Chapter 10

THE PARABOLA

196. Conic Section. Def. The locus of a point P, which moves so that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed straight line, is called a Conic Section.

The fixed point is called the **Focus** and is usually denoted by S.

The constant ratio is called the **Eccentricity** and is denoted by e.

The fixed straight line is called the **Directrix**.

The straight line passing through the Focus and perpendicular to the Directrix is called the **Axis**.

When the eccentricity e is equal to unity, the Conic Section is called a **Parabola**.

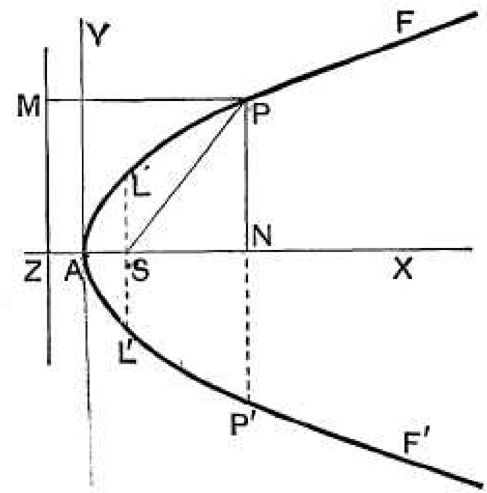
When e is less than unity, it is called an **Ellipse**.

When e is greater than unity, it is called a **Hyperbola.**

[The name Conic Section is derived from the fact that these curves were first obtained by cutting a cone in various ways.]

197. *To find the equation to a Parabola.*

Let S be the fixed point and ZM the directrix. We require therefore the locus of a point P which moves so that its distance from S is always equal to PM, its perpendicular distance from ZM.



Draw SZ perpendicular to the directrix and bisect SZ in the point A; produce ZA to X.

The point A is clearly a point on the curve and is called the **Vertex** of the Parabola.

Take A as origin, AX as the axis of x, and AY, perpendicular to it, as the axis of y.

Let the distance ZA, or AS, be called a, and let P be any point on the curve whose coordinates are x and y.

Join SP, and draw PN and PM perpendicular respectively to the axis and directrix.

We have then $\qquad SP^2 = PM^2$,

i.e. $\qquad (x-a)^2 + y^2 = ZN^2 = (a+x)^2$,

$\therefore \qquad \mathbf{y^2 = 4ax} \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$

This being the relation which exists between the coordinates of any point P on the parabola is, by Art. 42, the equation to the parabola.

Cor. The equation (1) is equivalent to the geometrical proposition

$$PN^2 = 4AS \,.\, AN.$$

198. The equation of the preceding article is the simplest possible equation to the parabola. Throughout this chapter this standard form of the equation is assumed unless the contrary is stated.

If instead of AX and AY we take the axis and the directrix ZM as the axes of coordinates, the equation would be

$$(x-2a)^2+y^2=x^2,$$

i.e. $$y^2=4a(x-a) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Similarly, if the axis SX and a perpendicular line SL be taken as the axes of coordinates, the equation is

$$x^2+y^2=(x+2a)^2,$$

i.e. $$y^2=4a(x+a) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

These two equations may be deduced from the equation of the previous article by transforming the origin, firstly to the point $(-a, 0)$ and secondly to the point $(a, 0)$.

199. The equation to the parabola referred to any focus and directrix may be easily obtained. Thus the equation to the parabola, whose focus is the point (2, 3) and whose directrix is the straight line $x-4y+3=0$, is

$$(x-2)^2+(y-3)^2=\left\{\frac{x-4y+3}{\sqrt{1^2+4^2}}\right\}^2,$$

i.e. $$17[x^2+y^2-4x-6y+13]=\{x^2+16y^2+9-8xy+6x-24y\},$$

i.e. $$16x^2+y^2+8xy-74x-78y+212=0.$$

200. *To trace the curve*

$$y^2=4ax \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If x be negative, the corresponding values of y are imaginary (since the square root of a negative quantity is unreal); hence there is no part of the curve to the left of the point A.

If y be zero, so also is x, so that the axis of x meets the curve at the point A only.

If x be zero, so also is y, so that the axis of y meets the curve at the point A only.

For every positive value of x we see from (1), by taking the square root, that y has two equal and opposite values.

Hence corresponding to any point P on the curve there is another point P' on the other side of the axis which is obtained by producing PN to P' so that PN and NP' are equal in magnitude. The line PP' is called a double ordinate.

As x increases in magnitude, so do the corresponding values of y; finally, when x becomes infinitely great, y becomes infinitely great also.

By taking a large number of values of x and the corresponding values of y it will be found that the curve is as in the figure of Art. 197.

The two branches never meet but are of infinite length.

201. *The quantity* $y'^2 - 4ax'$ *is negative, zero, or positive according as the point* (x', y') *is within, upon, or without the parabola.*

Let Q be the point (x', y') and let it be within the curve, *i.e.* be between the curve and the axis AX. Draw the ordinate QN and let it meet the curve in P.

Then (by Art. 197), $PN^2 = 4a \,.\, x'$.

Hence y'^2, *i.e.* QN^2, is $< PN^2$, and hence is $< 4ax'$.

$$\therefore \; y'^2 - 4ax' \text{ is negative.}$$

Similarly, if Q be without the curve, then y'^2, *i.e.* QN^2, is $> PN^2$, and hence is $> 4ax'$.

Hence the proposition.

202. Latus Rectum. Def. The latus rectum of any conic is the double ordinate LSL' drawn through the focus S.

In the case of the parabola we have SL = distance of L from the directrix $= SZ = 2a$.

Hence the latus rectum $= 4a$.

When the latus rectum is given it follows that the equation to the parabola is completely known in its standard form, and the size and shape of the curve determined.

The quantity $4a$ is also often called the **principal parameter** of the curve.

Focal Distance of any point. The focal distance of any point P is the distance SP.

This focal distance $= PM = ZN = ZA + AN = a + x$.

Ex. *Find the vertex, axis, focus, and latus rectum of the parabola*

$$4y^2+12x-20y+67=0.$$

The equation can be written

$$y^2-5y=-3x-\tfrac{67}{4},$$

i.e. $$(y-\tfrac{5}{2})^2=-3x-\tfrac{67}{4}+\tfrac{25}{4}=-3(x+\tfrac{7}{2}).$$

Transform this equation to the point $(-\frac{7}{2}, \frac{5}{2})$ and it becomes $y^2=-3x$, which represents a parabola, whose axis is the axis of x and whose concavity is turned towards the negative end of this axis. Also its latus rectum is 3.

Referred to the original axes the vertex is the point $(-\frac{7}{2}, \frac{5}{2})$, the axis is $y=\frac{5}{2}$, and the focus is the point $(-\frac{7}{2}-\frac{3}{4}, \frac{5}{2})$, *i.e.* $(-\frac{17}{4}, \frac{5}{2})$.

EXAMPLES XXV

Find the equation to the parabola with

1. focus $(3, -4)$ and directrix $6x-7y+5=0$.

2. focus (a, b) and directrix $\dfrac{x}{a}+\dfrac{y}{b}=1$.

Find the vertex, axis, latus rectum, and focus of the parabolas

3. $y^2=4x+4y$. **4.** $x^2+2y=8x-7$.

5. $x^2-2ax+2ay=0$. **6.** $y^2=4y-4x$.

7. Draw the curves

(1) $y^2=-4ax$, (2) $x^2=4ay$, and (3) $x^2=-4ay$.

8. Find the value of p when the parabola $y^2=4px$ goes through the point (i) $(3, -2)$, and (ii) $(9, -12)$.

9. For what point of the parabola $y^2=18x$ is the ordinate equal to three times the abscissa?

10. Prove that the equation to the parabola, whose vertex and focus are on the axis of x at distances a and a' from the origin respectively, is $$y^2=4(a'-a)(x-a).$$

11. In the parabola $y^2=6x$, find (1) the equation to the chord through the vertex and the negative end of the latus rectum, and (2) the equation to any chord through the point on the curve whose abscissa is 24.

12. Prove that the equation $y^2+2Ax+2By+C=0$ represents a parabola, whose axis is parallel to the axis of x, and find its vertex and the equation to its latus rectum.

13. Prove that the locus of the middle points of all chords of the parabola $y^2=4ax$ which are drawn through the vertex is the parabola $y^2=2ax$.

14. Prove that the locus of the centre of a circle, which intercepts a chord of given length $2a$ on the axis of x and passes through a given point on the axis of y distant b from the origin, is the curve

$$x^2 - 2yb + b^2 = a^2.$$

Trace this parabola.

15. PQ is a double ordinate of a parabola. Find the locus of its points of trisection.

16. Prove that the locus of a point, which moves so that its distance from a fixed line is equal to the length of the tangent drawn from it to a given circle, is a parabola. Find the position of the focus and directrix.

17. If a circle be drawn so as always to touch a given straight line and also a given circle, prove that the locus of its centre is a parabola.

18. The vertex A of a parabola is joined to any point P on the curve and PQ is drawn at right angles to AP to meet the axis in Q. Prove that the projection of PQ on the axis is always equal to the latus rectum.

19. If on a given base triangles be described such that the sum of the tangents of the base angles is constant, prove that the locus of the vertices is a parabola.

20. A double ordinate of the curve $y^2 = 4px$ is of length $8p$; prove that the lines from the vertex to its two ends are at right angles.

21. Two parabolas have a common axis and concavities in opposite directions; if any line parallel to the common axis meet the parabolas in P and P', prove that the locus of the middle point of PP' is another parabola, provided that the latera recta of the given parabolas are unequal.

