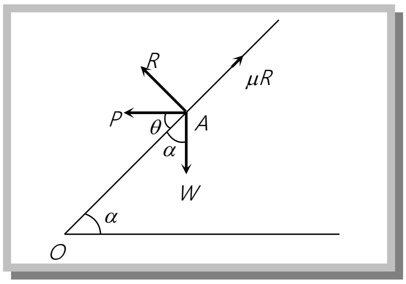
(1) **Least force required to pull a body up an inclined rough plane :** Let a body of weight W be at point A, α be the inclination of rough inclined plane to the horizontal and λ be the angle of friction. Let P be the force acting at an angle θ with the plane required just to move body up the plane.



$$P = W\frac{\sin(\alpha+\lambda)}{\cos(\theta-\lambda)} \qquad \{\because \mu = \tan\lambda\}$$

Clearly, the force P is least when $\cos(\theta-\lambda)$ is maximum, *i.e.* when $\cos(\theta-\lambda)=1$, *i.e.* $\theta-\lambda=0$ or $\theta=\lambda$. The least value of P is $W\sin(\alpha+\lambda)$

(2) **Least force required to pull a body down an inclined plane :** Let a body of weight W be at the point A, α be the inclination of rough inclined plane to the horizontal and λ be the angle of friction. Let P be the force acting an angle θ with the plane, required just to move the body up the plane.



$$P = \frac{W\sin(\lambda-\alpha)}{\cos(\theta-\lambda)} \qquad [\because \mu = \tan\lambda]$$

Clearly, P is least when $\cos(\theta-\lambda)$ is maximum, *i.e.* when $\theta-\lambda=0$ or $\theta=\lambda$. The least value of P is $W\sin(\lambda-\alpha)$.

Note : ❑ If $\alpha = \lambda$, then the body is in limiting equilibrium and is just on the point of moving downwards.

❑ If $\alpha < \lambda$, then the least force required to move the body down the plane is $W\sin(\lambda - \alpha)$.

❑ If $\alpha = \lambda, \alpha > \lambda$ or $\alpha < \lambda$, then the least force required to move the body up the plane is $W\sin(\alpha + \lambda)$.

❑ If $\alpha > \lambda$, then the body will move down the plane under the action of its weight and normal reaction.

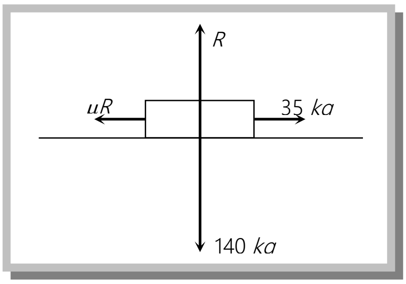
Important Tips

☞ ***Least force on the horizontal plane :*** *Least force required to move the body with weight W on the rough horizontal plane is W sin λ .*

Example: 20 A force of 35 *Kg* is required to pull a block of wood weighing 140 *Kg* on a rough horizontal surface. The coefficient of friction is **[BIT Ranchi 1995]**

(a) 1 (b) 0 (c) 4 (d) $\frac{1}{4}$

Solution: (d) In the position of limiting equilibrium, we have $\mu R = 35$ and $R = 140 \Rightarrow \mu = \frac{35}{140} = \frac{1}{4}$



Example: 21 A uniform ladder rests in limiting equilibrium, its lower end on a rough horizontal plane and its upper end against a smooth vertical wall. If θ is the angle of inclination of the ladder to the vertical wall and μ is the coefficient of friction, then tanθ is equal to **[MNR 1991; UPSEAT 2000]**

(a) μ (b) 2μ (c) $\frac{3\mu}{2}$ (d) $\mu + 1$

Solution: (b) Resolving the forces horizontally and vertically, we get

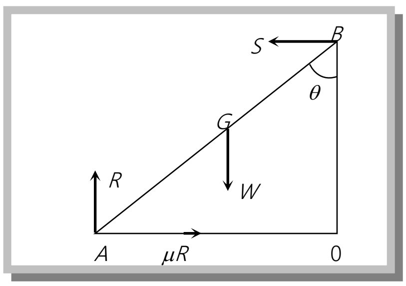
$S = \mu R$ and $R = W$

$\Rightarrow S = \mu W$ .....(i)

Taking moments about *A*, we get

$-W.AG\sin\theta + S.AB\cos\theta = 0$

$\Rightarrow W.AG\sin\theta = S.AB\cos\theta \Rightarrow W.\frac{AB}{2}\sin\theta = S.AB\cos\theta$ $\left[\because AG = \frac{AB}{2}\right]$



$\Rightarrow \frac{W}{2}.AB \sin\theta = \mu W.AB \cos\theta$ [from (i)]

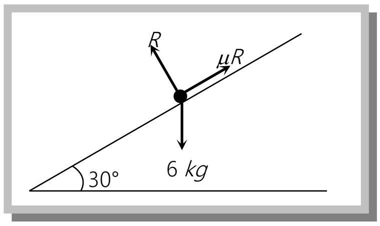
$\Rightarrow \tan\theta = 2\mu$.

