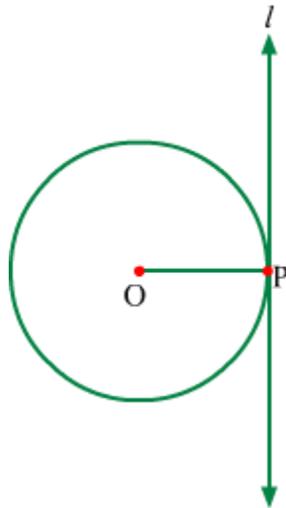


# Circles

## Concept Of Tangent At Any Point Of The Circle

A circle has important elements such as chords, secants, and tangents.

**A tangent at any point of a circle is perpendicular to the radius through the point of contact.**

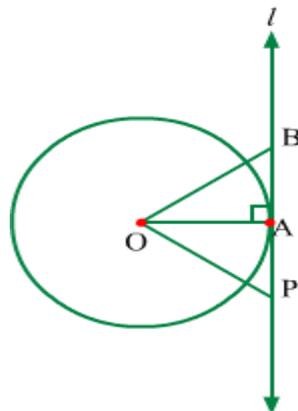


In the above figure O is the centre of circle, line  $l$  is the tangent and P is point of contact.

$\angle l \perp OP$

**Proof:**

It is given that O is the centre of the circle,  $l$  is the tangent to this circle and P is the point of contact.



Let us assume  $l$  is not perpendicular to the radius of the circle.

In this case, let us draw perpendicular  $OA$  to tangent  $l$ . Thus, point  $A$  is distinct from point  $P$ .

Let  $B$  be any point on tangent such that  $BAP$  is a line and  $BA = AP$ .

Now, in  $\triangle OAB$  and  $\triangle OAP$ , we have

$$OA = OA \quad (\text{Common side})$$

$$\angle OAB = \angle OAP \quad (OA \perp \text{tangent } l)$$

$$BA = AP \quad (\text{By construction})$$

$$\therefore \triangle OAB \cong \triangle OAP$$

$$\therefore OB = OP \quad (\text{By CPCT})$$

Since  $OB = OP$ , point  $B$  also lies on the circle.

Also, point  $B$  is different from point  $P$ .

Thus, tangent  $l$  touches the circle at two distinct points. This contradicts the definition of tangent.

Hence, our assumption is wrong.

Therefore, tangent  $l \perp OP$ .

Hence proved.

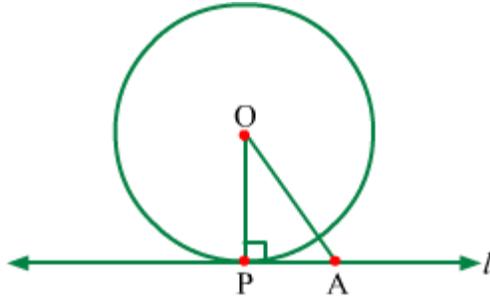
The converse of this theorem is also true which states that:

**The line perpendicular to the radius of a circle at its outer end is tangent to the circle.**

**Proof:**

Let  $O$  be the centre of the circle,  $OP$  be the radius and  $l$  be the line perpendicular to  $OP$  such as it passes through point  $P$  on the circle.

Also, let  $A$  be any point on line  $l$  distinct from  $P$ .



From the figure, it can be observed that  $\triangle OAP$  is a right angled triangle.

$\therefore$  OA is hypotenuse for  $\triangle OAP$ .

$\therefore$   $OA > OP$  (Radius)

$\therefore$  OA is not radius.

$\therefore$  Point A does not lie on the circle.

$\therefore$  No point of line  $l$  other than P lies on the circle.

$\therefore$  P is the only point common to the circle and line  $l$ .

$\therefore$  Line  $l$  is tangent to the circle at point P.

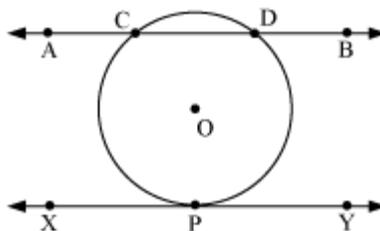
Hence proved.

Let us now solve some examples related to the tangents of the circle.

**Example 1: Draw a circle with centre O, and two lines such that one is a tangent and other is a secant.**

**Solution:**

The figure can be drawn as follows.



Here,  $\overline{AB}$  is the secant, which intersects the circle at C and D and  $\overline{XY}$  is a tangent whose point of contact with the circle is P.