22. A parabola is drawn to pass through A and B, the ends of a diameter of a given circle of radius a, and to have as directrix a tangent to a concentric circle of radius b; the axes being AB and a perpendicular diameter, prove that the locus of the focus of the parabola is $\frac{x^2}{b^2} + \frac{y^2}{b^2 - a^2} = 1$.

ANSWERS

1. $(7x + 6y)^2 - 570x + 750y + 2100 = 0$.

2. $(ax - by)^2 - 2a^3x - 2b^3y + a^4 + a^2b^2 + b^4 = 0$.

3. $(-1, 2)$; $y = 2$; 4; $(0, 2)$.

4. $(4, \frac{9}{2})$; $x = 4$; 2; $(4, 4)$.

5. $\left(a, \frac{a}{2}\right)$; $x=a$; $2a$; $(a, 0)$. **6.** $(1, 2)$; $y=2$; 4; $(0, 2)$.

8. (i) $\frac{1}{3}$; (ii) 4. **9.** $(2, 6)$. **11.** $y=-2x$; $y-12=m(x-24)$.

12. $\left(\frac{B^2-C}{2A}, -B\right)$; $x=\frac{B^2-A^2-C}{2A}$. **15.** $9y^2=4ax$.

SOLUTIONS/HINTS

1. $(x-3)^2+(y+4)^2=\left\{\frac{6x-7y+5}{\sqrt{6^2+7^2}}\right\}^2$, etc.

2. $(x-a)^2+(y-b)^2=\left\{\frac{bx+ay-ab}{\sqrt{a^2+b^2}}\right\}^2$, etc.

3. The equation may be written $(y-2)^2=4(x+1)$.

As in Art. 202, the vertex is $(-1, 2)$; the latus rectum is 4; the focus is $(0, 2)$; and the axis is $y=2$.

4. The equation may be written $(x-4)^2=-2\left(y-\frac{9}{2}\right)$.

As in Art. 202, the vertex is $\left(4, \frac{9}{2}\right)$; the latus rectum is 2; the focus is $(4, 4)$; and the axis is $x=4$.

5. The equation may be written in the form

$$(x-a)^2=-2a\left(y-\frac{a}{2}\right).$$

The vertex is $\left(a, \frac{a}{2}\right)$; the latus rectum is $2a$; the focus is $(a, 0)$; and the axis is $x=a$.

6. The equation may be written $(y-2)^2=-4(x-1)$.

The vertex is $(1, 2)$; the latus rectum is 4; the focus is $(0, 2)$; and the axis is $y=2$.

8. (i) We have $(-2)^2=4p\,.\,3$; $\therefore p=\frac{1}{3}$.

(ii) We have $(-12)^2=4p\,.\,9$; $\therefore p=4$.

9. Putting $y=3x$, we have $(3x)^2=18x$.

$$\therefore x=2, \text{ and } y=6.$$

10. See equation (i) of Art. 198; the latus rectum

$$=4SA=4(a'-a).$$

11. (i) The coordinates of the negative end of the latus rectum are $(\frac{6}{4}, -\frac{6}{2})$, *i.e.* $(\frac{3}{2}, -3)$. Hence the required equation is $y + 2x = 0$.

(ii) If $x = 24$, $y^2 = 6 \times 24$; $\therefore$ $y = \pm 12$. $\therefore$ the required equation is $y - 12 = m(x - 24)$.

12. The equation may be written

$$(y + B)^2 = -2A\left(x - \frac{B^2 - C}{2A}\right).$$

As in Art. 202, the equation to the axis is $y + B = 0$; the vertex is $\left(\frac{B^2 - C}{2A}, -B\right)$; and the abscissa of the focus is $\frac{B^2 - C}{2A} - \frac{A}{2}$, *i.e.* $\frac{B^2 - A^2 - C}{2A}$; $\therefore$ the latus rectum is $x = \frac{B^2 - A^2 - C}{2A}$.

13. If (h, k) be any point on the locus, we have

$$(2k)^2 = 4a(2h); \quad \therefore k^2 = 2ah, \text{ etc.}$$

14. Let (h, k) be the centre and r the radius of the circle. It is easily seen from a figure that

$$h^2 + (k - b)^2 = r^2 = k^2 + a^2;$$

hence the required locus is $x^2 - 2by + b^2 = a^2$.

The equation may be written, $x^2 = 2b\left(y + \frac{a^2 - b^2}{2b}\right)$, which is a parabola whose vertex is $\left(0, -\frac{a^2 - b^2}{2a}\right)$, whose axis is the axis of y, and whole latus rectum is $2b$, the concavity being turned towards the positive end of the axis of y.

15. If $y^2 = 4ax$ be the equation of the parabola and (h, k) a point on the locus; then the point $(h, 3k)$ lies on the original curve; $\therefore$ $(3k)^2 = 4ah$.

Hence the required equation is $9y^2 = 4ax$.

16. Take the centre of the given circle as origin, and the lines through it, which are respectively parallel and perpendicular to the given line, as axes.

Let $x^2 + y^2 = a^2$ be the equation to the circle and $y + b = 0$ the equation to the line. Then if (h, k) be any point on the locus we have $(k + b)^2 = h^2 + k^2 - a^2$. [Art. 168.]

Hence the required equation is $x^2 = 2b\left(y + \frac{a^2 + b^2}{2b}\right)$. This is a parabola whose vertex is the point $\left(0, -\frac{a^2 + b^2}{2b}\right)$, and whose latus rectum is $2b$. Hence the coordinates of the focus are $\left(0, -\frac{a^2 + b^2}{2b} + \frac{b}{2}\right)$, *i.e.* $\left(0, -\frac{a^2}{2b}\right)$, and the directrix is the line through $\left(0, -\frac{a^2 + b^2}{2b} - \frac{b}{2}\right)$, and parallel to the axis of x, viz. $y + b + \frac{a^2}{2b} = 0$.

17. Taking the same equations as in No. 16, it is easily seen from a figure that, if (h, k) be the centre of any such circle, then its radius must be equal to both $k + b$ and to $\sqrt{h^2 + k^2} - a$. Hence $\{k + b + a\}^2 = h^2 + k^2$.

Hence the required equation to the locus is

$$x^2 - 2y(a + b) - (a + b)^2 = 0, \text{ which is a parabola.}$$

18. Let $y^2 = 4ax$ be the equation to the parabola, and $my = x$ the equation of AP. Solving, the coordinates of P are $(4am^2, 4am)$. The equation of the line through P perpendicular to AP is $y - 4am + m(x - 4am^2) = 0$. Putting

$$y = 0, \text{ we have } AQ = 4a + 4am^2. \quad \therefore \ NQ = 4a.$$

19. Let $(a, 0), (-a, 0)$ be the coordinates of the extremities of the base, and (x, y) the coordinates of the vertex. Then $\frac{y}{a - x} + \frac{y}{a + x} = \text{cons.} = c$ (say).

$$\therefore \ 2ay = c(a^2 - x^2), \text{ which is a parabola.}$$

20. If PNP' be the double ordinate, then

$$PN = 4p; \therefore AN = 4p. \quad \therefore P\hat{A}N = 45°; \therefore P\hat{A}P' = 90°.$$

21. Take the common axis for the axis of x and the origin midway between the vertices. Let $y^2 = 4a(x-c)$ and $y^2 + 4b(x+c) = 0$, be the equations to the parabolas. The abscissae of the points where $y = k$ cuts these curves are $\frac{k^2}{4a} + c, -\frac{k^2}{4b} - c$. Hence, if (x, y) be the middle point,

$$2x = \frac{k^2}{4}\left\{\frac{1}{a} - \frac{1}{b}\right\}, \text{ and } y = k.$$

Eliminating k, we have $8abx = y^2(b-a)$, which is a parabola, unless $a = b$.

22. Let (h, k) be the coordinates of the focus and $x\cos\theta + y\sin\theta = b$ (Art. 178) the equation of the directrix.

The equation of the parabola will therefore be

$$(x-h)^2 + (y-k)^2 = (x\cos\theta + y\sin\theta - b)^2.$$

Since it passes through the point $(a, 0)$,

$$\therefore (a-h)^2 + k^2 = (a\cos\theta - b)^2.$$

$$\therefore a^2\sin^2\theta - 2ah + h^2 + k^2 = b^2 - 2ab\cos\theta.$$

So, since it passes through the point $(-a, 0)$, we have

$$a^2\sin^2\theta + 2ah + h^2 + k^2 = b^2 + 2ab\cos\theta;$$

Adding, $a^2\sin^2\theta + h^2 + k^2 = b^2$; subtracting, $h = b\cos\theta$.

Eliminating θ; $\quad a^2\left(1 - \frac{h^2}{b^2}\right) + h^2 + k^2 = b^2,$

i.e. $\quad h^2(b^2 - a^2) + b^2k^2 = b^2(b^2 - a^2)$. Hence, etc.

203. *To find the points of intersection of any straight line with the parabola*

$$y^2 = 4ax \quad \text{.......................(1).}$$

The equation to any straight line is

$$y = mx + c \quad \text{.......................(2).}$$

The coordinates of the points common to the straight line and the parabola satisfy both equations (1) and (2), and are therefore found by solving them.

Substituting the value of y from (2) in (1), we have

$$(mx + c)^2 = 4ax,$$

i.e.
$$m^2x^2 + 2x(mc - 2a) + c^2 = 0 \quad \text{............(3).}$$

This is a quadratic equation for x and therefore has two roots, real, coincident, or imaginary.

The straight line therefore meets the parabola in two points, real, coincident, or imaginary.