Example: 22 A body of 6 *Kg*. rests in limiting equilibrium on an inclined plane whose slope is 30°. If the plane is raised to slope of 60°, the force in *Kg*. weight along the plane required to support it is

(a) 3 (b) $2\sqrt{3}$

(c) $\sqrt{3}$ (d) $3\sqrt{3}$

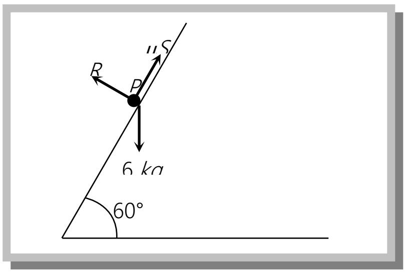
Solution: (b) In case (i),



$R = 6\cos 30°$, $\mu R = 6\sin 30°$.

$\therefore \mu = \tan 30° = \frac{1}{\sqrt{3}}$

In case (ii),



$S = 6\cos 60°$

$P + \mu S = 6\sin 60°$

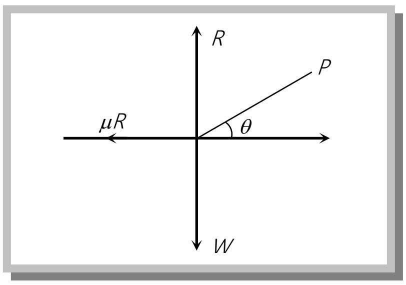
$\therefore P + \frac{1}{\sqrt{3}}(6\cos 60°) = 6\sin 60° = 6\frac{\sqrt{3}}{2} = 3\sqrt{3}$.

$\therefore P = 3\sqrt{3} - \frac{1}{\sqrt{3}}6 \times \frac{1}{2} = 3\sqrt{3} - \frac{3}{\sqrt{3}} = 3\sqrt{3} - \sqrt{3} = 2\sqrt{3}$.

Example: 23 The coefficient of friction between the floor and a box weighing 1 ton if a minimum force of 600 *Kgf* is required to start the box moving is **[SCRA 1995]**

(a) $\frac{1}{4}$ (b) $\frac{3}{4}$ (c) $\frac{1}{2}$ (d) 1

Solution: (b) Resolving horizontally and vertically



$P\cos\theta = \mu R; P\sin\theta + R = W$

$\therefore P\cos\theta = \mu[W - P\sin\theta]$

or $P[\cos\theta + \mu\sin\theta] = \mu W$

or $P = \frac{\mu W}{\cos\theta + \frac{\sin\lambda}{\cos\lambda}.\sin\theta} = \frac{\mu W\cos\lambda}{\cos(\theta-\lambda)} = \frac{W\sin\lambda}{\cos(\theta-\lambda)}$

Now *P* is minimum when $\cos(\theta-\lambda)$ is maximum, *i.e.* when $\cos(\theta-\lambda) = 1$

$\therefore \text{Min} P = W\sin\lambda$

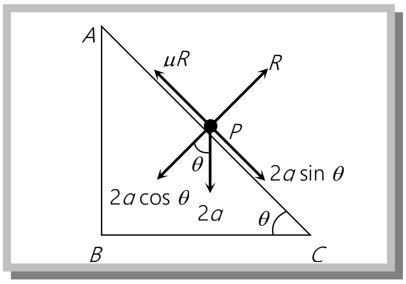
But $W = 1\ ton\ wt. = 1000\ Kg.$ and $P = 600\ kg$

$\therefore \sin\lambda = \frac{P}{W} = \frac{600}{1000} = \frac{3}{5}$; $\therefore \tan\lambda = \frac{3}{4}, \therefore \mu = \frac{3}{4}$

Example: 24 A block of mass 2 *Kg.* slides down a rough inclined plane starting from rest at the top. If the inclination of the plane to the horizontal is θ with $\tan\theta = \frac{4}{5}$, the coefficient of friction is 0.3 and the acceleration due to gravity is g = 9.8. The velocity of the block when it reaches the bottom is

(a) 6.3 (b) 5.2 (c) 7 (d) 8.1

Solution: (c) Let *P* be the position of the man at any time.



