Constant Electric Field In Vacuum (Part - 1)

Q. 1. Calculate the ratio of the electrostatic to gravitational interaction forces between two electrons, between two protons. At what value of the specific charge q/m of a particle would these forces become equal (in their absolute values) in the case of interaction of identical particles?

Solution. 1.

$$F_{el} \text{ (for electorns)} = \frac{q^2}{4\pi\varepsilon_0 r^2} \text{ and } F_{gr} = \frac{\gamma m^2}{r^2}$$

Thus $\frac{F_{el}}{F_{gr}} \text{ (for electrons)} = \frac{q^2}{4\pi\varepsilon_0 \gamma m^2}$

$$= \frac{(1.602\times10^{-19}\,C)^2}{\left(\frac{1}{9\times10^9}\right)\times 6.67\times10^{-11}\,m^3/(kg\cdot s^2)\times(9.11\times10^{-31}\,kg)^2} = 4\times10^{42}$$

Similarly $\frac{F_{el}}{F_{gr}} \text{ (for proton)} = \frac{q^2}{4\pi\varepsilon_0 \gamma m^2}$

$$= \frac{(1.602\times10^{-19}\,C)^2}{\left(\frac{1}{9\times10^9}\right)\times 6.67\times10^{-11}\,m^3/(kg\cdot s^2)\times(1.672\times10^{-27}\,kg)^2} = 1\times10^{36}$$

For $F_{el} = F_{gr}$

$$\frac{q^2}{4\pi\varepsilon_0 r^2} = \frac{\gamma m^2}{r^2} \text{ or } \frac{q}{m} = \sqrt{4\pi\varepsilon_0\gamma}$$

$$= \sqrt{\frac{6.67\times10^{-11}\,m^3\,(kg - s^2)}{9\times10^9}} = 0.86\times10^{-10}\,C/kg$$

Q. 2. What would be the interaction force between two copper spheres, each of mass 1m, separated by the distance 1 m, if the total electronic charge in them differed from the total charge of the nuclei by one per cent?

Solution. 2. Total number of atoms in the sphere of mass 1 gm $= \frac{1}{63.54}\times 6.023\times10^{23}$

So the total nuclear charge $\lambda = \frac{6{\cdot}023 \times 10^{23}}{63{\cdot}54} \times 1{\cdot}6 \times 10^{-19} \times 29$

Now the charge on the sphere = Total nuclear charge - Total electronic charge

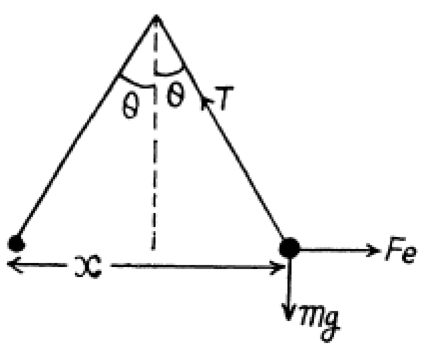
$$= \frac{6{\cdot}023 \times 10^{23}}{63{\cdot}54} \times 1{\cdot}6 \times 10^{-19} \times \frac{29 \times 1}{100} = 4{\cdot}298 \times 10^{2}\,C$$

Hence force of interaction between these two spheres,

$$F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{[4{\cdot}398 \times 10^{2}]^2}{1^2} N = 9 \times 10^{9} \times 10^{4} \times 19{\cdot}348\,N = 1{\cdot}74 \times 10^{15}\,N$$

Q. 3. Two small equally charged spheres, each of mass m, are suspended from the same point by silk threads of length 1. The distance between the spheres x << 1. Find the rate dolt with which the charge leaks off each sphere if their approach velocity varies as $v = a/\sqrt{x},$ where a is a constant.

Solution. 3. Let the balls be deviated by an angle θ, from the vertical when separation between them equals x.



Applying Newton's second law of motion for any one of the sphere, we get,

T cos θ = mg (1)

And T sin θ = F_e (2)

From the Eqs. (1) and (2)

$$\tan\theta = \frac{F_e}{mg}$$

$$\tan\theta = \frac{x}{2\sqrt{l^2 - \left(\frac{x}{2}\right)^2}} = \frac{x}{2l} \text{ as } x << l \qquad (4)$$

From Eqs. (3) and (4)

$$F_e = \frac{mg\,x}{2l} \text{ or } \frac{q^2}{4\pi\varepsilon_0 x^2} = \frac{mg\,x}{2l}$$

Thus $q^2 = \dfrac{2\pi\varepsilon_0\, mg\, x^3}{l}$ (5)

Differentiating Eqn. (5) with respect to time

$$2q\frac{dq}{dt} = \frac{2\pi\varepsilon_0\, mg}{l} 3x^2 \frac{dx}{dt}$$

According to the problem $\frac{dx}{dt} = v = a/\sqrt{x}$ (approach velocity is $\frac{dx}{dt}$)

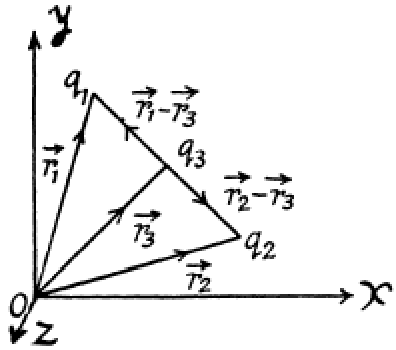
$$\text{so, } \left(\frac{2\pi\varepsilon_0\, mg}{l} x^3\right)^{1/2} \frac{dq}{dt} = \frac{3\pi\varepsilon_0\, mg}{l} x^2 \frac{a}{\sqrt{x}}$$

$$\text{Hence, } \frac{dq}{dt} = \frac{3}{2} a \sqrt{\frac{2\pi\varepsilon_0\, mg}{l}}$$

Q. 4. Two positive charges q_1 and q_2 are located at the points with radius vectors r_1 and r_2. Find a negative charge q_3 and a radius vector r_3 of the point at which it has to be placed for the force acting on each of the three charges to be equal to zero.

Solution. 4. Let us choose coordinate axes as shown in the figure and fix three charges,

q_1 , q_2 and q_3 having position vectors $\vec{r_1}, \vec{r_2}$ and $\vec{r_3}$ respectively.



Now, for the equilibrium of q3

$$\frac{+q_2 q_3(\vec{r_2}-\vec{r_3})}{|\vec{r_2}-\vec{r_3}|^3}+\frac{q_1 q_3(\vec{r_1}-\vec{r_3})}{|\vec{r_1}-\vec{r_3}|^3}=0$$

Or, $\frac{q_2}{|\vec{r_2}-\vec{r_3}|^2}=\frac{q_1}{|\vec{r_1}-\vec{r_3}|^2}$

Because $\frac{\vec{r_2}-\vec{r_3}}{|\vec{r_2}-\vec{r_3}|}=-\frac{\vec{r_1}-\vec{r_3}}{|\vec{r_1}-\vec{r_3}|}$

Or, $\sqrt{q_2}\,(\vec{r_1}-\vec{r_3})=\sqrt{q_1}\,(\vec{r_3}-\vec{r_2})$

Or, $\vec{r_3}=\frac{\sqrt{q_2}\,\vec{r_1}+\sqrt{q_1}\,\vec{r_2}}{\sqrt{q_1}+\sqrt{q_2}}$

Also for the equilibrium of q_1

$$\frac{q_3(\vec{r_3}-\vec{r_1})}{|\vec{r_3}-\vec{r_1}|^3}+\frac{q_2(\vec{r_2}-\vec{r_1})}{|\vec{r_2}-\vec{r_1}|^3}=0$$

Or, $q_3=\frac{-q_2}{|\vec{r_2}-\vec{r_1}|^2}|\vec{r_1}-\vec{r_3}|^2$

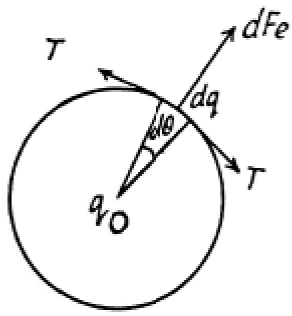
Substituting the value of $\vec{r_3}$, we get,

$$q_3=\frac{-q_1 q_2}{(\sqrt{q_1}+\sqrt{q_2})^2}$$

Q. 5. A thin wire ring of radius r has an electric charge q. What will be the

increment of the force stretching the wire if a point charge q_0 is placed at the ring's centre?

Solution. 5. When the charge q_0 is placed at the centre of the ring, the wire get stretched and the extra tension, produced in the wire, will balance the electric force due to the charge q_0. Let the tension produced in the wire, after placing the charge q_0, be T. From Newton's second law in projection form $F_n = mw_n$.



$$Td\theta - \frac{1}{4\pi\varepsilon_0}\frac{q_0}{r^2}\left(\frac{q}{2\pi r}r\,d\theta\right) = (dm)\,0,$$

Or, $$T = \frac{q q_0}{8\pi^2\varepsilon_0 r^2}$$

Q.6. A positive point charge 50 µC is located in the plane xy at the point with radius vector r_0 = 2i + 3j, where i and j are the unit vectors of the x and y axes. Find the vector of the electric field strength E and its magnitude at the point with radius vector r = 8i — 5j. Here r_0 and r are expressed in metres.

Solution. 6. Sought field strength

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r_0}|^2}$$

= 4.5 kV/m on putting the values.

Q.7. Point charges q and —q are located at the vertices of a square with diagonals 2l as shown in Fig. 3.1. Find the magnitude of the electric field strength at a point located symmetrically with respect to the vertices of the square at a distance x from its centre.

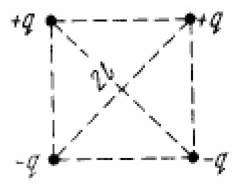
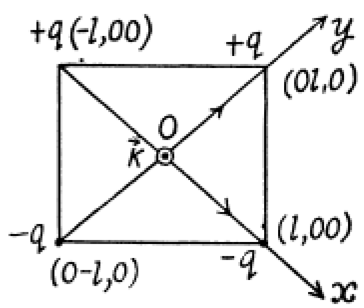


Fig. 3.1.

Solution. 7. Let us fix the coordinate system by taking the point of intersection of the diagonals as the origin and let $\vec{k}$ be directed normally, emerging from the plane of figure.