The roots of (3) are real or imaginary according as

$$\{2(mc - 2a)\}^2 - 4m^2c^2$$

is positive or negative, *i.e.* according as $-amc + a^2$ is positive or negative, *i.e.* according as mc is $\lesseqgtr 4a$.

204. *To find the length of the chord intercepted by the parabola on the straight line*

$$y = mx + c \quad \text{.............................. (1).}$$

If (x_1, y_1) and (x_2, y_2) be the common points of intersection, then, as in Art. 154, we have, from equation (3) of the last article,

$$(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$$

$$= \frac{4(mc - 2a)^2}{m^4} - \frac{4c^2}{m^2} = \frac{16a(a - mc)}{m^4},$$

and
$$y_1 - y_2 = m(x_1 - x_2).$$

Hence the required length $= \sqrt{(y_1 - y_2)^2 + (x_1 - x_2)^2}$

$$= \sqrt{1 + m^2}\,(x_1 - x_2) = \frac{4}{m^2}\sqrt{1 + m^2}\,\sqrt{a(a - mc)}.$$

205. *To find the equation to the tangent at any point (x', y') of the parabola $y^2 = 4ax$.*

The definition of the tangent is given in Art. 149.

Let P be the point (x', y') and Q a point (x'', y'') on the parabola.

The equation to the line PQ is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \quad \dots\dots\dots\dots\dots(1).$$

Since P and Q both lie on the curve, we have

$$y'^2 = 4ax' \quad \dots\dots\dots\dots\dots\dots\dots\dots(2),$$

and

$$y''^2 = 4ax'' \quad \dots\dots\dots\dots\dots\dots\dots\dots(3).$$

Hence, by subtraction, we have

$$y''^2 - y'^2 = 4a(x'' - x'),$$

i.e.

$$(y'' - y')(y'' + y') = 4a(x'' - x'),$$

and hence

$$\frac{y'' - y'}{x'' - x'} = \frac{4a}{y'' + y'}.$$

Substituting this value in equation (1), we have, as the equation to any secant PQ,

$$y - y' = \frac{4a}{y'' + y'}(x - x')$$

i.e.

$$y(y' + y'') = 4ax + y'y'' + y'^2 - 4ax'$$

$$= 4ax + y'y'' \quad \dots\dots\dots\dots\dots\dots\dots(4).$$

To obtain the equation of the tangent at (x', y') we take Q indefinitely close to P, and hence, in the limit, put $y'' = y'$.

The equation (4) then becomes

$$2yy' = y'^2 + 4ax = 4ax + 4ax',$$

i.e.

$$\mathbf{yy' = 2a(x + x')}.$$

Cor. It will be noted that the equation to the tangent is obtained from the equation to the curve by the rule of Art. 152.

Exs. The equation to the tangent at the point $(2, -4)$ of the parabola $y^2 = 8x$ is

$$y(-4) = 4(x+2),$$

i.e. $$x + y + 2 = 0.$$

The equation to the tangent at the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ of the parabola $y^2 = 4ax$ is

$$y \cdot \frac{2a}{m} = 2a\left(x + \frac{a}{m^2}\right),$$

i.e. $$y = mx + \frac{a}{m}.$$

206. *To find the condition that the straight line*

$$y = mx + c \quad \text{.......................(1)}$$

may touch the parabola $$y^2 = 4ax \quad \text{.......................(2).}$$

The abscissæ of the points in which the straight line (1) meets the curve (2) are as in Art. 203, given by the equation

$$m^2x^2 + 2x(mc - 2a) + c^2 = 0 \quad \text{...........(3).}$$

The line (1) will touch (2) if it meet it in two points which are indefinitely close to one another, *i.e.* in two points which ultimately coincide.

The roots of equation (3) must therefore be equal.

The condition for this is

$$4(mc - 2a)^2 = 4m^2c^2,$$

i.e. $$a^2 - amc = 0,$$

so that $$c = \frac{a}{m}.$$

Substituting this value of c in (1), we have as the equation to a tangent,

$$\mathbf{y = mx + \frac{a}{m}}.$$

In this equation m is the tangent of the angle which the tangent makes with the axis of x.

The foregoing proposition may also be obtained from the equation of Art. 205.

For equation (4) of that article may be written

$$y = \frac{2a}{y'} x + \frac{2ax'}{y'} \quad \dots\dots\dots\dots\dots\dots (1).$$

In this equation put $\frac{2a}{y'} = m$, *i.e.* $y' = \frac{2a}{m}$,

and hence $\quad x' = \frac{y'^2}{4a} = \frac{a}{m^2}$, and $\frac{2ax'}{y'} = \frac{a}{m}$.

The equation (1) then becomes $y = mx + \frac{a}{m}$.

Also it is the tangent at the point (x', y'), *i.e.* $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

207. *Equation to the normal at (x', y').* The required normal is the straight line which passes through the point (x', y') and is perpendicular to the tangent, *i.e.* to the straight line

$$y = \frac{2a}{y'}(x + x').$$

Its equation is therefore

$$y - y' = m'(x - x'),$$

where $\quad m' \times \frac{2a}{y'} = -1, \quad$ *i.e.* $\quad m' = -\frac{y'}{2a}, \qquad$ (Art. 69.)

and the equation to the normal is

$$\mathbf{y - y' = \frac{-y'}{2a}(x - x')} \quad \dots\dots\dots\dots\dots (1).$$

208. *To express the equation of the normal in the form*

$$y = mx - 2am - am^3.$$

In equation (1) of the last article put

$$\frac{-y'}{2a} = m, \;\; i.e. \;\; y' = -2am.$$

Hence $\quad x' = \frac{y'^2}{4a} = am^2.$

The normal is therefore

$$y + 2am = m(x - am^2),$$

i.e. $$\mathbf{y = mx - 2am - am^3},$$

and it is a normal at the point $(am^2, -2am)$ of the curve.

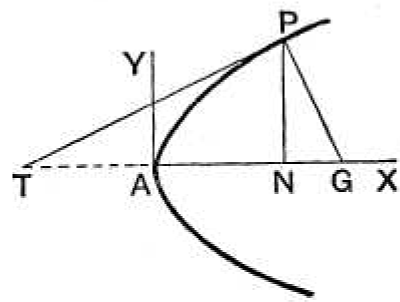
In this equation m is the tangent of the angle which the *normal* makes with the axis. It must be carefully distinguished from the m of Art. 206 which is the tangent of the angle which the *tangent* makes with the axis. The "m" of this article is -1 divided by the "m" of Art. 206.

209. Subtangent and Subnormal. Def. If the tangent and normal at any point P of a conic section meet the axis in T and G respectively and PN be the ordinate at P, then NT is called the Subtangent and NG the Subnormal of P.

To find the length of the subtangent and subnormal.

If P be the point (x', y') the equation to TP is, by Art. 205,

$$yy' = 2a(x + x') \ldots\ldots(1).$$



To obtain the length of AT, we have to find the point where this straight line meets the axis of x, *i.e.* we put $y = 0$ in (1) and we have

$$x = -x' \ldots\ldots(2).$$

Hence $$AT = AN.$$

[The negative sign in equation (2) shews that T and N always lie on opposite sides of the vertex A.]

Hence the subtangent $NT = 2AN$ = twice the abscissa of the point P.

Since TPG is a right-angled triangle, we have (Euc. VI. 8)

$$PN^2 = TN \cdot NG.$$

Hence the subnormal NG

$$=\frac{PN^2}{TN}=\frac{PN^2}{2AN}=2a.$$

The subnormal is therefore constant for all points on the parabola and is equal to the semi-latus rectum.

210. Ex. 1. *If a chord which is normal to the parabola at one end subtend a right angle at the vertex, prove that it is inclined at an angle* $\tan^{-1}\sqrt{2}$ *to the axis.*

The equation to any chord which is normal is

$$y=mx-2am-am^3,$$

i.e.

$$mx-y=2am+am^3.$$

The parabola is $y^2=4ax.$

The straight lines joining the origin to the intersections of these two are therefore given by the equation

$$y^2(2am+am^3)-4ax(mx-y)=0.$$

If these be at right angles, then

$$2am+am^3-4am=0,$$

i.e.

$$m=\pm\sqrt{2}.$$

Ex. 2. *From the point where any normal to the parabola* $y^2=4ax$ *meets the axis is drawn a line perpendicular to this normal; prove that this line always touches an equal parabola.*

The equation of any normal to the parabola is

$$y=mx-2am-am^3.$$

This meets the axis in the point $(2a+am^2,\ 0)$.

The equation to the straight line through this point perpendicular to the normal is

$$y=m_1(x-2a-am^2),$$

where

$$m_1m=-1.$$

The equation is therefore

$$y=m_1\left(x-2a-\frac{a}{m_1{}^2}\right),$$

i.e.

$$y=m_1(x-2a)-\frac{a}{m_1}.$$

This straight line, as in Art. 206, always touches the equal parabola

$$y^2=-4a(x-2a),$$

whose vertex is the point $(2a,\ 0)$ and whose concavity is towards the negative end of the axis of x.

EXAMPLES XXVI

Write down the equations to the tangent and normal

1. at the point (4, 6) of the parabola $y^2=9x$,

2. at the point of the parabola $y^2=6x$ whose ordinate is 12,

3. at the ends of the latus rectum of the parabola $y^2=12x$,

4. at the ends of the latus rectum of the parabola $y^2=4a(x-a)$.

5. Find the equation to that tangent to the parabola $y^2=7x$ which is parallel to the straight line $4y-x+3=0$. Find also its point of contact.

6. A tangent to the parabola $y^2=4ax$ makes an angle of $60°$ with the axis; find its point of contact.

7. A tangent to the parabola $y^2=8x$ makes an angle of $45°$ with the straight line $y=3x+5$. Find its equation and its point of contact.