**Example 2: Which of the following statements are correct?**

**(i) There can be only one tangent at a point on the circle.**

**(ii) Diameter is also a secant of the circle.**

**Solution:**

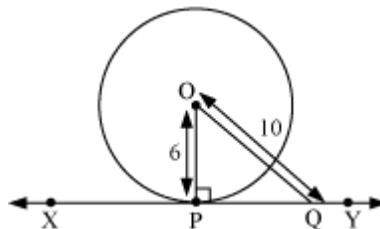
**(i)** Correct

**(ii)** Incorrect

**Example 3: A line XY is a tangent of the circle with centre O and radius 6 cm. The point of contact is P, and Q is any point on the tangent XY. If OQ = 10 cm, then what is the length of PQ?**

**Solution:**

The figure can be drawn as follows.



We know that the radius through the point of contact is perpendicular to the tangent.

$\therefore OP \perp XY$

Using Pythagoras theorem in right-angled triangle OPQ, we obtain

$$(OQ)^2 = (OP)^2 + (PQ)^2$$

$$\Rightarrow (PQ)^2 = (OQ)^2 - (OP)^2$$

$$\Rightarrow (PQ)^2 = (10)^2 - (6)^2$$

$$\Rightarrow (PQ)^2 = 100 - 36$$

$$\Rightarrow (PQ)^2 = 64$$

$$\Rightarrow PQ = 8 \text{ cm}$$

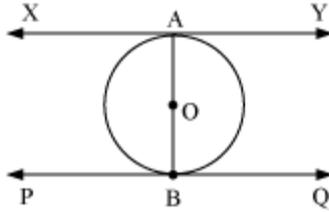
Thus, the length of PQ is 8 cm.

**Example 4: Two tangents XY and PQ are drawn at the ends of a diameter AB of the circle**

**with centre O. Show that  $XY \parallel PQ$ .**

**Solution:**

The figure can be drawn as follows.



XY is a tangent at A and OA is the radius.

We know that the radius through the point of contact is perpendicular to the tangent.

$$\therefore XY \perp OA$$

$$\Rightarrow XY \perp AB \dots (1)$$

$$\text{Similarly, } PQ \perp AB \dots (2)$$

Now, the lines perpendicular to the same line are parallel.

$\therefore$  From equations (1) and (2), we obtain

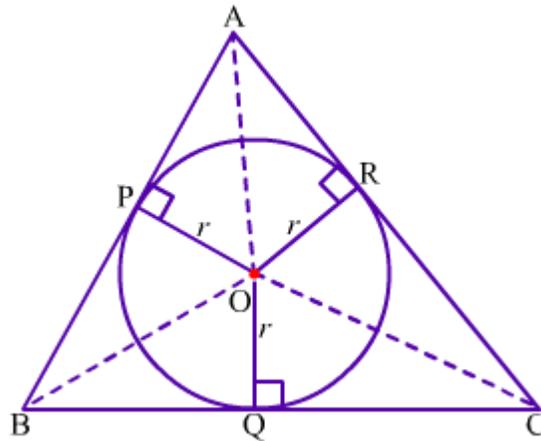
$$XY \parallel PQ$$

Hence, proved

**Example 5: A circle is inscribed in a triangle such as it touches all the three sides of the triangle. Prove that the area of the triangle is half the product of its perimeter and the radius of circle.**

**Solution:**

Let  $O$  be the centre of the circle inscribed in  $\triangle ABC$ .



It can be seen that  $AB$ ,  $BC$  and  $CA$  are tangents which touches the circle at  $P$ ,  $Q$  and  $R$  respectively.

Also,  $OP$ ,  $OQ$  and  $OR$  are the radii of the circle.

We know that a tangent at any point of a circle is perpendicular to the radius through the point of contact.

Therefore,  $AB \perp OP$ ,  $BC \perp OQ$  and  $CA \perp OR$ .