Hence the sought field strength:



$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{l\vec{i}+x\vec{k}}{(l^2+x^2)^{3/2}} + \frac{-q}{4\pi\varepsilon_0}\frac{l(-\vec{i})+x\vec{k}}{(l^2+x^2)^{3/2}}$$

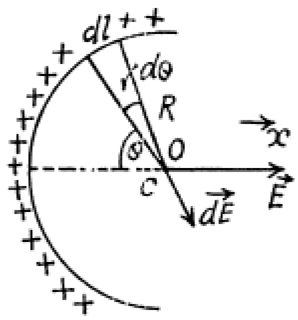
$$+\frac{-q}{4\pi\varepsilon_0}\frac{l\vec{j}+x\vec{k}}{(l^2+x^2)^{3/2}} + \frac{q}{4\pi\varepsilon_0}\frac{l(-\vec{j})+x\vec{k}}{(l^2+x^2)^{3/2}}$$

$$= \frac{q}{4\pi\varepsilon_0(l^2+x^2)^{3/2}}[2l\vec{i} - 2l\vec{j}]$$

Thus $E = \dfrac{ql}{\sqrt{2}\,\pi\varepsilon_0(l^2+x^2)^{3/2}}$

Q.8. A thin half-ring of radius R = 20 cm is uniformly charged with a total charge q = 0.70 nC. Find the magnitude of the electric field strength at the curvature centre of this half-ring.

Solution. 8. From the symmetry of the problem the sought field.



$$E = \int dE_x$$

Where the projection of field strength along x - axis due to an elemental charge is

$$dE_x = \frac{dq \cos\theta}{4\pi\varepsilon_0 R^2} = \frac{q R \cos\theta \, d\theta}{4\pi^2 \varepsilon_0 R^3}$$

Hence

$$E = \frac{q}{4\pi^2 \varepsilon_0 R^2} \int_{\pi/2}^{\pi/2} \cos\theta \, d\theta \;\; \frac{q}{2\pi^2 \varepsilon_0 R^2}$$

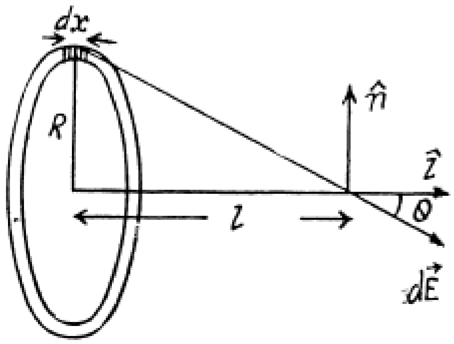
Q.9. A thin wire ring of radius r carries a charge q. Find the magnitude of the electric field strength on the axis of the ring as a function of distance l from its centre. Investigate the obtained function at 1 ≫ r. Find the maximum strength magnitude and the corresponding distance l. Draw the approximate plot of the function E(l).

Solution. 9. From the symmetry of the condition, it is clear that, the field along the normal will be zero

i.e. $E_n = 0$ and $E = E_l$

Now $$dE_l = \frac{dq}{4\pi\varepsilon_0 (R^2 + l^2)} \cos\theta$$

But $$dq = \frac{q}{2\pi R} dx \text{ and } \cos\theta = \frac{l}{(R^2 + l^2)^{1/2}}$$



Hence
$$E = \int dE_l = \int_0^{2\pi R} \frac{ql}{2\pi R} \cdot \frac{dx}{4\pi\varepsilon_0 (R^2 + l^2)^{3/2}}$$

$$\text{or } E = \frac{1}{4\pi\varepsilon_0} \frac{ql}{(l^2 + R^2)^{3/2}}$$

And for l ≫ R, the ring behaves like a point charge, reducing the field to the value,
$$E \approx \frac{1}{4\pi\varepsilon_0} \frac{q}{l^2}$$

For E_{max}, we should have $\frac{dE}{dl} = 0$

$$\text{So,} (l^2 + R^2)^{3/2} - \frac{3}{2} l (l^2 + R^2)^{1/2} 2l = 0 \quad \text{or} \quad l^2 + R^2 - 3l^2 = 0$$

$$\text{Thus } l = \frac{R}{\sqrt{2}} \text{ and } E_{max} = \frac{q}{6\sqrt{3}\,\pi\varepsilon_0 R^2}$$

Q.10. A point charge q is located at the centre of a thin ring of radius R with uniformly distributed charge —q. Find the magnitude of the electric field strength vector at the point lying on the axis of the ring at a distance x from its centre, if x ≫ R.

Solution. 10. The electric potential at a distance a: from the given ring is given by,

$$\varphi(x) = \frac{q}{4\pi\varepsilon_0 x} - \frac{q}{4\pi\varepsilon_0 (R^2 + x^2)^{1/2}}$$

Hence, the field strength along x-axis (which is the net field strength in 'our case),

$$E_x = -\frac{d\varphi}{dx} = \frac{q}{4\pi\varepsilon_0}\frac{1}{x^2} - \frac{qx}{4\pi\varepsilon_0 (R^2+x^2)^{3/2}}$$

$$= \frac{\frac{q}{4\pi\varepsilon_0} x^3 \left[\left(1+\frac{R^2}{x^2}\right)^{3/2} - 1\right]}{x^2 (R^2+x^2)^{3/2}}$$

$$= \frac{\frac{q}{4\pi\varepsilon_0} x^3 \left[1 + \frac{3}{2}\frac{R^2}{x^2} + \frac{3}{8}\frac{R^4}{x^4} + \dots\right]}{x^2 (R^2+x^2)^{3/2}}$$

Neglecting the higher power of R/x, as $x >> R$.

$$E = \frac{3qR^2}{8\pi\varepsilon_0 x^4}.$$

Note: Instead of φ (x), we may write E (x) directly using 3.9

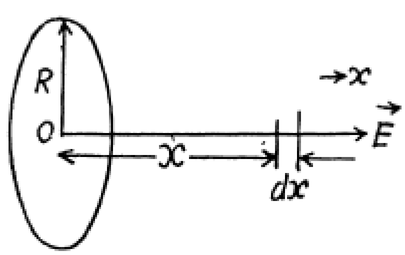
Q.11. A system consists of a thin charged wire ring of radius R and a very long uniformly charged thread oriented along the axis of the ring, with one of its ends coinciding with the centre of the ring. The total charge of the ring is equal to q. The charge of the thread (per unit length) is equal to λ. Find the interaction force between the ring and the thread.

Solution. 11. From the solution of 3.9, the electric field strength due to ring at a point on its axis (say x-axis) at distance x from the centre of the ring is given by:

$$E(x) = \frac{qx}{4\pi\varepsilon_0 (R^2+x^2)^{3/2}}$$

And from symmetry $\vec{E}$ at every point on the axis is directed along the x-axis (Fig.). Let us consider an element (dx) on thread which carries the charge (λdx). The electric force experienced by the element in the field of ring.

$$dF = (\lambda\, dx) E(x) = \frac{\lambda q x\, dx}{4\pi\varepsilon_0 (R^2+x^2)^{3/2}}$$



Thus the sought interaction

$$F = \int_0^\infty \frac{\lambda\, q\, x\, dx}{4\pi\varepsilon_0 (R^2 + x^2)^{3/2}}$$

On integrating we get, $F = \frac{\lambda q}{4\pi\varepsilon_0 R}$

Q.12. A thin non conducting ring of radius R has a linear charge density $\lambda = \lambda_0 \cos\varphi$, where λ_0 is a constant, φ is the azimuthal angle. Find the magnitude of the electric field strength
(a) at the centre of the ring;
(b) on the axis of the ring as a function of the distance x from its centre. Investigate the obtained function at $x \gg R$.

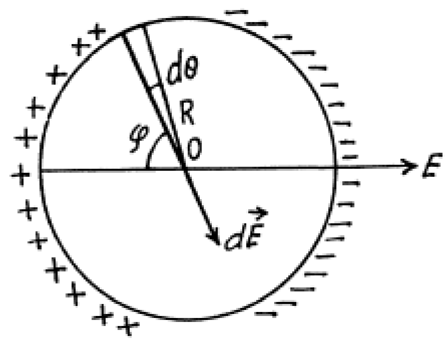
Solution. 12. (a) The given charge distribution is shown in Fig. The symmetry of this distribution implies that vector $\vec{E}$ at the point O is directed to the right, and its magnitude is equal to the sum of the projection onto the direction of $\vec{E}$ of vectors $d\vec{E}$ from elementary charges dq. The projection of vector $d\vec{E}$ onto vector $\vec{E}$ is

$$dE \cos\varphi = \frac{1}{4\pi\varepsilon_0} \frac{dq}{R^2} \cos\varphi,$$

Where $dq = \lambda R\, d\varphi = \lambda_0 R \cos\varphi\, d\varphi.$

Integrating (1) over φ between 0 and 2π we find the magnitude of the vector E:

$$E = \frac{\lambda_0}{4\pi\varepsilon_0 R} \int_0^{2\pi} \cos^2\varphi\, d\varphi = \frac{\lambda_0}{4\varepsilon_0 R}.$$



It should be noted that this integral is evaluated in the most simple way if we take into account that

$\langle\cos^2\varphi\rangle = 1/2$. Then

$$\int_0^{2\pi} \cos^2\varphi \, d\varphi = \langle\cos^2\varphi\rangle \, 2\pi = \pi.$$

(b) Take an element S at an azimuthal angle φ from the x-axis, the element subtending an angle dφ at the centre.

The elementary field at P due to the element is

$$\frac{\lambda_0 \cos\varphi \, d\varphi \, R}{4\pi\varepsilon_0 (x^2 + R^2)} \text{ along } SP \text{ with components}$$

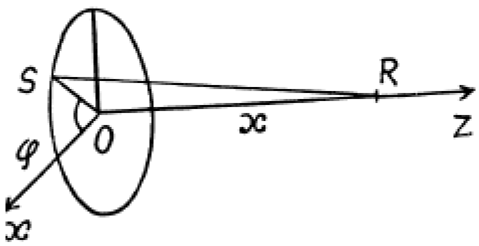
$$\frac{\lambda_0 \cos\varphi \, d\varphi \, R}{4\pi\varepsilon_0 (x^2 + R^2)} \times \{ \cos\theta \text{ along } OP, \sin\theta \text{ along } OS \}$$

Where $\cos\theta = \dfrac{x}{(x^2 + R^2)^{1/2}}$

The component along OP vanishes on integration as $\int_0^{2\pi} \cos\varphi \, d\varphi = 0$

The component along OS can be broken into the parts along OX and OY with

$$\frac{\lambda_0 R^2 \cos\varphi \, d\varphi}{4\pi\varepsilon_0 (x^2 + R^2)^{3/2}} \times \{ \cos\varphi \text{ along } OX, \sin\varphi \text{ along } OY \}$$



On integration, the part along OY vanishes. Finally

$$E = E_x = \frac{\lambda_0 R^2}{4\,\varepsilon_0\,(x^2 + R^2)^{3/2}}$$

For x ≫ R

$$E_x = \frac{p}{4\pi\varepsilon_0 x^3} \text{ where } p = \lambda_0 \pi R^2$$

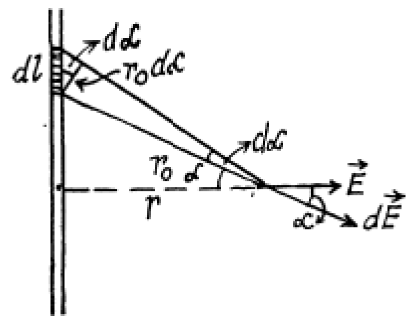
Q.13. A thin straight rod of length 2a carrying a uniformly distributed charge q is located in vacuum. Find the magnitude of the electric field strength as a function of the distance r from the rod's centre along the straight line
(a) perpendicular to the rod and passing through its centre;
(b) coinciding with the rod's direction (at the points lying outside the rod).
Investigate the obtained expressions at r ≫ a.