8. Find the points of the parabola $y^2=4ax$ at which (i) the tangent, and (ii) the normal is inclined at $30°$ to the axis.

9. Find the equation to the tangents to the parabola $y^2=9x$ which goes through the point (4, 10).

10. Prove that the straight line $x+y=1$ touches the parabola $y=x-x^2$.

11. Prove that the straight line $y=mx+c$ touches the parabola $y^2=4a(x+a)$ if $c=ma+\frac{a}{m}$.

12. Prove that the straight line $lx+my+n=0$ touches the parabola $y^2=4ax$ if $ln=am^2$.

13. For what point of the parabola $y^2=4ax$ is (1) the normal equal to twice the subtangent, (2) the normal equal to the difference between the subtangent and the subnormal?

Find the equations to the common tangents of

14. the parabolas $y^2=4ax$ and $x^2=4by$,

15. the circle $x^2+y^2=4ax$ and the parabola $y^2=4ax$.

16. Two equal parabolas have the same vertex and their axes are at right angles; prove that the common tangent touches each at the end of a latus rectum.

17. Prove that two tangents to the parabolas $y^2=4a(x+a)$ and $y^2=4a'(x+a')$, which are at right angles to one another, meet on the straight line $x+a+a'=0$.

Shew also that this straight line is the common chord of the two parabolas.

18. PN is an ordinate of the parabola; a straight line is drawn parallel to the axis to bisect NP and meets the curve in Q; prove that NQ meets the tangent at the vertex in a point T such that $AT=\frac{2}{3}NP$.

19. Prove that the chord of the parabola $y^2=4ax$, whose equation is $y-x\sqrt{2}+4a\sqrt{2}=0$, is a normal to the curve and that its length is $6\sqrt{3}a$.

20. If perpendiculars be drawn on any tangent to a parabola from two fixed points on the axis, which are equidistant from the focus, prove that the difference of their squares is constant.

21. If P, Q, and R be three points on a parabola whose ordinates are in geometrical progression, prove that the tangents at P and R meet on the ordinate of Q.

22. Tangents are drawn to a parabola at points whose abscissæ are in the ratio $\mu : 1$; prove that they intersect on the curve

$$y^2=(\mu^{\frac{1}{4}}+\mu^{-\frac{1}{4}})^2\,ax.$$

23. If the tangents at the points (x', y') and (x'', y'') meet at the point (x_1, y_1) and the normals at the same points in (x_2, y_2), prove that

$$(1)\quad x_1=\frac{y'y''}{4a} \text{ and } y_1=\frac{y'+y''}{2},$$

$$(2)\quad x_2=2a+\frac{y'^2+y'y''+y''^2}{4a} \text{ and } y_2=-y'y''\frac{y'+y''}{8a^2},$$

and hence that

$$(3)\quad x_2=2a+\frac{{y_1}^2}{a}-x_1 \text{ and } y_2=-\frac{x_1y_1}{a}.$$

24. From the preceding question prove that, if tangents be drawn to the parabola $y^2=4ax$ from any point on the parabola $y^2=a(x+b)$, then the normals at the points of contact meet on a fixed straight line.

25. Find the lengths of the normals drawn from the point on the axis of the parabola $y^2=8ax$ whose distance from the focus is $8a$.

26. Prove that the locus of the middle point of the portion of a normal intersected between the curve and the axis is a parabola whose vertex is the focus and whose latus rectum is one quarter of that of the original parabola.

27. Prove that the distance between a tangent to the parabola and the parallel normal is $a \operatorname{cosec}\theta \sec^2\theta$, where θ is the angle that either makes with the axis.

28. PNP' is a double ordinate of the parabola; prove that the locus of the point of intersection of the normal at P and the diameter through P' is the equal parabola $y^2 = 4a(x - 4a)$.

29. The normal at any point P meets the axis in G and the tangent at the vertex in G'; if A be the vertex and the rectangle $AGQG'$ be completed, prove that the equation to the locus of Q is

$$x^3 = 2ax^2 + ay^2.$$

30. Two equal parabolas have the same focus and their axes are at right angles; a normal to one is perpendicular to a normal to the other; prove that the locus of the point of intersection of these normals is another parabola.

31. If a normal to a parabola make an angle ϕ with the axis, shew that it will cut the curve again at an angle $\tan^{-1}(\frac{1}{2}\tan\phi)$.

32. Prove that the two parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ cannot have a common normal, other than the axis, unless $\frac{b}{a - c} > 2$.

33. If $a^2 > 8b^2$, prove that a point can be found such that the two tangents from it to the parabola $y^2 = 4ax$ are normals to the parabola $x^2 = 4by$.

34. Prove that three tangents to a parabola, which are such that the tangents of their inclinations to the axis are in a given harmonical progression, form a triangle whose area is constant.

35. Prove that the parabolas $y^2 = 4ax$ and $x^2 = 4by$ cut one another at an angle $\tan^{-1}\frac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2(a^{\frac{2}{3}} + b^{\frac{2}{3}})}$.

36. Prove that two parabolas, having the same focus and their axes in opposite directions, cut at right angles.

37. Shew that the two parabolas

$$x^2 + 4a(y - 2b - a) = 0 \text{ and } y^2 = 4b(x - 2a + b)$$

intersect at right angles at a common end of the latus rectum of each.

38. A parabola is drawn touching the axis of x at the origin and having its vertex at a given distance k from this axis. Prove that the axis of the parabola is a tangent to the parabola $x^2 = -8k(y - 2k)$.

ANSWERS

1. $4y=3x+12$; $4x+3y=34$.
2. $4y-x=24$; $4x+y=108$.
3. $y-x=3$; $y+x=9$; $x+y+3=0$; $x-y=9$.
4. $y=x$; $x+y=4a$; $y+x=0$; $x-y=4a$.
5. $4y=x+28$; $(28, 14)$.
6. $\left(\frac{a}{3}, \frac{2a}{\sqrt{3}}\right)$.
7. $y+2x+1=0$; $(\frac{1}{2}, -2)$; $2y=x+8$; $(8, 8)$.
8. $(3a, 2\sqrt{3}a)$; $\left(\frac{a}{3}, -\frac{2\sqrt{3}}{3}a\right)$.
9. $4y=9x+4$; $4y=x+36$.

13. $\left(\frac{\sqrt{5}+1}{2}a, a\sqrt{2\sqrt{5}+2}\right)$; $(3a, 2\sqrt{3}a)$.

14. $b^{\frac{1}{3}}y+a^{\frac{1}{3}}x+a^{\frac{2}{3}}b^{\frac{2}{3}}=0$.
15. $x=0$.

SOLUTIONS/HINTS

1. Put $x'=4$, $y'=6$, $a=\frac{9}{2}$ in the equations of Arts. 205, 207.

2. Put $x'=24$, $y'=12$, $a=\frac{3}{2}$ in the equations of Arts. 205, 207.

3. Put $x'=3$, $y'=\pm 6$, $a=3$ in the equations of Arts. 205, 207.

4. The coordinates of the ends of the latus rectum of the parabola $y^2=4ax$, are $(a, \pm 2a)$.

Hence, by Arts. 205, 207, the equations of the tangents and normals are

$$x+a=\pm y, \text{ and } x\pm y=3a.$$

Changing x into $x-a$, we have for the required equations $x=\pm y$, $x\pm y=4a$.

5. Put $m=\frac{1}{4}$, $a=\frac{7}{4}$ in the equations of Art. 206. $\therefore$ the required tangent is $4y=x+28$, and the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$, *i.e.* $(28, 14)$.

6. By Art. 206, it is the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$, where $m = \tan 60° = \sqrt{3}$.

7. The "m's" of the tangents are (Art. 72)

$$\frac{3+1}{1-3} \text{ and } \frac{3-1}{1+3}, \text{ i.e. } -2 \text{ and } \tfrac{1}{2}.$$

Also $a = 2$.

Hence the required tangents are, by Art. 206,

$$y = -2x - 1 \text{ and } y = \tfrac{1}{2}x + 4,$$

and the points of contact are

$$\left(\frac{2}{4}, \frac{4}{-2}\right) \text{ and } \left(\frac{2}{\frac{1}{4}}, \frac{4}{\frac{1}{2}}\right), \text{ or } \left(\frac{1}{2}, -2\right) \text{ and } (8, 8).$$

8. Any point on the parabola is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$. By Art. 206 in the first case $m = \tan 30° = \frac{1}{\sqrt{3}}$, and in the second case $m = \tan 60° = \sqrt{3}$.

9. Any tangent to the parabola $y^2 = 9x$ is

$$y = mx + \frac{9}{4m}.$$

If this passes through the point (4, 10) we have $10 = 4m + \frac{9}{4m}$, whence $m = \frac{1}{4}$ or $\frac{9}{4}$.

Substituting for m, we have the required equations.

10. The common points are given by $1 - x = x - x^2$, or $x^2 - 2x + 1 = 0$, which is a perfect square.

11. The common points are given by

$$(mx + c)^2 = 4a(x + a), \text{ or } m^2x^2 + 2x(mc - 2a) + c^2 - 4a^2 = 0.$$

The condition for equal roots is

$$(mc - 2a)^2 = m^2(c^2 - 4a^2), \text{ so that } c = ma + \frac{a}{m}.$$

12. In the condition of Art. 206, for m put $-\frac{l}{m}$ and for c put $-\frac{n}{m}$, and it becomes $\frac{ln}{m^2}=a$.