Let  $s$  be the perimeter of  $\triangle ABC$ .

$$\therefore s = AB + BC + CA$$

Now,

$$\text{Area of } \triangle OAB = \frac{1}{2}(AB \times OP) = \frac{1}{2}(AB \times r)$$

$$\text{Area of } \triangle OBC = \frac{1}{2}(BC \times OQ) = \frac{1}{2}(BC \times r)$$

$$\text{Area of } \triangle OCA = \frac{1}{2}(CA \times OR) = \frac{1}{2}(CA \times r)$$

$$\therefore \text{Area of } \triangle ABC = \text{Area of } \triangle OAB + \text{Area of } \triangle OBC + \text{Area of } \triangle OCA$$

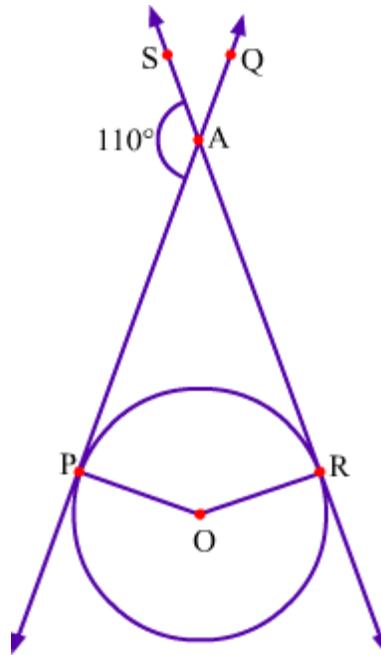
$$\Rightarrow \text{Area of } \triangle ABC = \frac{1}{2}(AB \times r) + \frac{1}{2}(BC \times r) + \frac{1}{2}(CA \times r)$$

$$\Rightarrow \text{Area of } \triangle ABC = \frac{1}{2}r(AB + BC + CA)$$

$$\Rightarrow \text{Area of } \triangle ABC = \frac{1}{2}rs$$

Hence proved.

**Example 6: Observe the given figure.**



**Find the angle between radii.**

**Solution:**

In the given figure, O is the centre of the circle and OP and OR are the radii. Also, QP and SR are the tangents to the circle at points P and R respectively which intersect each other at point A.

It is given that,

$$\angle PAS = 110^\circ$$

We need to find the angle between radii OP and OR i.e.,  $\angle POR$ .

From the figure, we have

$$\angle PAS + \angle PAR = 180^\circ$$

$$\Rightarrow 110^\circ + \angle PAR = 180^\circ$$

$$\Rightarrow \angle PAR = 180^\circ - 110^\circ$$

$$\Rightarrow \angle PAR = 70^\circ$$

We know that a tangent at any point of a circle is perpendicular to the radius through the point of contact.

Therefore,  $QP \perp OP$  and  $SR \perp OR$ .

$$\therefore \angle APO = 90^\circ \text{ and } \angle ARO = 90^\circ$$

Now, in quadrilateral APOR, we have

$$\angle POR + \angle APO + \angle ARO + \angle PAR = 360^\circ \quad (\text{By angle sum property of quadrilaterals})$$

$$\Rightarrow \angle POR + 90^\circ + 90^\circ + 70^\circ = 360^\circ$$

$$\Rightarrow \angle POR + 250^\circ = 360^\circ$$

$$\Rightarrow \angle POR = 110^\circ$$

Thus, the angle between the radii is  $110^\circ$ .

### **Tangents Drawn From An External Point To A Circle**

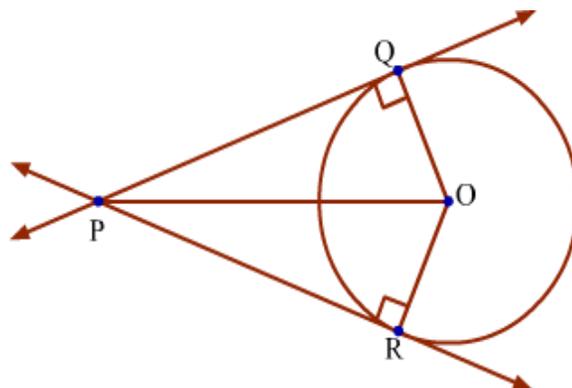
We are very well aware of what a tangent is. Now let us look at the following video and find out how many tangents can be drawn to a circle from an external point and if there is any relation between the lengths of these tangents.

**The lengths of the two tangent segments to a circle drawn from an external point are equal.**

**Proof:**

Let P be the point outside the circle having centre O from which the tangents PQ and PR are drawn touching the circle at Q and R respectively.

We have to prove that  $PQ = PR$ .



From the figure, it can be observed that OQ and OR are the radii of the circle.