Solution. 13. (a) It is clear from symmetry considerations that vector $\vec{E}$ must be directed as shown in the figure. This shows the way of solving this problem: we must find the component dEr of the field created by the element dl of the rod, having the charge dq and then integrate the result over all the elements of the rod. In this cas

$$dE_r = dE\cos\alpha = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,dl}{r_0^2}\cos\alpha,$$

Where $\lambda = \frac{q}{2a}$ is the linear charge density. Let us reduce this equation of the form

convenient for integration. Figure show s that dl cos $\alpha = r_0\,d\alpha$ and $r_0 = \frac{r}{\cos\alpha}$;



Consequently,

$$dE_r = \frac{1}{4\pi\varepsilon_0}\frac{\lambda r_0 \, d\alpha}{r_0^2} = \frac{\lambda}{4\pi\varepsilon_0 r}\cos\alpha \, d\alpha$$

This expression can be easily integrated :

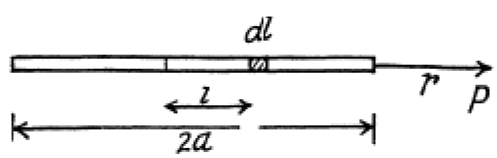
$$E = \frac{\lambda}{4\pi\varepsilon_0 r}\, 2\int_0^{\alpha_0} \cos\alpha \, d\alpha = \frac{\lambda}{4\pi\varepsilon_0 r}\, 2\sin\alpha_0$$

$$\sin\alpha_0 = a \Big/ \sqrt{a^2 + r^2}$$

Thus, $E = \frac{q/2a}{4\pi\varepsilon_0 r}\, 2\frac{a}{\sqrt{a^2+r^2}} = \frac{q}{4\pi\varepsilon_0 r\sqrt{a^2+r^2}}$

Note that in this case also $E \approx \frac{q}{4\pi\varepsilon_0 r^2}$ for $r >> a$ of the field of appoint charge.

(b) Let, us consider the element of length dl at a distance l from the centre of the rod, as shown in the figure.



Then field at P, due to this element.

$$dE = \frac{\lambda \, dl}{4\pi\varepsilon_0 (r-l)^2},$$

If the element lies on the side, shown in the diagram, and $dE = \frac{\lambda \, dl}{4\pi\varepsilon_0 (r+l)^2},$ if it lies on other side.

Hence $$E = \int dE = \int_0^a \frac{\lambda\, dl}{4\pi\varepsilon_0 (r-l)^2} + \int_0^a \frac{\lambda\, dl}{4\pi\varepsilon_0 (r+l)^2}$$

On integrating and putting $\lambda = \frac{q}{2a}$, we get, $E = \frac{q}{4\pi\varepsilon_0} \frac{1}{(r^2 - a^2)}$

For $r >> a, \quad E \approx \frac{q}{4\pi\varepsilon_0 r^2}$

Q.14. A very long straight uniformly charged thread carries a charge λ. per unit length. Find the magnitude and direction of the electric field strength at a point which is at a distance y from the thread and lies on the perpendicular passing through one of the thread's ends.

Solution. 14. The problem is reduced to finding E_x and E_y viz . the projections

of $\vec{E}$ in Fig, where it is assumed that $\lambda > 0$.

Let us start with E_x. The contribution to E_x. from the charge element of the segment

d_x is

$$dE_x = \frac{1}{4\pi\varepsilon_0} \frac{\lambda\, dx}{r^2} \sin\alpha \qquad (1)$$

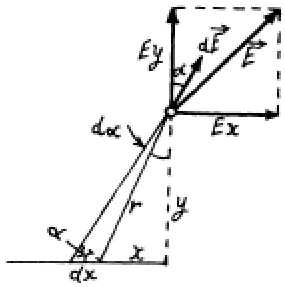
Let us reduce this expression to the form convenient for integration. In our case, dx = r

Ja/cos α, r = y/cosα. Then

$$dE_x = \frac{\lambda}{4\pi\varepsilon_0 y} \sin\alpha\, d\alpha.$$

Integrating this expression over α between 0 and π/2, we find

$$E_x = \lambda / 4\pi\varepsilon_0 y.$$



In order to find the projection E_y it is sufficient to recall that differs from dE_x in that sin α in (1) is simply replaced
by cos α.
This gives

$dE_y = (\lambda \cos\alpha \, d\alpha)/4\pi\varepsilon_0 y$ and $E_y = \lambda/4\pi\varepsilon_0 y$. We have obtained an in teresting result :

$E_x = E_y$ in dependently of y,

i.e. $\vec{E}$ is Oriented at the angle of 45° to the rod. The modulus of $\vec{E}$ is

$$E = \sqrt{E_x^2 + E_y^2} = \lambda\sqrt{2}/4\pi\varepsilon_0 y.$$

Q.15. A thread carrying a uniform charge λ per unit length has the configurations shown in Fig. 3.2 a and b. Assuming a curvature radius R to be considerably less than the length of the thread, find the magnitude of the electric field strength at the point O.

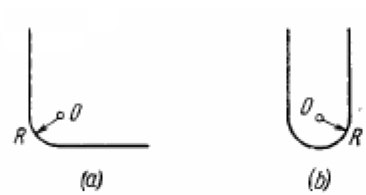
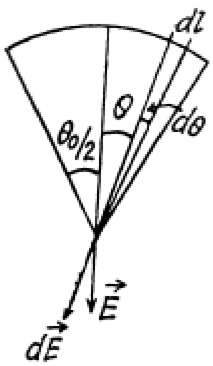


Fig. 3.2.

Solution. 15. (a) Using the solution of Q.14, the net electric field strength at the point O due to straight parts of the thread equals zero. For the curved part (arc) let us derive a general expression i.e. let us calculate the field strength at the centre of arc of radius R and linear charge density λ and which subtends angle θ_0 at the centre.



From the symmetry the sought field strength will be directed along the bisector of the angle θ_0 and is given by

$$E = \int_{-\theta_0/2}^{+\theta_0/2} \frac{\lambda\,(R\,d\theta)}{4\pi\varepsilon_0 R^2}\cos\theta = \frac{\lambda}{2\pi\varepsilon_0 R}\sin\frac{\theta_0}{2}$$

In our problem $\theta_0 = \pi/2$, thus the field strength due to the turned part at the

point $E_0 = \frac{\sqrt{2}\,\lambda}{4\pi\varepsilon_0 R}$ which is also the sought result.

(b) Using the solution of 3.14 (a) , net field strength at O due to straight parts

equals $\sqrt{2}\left(\frac{\sqrt{2}\,\lambda}{4\pi\varepsilon_0 R}\right) = \frac{\lambda}{2\pi\varepsilon_0 R}$ and is directed vertically down. Now using the solution of

(a), field strength due to the given curved part (semi-circle) at the point O

becomes $\frac{\lambda}{2\pi\varepsilon_0 R}$ and is directed vertically upward. Hence the sought net field strength

becomes zero.

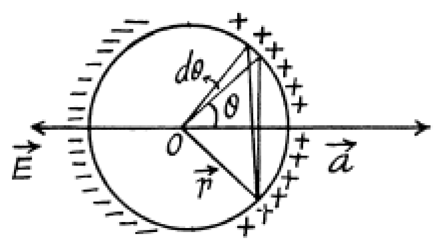
Q.16. A sphere of radius r carries a surface charge of density σ = ar, where a is a constant vector, and r is the radius vector of a point of the sphere relative to its centre. Find the electric field strength vector at the centre of the sphere.

Solution. 16. Given charge distribution on the surface $\sigma = \vec{a}\cdot\vec{r}$ is shown in the figure. Symmetry of this distribution implies that the sought $\vec{E}$ at the centre O of the sphere is opposite to $\vec{a}$

$$dq = \sigma (2\pi r \sin\theta) r\, d\theta = (\vec{a}\cdot\vec{r})\, 2\pi r^2 \sin\theta\, d\theta = 2\pi a r^3 \sin\theta \cos\theta\, d\theta$$

Again from symmetry, field strength due to any ring element $\vec{dE}$ is also opposite

to $\vec{a}$ i.e. $\vec{dE} \uparrow\downarrow \vec{a}$. Hence



$$\vec{dE} = \frac{dq\, r\cos\theta}{4\pi\varepsilon_0 (r^2\sin^2\theta + r^2\cos^2\theta)^{3/2}} \frac{-\vec{a}}{a} \text{(Using the result of 3.9)}$$

$$= \frac{(2\pi a r^3 \sin\theta\cos\theta\, d\theta)\, r\cos\theta}{4\pi\varepsilon_0 r^3} \frac{(-\vec{a})}{a}$$

$$= \frac{-\vec{a}\, r}{2\varepsilon_0} \sin\theta \cos^2 d\theta$$

Thus

$$\vec{E} = \int \vec{dE} = \frac{(-\vec{a})\, r}{2\varepsilon_0} \int_0^{\pi} \sin\theta\cos^2\theta\, d\theta$$

Integrating, we get $\vec{E} = -\frac{\vec{a}\, r}{2\varepsilon_0}\frac{2}{3} = -\frac{\vec{a}\, r}{3\varepsilon_0}$

Q.17. Suppose the surface charge density over a sphere of radius R depends on a polar angle θ as $\sigma = \sigma_0 \cos\theta$, where σ_0 is a positive constant. Show that such a charge distribution can be represented as a result of a small relative shift of two uniformly charged balls of radius R whose charges are equal in magnitude and opposite in sign. Resorting to this representation, find the electric field strength vector inside the given sphere.