13. (1) In the figure of Art. 209, we have

$$PN^2+NG^2=PG^2=4NT^2.$$

$$\therefore\ y^2+4a^2=16x^2,\ \text{i.e.}\ ax+a^2=4x^2.$$

$$\therefore\ x=\frac{\sqrt{17}+1}{8}a,\ \text{and}\ y^2=\frac{a^2}{2}(\sqrt{17}+1).$$

(2) $PG=NT-NG=2AN-NG.$

$$\therefore\ y^2+4a^2=4(x-a)^2.\quad \therefore\ 4ax=4x^2-8ax.$$

$$\therefore\ x=3a,\ \text{and}\ y^2=4ax=12a^2.$$

14. The common points of $y=mx+\frac{a}{m}$ (a tangent to the first parabola), and $x^2=4by$ are given by

$$x^2=4b\left(mx+\frac{a}{m}\right),$$

which has equal roots if $m=-\frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$. Substitute for m, etc.

15. Solving, the common points are given by $x^2=0$; and the circle is entirely within the parabola. Hence the only common tangent is $x=0$.

16. Let $y^2=4ax$, $x^2=4ay$ be the equations to the parabolas.

The common points of $y=mx+\frac{a}{m}$, a tangent to the first parabola, and $x^2=4ay$ are given by

$$x^2=4a\left(mx+\frac{a}{m}\right),$$

which has equal roots if $m=-1$.

$\therefore$ the equation of the common tangent is $y+x+a=0$, which passes through the points $(a,-2a)$ and $(-2a, a)$.

17. Any tangent to the first parabola is

$$y = m(x+a) + \frac{a}{m};$$

A perpendicular tangent to the second parabola is

$$y = -\frac{1}{m}(x+a') - a'm.$$

Subtracting, the point of intersection lies on

$$x\left(m+\frac{1}{m}\right) + a\left(m+\frac{1}{m}\right) + a'\left(m+\frac{1}{m}\right) = 0, \text{ or } x + a + a' = 0.$$

The same equation is obtained by subtracting the equations to the two parabolas. Hence etc.

18. Let $(am^2, 2am)$ be the coordinates of P. The line $y = am$ cuts the curve where $a^2m^2 = 4ax$, or $x = \frac{1}{4}am^2$. $\therefore$ the coordinates of Q are $(\frac{1}{4}am^2, am)$ and the coordinates of N are $(am^2, 0)$. Hence the equation of QN is

$$\frac{y}{am} = \frac{x - am^2}{-\frac{3}{4}am^2}.$$

Putting $x = 0$, we have $AT = \frac{4}{3}am = \frac{2}{3}NP$.

19. Whatever be the value of m, the line

$$y = mx - 2am - am^3$$

is a normal to the curve. (Art. 208.)

Put $m = \sqrt{2}$, and the equation becomes $y = \sqrt{2}x - 4a\sqrt{2}$. In the result of Art. 204, put $m = \sqrt{2}$ and $c = -4a\sqrt{2}$. Hence the required length

$$= \frac{4}{2}\sqrt{1+2}\,\sqrt{a(a+8a)} = 6\sqrt{3}\,.\,a.$$

20. The difference of the squares on the perpendiculars from $(a \pm c, 0)$ upon any tangent $\left(mx - y + \frac{a}{m} = 0\right)$

$$= \frac{\left\{m(a+c) + \frac{a}{m}\right\}^2 - \left\{m(a-c) + \frac{a}{m}\right\}^2}{1+m^2} = \frac{4ac\,m\left(m+\frac{1}{m}\right)}{1+m^2} = 4ac.$$

21. Let the coordinates of P, Q, R be

$$\left(\frac{a}{m_1^2}, \frac{2a}{m_1}\right), \left(\frac{a}{m_2^2}, \frac{2a}{m_2}\right) \text{ and } \left(\frac{a}{m_3^2}, \frac{2a}{m_3}\right),$$

so that $\frac{1}{m_2^2} = \frac{1}{m_1 m_3}$, since the ordinates are in geometrical progression. By Art. 206,

$$y = m_1 x + \frac{a}{m_1} \text{ and } y = m_3 x + \frac{a}{m_3}$$

are the equations of the tangents at P and R.

Solving, their point of intersection is given by

$$x = \frac{a}{m_1 m_3} = \frac{a}{m_2^2},$$

which is the equation of the ordinate of Q.

22. Let $y = m_1 x + \frac{a}{m_1}$ and $y = m_2 x + \frac{a}{m_2}$ be the equation of the tangents, so that $m_2^2 = \mu \,.\, m_1^2$, by Art. 206.

Solving $$x = \frac{a}{m_1 m_2} = \frac{a}{\mu^{\frac{1}{2}} m_1^2}, \text{ ....(1)}$$

and $$y = a\left\{\frac{1}{m_1} + \frac{1}{m_2}\right\} = a\left\{\frac{1}{m_1} + \frac{1}{\mu^{\frac{1}{2}} m_1}\right\}.$$

$$\therefore m_1 = \frac{a}{y} \,.\, \frac{\mu^{\frac{1}{2}} + 1}{\mu^{\frac{1}{2}}}. \text{ ....(2)}$$

Eliminating m_1 between (1) and (2), we have, as the required locus,

$$\left\{\frac{a}{y} \,.\, \frac{\mu^{\frac{1}{2}} + 1}{\mu^{\frac{1}{2}}}\right\}^2 = \frac{a}{\mu^{\frac{1}{2}} \,.\, x}. \quad \therefore y^2 = ax \,.\left\{\frac{\mu^{\frac{1}{2}} + 1}{\mu^{\frac{1}{4}}}\right\}^2 = ax\,(\mu^{\frac{1}{4}} + \mu^{-\frac{1}{4}})^2.$$

23. (1) Solving $yy' = 2a(x + x')$, and $yy'' = 2a(x + x'')$, we have

$$y_1 = \frac{2a(x' - x'')}{y' - y''} = \frac{1}{2}\,\frac{y'^2 - y''^2}{y' - y''} = \frac{y' + y''}{2},$$

and $y'^2 + y'y'' = 4ax_1 + 4ax'$, so that $x_1 = \dfrac{y'y''}{4a}$.

(2) Solving

$2a(y-y') = x'y' - xy'$, and $2a(y-y'') = x''y'' - xy''$, (Art. 207),

we have $$2a(y''-y') = \frac{y'^3 - y''^3}{4a} + x(y''-y').$$

$$\therefore\ x_2 = 2a + \frac{y'^2 + y'y'' + y''^2}{4a}.$$

Hence $$2ay_2 - 2ay' = \frac{y'^3}{4a} - 2ay' - \frac{y'(y'^2 + y'y'' + y''^2)}{4a}.$$

$$\therefore\ y_2 = -y'y''\frac{y'+y''}{8a^2}.$$

(3) This follows at once from (1) and (2).

24. The condition that (x_1, y_1) should lie on the parabola $y^2 = a(x+b)$ is $\left(\dfrac{y'+y''}{2}\right)^2 = a\left(\dfrac{y'y''}{4a} + b\right)$, or

$$\frac{y'^2 + y''^2 + y'y''}{4a} = b.\quad \text{Hence } x_2 = 2a + b, \text{ etc.}$$

25. The axis is one normal, and, if G be the given point, $AG = AS + SG = 2a + 8a = 10a$. Also

$$AG = AN + NG = AN + 4a;$$

$$\therefore\ AN = 6a, \text{ so that } PN^2 = 48a^2.$$

$$\therefore\ PG^2 = PN^2 + NG^2 = 48a^2 + 16a^2 = 64a^2. \quad \therefore\ PG = 8a.$$

26. Taking $y^2 = 4ax$ as the parabola, if the coordinates of P are $(am^2, 2am)$, the coordinates of G (Art. 209) are $(am^2 + 2a, 0)$. If (x, y) be a point on the locus,

$$2x = 2am^2 + 2a, \text{ and } 2y = 2am.$$

Eliminating m, we have $y^2 = a(x-a)$,

which is a parabola whose vertex is $(a, 0)$, and the length of whose latus rectum is a.

27. Let $y = mx + \frac{a}{m}$ and $y = mx - 2am - am^3$ be the equation of the tangent and parallel normal. The portion they intercept on the axis of x

$$= (2a + am^2) - \left(-\frac{a}{m^2}\right) = 2a + a\tan^2\theta + a\cot^2\theta$$

(where $m = \tan\theta$)

$$= a(\operatorname{cosec}^2\theta + \sec^2\theta) = a\operatorname{cosec}^2\theta \,.\, \sec^2\theta.$$

Hence the required length

$$= a\operatorname{cosec}^2\theta \,.\, \sec^2\theta \,.\, \sin\theta = a\operatorname{cosec}\theta \,.\, \sec^2\theta.$$

28. If $y = mx - 2am - am^3$ be the equation to the normal at P, the coordinates of P will be $(am^2, -2am)$. [Art. 208.] Hence P' is the point $(am^2, 2am)$, so that the equation of the diameter through P' is $y = 2am$.

Eliminating m, $\quad y^2 = 4a(x - 4a)$.

29. Putting $y = 0$ and $x = 0$ successively in the equation $y = mx - 2am - am^3$, the coordinates of Q are seen to be

$$x = 2a + am^2, \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$y = -2am - am^3. \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Multiply (i) by m, and add (ii); $\therefore$ $y + mx = 0$, *i.e.* $m = -\frac{y}{x}$.

Eliminating m, we have $x^3 = 2ax^2 + ay^2$.

30. Let $y^2 = 4a(x + a)$ and $x^2 = 4a(y + a)$ be the equation of the parabolas, the common focus being the origin. The equations of perpendicular normals will be

$y = m(x + a) - 2am - am^3$, and $x = -m(y + a) + 2am + am^3$.