Clearly, $R = 2g\cos\theta$

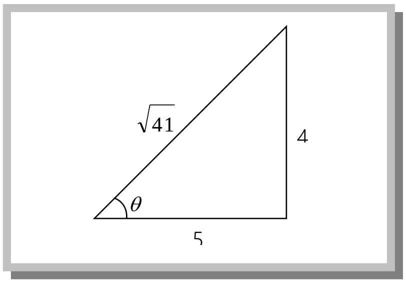
Let *f* be acceleration down the plane.

Equation of motion is $2f = 2g\sin\theta - \mu R$

$2f = 2g\sin\theta - \mu(2g\cos\theta)$

$2f = 2g(\sin\theta - \mu\cos\theta)$

Here, $\tan\theta = \frac{4}{5}$, $\sin\theta = \frac{4}{\sqrt{41}}$, $\cos\theta = \frac{5}{\sqrt{41}}$



Now, $2f = 2g\left(\frac{4}{\sqrt{41}} - \frac{3}{10}.\frac{5}{\sqrt{41}}\right)$

$2f = \frac{2g}{\sqrt{41}}\left(4 - \frac{3}{2}\right) = \frac{2g}{\sqrt{41}}.\frac{5}{2} = \frac{5g}{\sqrt{41}}$, $\therefore f = \frac{5g}{2\sqrt{41}}$

Let *v* be the velocity at *C*.

Then, $v^2 = u^2 + 2fS = 0 + 2\frac{5g}{2\sqrt{41}}AC$

$v^2 = \frac{5g}{\sqrt{41}}.\sqrt{41}$ $\left\{\text{we can take } AC = \sqrt{41}, \text{since } \tan\theta = \frac{4}{5}\right\}$

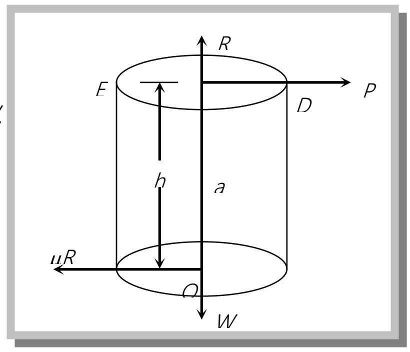
$v^2 = 5g = 5\times 9.8 = 49.0$, *i.e.*, $v^2 = 7m/\sec$

Example: 25 A circular cylinder of radius *r* and height *h* rests on a rough horizontal plane with one of its flat ends on the plane. A gradually increasing horizontal force is applied through the centre of the upper end. If the coefficient of friction is μ. The cylinder will topple before sliding of [UPSEAT 1994]

(a) $r < \mu h$ (b) $r \geq \mu h$ (c) $r \geq 2\mu h$ (d) $r = 2\mu h$

Solution: (b) Let base of cylinder is *AB*.

$\therefore BC = r$



Let force *P* is applied at *O*.

Let reaction of plane is *R* and force of friction is μR. Let weight of cylinder is *W*.

In equilibrium condition,

$R = W$ .....(i) and $P = \mu R$ .....(ii)

From (i) and (ii), we have $P = \mu W$

Taking moment about the point *O*,

We have $W \times BC - P \times OC = 0 \Rightarrow P = \frac{W \times BC}{OC} = \frac{W \times r}{h}$

If $\frac{W \times r}{h} \geq \mu W$ or $r \geq \mu h$

The cylinder will be topple before sliding.

3.14 Centre of Gravity

The centre of gravity of a body or a system of particles rigidly connected together, is that point through which the line of action of the weight of the body always passes in whatever position the body is placed and this point is called centroid. A body can have one and only one centre of gravity.

If $w_1, w_2, \dots\dots\dots, w_n$ are the weights of the particles placed at the points $A_1(x_1, y_1), A_2(x_2, y_2), \dots\dots\dots, A_n(x_n, y_{n)}$ respectively, then the centre of gravity $G(\bar{x}, \bar{y})$ is given by $\bar{x} = \frac{\sum w_1 x_1}{\sum w_1}, \bar{y} = \frac{\sum w_1 y_1}{\sum w_1}$.

(1) **Centre of gravity of a number of bodies of different shape :**

(i) **C.G. of a uniform rod :** The C.G. of a uniform rod lies at its mid-point.

(ii) **C.G. of a uniform parallelogram :** The C.G. of a uniform parallelogram is the point of inter-section of the diagonals.

(iii) **C.G. of a uniform triangular lamina :** The C.G. of a triangle lies on a median at a distance from the base equal to one third of the medians.