Solution. 17. We start from two charged spherical balls each of radius R with equal and opposite charge densities + p and - p. The centre of the balls are

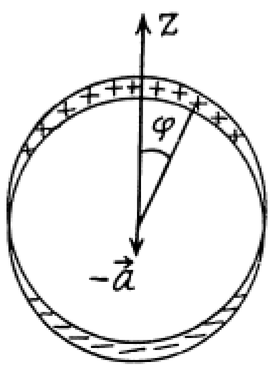
at $+\frac{\vec{a}}{2}$ and $-\frac{\vec{a}}{2}$ respectively so the equation of their surfaces

are $\left|\vec{r} - \frac{\vec{a}}{2}\right| = R$ or $r - \frac{a}{2}\cos\theta = R$ and $r + \frac{a}{2}\cos\theta = R,$ considering a to be small. The distance between the two surfaces in the radial direction at angle θ is $|a\cos\theta|$ and does not depend on the azimuthal angle. It is seen from the diagram that the surface of the sphere has in effect a surface density $\sigma = \sigma_0 \cos\theta$ when
$\sigma_0 = \rho a.$

Inside any uniformly charged spherical ball, the field is radial and has the magnitude given by Gauss's theorem

$$4\pi r^2 E = \frac{4\pi}{3} r^3 \rho/\varepsilon_0$$

Or $E = \frac{\rho r}{3\varepsilon_0}$



In vector notation, using the fact the V must be measured from the centre of the ball, we get, for the present case

$$\vec{E} = \frac{\rho}{3\varepsilon_0}\left(\vec{r} - \frac{\vec{a}}{2}\right) - \frac{\rho}{3\varepsilon_0}\left(\vec{r} + \frac{\vec{a}}{2}\right)$$

$$= -\rho\vec{a}/3\varepsilon_0 = \frac{\sigma_0}{3\varepsilon_0}\vec{k}$$

When $\vec{k}$ is is the unit vector along the polar axis from which θ is measured.

Constant Electric Field In Vacuum (Part - 2)

Q. 18. Find the electric field strength vector at the centre of a ball of radius R with volume charge density p = ar, where a is a constant vector, and r is a radius vector drawn from the ball's centre.

Solution.18. Let us consider an elemental spherical shell of thickness dr. Thus surface charge density of the shell $\sigma = \rho\, dr = (\vec{a} \cdot \vec{r})\, dr.$

Thus using the solution of Q.16, field strength due to this spherical shell

$$d\vec{E} = -\frac{\vec{a}\, r}{3\,\varepsilon_0}\, dr$$

Hence the sought field strength

$$\vec{E} = -\frac{\vec{a}}{3\,\varepsilon_0} \int_0^R r\, dr = -\frac{\vec{a} R^2}{6\,\varepsilon_0}.$$

Q. 19. A very long uniformly charged thread oriented along the axis of a circle of radius R rests on its centre with one of the ends. The charge of the thread per unit length is equal to λ. Find the flux of the vector E across the circle area.

Solution.19. From the solution of Q.14 field strength at a perpendicular distance r < R from its left end

$$\vec{E}(r) = \frac{\lambda}{4\pi\varepsilon_0 r}(-\vec{i}) + \frac{\lambda}{4\pi\varepsilon_0 r}(\hat{e}_r)$$

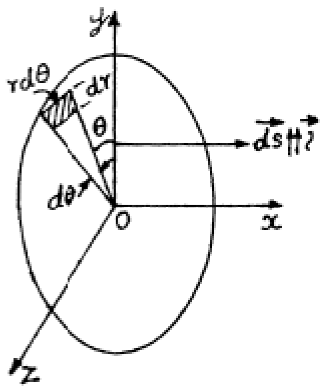
Here $\hat{e}_r$ is a unit vector along radial direction.

Let us consider an elemental surface, $dS = dy\, dz = dz\,(r\, d\theta)$ a figure. Thus flux

of $\vec{E}(r)$ over the element $d\vec{S}$ is given by

$$d\Phi = \vec{E}\cdot d\vec{S} = \left[\frac{\lambda}{4\pi\varepsilon_0 r}(-\vec{i}) + \frac{\lambda}{4\pi\varepsilon_0 r}(\hat{e}_r)\right]\cdot dr\,(r\,d\theta)\,\vec{i}$$

$$= -\frac{\lambda}{4\pi\varepsilon_0}dr\,d\theta\left(\text{as } \vec{e}_r \perp \vec{i}\right)$$



The sought flux, $\Phi = -\frac{\lambda}{4\pi\varepsilon_0}\int_0^R dr \int_0^{2\pi} d\theta = -\frac{\lambda R}{2\varepsilon_0}.$

If we have taken $d\vec{S}\uparrow\uparrow(-\vec{i})$, then Φ were $\frac{\lambda R}{2\varepsilon_0}$

Hence $|\Phi| = \frac{\lambda R}{2\varepsilon_0}$

Q. 20. Two point charges q and —q are separated by the distance 21 (Fig. 3.3). Find the flux of the electric field strength vector across a circle of radius R.

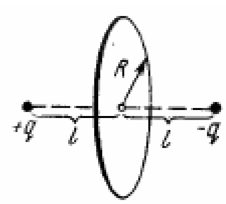


Fig. 3.3.

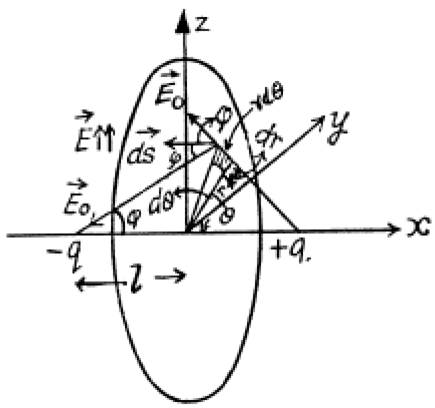
Solution. 20. Let us consider an elemental surface area as shown in the figure. Then

flux of the vector $\vec{E}$ through the elemental area,

$$d\Phi = \vec{E}\cdot d\vec{S} = E\,dS = 2E_0\cos\varphi\,dS\ (\text{as } \vec{E}\uparrow\uparrow d\vec{S})$$

$$= \frac{2q}{4\pi\varepsilon_0(l^2+r^2)} \frac{l}{(l^2+r^2)^{1/2}}(r\,d\theta)\,dr = \frac{2ql\,r\,dr\,d\theta}{4\pi\varepsilon_0(r^2+l^2)^{3/2}}$$

where $E_0 = \frac{q}{4\pi\varepsilon_0(l^2+r^2)}$ is magnitude of field strength due to any point charge at the point of location of considered elemental area.



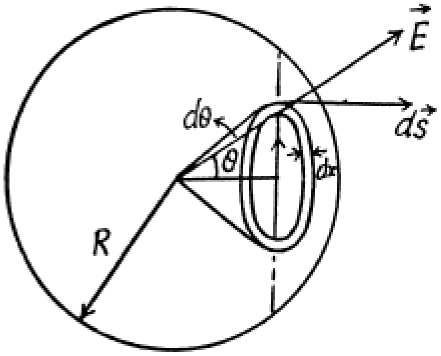
$$\text{Thus } \Phi = \frac{2ql}{4\pi\varepsilon_0}\int_0^R \frac{r\,dr}{(r^2+l^2)^{3/2}}\int_0^{2\pi} d\theta$$

$$= \frac{2ql\times 2\pi}{4\pi\varepsilon_0}\int_0^R \frac{r\,dr}{(r^2+l^2)^{3/2}} = \frac{q}{\varepsilon_0}\left[1-\frac{l}{\sqrt{l^2+R^2}}\right]$$

It can also be solved by considering a ring element or by using solid angle.

Q. 21. A ball of radius R is uniformly charged with the volume density p. Find the flux of the electric field strength vector across the ball's section formed by the plane located at a distance r_0 < R from the centre of the ball.

Solution. 21. Let us consider a ring element of radius x and thickness dx, as shown in the figure. Now, flux over the considered element,



$$d\Phi = \vec{E}\cdot\vec{dS} = E_r\, dS\cos\theta$$

But $E_r = \frac{\rho r}{3\varepsilon_0}$ from Gauss's theorem, and $dS = 2\pi x\, dx$, $\cos\theta = \frac{r_0}{r}$

Thus $d\Phi = \frac{\rho r}{3\varepsilon_0} 2\pi x\, dx \frac{r_0}{r} = \frac{\rho r_0}{3\varepsilon_0} 2\pi x\, dx$

Hence sought flux

$$\Phi = \frac{2\pi\rho r_0}{3\varepsilon_0}\int_0^{\sqrt{R^2 - r_0^2}} x\, dx = \frac{2\pi\rho r_0}{3\varepsilon_0}\frac{(R^2 - r_0^2)}{2} = \frac{\pi\rho r_0}{3\varepsilon_0}(R^2 - r_0^2)$$

Q. 22. Each of the two long parallel threads carries a uniform charge per unit length. The threads are separated by a distance l. Find the maximum magnitude of the electric field strength in the symmetry plane of this system located between the threads.

Solution. 22. The field at P due to the threads at A and B are both of

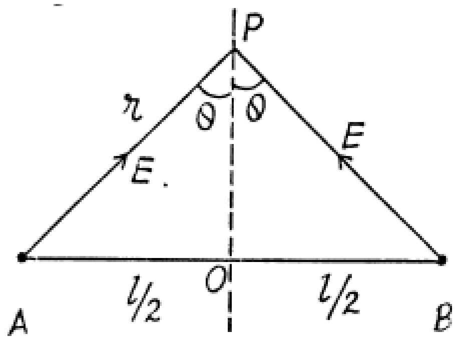
magnitude $\frac{\lambda}{2\pi\varepsilon_0 (x^2 + l^2/4)^{1/2}}$ and directed along AP and BP. The resultant is along OP

with

$$E = \frac{2\lambda\cos\theta}{2\pi\varepsilon_0(\pi^2+\pi^{1/2})^{1/2}} = \frac{\lambda x}{\pi\varepsilon_0(x^2+l^2/4)}$$

$$= \frac{\lambda}{\pi\varepsilon_0\left[x + \frac{l^2}{4x} - 2\cdot\frac{l}{2\sqrt{x}}\cdot\sqrt{x} + l\right]}$$

$$= \frac{\lambda}{\pi\varepsilon_0\left[\left(\sqrt{x} - \frac{l}{2\sqrt{x}}\right)^2 + l\right]}$$

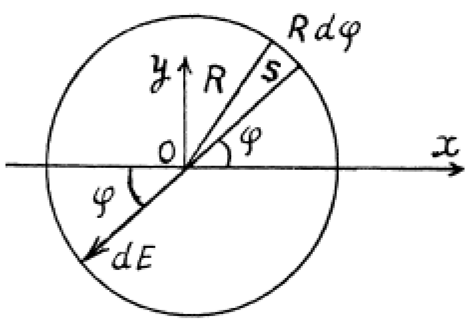


This is maximum when $x = l/2$ and then $E = E_{max} = \frac{\lambda}{\pi\varepsilon_0 l}$

Q. 23. An infinitely long cylindrical surface of circular cross-section is uniformly charged lengthwise with the surface density $\sigma = \sigma_0\cos\varphi$, where φ is the polar angle of the cylindrical coordinate system whose z axis coincides with the axis of the given surface. Find the magnitude and direction of the electric field strength vector on the z axis.