By adding, $\quad m = \frac{x + y}{x - y}$;

Substituting for m, we have $y = \frac{x + y}{x - y}x - a\frac{x + y}{x - y} \cdot \frac{2(x^2 + y^2)}{(x - y)^2}$,

whence $(x - y)^2 = 2a(x + y)$, which is a parabola.

31. If the lines

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \text{ (Art. 207)}$$

and $\qquad y(y_1+y_2)=4ax+y_1y_2$ (Art. 205)

are coincident, then $\dfrac{4a}{y_1+y_2}=-\dfrac{y_1}{2a}=\tan\phi$.

Eliminating y_1, we have $y_2\tan\phi=2a\tan^2\phi+4a$.

Also the tangent at the point (x_2, y_2) makes an angle $\tan^{-1}\left(\dfrac{2a}{y_2}\right)$ with the axes.

$$\therefore \text{ the required angle}=\tan^{-1}\frac{\dfrac{2a}{y_2}\sim\tan\phi}{1+\dfrac{2a}{y_2}\tan\phi}$$

$$=\tan^{-1}\frac{2a(\tan^2\phi+1)}{4a(1+\tan^2\phi)}.\tan\phi=\tan^{-1}\left(\frac{1}{2}\tan\phi\right).$$

32. The lines

$$y=mx-2am-am^3 \text{ and } y=m(x-b)-2cm-cm^3$$

are coincident if

$$2a+am^2=b+2c+cm^2, \text{ i.e. if } m^2(a-c)=b-2(a-c).$$

$$\therefore \frac{b}{a-c}-2>0, \text{ if } m \text{ is real.}$$

33. If $x=\dfrac{1}{m}y-\dfrac{a}{m^2}$, and $x=\dfrac{1}{m}y-\dfrac{2b}{m}-\dfrac{b}{m^3}$ (viz. a tangent to the first and a normal to the second parabola), are coincident, then

$$\frac{a}{m}=2b+\frac{b}{m^2}.$$

$\therefore 2bm^2-am+b=0$, which has real roots if $a^2>8b^2$.

34. If $\dfrac{1}{m_1}-\dfrac{1}{m_2}=b$, and $\dfrac{1}{m_2}-\dfrac{1}{m_3}=b$, then $\dfrac{1}{m_1}-\dfrac{1}{m_3}=2b$. Hence, by Ex. x. 16,

$$\Delta=\frac{a^2}{2}\Pi\left(\frac{1}{m_2}-\frac{1}{m_3}\right)=a^2b^3.$$

35. By solving, the common point is $(4a^{\frac{1}{3}}b^{\frac{2}{3}}, 4a^{\frac{2}{3}}b^{\frac{1}{3}})$ and the m's of the tangents at this point are

$$\frac{2a}{4a^{\frac{2}{3}}b^{\frac{1}{3}}} \text{ and } \frac{4a^{\frac{1}{3}}b^{\frac{2}{3}}}{2b}, \text{ i.e. } \frac{a^{\frac{1}{3}}}{2b^{\frac{1}{3}}} \text{ and } \frac{2a^{\frac{1}{3}}}{b^{\frac{1}{3}}}.$$

$$\therefore \text{ the required angle} = \tan^{-1}\frac{\dfrac{2a^{\frac{1}{3}}}{b^{\frac{1}{3}}} - \dfrac{a^{\frac{1}{3}}}{2b^{\frac{1}{3}}}}{1 + \dfrac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}}} = \tan^{-1}\frac{3a^{\frac{1}{3}}\,.\,b^{\frac{1}{3}}}{2\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)}.$$

36. Taking the common focus as origin, let

$$y^2 = 4a(x+a) \quad \text{and} \quad y^2 = -4b(x-b)$$

be the equations to the parabolas.

Solving $x = b - a$, $y = 2\sqrt{ab}$. Hence the m's of the tangents at these points are $\sqrt{\dfrac{a}{b}}$ and $-\sqrt{\dfrac{b}{a}}$.

Aliter. Let P be one of the points of intersection of the two parabolas, and let the tangents at P to them meet the axis in T and T'. By Art. 211, both ST and ST' are equal to SP. Hence S is the centre of the circumcircle of TPT'. $\therefore$ TPT' is a right angle, *i.e.* the tangents at P are perpendicular.

37. By drawing the curves, it is easily seen that the foci are $(2a, 0)$ and $(0, 2b)$. Hence the point $(2a, 2b)$ is a common end of the latus rectum of each parabola. The m's of the tangents at this point are 1 and -1.

38. Let A be the vertex, and let the axis of the parabola cut the axis of x in T. Draw ON perpendicular to AT, and NR and AH perpendicular to the axis of x. Let $N\hat{O}R = \theta$.

Since $AN = AT$ (Art. 209),

$\therefore$ $NR = 2AH = 2k$. $\therefore$ $ON = NR \operatorname{cosec}\theta = 2k \operatorname{cosec}\theta$.

$\therefore$ the equation to the axis is $x\cos\theta + y\sin\theta = 2k\operatorname{cosec}\theta$, or

$$x = -y\tan\theta + 2k(\tan\theta + \cot\theta) = -\tan\theta\,(y - 2k) + \frac{2k}{\tan\theta},$$

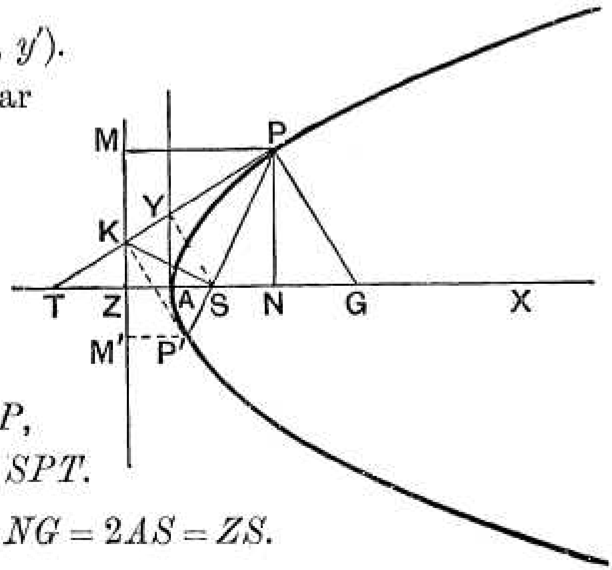
which always touches the parabola $x^2 = -8k(y - 2k)$.

211. Some properties of the Parabola.

(α) *If the tangent and normal at any point P of the parabola meet the axis in T and G respectively, then*

$$ST = SG = SP,$$

and the tangent at P is equally inclined to the axis and the focal distance of P.



Let P be the point (x', y').
Draw PM perpendicular to the directrix.
By Art. 209, we have $AT = AN$.

$$\therefore \; TS = TA + AS$$
$$= AN + ZA$$
$$= ZN = MP = SP,$$

and hence $\angle STP = \angle SPT$.

By the same article, $NG = 2AS = ZS$.

$$\therefore \; SG = SN + NG = ZS + SN = MP = SP.$$

(β) *If the tangent at P meet the directrix in K, then KSP is a right angle.*

For $\quad \angle SPT = \angle PTS = \angle KPM.$

Hence the two triangles KPS and KPM have the two sides KP, PS and the angle KPS equal respectively to the two sides KP, PM and the angle KPM.

Hence $\quad \angle KSP = \angle KMP =$ a right angle.

Also $\quad \angle SKP = \angle MKP.$

(γ) *Tangents at the extremities of any focal chord intersect at right angles in the directrix.*

For, if PS be produced to meet the curve in P', then, since $\angle P'SK$ is a right angle, the tangent at P' meets the directrix in K.

Also, by (β), $\angle MKP = \angle SKP$,
and, similarly, $\angle M'KP' = \angle SKP'$.

Hence

$$\angle PKP' = \tfrac{1}{2} \angle SKM + \tfrac{1}{2} \angle SKM' = \text{a right angle.}$$

(δ) *If SY be perpendicular to the tangent at P, then Y lies on the tangent at the vertex and $SY^2 = AS \,.\, SP$.*

For the equation to any tangent is

$$y = mx + \frac{a}{m} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equation to the perpendicular to this line passing through the focus is

$$y = -\frac{1}{m}(x - a) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The lines (1) and (2) meet where

$$mx + \frac{a}{m} = -\frac{1}{m}(x - a) = -\frac{1}{m}x + \frac{a}{m},$$

i.e. where $x = 0$.

Hence Y lies on the tangent at the vertex.

Also, by Euc. VI. 8, Cor.,

$$SY^2 = SA \,.\, ST = AS \,.\, SP.$$

212. *To prove that through any given point (x_1, y_1) there pass, in general, two tangents to the parabola.*

The equation to any tangent is (by Art. 206)

$$y = mx + \frac{a}{m} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If this pass through the fixed point (x_1, y_1), we have

$$y_1 = mx_1 + \frac{a}{m},$$

i.e.

$$m^2x_1 - my_1 + a = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

For any given values of x_1 and y_1 this equation is in general a quadratic equation and gives two values of m (real or imaginary).

Corresponding to each value of m we have, by substituting in (1), a different tangent.

The roots of (2) are real and different if $y_1^2 - 4ax_1$ be positive, *i.e.*, by Art. 201, if the point (x_1, y_1) lie without the curve.

They are equal, *i.e.* the *two* tangents coalesce into one tangent, if $y_1^2 - 4ax_1$ be zero, *i.e.* if the point (x_1, y_1) lie on the curve.