Therefore,  $\angle PQO = \angle PRO = 90^\circ$

Now, in  $\triangle POQ$  and  $\triangle POR$ , we have

$\angle PQO = \angle PRO = 90^\circ$

$PO = PO$  (Common hypotenuse)

$OQ = OR$  (Radii of same circle)

Using RHS (Right-Hypotenuse-Side) congruence rule, we get

$\triangle POQ \cong \triangle POR$

$\therefore PQ = PR$  (By CPCT)

Thus, the lengths of the two tangent segments to a circle drawn from an external point are equal.

**Note:** Since  $\triangle POQ \cong \triangle POR$ , we have

$\angle OPQ = \angle OPR$  (By CPCT)

$\angle POQ = \angle POR$  (By CPCT)

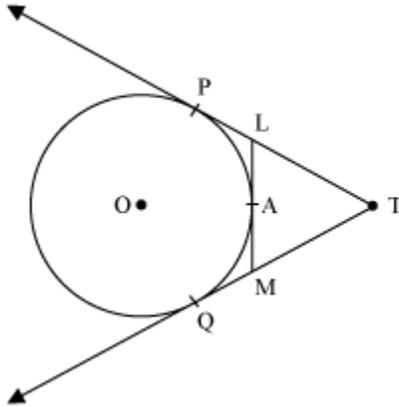
**Thus, the above theorem can be extended as,**

**(1) The tangents drawn to a circle from an external point are equally inclined to the line joining the external point and the centre.**

**(2) The tangents drawn to a circle from an external point subtend equal angles at the centre.**

Now, let us solve some examples to understand the concept.

**Example 1: In the given figure, prove that  $TL + AL = TM + AM$**

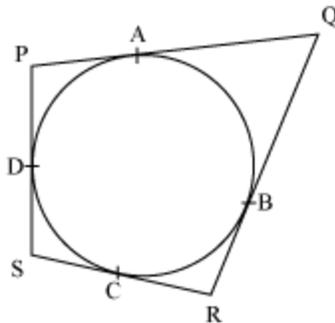


**Example 2: A circle is circumscribed by a quadrilateral PQRS such that the circle touches all the sides of quadrilateral PQRS at points A, B, C, and D respectively. Show that**

$$PQ + RS = QR + PS$$

**Solution:**

The figure can be drawn as follows.



Now, applying the theorem “The tangents drawn from an external point to the circle are equal in length”, we obtain

$$\left. \begin{array}{l} PA = PD \\ QA = QB \\ RB = RC \\ \text{and } SC = SD \end{array} \right\} \dots(i)$$

Therefore,

$$PQ + RS = (PA + QA) + (RC + SC)$$

$$= (PD + QB) + (RB + SD) \text{ [Using (i)]}$$

$$= (PD + SD) + (QB + RB)$$

$$= PS + QR$$

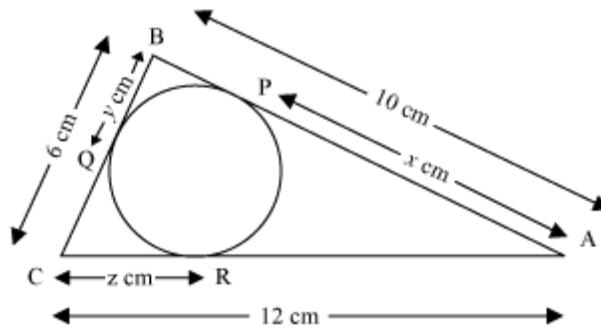
$$\text{Thus, } PQ + RS = PS + QR$$

Hence, proved

**Example 3:** A circle is inscribed in a triangle ABC such that the circle touches the sides AB, BC, and AC of the triangle at P, Q, and R respectively. What are the lengths of AP, BQ, and CR if AB = 10 cm, BC = 6 cm, and AC = 12 cm.

**Solution:**

The figure can be drawn as follows.



Let AP = x cm, BQ = y cm, and CR = z cm.

Now, applying the theorem “The tangents drawn from an external point to the circle are equal in length”, we obtain