Solution. 23. Take a section of the cylinder perpendicular to its axis through the point where the electric field is to be calculated. (All points on the axis are equivalent.) Consider an element S with azimuthal angle φ. The length of the element is R,dφ, R being the radius of cross section of the cylinder. The element itself is a section of an infinite strip. The electric field at O due to this strip is

$$\frac{\sigma_0\cos\varphi\,(R\,d\varphi)}{2\pi\varepsilon_0 R} \text{ along } SO$$



This can be resolved into

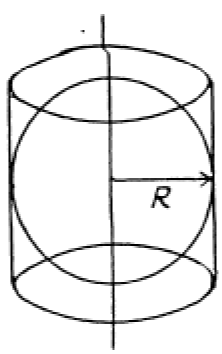
$$\frac{\sigma_0 \cos \varphi \, d\varphi}{2\pi \varepsilon_0} \begin{cases} \cos \varphi \text{ along } OX \text{ towards } O \\ \sin \varphi \text{ along } YO \end{cases}$$

On integration the component along YO vanishes. What remains is

$$\int_0^{2\pi} \frac{\sigma_0 \cos^2 \varphi \, d\varphi}{2\pi \varepsilon_0} = \frac{\sigma_0}{2\varepsilon_0} \text{ along } XO$$

Q. 24. The electric field strength depends only on the x and y coordinates according to the law E = a (xi + yj)/(x^2 + y^2), where a is a constant, i and j are the unit vectors of the x and y axes. Find the flux of the vector E through a sphere of radius R with its centre at the origin of coordinates.

Solution. 24. Since the field is axisymmetric (as the field of a uniformly charged filament), we conclude that the flux through the sphere of radius R is equal to the flux through the lateral surface of a cylinder having the same radius and the height 2R, as arranged in the figure.



Now, $\Phi = \oint \vec{E} \cdot d\vec{S} = E_r S$

But $E_r = \frac{a}{R}$

Thus $\Phi = \frac{a}{R} S = \frac{a}{R} 2 \pi R \cdot 2 R = 4 \pi a R$

Q. 25. A ball of` radius R carries a positive charge whose volume density depends only on a separation r from the ball's centre as P = p_0 (1 — r/R), where P_0 is a constant. Assuming the permittivity's of the ball and the environment to be equal to unity, find:
(a) the magnitude of the electric field strength as a function of the distance r both inside and outside the ball;
(b) the maximum intensity E_{max} and the corresponding distance r_m.

Solution. 25. (a) Let us consider a sphere of radius r < R then charge, inclosed by the considered sphere,

$$q_{inclosed} = \int_0^r 4 \pi r^2 \, dr \, \rho = \int_0^r 4 \pi r^2 \rho_0 \left(1 - \frac{r}{R}\right) dr \qquad (1)$$

Now, applying Gauss' theorem,

$E_r 4 \pi r^2 = \frac{q_{inclosed}}{\varepsilon_0}$, (where E_r is the projection of electric field along the radial line.)

$$= \frac{\rho_0}{\varepsilon_0} \int_0^r 4 \pi r^2 \left(1 - \frac{r}{R}\right) dr$$

or, $E_r = \frac{\rho_0}{3 \varepsilon_0} \left[r^2 - \frac{3 r^2}{4 R} \right]$

And for a point, outside the sphere r > R.

$q_{inclosed} = \int_0^R 4 \pi r^2 \, dr \, \rho_0 \left(1 - \frac{r}{R}\right)$ (as there is no charge outside the ball)

Again from Gauss' theorem,

$$E_r \, 4 \pi r^2 = \int_0^R \frac{4 \pi r^2 \, dr \, \rho_0 \left(1 - \frac{r}{R}\right)}{\varepsilon_0}$$

or, $$E_r = \frac{\rho_0}{r^2 \varepsilon_0} \left[\frac{R^3}{3} - \frac{R^4}{4R}\right] = \frac{\rho_0 R^3}{12 r^2 \varepsilon_0}$$

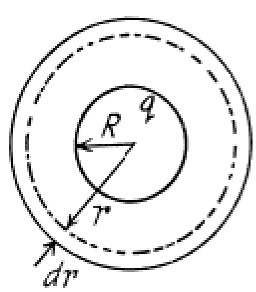
(b) As 'magnitude of electric field decreases with increasing r for r > R, field will be maximum for r < R. Now, for Er to be maximum

$$\frac{d}{dr}\left(r - \frac{3 r^2}{4 R}\right) = 0 \quad \text{or} \quad 1 - \frac{3r}{2R} = 0 \quad \text{or} \quad r = r_m = \frac{2R}{3}$$

Hence $$E_{max} = \frac{\rho_0 R}{9 \varepsilon_0}$$

Q. 26. A system consists of a ball of radius R carrying a spherically symmetric charge and the surrounding space filled with a charge of volume density p = α/r, where α is a constant, r is the distance from the centre of the ball. Find the ball's charge at which the magnitude of the electric field strength vector is independent of r outside the ball. How high is this strength? The permittivity's of the ball and the surrounding space are assumed to be equal to unity.

Solution. 26. Let the charge carried by the sphere be q, then using Gauss' theorem for a spherical surface having radius r > R, we can write.



$$E \, 4 \pi r^2 = \frac{q_{inclosed}}{\varepsilon_0} = \frac{q}{\varepsilon_0} + \frac{1}{\varepsilon_0} \int_R^r \frac{\alpha}{r} 4 \pi r^2 \, dr$$

On integrating we get,

$$E \, 4 \pi r^2 = \frac{(q - 2 \pi \alpha R^2)}{\varepsilon_0} + \frac{4 \pi \alpha r^2}{2 \varepsilon_0}$$

The intensity E does not depend on r when the expression in the parentheses is equal to zero. Hence

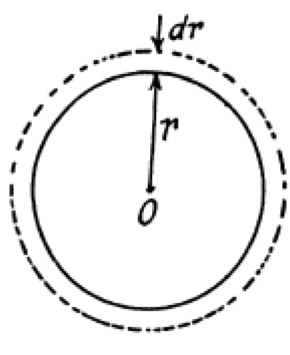
$$q = 2\pi\alpha R^2 \text{ and } E = \frac{\alpha}{2\varepsilon_0}$$

Q. 27. A space is filled up with a charge with volume density $\rho = \rho_0 e^{-\alpha r^3}$, where P_0 and α are positive constants, r is the distance from the centre of this system. Find the magnitude of the electric field strength vector as a function of r. Investigate the obtained expression for the small and large values of r, i.e. at $\alpha r^3 \ll 1$ and $\alpha r3 \gg 1$.

Solution. 27. Let us consider a spherical layer of radius r and thickness dr, having its centre coinciding with the centre of the system. Then using Gauss' theorem for this surface,

$$E_r 4\pi r^2 = \frac{q_{inclosed}}{\varepsilon_0} = \int_0^r \frac{\rho\, dV}{\varepsilon_0}$$

$$= \frac{1}{\varepsilon_0}\int_0^r \rho_0 e^{-\alpha r^3} 4\pi r^2\, dr$$



After integration

$$E_r 4\pi r^2 = \frac{\rho 4\pi}{3\varepsilon_0\alpha}[1 - e^{-\alpha r^3}]$$

$$\text{or,} \quad E_r = \frac{\rho_0}{3\varepsilon\alpha r^2}[1 - e^{-\alpha r^3}]$$

Now when $\alpha r^3 << 1$, $E_r \approx \frac{\rho_0 r}{3\varepsilon_0}$

And when $\alpha r^3 >> 1$, $E_r \approx \frac{\rho_0}{3\varepsilon_0 \alpha r^2}$

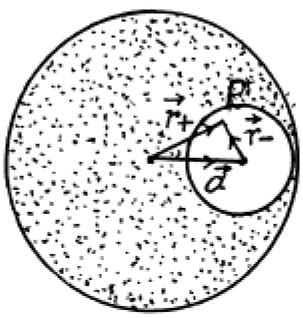
Q. 28. Inside a ball charged uniformly with volume density p there is a spherical cavity. The centre of the cavity is displaced with respect to the centre of the ball by a distance a. Find the field strength E inside the cavity, assuming the permittivity equal to unity.

Solution. 28. Using Gauss theorem we can easily show that the electric field strength

within a uniformly charged sphere is $\vec{E} = \left(\frac{\rho}{3\varepsilon_0}\right)\vec{r}$

The cavity, in our problem, may be considered as the superposition of two balls, one

with the charge density p and the other with - p.



Let P be a point inside the cavity such that its position vector with respect to the centre

of cavity be $\vec{r}_-$ and with respect to the centre of the ball $\vec{r}_+$ Then then from the principle

of superposition, field inside the cavity, at an arbitrary point P,

$$\vec{E} = \vec{E}_+ + \vec{E}_-$$

$$= \frac{\rho}{3\varepsilon_0}(\vec{r}_+ - \vec{r}_-) = \frac{\rho}{3\varepsilon_0}\vec{a}$$

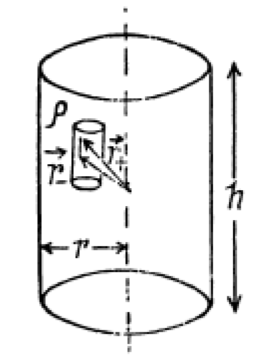
Note: Obtained expression for $\vec{E}$ shows that it is valid regardless of the ratio between

the radii of the sphere and the distance between their centres.

Q. 29. Inside an infinitely long circular cylinder charged uniformly with volume density p there is a circular cylindrical cavity. The distance between the axes of the

cylinder and the cavity is equal to a. Find the electric field strength E inside the cavity. The permittivity is assumed to be equal to unity.

Solution. 29. Let us consider a cylindrical Gaussian surface of radius r and height h inside an infinitely long charged cylinder with charge density p. Now from Gauss theorem :



$$E_r\, 2\pi r h = \frac{q_{inclosed}}{\varepsilon_0}$$

(Where E_r is the field inside the cylinder at a distance r from its axis.)

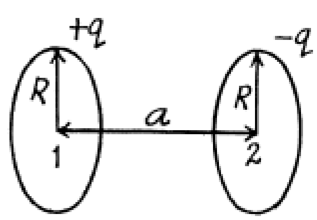
Or, $$\text{or,}\quad E_r\, 2\pi r h = \frac{\rho \pi r^2 h}{\varepsilon_0} \quad \text{or} \quad E_r = \frac{\rho r}{2\varepsilon_0}$$

Now, using the method of Q.28 field at a point P, inside the cavity, is

$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho}{2\varepsilon_0}(\vec{r}_+ - \vec{r}_-) = \frac{\rho}{2\varepsilon_0}\vec{a}$$

Q. 30. There are two thin wire rings, each of radius R, whose axes coincide. The charges of the rings are q and —q. Find the potential difference between the centres of the rings separated by a distance a.