The two roots are imaginary if $y_1^2 - 4ax_1$ be negative, *i.e.* if the point (x_1, y_1) lie within the curve.

213. *Equation to the* **chord of contact** *of tangents drawn from a point* (x_1, y_1).

The equation to the tangent at any point Q, whose coordinates are x' and y', is

$$yy' = 2a(x + x').$$

Also the tangent at the point R, whose coordinates are x'' and y'', is

$$yy'' = 2a(x + x'').$$

If these tangents meet at the point T, whose coordinates are x_1 and y_1, we have

$$y_1y' = 2a(x_1 + x') \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and $$y_1y'' = 2a(x_1 + x'') \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The equation to QR is then

$$\mathbf{yy_1 = 2a(x + x_1)} \ldots\ \ldots\ldots\ldots\ldots(3).$$

For, since (1) is true, the point (x', y') lies on (3).

Also, since (2) is true, the point (x'', y'') lies on (3).

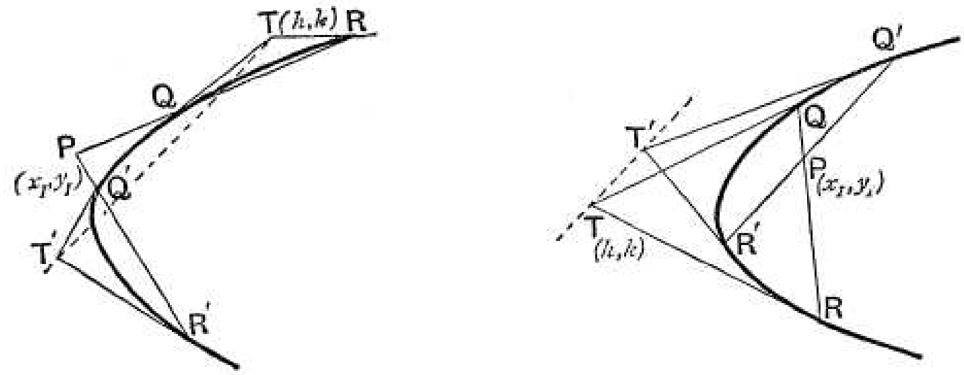
Hence (3) must be the equation to the straight line joining (x', y') to the point (x'', y''), *i.e.* it must be the equation to QR the chord of contact of tangents from the point (x_1, y_1).

214. The polar of any point with respect to a parabola is defined as in Art. 162.

To find the equation of the **polar** *of the point* (x_1, y_1) *with respect to the parabola* $y^2 = 4ax$.

Let Q and R be the points in which any chord drawn through the point P, whose coordinates are (x_1, y_1), meets the parabola.

Let the tangents at Q and R meet in the point whose coordinates are (h, k).



We require the locus of (h, k).

Since QR is the chord of contact of tangents from (h, k) its equation (Art. 213) is

$$ky = 2a(x + h).$$

Since this straight line passes through the point (x_1, y_1) we have

$$ky_1 = 2a(x_1 + h) \qquad (1).$$

Since the relation (1) is true, it follows that the point (h, k) always lies on the straight line

$$\mathbf{yy_1 = 2a(x + x_1)} \qquad (2).$$

Hence (2) is the equation to the polar of (x_1, y_1).

Cor. The equation to the polar of the focus, viz. the point $(a, 0)$, is $0 = x + a$, so that the polar of the focus is the directrix.

215. When the point (x_1, y_1) lies without the parabola the equation to its polar is the same as the equation to the chord of contact of tangents drawn from (x_1, y_1).

When (x_1, y_1) is on the parabola the polar is the same as the tangent at the point.

As in Art. 164 the polar of (x_1, y_1) might have been defined as the chord of contact of the tangents (real or imaginary) that can be drawn from it to the parabola.

216. *Geometrical construction for the polar of a point* (x_1, y_1).

Let T be the point (x_1, y_1), so that its polar is

$$yy_1 = 2a(x + x_1) \qquad \text{.....................(1).}$$

Through T draw a straight line parallel to the axis; its equation is therefore

$$y = y_1 \qquad \text{..............(2).}$$

Let this straight line meet the polar in V and the curve in P.

The coordinates of V, which is the intersection of (1) and (2), are therefore

$$\frac{y_1^2}{2a} - x_1 \text{ and } y_1 \qquad \text{........(3).}$$

Also P is the point on the curve whose ordinate is y_1, and whose coordinates are therefore

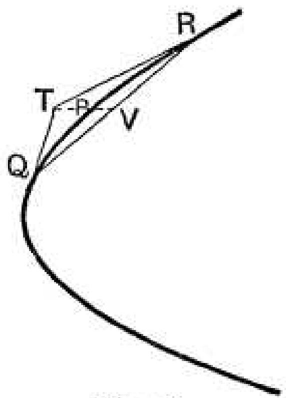


Fig. 1.

$$\frac{y_1^2}{4a} \text{ and } y_1.$$

Since abscissa of $P = \dfrac{\text{abscissa of } T + \text{abscissa of } V}{2}$, therefore, by Art. 22, Cor., P is the middle point of TV.

Also the tangent at P is

$$yy_1 = 2a\left(x + \frac{y_1^2}{4a}\right),$$

which is parallel to (1).

Hence the polar of T is parallel to the tangent at P.

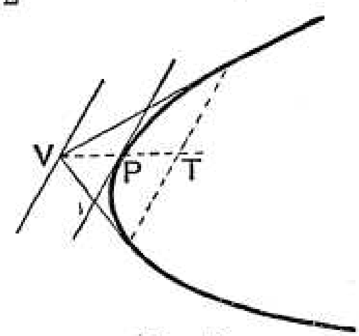


Fig. 2.

To draw the polar of T we therefore draw a line through T, parallel to the axis, to meet the curve in P and produce it to V so that $TP = PV$; a line through V parallel to the tangent at P is then the polar required.

217. *If the polar of a point P passes through the point T, then the polar of T goes through P.* (Fig. Art. 214).

Let P be the point (x_1, y_1) and T the point (h, k).

The polar of P is $yy_1 = 2a(x + x_1)$.

Since it passes through T, we have

$$y_1 k = 2a(x_1 + h) \qquad \text{..............................(1).}$$

The polar of T is $yk = 2a(x+h)$.

Since (1) is true, this equation is satisfied by the coordinates x_1 and y_1.

Hence the proposition.

Cor. The point of intersection, T, of the polars of two points, P and Q, is the pole of the line PQ.

218. *To find the pole of a given straight line with respect to the parabola.*

Let the given straight line be

$$Ax + By + C = 0.$$

If its pole be the point (x_1, y_1), it must be the same straight line as

$$yy_1 = 2a(x + x_1),$$

i.e.
$$2ax - yy_1 + 2ax_1 = 0.$$

Since these straight lines are the same, we have

$$\frac{2a}{A} = \frac{-y_1}{B} = \frac{2ax_1}{C},$$

i.e.
$$x_1 = \frac{C}{A} \text{ and } y_1 = -\frac{2Ba}{A}.$$

219. *To find the equation to the pair of tangents that can be drawn to the parabola from the point (x_1, y_1).*

Let (h, k) be any point on *either* of the tangents drawn from (x_1, y_1). The equation to the line joining (x_1, y_1) to (h, k) is

$$y - y_1 = \frac{k - y_1}{h - x_1}(x - x_1),$$

i.e.
$$y = \frac{k - y_1}{h - x_1}x + \frac{hy_1 - kx_1}{h - x_1}.$$

If this be a tangent it must be of the form

$$y = mx + \frac{a}{m},$$

so that
$$\frac{k - y_1}{h - x_1} = m \text{ and } \frac{hy_1 - kx_1}{h - x_1} = \frac{a}{m}.$$

Hence, by multiplication,

$$a = \frac{k - y_1}{h - x_1}\,\frac{hy_1 - kx_1}{h - x_1},$$

i.e.
$$a(h - x_1)^2 = (k - y_1)(hy_1 - kx_1).$$

The locus of the point (h, k) (*i.e.* the *pair* of tangents required) is therefore

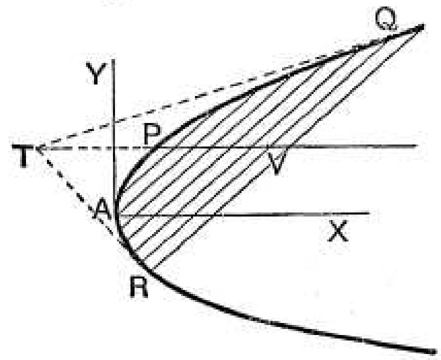
$$a(x - x_1)^2 = (y - y_1)(xy_1 - yx_1) \;\ldots\ldots\ldots\; (1).$$

It will be seen that this equation is the same as

$$(y^2 - 4ax)(y_1^2 - 4ax_1) = \{yy_1 - 2a(x + x_1)\}^2.$$

220. *To prove that the middle points of a system of parallel chords of a parabola all lie on a straight line which is parallel to the axis.*

Since the chords are all parallel, they all make the same angle with the axis of x. Let the tangent of this angle be m.



The equation to QR, any one of these chords, is therefore

$$y = mx + c \;\ldots\ldots\; (1),$$

where c is different for the several chords, but m is the same.

This straight line meets the parabola $y^2 = 4ax$ in points whose ordinates are given by

$$my^2 = 4a(y - c),$$

i.e.
$$y^2 - \frac{4a}{m}y + \frac{4ac}{m} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

Let the roots of this equation, *i.e.* the ordinates of Q and R, be y' and y'', and let the coordinates of V, the middle point of QR, be (h, k).

Then, by Art. 22,

$$k = \frac{y' + y''}{2} = \frac{2a}{m},$$

from equation (2).