(2) **Some Important points to remember :**

(i) The C.G. of a uniform tetrahedron lies on the line joining a vertex to the C.G. of the opposite face, dividing this line in the ratio 3 : 1.

(ii) The C.G. of a right circular solid cone lies at a distance $h/4$ from the base on the axis and divides it in the ratio 3 : 1.

(iii) The C.G. of the curved surface of a right circular hollow cone lies at a distance $h/3$ from the base on the axis and divides it in the ratio 2 : 1

(iv) The C.G. of a hemispherical shell at a distance $a/2$ from the centre on the symmetrical radius.

(v) The C.G. of a solid hemisphere lies on the central radius at a distance $3a/8$ from the centre where a is the radius.

(vi) The C.G. of a circular arc subtending an angle 2α at the centre is at a distance $\frac{a\sin\alpha}{\alpha}$ from the centre on the symmetrical radius, *a* being the radius, and α in radians.

(vii) The C.G. of a sector of a circle subtending an angle 2α at the centre is at a distance $\frac{2a}{3}\frac{\sin\alpha}{\alpha}$ from the centre on the symmetrical radius, *a* being the radius and α in radians.

(viii) The C.G. of the semi circular arc lies on the central radius at a distance of $\frac{2a}{\pi}$ from the boundry diameter, where *a* is the radius of the arc.

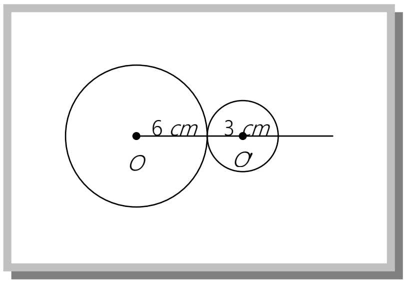
Important Tips

- *Let there be a body of weight w and x be its C.G. If a portion of weight w_1 is removed from it and x_1 be the C.G. of the removed portion. Then, the C.G. of the remaining portion is given by* $x_2 = \frac{wx - w_1x_1}{w - w_1}$
- *Let x be the C.G. of a body of weight w. If x_1, x_2, x_3 are the C.G. of portions of weights w_1, w_2, w_3 respectively, which are removed from the body, then the C.G. of the remaining body is given by* $x_4 = \frac{wx - w_1x_1 - w_2x_2 - w_3x_3}{w - w_1 - w_2 - w_3}$

Example: 26 Two uniform solid spheres composed of the same material and having their radii 6 *cm* and 3 *cm* respectively are firmly united. The distance of the centre of gravity of the whole body from the centre of the larger sphere is **[MNR 1980]**

(a) 1 *cm.* (b) 3 *cm.* (c) 2 *cm.* (d) 4 *cm.*

Solution: (a) Weights of the spheres are proportional to their volumes.



Let *P* be the density of the material, then

w_1 = Weight of the sphere of radius $6cm = \frac{4}{3}\pi(6^3)\rho = 288\pi\rho$

w_2 = Weight of the sphere of radius $3cm = \frac{4}{3}\pi(3^3)\rho = 36\pi\rho$

x_1 = Distance of the *C.G.* of the larger sphere from its centre *O* = 0

x_2 = Distance of the *C.G.* of smallar sphere from *O* = 9 *cm.*

$\bar{x}$ = Distance of the *C.G.* of the whole body from *O*

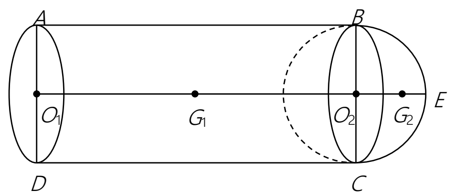
Now $\bar{x} = \frac{w_1x_1 + w_2x_2}{w_1 + w_2} = \frac{288\pi\rho \times 0 + 36\pi\rho \times 9}{288\pi\rho + 36\pi\rho}$

$\bar{x} = \frac{36 \times 9}{324} = 1$

Example: 27 A solid right circular cylinder is attached to a hemisphere of equal base. It the C.G. of combined solid is at the centre of the base, then the ratio of the radius and height of cylinder is

(a) 1 : 2 (b) $\sqrt{2}:1$ (c) 1 : 3 (d) None of these

Solution: (b) Let *a* be the radius of the base of the cylinder and *h* be the height of the cylinder. Let w_1 and w_2 be the weight of the cylinder and hemisphere respectively. These weights act at their centres of gravity G_1 and G_2 respectively.