Solution. 30. The arrangement of the rings are as shown in the figure. Now, potential at the point 1, φ_1= potential at 1 due to the ring 1 + potential at 1 due to the ring 2.



$$= \frac{q}{4\pi\varepsilon_0 R} + \frac{-q}{4\pi\varepsilon_0 (R^2 + a^2)^{1/2}}$$

Similarly, the potential at point 2,

$$\varphi_2 = \frac{-q}{4\pi\varepsilon_0 R} + \frac{q}{4\pi\varepsilon_0 (R^2 + a^2)^{1/2}}$$

Hence, the sought potential difference,

$$\varphi_1 - \varphi_2 = \Delta\varphi = 2\left(\frac{q}{4\pi\varepsilon_0 R} + \frac{-q}{4\pi\varepsilon_0 (R^2 + a^2)^{1/2}}\right)$$

$$= \frac{q}{2\pi\varepsilon_0 R}\left(1 - \frac{1}{\sqrt{1 + (a/R)^2}}\right)$$

Q. 31. There is an infinitely long straight thread carrying a charge with linear density λ, = 0.40 µC/m. Calculate the potential difference between points 1 and 2 if point 2 is removed η = 2.0 times farther from the thread than point I.

Solution. 31. We know from Gauss theorem that the electric field due to an infinitely long straight wire, at a perpendicular distance r from it equals, $E_r = \frac{\lambda}{2\pi\varepsilon_0 r}$. so, the work done is

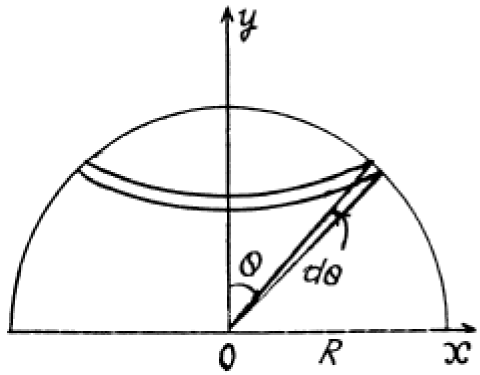
$$\int_1^2 E_r\, dr = \int_x^{\eta x} \frac{\lambda}{2\pi\varepsilon_0 r}\, dr$$

(Where x is perpendicular distance from the thread by which point 1 is removed from it.)

Hence $\Delta\varphi_{12} = \frac{\lambda}{2\pi\varepsilon_0} \ln \eta$

Q. 32. Find the electric field potential and strength at the centre of a hemisphere of radius R charged uniformly with the surface density σ.

Solution. 32. Let us consider a ring element as shown in the figure. Then the charge, carried by the element, $dq = (2\pi R \sin\theta) R\, d\theta\, \sigma,$



Hence, the potential due to the considered element at the centre of the hemisphere,

$$d\varphi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R} = \frac{2\pi\sigma R\sin\theta\, d\theta}{4\pi\varepsilon_0} = \frac{\sigma R}{2\varepsilon_0}\sin\theta\, d\theta$$

So potential due to the whole hemisphere

$$\varphi = \frac{R\sigma}{2\varepsilon_0}\int_0^{\pi/2}\sin\theta\, d\theta = \frac{\sigma R}{2\varepsilon_0}$$

Now from the symmetry of the problem, net electric field of the hemisphere is directed towards the negative y - axis. We have

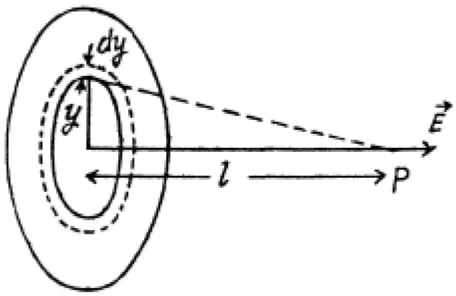
$$dE_y = \frac{1}{4\pi\varepsilon_0}\frac{dq\cos\theta}{R^2} = \frac{\sigma}{2\varepsilon_0}\sin\theta\cos\theta\, d\theta$$

$$\text{Thus}\quad E = E_y' = \frac{\sigma}{2\varepsilon_0}\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta = \frac{\sigma}{4\varepsilon_0}\int_0^{\pi/2}\sin 2\theta\, d\theta = \frac{\sigma}{4\varepsilon_0},\ \text{along } YO$$

Q. 33. A very thin round plate of radius R carrying a uniform surface charge density a is located in vacuum. Find the electric field potential and strength along the plate's axis as a function of a distance l from its centre. Investigate the obtained expression at l → 0 and l ≫ R.

Solution. 33. Let us consider an elementary ring of thickness dy and radius y as shown in the figure. Then potential at a point P, at distance l from the centre of the disc, is

$$d\varphi = \frac{\sigma\, 2\pi y\, dy}{4\pi\varepsilon_0 (y^2 + l^2)^{1/2}}$$



Hence potential due to the whole disc,

$$\varphi = \int_0^R \frac{\sigma 2\pi y\,dy}{4\pi\varepsilon_0 (y^2 + l^2)^{1/2}} = \frac{\sigma l}{2\varepsilon_0}\left(\sqrt{1+(R/l)^2} - 1\right)$$

From symmetry

$$E = E_l = -\frac{d\varphi}{dl}$$

$$= -\frac{\sigma}{2\varepsilon_0}\left[\frac{2l}{2\sqrt{R^2+l^2}} - 1\right] = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{1}{\sqrt{1+(R/l)^2}}\right]$$

when $l \to 0$, $\varphi = \frac{\sigma R}{2\varepsilon_0}$, $E = \frac{\sigma}{2\varepsilon_0}$ and when $l >> R$,

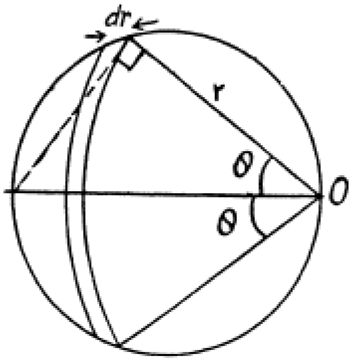
$$\varphi = \frac{\sigma R^2}{4\varepsilon_0 l},\ E = \frac{\sigma R^2}{4\varepsilon_0 l^2}$$

Q. 34. Find the potential φ at the edge of a thin disc of radius R carrying the uniformly distributed charge with surface density σ.

Solution. 34. By definition, the potential in the case of a surface charge distribution is defined by integral $\varphi = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\,dS}{r}$. In order to simplify integration, we shall choose the area element ds in the form of a part of the ring of radius r and width dr in (Fig.). Then dS - 2θ r dr, r = 2R cos θ and dr = - 2R sin θ d θ. After substituting these expressions into integral

$$\varphi = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\, dS}{r},$$ We obtain the expression for cp at the point O:

$$\varphi = -\frac{\sigma R}{\pi\varepsilon_0}\int_{\pi/2}^{0} \theta \sin\theta\, d\theta.$$



We integrate by parts, denoting $\theta = u$ and $\sin\theta\, d\theta = dv$:

$$\int \theta \sin\theta\, d\theta = -\theta\cos\theta + \int \cos\theta\, d\theta = -\theta\cos\theta + \sin\theta$$

Which gives -1 after substituting the limits of integration. As a result, we obtain $\varphi = \sigma R/\pi\varepsilon_0$.

Constant Electric Field In Vacuum (Part - 3)

Q. 35. Find the electric field strength vector if the potential of this field has the form φ = ar, where a is a constant vector, and r is the radius vector of a point of the field.

Solution. 35. In accordance with the problem $\varphi = \vec{a} \cdot \vec{r}$

Thus from the equation : $\vec{E} = -\vec{\nabla}\varphi$

$$\vec{E} = -\left[\frac{\partial}{\partial x}(a_x x)\vec{i} + \frac{\partial}{\partial_y}(a_y y)\vec{j} + \frac{\partial}{\partial_z}(a_z z)\vec{k}\right] = -[a_x\vec{i} + a_y\vec{j} + {}_z\vec{k}] = -\vec{a}$$

Q. 36. Determine the electric field strength vector if the potential of this field depends on x, y coordinates as
(a) $\varphi = a(x^2 — y^2)$;
(b) φ = axy,

where a is a constant. Draw the approximate shape of these fields' .using lines of force (in the x, y plane).

Solution. 36. (a) Given, $\varphi = a(x^2 - y^2)$

So, $\vec{E} = -\vec{\nabla}\varphi = -2a(x\vec{i} - y\vec{j})$

The sought shape of field lines is as shown in the figure (a) of answer sheet assuming a > 0:

(b) Since φ = axy

So, $\vec{E} = -\vec{\nabla}\varphi = -ay\vec{i} - ax\vec{j}$

Plot as shown in the figure (b) of answer sheet

Q. 37. The potential of a certain electrostatic field has the form cp = a $(x^2 + y^2) + bz^2$, where a and b are constants. Find the magnitude and direction of the electric field strength vector. What shape have the equipotential surfaces in the following cases:
(a) a > 0, b> 0; (b) a > 0, b < 0?

Solution. 37. Given, $\varphi = a(x^2 + y^2) + bz^2$

So, $\vec{E} = -\vec{\nabla}\varphi = -[2ax\vec{i} + 2ay\vec{j} + 2bz\vec{k}]$

Hence $|\vec{E}| = 2\sqrt{a^2(x^2+y^2) + b^2z^2}$

Shape of the equipotential surface:

Put $\vec{\rho} = x\vec{i} + y\vec{j}$ or $\rho^2 = x^2 + y^2$

Then the equipotential surface has the equation

$$a\rho^2 + bz^2 = \text{constant} = \varphi$$

If $a > 0,\ b > 0$ then $\varphi > 0$ and the equation of the equipotential surface is

$$\frac{\rho}{\varphi/a} + \frac{z^2}{\varphi/b} = 1$$

Which is an ellipse in p , z coordinates. In three dimensions the surface is an ellipsoid

of revolution with semi- axis $\sqrt{\varphi/a}$, $\sqrt{\varphi/a}$, $\sqrt{\varphi/b}$.

If $a > 0,\ b < 0$ then φ can be ≥ 0. If $\varphi > 0$ then the equation is

$$\frac{\rho^2}{\varphi/a} - \frac{z^2}{\varphi/|b|} = 1$$

This is a single cavity hyperboloid of revolution about z axis. If $\varphi = 0$ then

$$a\rho^2 - |b|z^2 = 0$$

or $$z = \pm\sqrt{\frac{a}{|b|}}\,\rho$$

Is the equation of a right circular cone.

$$|b|z^2 - a\rho^2 = |\varphi|$$

or $\frac{z^2}{|\varphi|/|b|} - \frac{\rho^2}{|\varphi|/a} = 1$

This is a two cavity hyperboloid of revolution about z-axis.