$$AP = AR = x$$

$$BQ = BP = y$$

$$CR = CQ = z$$

Therefore, we can write

$$AP + BP = AB$$

$$\Rightarrow x + y = 10 \text{ cm ... (i)}$$

$$BQ + QC = BC$$

$$\Rightarrow y + z = 6 \text{ cm ... (ii)}$$

$$CR + AR = AC$$

$$\Rightarrow z + x = 12 \text{ cm ... (iii)}$$

On adding (i), (ii), and (iii), we obtain

$$2x + 2y + 2z = 28 \text{ cm}$$

$$\Rightarrow x + y + z = 14 \text{ cm ... (iv)}$$

Subtracting (i) from (iv), we obtain

$$(x + y + z) - (x + y) = 14 - 10$$

$$\Rightarrow z = 4 \text{ cm}$$

Similarly, subtracting (ii) and (iii) respectively from (iv), we obtain

$$(x + y + z) - (y + z) = 14 - 6$$

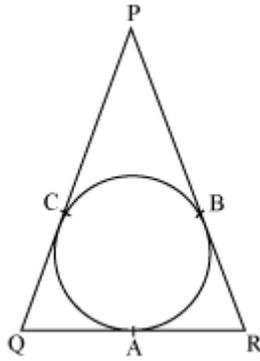
$$\Rightarrow x = 8 \text{ cm}$$

$$(x + y + z) - (z + x) = 14 - 12$$

$$\Rightarrow y = 2 \text{ cm}$$

Thus,  $AP = 8 \text{ cm}$ ,  $BQ = 2 \text{ cm}$ , and  $CR = 4 \text{ cm}$

**Example 4:** In the given figure, PQR is an isosceles triangle with  $PQ = PR$ . A circle, which is inscribed in  $\Delta PQR$ , touches the sides of the triangle at A, B, and C. Show that  $AQ = AR$ .



**Solution:**

It is given that PQR is an isosceles triangle where

$$PQ = PR \dots (i)$$

Now, the tangents drawn from an external point to a circle are equal in length.

$$\therefore PC = PB \dots (ii)$$

On subtracting equation (ii) from equation (i), we obtain

$$PQ - PC = PR - PB$$

$$QC = RB \dots (iii)$$

Now, QA and QC are tangents to the circle from point Q.

$$\therefore QA = QC$$

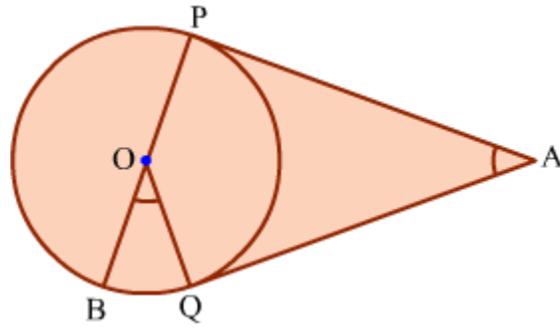
Similarly, RB = RA

Using the above relations in equation (iii), we obtain

$$QA = RA$$

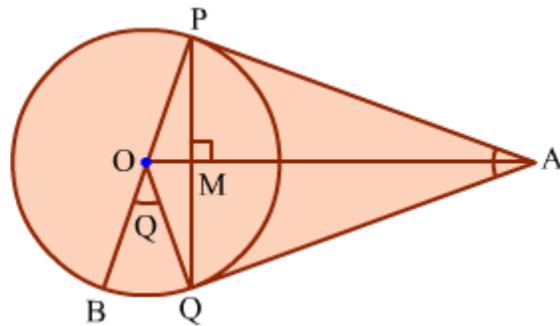
$$\therefore AQ = AR$$

**Example 5:** PA and QA are tangents drawn to a circle with centre O. Show that  $\angle BOQ = \angle PAQ$ .



**Solution:**

Join O with A, P with Q.



We know that, the tangents drawn to a circle from an external point are equal and they are equally inclined to the line joining the external point and the centre.

Therefore,  $\angle PAO = \angle QAO$  and  $\angle POA = \angle QOA$ .

It is clear that  $PQ \perp OA$ .

Now,  $\angle BOQ = 2\angle OPQ$  ... (1)

[Angle subtended by an arc at the centre is twice the angle subtended by the same arc at anywhere on the Circle]

Let  $\angle PAO = \angle QAO = \theta$ .

In  $\triangle PMA$ ,

$$\angle PAM + \angle APM + \angle PMA = 180^\circ$$

$$\Rightarrow \theta + \angle APM + 90^\circ = 180^\circ$$

$$\Rightarrow \angle APM = 90^\circ - \theta$$

$$\Rightarrow 90^\circ - \angle OPM = 90^\circ - \theta$$

$$\Rightarrow \angle OPM = \theta \quad \dots(2)$$

From (1) and (2),  $\angle BOQ = 2\theta$ .

Therefore,

$$\angle PAQ = \theta + \theta = 2\theta$$

$$\Rightarrow \angle PAQ = \angle BOQ$$

Hence, the result is proved.