Now, w_1 = weight of the cylinder = $\pi a^2 h\rho g$

w_2 = weight of the hemisphere = $\frac{2}{3}\pi a^3 \rho g$

$O_1G_1 = \frac{h}{2}$ and $O_1G_2 = h + \frac{3a}{8}$

Since the combined C.G. is at O_2. Therefore

$O_1O_2 = \frac{w_1 \times O_1G + w_2 + O_1G_2}{w_1 + w_2}$

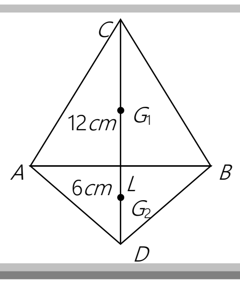
$\Rightarrow h = \frac{(\pi a^2 h\rho g)\times\frac{h}{2} + \left(\frac{2}{3}\pi a^3 \rho g\right)\times\left(h + \frac{3a}{8}\right)}{\pi a^2 h\rho g + \frac{2}{3}\pi a^3 \rho g} \Rightarrow h = \frac{\frac{h^2}{2} + \frac{2}{3}a\left(h + \frac{3a}{8}\right)}{h + \frac{2}{3}a} \Rightarrow h^2 + \frac{2ah}{3} = \frac{h^2}{2} + \frac{2ah}{3} + \frac{a^2}{4}$

$\Rightarrow 2h^2 = a^2 \Rightarrow \frac{a}{h} = \sqrt{2} \Rightarrow a : h = \sqrt{2} : 1$

Example: 28 On the same base *AB* and on opposite side of it, isosceles triangles *CAB* and *DAB* are described whose altitudes are 12 *cm* and 6 *cm* respectively. The distance of the centre of gravity of the quadrilateral *CADB* from *AB*, is

(a) 0.5 *cm* (b) 1 *cm* (c) 1.5 *cm* (d) 2 *cm*

Solution: (b) Let *L* be the midpoint of *AB*. Then $CL \perp AB$ and $DL \perp AB$.



Let G_1 and G_2 be the centres of gravity of triangular lamina *CAB* and *DAB* respectively.

Then, $LG_1 = \frac{1}{3}CL = 4cm.$ and $LG_2 = \frac{1}{3}DL = 2cm.$

The C.G. of the quadrilateral *ABCD* is at *G*, the mid point of G_1 G_2.

$\therefore G_1G_2 = GG_1 = 3cm.$

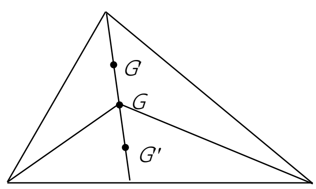
$\Rightarrow GL = G_1L - GG_1 = (4-3)cm = 1cm.$

Example: 29 *ABC* is a uniform triangular lamina with centre of gravity at *G*. If the portion *GBC* is removed, the centre of gravity of the remaining portion is at *G*. Then *GG* is equal to [UPSEAT 1994]

(a) $\frac{1}{3}AG$ (b) $\frac{1}{4}AG$ (c) $\frac{1}{5}AG$ (d) $\frac{1}{6}AG$

Solution: (d) Since *G* and *G* are the centroids of ΔABC and *GBD* respectively. Therefore $AG = \frac{2}{3}AD,$

$GD = \frac{1}{3}AD$ and $GG" = \frac{2}{3}GD = \frac{2}{3}\left(\frac{1}{3}AD\right) = \frac{2}{9}AD$



Now, $AG = \frac{2}{3}AD$ and $GD = \frac{1}{3}AD$

$\Rightarrow$ Area of $\Delta GBC = \frac{1}{3}$ Area of ΔABC

$\Rightarrow$ Weight of triangular lamina *GBC* = $\frac{1}{3}$ (weight of triangle lamina *ABC*)

Thus, if *W* is the weight of lamina *GBC*, then the weight of lamina *ABC* is $3W$.

Now, *G* is the *C.G.* of the remaining portion *ABGC*.

Therefore,

$$AG' = \frac{3W(AG) - W(AG'')}{3W - W}$$

$$= \frac{1}{2}(3AG - AG'') \qquad \left\{\because AG'' = AG + GG'' = \frac{2}{3}AD + \frac{2}{9}AD = \frac{8}{9}AD\right\}$$

$$= \frac{1}{2}\left(3 \times \frac{2}{3}AD - \frac{8}{9}AD\right) = \frac{5}{9}AD$$

$$\therefore GG' = AG - AG' = \frac{2}{3}AD - \frac{5}{9}AD = \frac{1}{9}AD = \frac{1}{9}\left(\frac{3}{2}AG\right) = \frac{1}{6}AG .$$