Q. 38. A charge q is uniformly distributed over the volume of a sphere of radius R. Assuming the permittivity to be equal to unity throughout, find the potential
(a) at the centre of the sphere;
(b) inside the sphere as a function of the distance r from its centre.

Solution. 38. From Gauss' theorem intensity at a point, inside the sphere at a distance r

from the centre is given by, $E_r = \frac{\rho r}{3\varepsilon_0}$ and outside it, is given by $E_r = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$.

(a) Potential at the centre of the sphere,

$$\varphi_0 = \int_0^\infty \vec{E}\cdot d\vec{r} = \int_0^R \frac{\rho r}{3\varepsilon_0} dr + \int_R^\infty \frac{q}{4\pi\varepsilon_0 r^2} \cdot dr = \frac{\rho}{3\varepsilon}\frac{R^2}{2} + \frac{q}{4\pi\varepsilon_0 R}$$

As $= \frac{q}{8\pi\varepsilon_0 R} + \frac{q}{4\pi\varepsilon_0 R} = \frac{3q}{8\pi\varepsilon_0 R} \quad \left(\text{as } \rho = \frac{3q}{4\pi R^3}\right)$

(b) Now, potential at any point, inside the sphere, at a distance r from it s centre.

$$\varphi(r) = \int_r^R \frac{\rho}{3\varepsilon_0} r\, dr \int_r^\infty \frac{q}{4\pi\varepsilon_0}\frac{dr}{r^2}$$

On integration: $\varphi(r) = \frac{3q}{8\pi\varepsilon_0 R}\left[1 - \frac{r^2}{3R^2}\right] = \varphi_0\left[1 - \frac{r^2}{3R^2}\right]$

Q. 39. Demonstrate that the potential of the field generated by a dipole with the electric moment p (Fig. 3.4) may be represented as $\varphi = \mathbf{p}\mathbf{r}/4\pi\varepsilon_0 r^3$, where r is the radius vector. Using this expression, find the magnitude of the electric field strength vector as a function of r and θ.

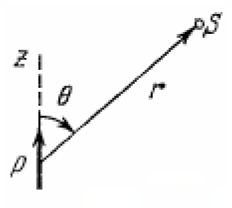
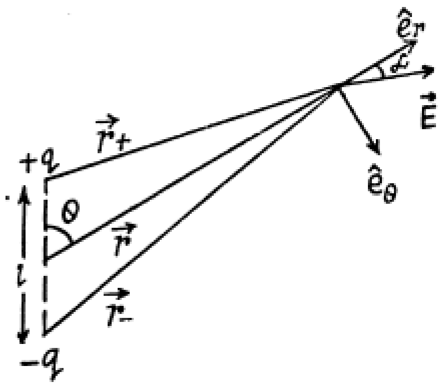


Fig. 3.4.

Solution. 39. Let two charges +q and - q be separated by a distance l. Then electric potential at a point at distance r > l from this dipole,



$$\varphi(r) = \frac{+q}{4\pi\varepsilon_0 r_+} + \frac{-q}{4\pi\varepsilon_0 r_-} = \frac{q}{4\pi\varepsilon_0}\left(\frac{r_- - r_+}{r_+ r_-}\right) \qquad (1)$$

But $r_- - r_+ = l\cos\theta$ and $r_+ r_- = r^2$

From Eqs. (1) and (2),

$$\varphi(r) = \frac{q\,l\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}\,\varphi = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3},$$

Where p is magnitude of electric moment vector.

Now, $E_r = -\frac{\partial\varphi}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$

and $E_\theta = -\frac{\partial\varphi}{r\,\partial\theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$

So $E = \sqrt{E_r^2 + E_0^2} = \frac{p}{4\pi\varepsilon_0 r^3}\sqrt{4\cos^2\theta + \sin^2\theta}$

Q. 40. A point dipole with an electric moment p oriented in the positive direction of the z axis is located at the origin of coordinates. Find the projections E_z and $E_\perp$ of the electric field strength vector (on the plane perpendicular to the z axis at the point S (see Fig. 3.4)). At which points is E perpendicular to p?

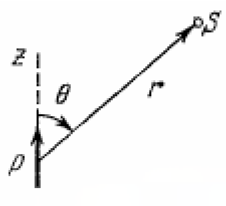


Fig. 3.4.

Solution. 40. From the results, obtained in the previous problem,

$$E_r = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3} \text{ and } E_\theta = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

From the given figure, it is clear that,

$$E_z = E_r\cos\theta - E_\theta\sin\theta = \frac{p}{4\pi\varepsilon_0 r^3}(3\cos^2\theta - 1)$$

And $$E_\perp = E_r\sin\theta + E_\theta\cos\theta = \frac{3p\sin\theta\cos\theta}{4\pi\varepsilon_0 r^3}$$

When $\vec{E}\perp\vec{p},\ |\vec{E}| = E_\perp$ and $E_z = 0$

So $$3\cos^2\theta = 1 \text{ and } \cos\theta = \frac{1}{\sqrt{3}}$$

Thus $\vec{E}\perp\vec{p}$ at the points located on the lateral surface of the cone, having its axis,

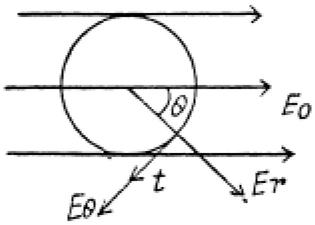
coinciding with the direction of z-axis and semi vertex angle $\theta = \cos^{-1} 1/\sqrt{3}$.

Q. 41. A point electric dipole with a moment p is placed in the external uniform electric field whose strength equals E_0, with $\mathbf{p} \uparrow\uparrow \mathbf{E}_0$. In this case one of the equipotential surfaces enclosing the dipole forms a sphere. Find the radius of this sphere.

Solution. 41. Let us assume that the dipole is at the centre of the one equipotential surface which is spherical (Fig.)* On an equipotential surface the net electric field strength along the tangent of it becomes zero. Thus

$$-E_0 \sin\theta + E_\theta = 0 \quad \text{or} \quad -E_0 \sin\theta + \frac{p \sin\theta}{4\pi\varepsilon_0 r^3} = 0$$

Hence $\quad r = \left(\dfrac{p}{4\pi\varepsilon_0 E_0}\right)^{1/3}$



Alternate: Potential at the point, near the dipole is given by,

$$\varphi = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} - \vec{E}_0 \cdot \vec{r} + \text{constant},$$

$$= \left(\frac{p}{4\pi\varepsilon_0 r^3} - E_0\right)\cos\theta + \text{Const}$$

For φ to be constant,

$$\frac{p}{4\pi\varepsilon_0 r^3} - E_0 = 0 \quad \text{or,} \quad \frac{p}{4\pi\varepsilon_0 r^3} = E_0$$

Thus $\quad r = \left(\dfrac{p}{4\pi\varepsilon_0 E_0}\right)^{1/3}$

Q. 42. Two thin parallel threads carry a uniform charge with linear densities λ and —λ. The distance between the threads is equal to l. Find the potential of the electric field and the magnitude of its strength vector at the distance r ≫ l at the angle θ to the vector l (Fig. 3.5).

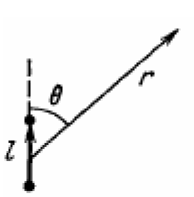
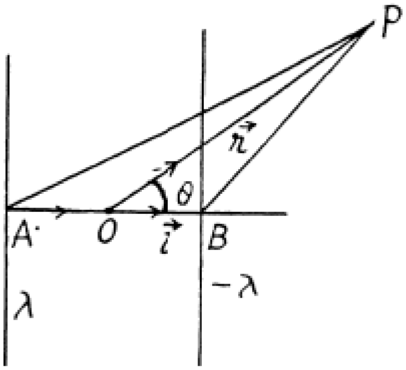


Fig. 3.5.

Solution. 42. Let P be a point, at distance r >> l and at an angle to θ the vector $\vec{l}$ (Fig.).



Thus $\vec{E}$ at $P = \frac{\lambda}{2\pi\varepsilon_0}\frac{\vec{r}+\frac{\vec{l}}{2}}{\left|\vec{r}+\frac{\vec{l}}{2}\right|^2} - \frac{\lambda}{2\pi\varepsilon_0}\frac{\vec{r}-\frac{\vec{l}}{2}}{\left|\vec{r}-\frac{\vec{l}}{2}\right|^2}$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{\vec{r}+\vec{l}/2}{r^2+\frac{l^2}{4}+r\,l\cos\theta} - \frac{\vec{r}-\vec{l}/2}{r^2+\frac{l^2}{4}-r\,l\cos\theta}\right]$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{\vec{l}}{r^2} - \frac{2\,l\,\vec{r}}{r^3}\cos\theta\right)$$

Hence $E = |\vec{E}| = \frac{\lambda\,l}{2\pi\varepsilon_0 r^2},\ r >> l$

Also, $\varphi = \frac{\lambda}{2\pi\varepsilon_0}\ln|\vec{r}+\vec{l}/2| - \frac{\lambda}{2\pi\varepsilon_0}\ln|\vec{r}-\vec{l}/2]$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{r^2+r\,l\cos\theta+l^2/4}{r^2-r\,l\cos\theta+l^2/4} = \frac{\lambda\,l\cos\theta}{2\pi\varepsilon_0 r},\ r >> l$$

Q. 43. Two coaxial rings, each of radius R, made of thin wire are separated by a small distance l (l ≪ R) and carry the charges q and —q. Find the electric field potential and strength at the axis of the system as a function of the x coordinate (Fig. 3.6). Show in the same drawing the approximate plots of the functions obtained. Investigate these functions at |x| ≫ R.

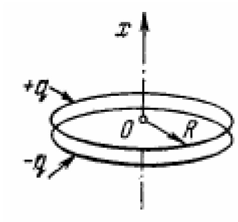


Fig. 3.6.

Solution. 43. The potential can be calculated by superposition. Choose the plane of the upper ring as x = l/2 and that of the lower ring as x = - l/2.

Then $$\varphi = \frac{q}{4\pi\varepsilon_0\left[R^2+(x-l/2)^2\right]^{1/2}} - \frac{q}{4\pi\varepsilon_0\left[R^2+(x+l/2)^2\right]^{1/2}}$$

$$= \frac{q}{4\pi\varepsilon_0\left[R^2+x^2-lx\right]^{1/2}} - \frac{q}{4\pi\varepsilon_0\left[R^2+x^2+lx\right]^{1/2}}$$

$$= \frac{q}{4\pi\varepsilon_0(R^2+x^2)^{1/2}}\left(1+\frac{lx}{2(R^2+x^2)}\right) - \frac{q}{4\pi\varepsilon_0(R^2+x^2)^{1/2}}\left(1-\frac{lx}{2(R^2+x^2)}\right)$$

$$= \frac{q\,lx}{4\pi\varepsilon_0(R^2+x^2)^{3/2}}$$

For $|x| >> R,\ \varphi = \frac{q\,l}{4\pi x^2}$

The electric field is $E = -\frac{\partial\varphi}{\partial x}$

$$= -\frac{ql}{4\pi\varepsilon_0(R^2+x^2)^{3/2}} + \frac{3}{2}\frac{ql}{(R^2+x^2)^{5/2}\,4\pi\varepsilon_0}\times 2x = \frac{ql\,(2x^2-R^2)}{4\pi\varepsilon_0(R^2+x^2)^{5/2}}$$

For $|x| >> R,\ E = \frac{ql}{2\pi\varepsilon_0 x^3}$. The plot is as given in the book.

Q. 44. Two infinite planes separated by a distance l carry a uniform surface charge of densities σ and —σ (Fig. 3.7). The planes have round coaxial holes of radius R, with 1 ≪ R. Taking the origin O and the x coordinate axis as shown in the figure, find the potential of the electric field and the projection of its strength vector E_x on the axes of the system as functions of the x coordinate. Draw the approximate plot φ (x).

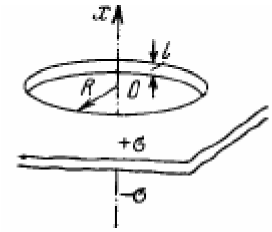
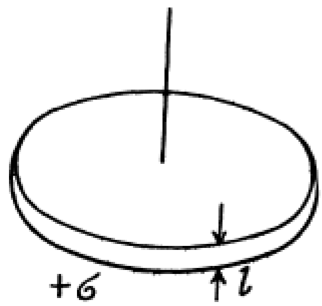


Fig. 3.7.

Solution. 44. The field of a pair of oppositely chaiged sheets with holes can by superposition be reduced to that of a pair of uniform opposite charged sheets and discs with opposite charges. Now the charged sheets do not contribute any field outside them. Thus using the result of the previous problem



$$\varphi = \int_0^R \frac{(-\sigma)\, l\, 2\pi r\, dr\, x}{4\pi\varepsilon_0 (r^2+x^2)^{3/2}}$$

$$= -\frac{\sigma x l}{4\varepsilon_0}\int_{x^2}^{R^2+x^2} \frac{dy}{y^{3/2}} = -\frac{\sigma x l}{2\varepsilon_0\sqrt{R^2+x^2}}$$

$$E_x = -\frac{\partial\varphi}{\partial x} = -\frac{\sigma l}{2\varepsilon_0}\left[\frac{1}{\sqrt{R^2+x^2}} - \frac{x^2}{(R^2+x^2)^{3/2}}\right] = -\frac{\sigma l R^2}{2\varepsilon_0 (R^2+x^2)^{3/2}}.$$

The plot is as shown in the answer sheet.

Q. 45. An electric capacitor consists of thin round parallel plates, each of radius R, separated by a distance l (l ≪ R) and uniformly charged with surface densities σ and —σ. Find the potential of the electric field and the magnitude of its strength vector at the axes of the capacitor as functions of a distance x from the plates if x ≫ l. Investigate the obtained expressions at x ≫ R.

Solution. 45. For x > 0 we can use the result as given above and write

$$\varphi = \pm \frac{\sigma l}{2\varepsilon_0}\left(1 - \frac{|x|}{(R^2 + x^2)^{1/2}}\right)$$

For the solution that vanishes at α. There is a discontinuity in potential for |x| = 0. The solution for negative x is obtained by σ → -σ. Thus

$$\varphi = -\frac{\sigma l x}{2\varepsilon_0 (R + x^2)^{1/2}} + \text{constant}$$

Hence ignoring the jump

$$E = -\frac{\partial \varphi}{\partial x} = \frac{\sigma l R^2}{2\varepsilon_0 (R^2 + x^2)^{3/2}}$$

For large $|x|$ $\varphi = \pm \frac{p}{4\pi\varepsilon_0 x^2}$ and $E = \frac{p}{2\pi\varepsilon_0 |x|^3}$ (where $p = \pi R^2 \sigma l$)

Q. 46. A dipole with an electric moment p is located at a distance r from a long thread charged uniformly with a linear density λ. Find the force F acting on the dipole if the vector p is oriented
(a) along the thread;
(b) along the radius vector r;
(c) at right angles to the thread and the radius vector r.

Solution. 46. Here $E_r = \frac{\lambda}{2\pi\varepsilon_0 r}$, $E_\theta = E_\varphi = 0$ and $\vec{F} = p\frac{\partial \vec{E}}{\partial l}$

(a) $\vec{p}$ along the thread.

$\vec{E}$ does not change as the point of observation is moved along the thread.|

$\vec{F} = 0$

(b) $\vec{p}$ along $\vec{r}$,

$$\vec{F} = F_r \vec{e}_r = -\frac{\lambda p}{2\pi\varepsilon_0 r^2}\vec{e}_r = -\frac{\lambda \vec{p}}{2\pi\varepsilon_0 r^2}\left(\text{On using } \frac{\partial}{\partial r}\vec{e}_r = 0\right)$$

(c) $\vec{p}$ along $\vec{e}_\theta$

$$\vec{F} = p\frac{\partial}{r\,\partial\theta}\frac{\lambda}{2\pi\varepsilon_0 r}e_r$$

$$= \frac{p\lambda}{2\pi\varepsilon_0 r^2}\frac{\partial \vec{e}_r}{\partial\theta} = \frac{p\lambda}{2\pi\varepsilon_0 r^2}\vec{e}_\theta = \frac{\vec{p}\lambda}{2\pi\varepsilon_0 r^2}.$$

Q. 47. Find the interaction force between two water molecules separated by a distance l = 10 nm if their electric moments are oriented along the same straight line. The moment of each molecule equals p = 0.62.10^{-29} C • m.

Solution. 47. Force on a dipole of moment p is given by,

$$F = \left| \varphi \frac{\partial \vec{E}}{\partial l} \right|$$

In our problem, field, due to a dipole at a distance l, where a dipole is placed,

$$|\vec{E}| = \frac{p}{2\pi\varepsilon_0 l^3}$$

Hence, the force of interaction,

$$F = \frac{3p^2}{2\pi\varepsilon_0 l^4} = 2{\cdot}1 \times 10^{-16}\,\text{N}$$

Q. 48. Find the potential φ (x, y) of an electrostatic field E = a (yi xj), where a is a constant, i and j are the unit vectors of the x and y axes.

Solution. 48.

$$-d\varphi = \vec{E}\cdot d\vec{r} = a\,(y\,dx + x\,dy) = a\,d\,(xy)$$

On integrating, $\varphi = -a\,xy + C$

Q. 49. Find the potential φ (x, y) of an electrostatic field E = 2axyi + a (x^2 — y^2) j, where a is a constant, i and j are the unit vectors of the x and y axes.

Solution. 49.

$$-d\varphi = \vec{E}\cdot d\vec{r} = [2\,a\,xy\,\vec{i} + 2\,(x^2 - y^2)\,\vec{j}\,]\cdot[dx\,\vec{i} + dy\,\vec{j}\,]$$

Or $d\varphi = 2\,a\,x\,y\,dx + a\,(x^2 - y^2)\,dy = ad\,(x^2\,y) - ay^2\,dy$

On integrating, we get,

$$\varphi = ay\left(\frac{y^2}{3} - x^2\right) + C$$

Q. 50. Determine the potential φ (x, y, z) of an electrostatic field E = ayi (ax + bz) j + byk, where a and b are constants, i, j, k are the unit vectors of the axes x, y, z.

Solution. 50. Given, again

$$-d\varphi = \vec{E} \cdot d\vec{r} = (ay\vec{i} + (ax + bz)\vec{j} + by\vec{k}) \cdot (dx\,\vec{i} + dy\,\vec{j} + dx\,\vec{k})$$
$$= a\,(y\,dx + ax\,dy) + b\,(z\,dy + y\,dz) = ad\,(x\,y) + bd\,(yz)$$

On integrating,

$$\varphi = -(a\,xy + b\,yz) + C$$

Q. 51. The field potential in a certain region of space depends only on the x coordinate as φ = — ax^3 + b, where a and b are constants. Find the distribution of the space charge p (x).

Solution. 51. Field intensity along x-axis.

$$E_x = -\frac{\partial \varphi}{\partial x} = 3\,a\,x^2 \qquad (1)$$

Then using Gauss's theorem in differential from

$$\frac{\partial E_x}{\partial x} = \frac{\rho\,(x)}{\varepsilon_0} \quad \text{so,} \quad \rho\,(x) = 6a\,\varepsilon_0\,x.$$

Q. 52. A uniformly distributed space charge fills up the space between two large parallel plates separated by a distance d. The potential difference between the plates is equal to Δφ. At what value of charge density p is the field strength in the vicinity of one of the plates equal to zero? What will then be the field strength near the other plate?

Solution. 52. In the space between the plates we have the Poisson equation

$$\frac{\partial^2 \varphi}{\partial x^2} = -\frac{\rho_0}{\varepsilon_0}$$

Or, $$\varphi = -\frac{\rho_0}{2\,\varepsilon_0}x^2 + A\,x + B$$

Where p_0 is the constant space charge density between the plates.

We can choose $\varphi(0) = 0$ so $B = 0$

Then $\varphi(d) = \Delta\varphi = Ad - \frac{\rho_0 d^2}{2\varepsilon_0}$ or, $A = \frac{\Delta\varphi}{d} + \frac{\rho_0 d}{2\varepsilon_0}$

Now $E = -\frac{\partial\varphi}{\partial x} = \frac{\rho_0}{\varepsilon_0}x - A = 0$ for $x = 0$

If $A = \frac{\Delta\varphi}{d} + \frac{\rho_0 d}{2\varepsilon_0} = 0$

Then $\rho_0 = -\frac{2\varepsilon_0 \Delta\varphi}{d^2}$

Also $E(d) = \frac{\rho_0 d}{\varepsilon_0}$.

Q. 53. The field potential inside a charged ball depends only on the distance from its centre as $\varphi = ar^2 + b$, where a and b are constants. Find the space charge distribution p (r) inside the ball.

Solution. 53. Field intensity is along radial line and is

$$E_r = -\frac{\partial\varphi}{\partial r} = -2ar \quad (1)$$

From the Gauss' theorem,

$$4\pi r^2 E_r = \int \frac{dq}{\varepsilon_0,}$$

Where do is the charge contained between the sphere of radii r and r + dr.

Hence $4\pi r^2 E_r = 4\pi r^2 \times (-2ar) = \frac{4\pi}{\varepsilon_0}\int_0^r r'^2 \rho(r')\, dr'$ (2)

Differentiating (2) $\rho = -6\varepsilon_0 a$