The coordinates of V therefore satisfy the equation

$$y = \frac{2a}{m},$$

so that the locus of V is a straight line parallel to the axis of the curve.

The straight line $y = \frac{2a}{m}$ meets the curve in a point P, whose ordinate is $\frac{2a}{m}$ and whose abscissa is therefore $\frac{a}{m^2}$.

The tangent at this point is, by Art. 205,

$$y = mx + \frac{a}{m},$$

and is therefore parallel to each of the given chords.

Hence the locus of the middle points of a system of parallel chords of a parabola is a straight line which is parallel to the axis and meets the curve at a point the tangent at which is parallel to the given system.

221. *To find the equation to the chord of the parabola which is bisected at any point (h, k).*

By the last article the required chord is parallel to the tangent at the point P where a line through (h, k) parallel to the axis meets the curve.

Also, by Art. 216, the polar of (h, k) is parallel to the tangent at this same point P.

The required chord is therefore parallel to the polar $yk = 2a(x + h)$.

Hence, since it goes through (h, k), its equation is

$$k(y - k) = 2a(x - h) \qquad \text{(Art. 67).}$$

222. Diameter. Def. The locus of the middle points of a system of parallel chords of a parabola is called a diameter and the chords are called its double ordinates.

Thus, in the figure of Art. 220, PV is a diameter and QR and all the parallel chords are ordinates to this diameter.

The proposition of that article may therefore be stated as follows.

Any diameter of a parabola is parallel to the axis and the tangent at the point where it meets the curve is parallel to its ordinates.

223. *The tangents at the ends of any chord meet on the diameter which bisects the chord.*

Let the equation of QR (Fig., Art. 220) be

$$y = mx + c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and let the tangents at Q and R meet at the point T (x_1, y_1).

Then QR is the chord of contact of tangents drawn from T, and hence its equation is

$$yy_1 = 2a(x + x_1) \text{ (Art. 213).}$$

Comparing this with equation (1), we have

$$\frac{2a}{y_1} = m, \text{ so that } y_1 = \frac{2a}{m},$$

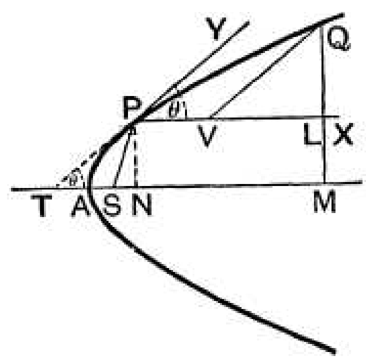
and therefore T lies on the straight line

$$y = \frac{2a}{m}.$$

But this straight line was proved, in Art. 220, to be the diameter PV which bisects the chord.

224. *To find the equation to a parabola, the axes being any diameter and the tangent to the parabola at the point where this diameter meets the curve.*

Let PVX be the diameter and PY the tangent at P meeting the axis in T.



Take any point Q on the curve, and draw QM perpendicular to the axis meeting the diameter PV in L.

Let PV be x and VQ be y.

Draw PN perpendicular to the axis of the curve, and let

$$\theta = \angle YPX = \angle PTM.$$

Then

$$4AS \cdot AN = PN^2 = NT^2 \tan^2\theta = 4AN^2 \cdot \tan^2\theta.$$

$$\therefore AN = AS \cdot \cot^2\theta = a\cot^2\theta,$$

and $$PN = \sqrt{4AS \cdot AN} = 2a\cot\theta.$$

Now $$QM^2 = 4AS \cdot AM = 4a \cdot AM \text{ .... (1).}$$

Also

$$QM = NP + LQ = 2a\cot\theta + VQ\sin\theta = 2a\cot\theta + y\sin\theta,$$

and $$AM = AN + PV + VL = a\cot^2\theta + x + y\cos\theta.$$

Substituting these values in (1), we have

$$(2a \cot \theta + y \sin \theta)^2 = 4a (a \cot^2 \theta + x + y \cos \theta),$$

i.e. $$y^2 \sin^2 \theta = 4ax.$$

The required equation is therefore

$$y^2 = 4px \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where

$$p = \frac{a}{\sin^2 \theta} = a (1 + \cot^2 \theta) = a + AN = SP \text{ (by Art. 202).}$$

The equation to the parabola referred to the above axes is therefore of the same form as its equation referred to the rectangular axes of Art. 197.

The equation (2) states that

$$QV^2 = 4SP \,.\, PV.$$

225. The quantity $4p$ is called the parameter of the diameter PV. It is equal in length to the chord which is parallel to PY and passes through the focus.

For if $Q'V'R'$ be the chord, parallel to PY and passing through the focus and meeting PV in V', we have

$$PV' = ST = SP = p,$$

so that $$Q'V'^2 = 4p \,.\, PV' = 4p^2,$$

and hence $$Q'R' = 2Q'V' = 4p.$$

226. Just as in Art. 205 it could now be shown that the tangent at any point (x', y') of the above curve is

$$yy' = 2p (x + x').$$

Similarly for the equation to the polar of any point.

EXAMPLES XXVII

1. Prove that the length of the chord joining the points of contact of tangents drawn from the point (x_1, y_1) is

$$\frac{\sqrt{y_1^2 + 4a^2} \sqrt{y_1^2 - 4ax_1}}{a}.$$

2. Prove that the area of the triangle formed by the tangents from the point (x_1, y_1) and the chord of contact is $(y_1^2 - 4ax_1)^{\frac{3}{2}} \div 2a$.

3. If a perpendicular be let fall from any point P upon its polar prove that the distance of the foot of this perpendicular from the focus is equal to the distance of the point P from the directrix.

4. What is the equation to the chord of the parabola $y^2=8x$ which is bisected at the point $(2, -3)$?

5. The general equation to a system of parallel chords in the parabola $y^2=\frac{25}{7}x$ is $4x-y+k=0$.

What is the equation to the corresponding diameter?

6. P, Q, and R are three points on a parabola and the chord PQ cuts the diameter through R in V. Ordinates PM and QN are drawn to this diameter. Prove that $RM \cdot RN = RV^2$.

7. Two equal parabolas with axes in opposite directions touch at a point O. From a point P on one of them are drawn tangents PQ and PQ' to the other. Prove that QQ' will touch the first parabola in P' where PP' is parallel to the common tangent at O.

ANSWERS

4. $4x+3y+1=0$. **5.** $56y=25$.

SOLUTIONS/HINTS

1. The polar of (x_1, y_1), viz. $yy_1=2a(x+x_1)$, meets the parabola $y^2=4ax$, where $2yy_1=y^2+4ax_1$. Hence if (x', y') and (x'', y'') be the points of contact, we have

$$y'+y''=2y_1, \text{ and } y'y''=4ax_1.$$

$$\therefore (\text{chord})^2=(y'-y'')^2+(x'-x'')^2=(y'-y'')^2+\left(\frac{y'^2-y''^2}{4a}\right)^2$$

$$=(y'-y'')^2\left\{1+\frac{(y'+y'')^2}{16a^2}\right\}=\{(y'+y'')^2-4y'y''\}\left\{1+\frac{(y'+y'')^2}{16a^2}\right\}$$

$$=\{4y_1^2-16ax_1\}\left\{1+\frac{y_1^2}{4a^2}\right\}=\frac{(y_1^2-4ax_1)(y_1^2+4a^2)}{a^2}.$$

2. The perpendicular from (x_1, y_1) upon $yy_1=2a(x+x_1)$ is $\dfrac{y_1^2-4ax_1}{\sqrt{y_1^2+4a^2}}$.

Hence, by the last example,

$$\Delta=\frac{1}{2}\cdot\frac{y_1^2-4ax_1}{\sqrt{y_1^2+4a^2}}\cdot\frac{(y_1^2-4ax_1)^{\frac{1}{2}}(y_1^2+4a^2)^{\frac{1}{2}}}{a}=(y_1^2-4ax_1)^{\frac{3}{2}}\div 2a.$$

3. The polar of (x_1, y_1) may be written

$$yy_1 - 2a(x-a) = 2ax_1 + 2a^2.$$

The perpendicular from (x_1, y_1) upon this line is

$$(x-a)y_1 + 2ay = x_1y_1 + ay_1.$$

The foot of the perpendicular is the common point of these two lines.

Square and add;

$$\therefore \{(x-a)^2 + y^2\}\{y^2 + 4a^2\} = (x_1^2 + a)^2(y^2 + 4a^2),$$

where (x, y) are now the coordinates of the foot of the perpendicular.

$$\therefore (x-a)^2 + y^2 = (x_1 + a)^2.$$

$\therefore$ distance of (x, y) from $(a, 0)$ = distance of (x_1, y_1) from $x + a = 0$.

4. By Art. 221, the required equation is

$$-3(y+3) = 4(x-2), \quad i.e. \quad 4x + 3y + 1 = 0.$$

5. See Art. 223;

$m = 4$, $a = \dfrac{25}{28}$. $\therefore$ the required equation is $y = \dfrac{2a}{m} = \dfrac{25}{56}$.

6. The diameter and tangent through R being the axes, the equation to the parabola is, by Art. 224,

$$y^2 = 4px. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Let the equation to PQ be $y = mx + c$, so that $RV = -\dfrac{c}{m}$.

This meets the parabola where $(mx + c)^2 - 4px = 0$.

$$\therefore RM \,.\, RN = x_1x_2 = \frac{c^2}{m^2} = RV^2.$$

Aliter. Take the diameter and tangent through R as axes, and let P and Q be the points t_1, t_2 respectively (Art. 229). Put $y = 0$ in the equation of PQ. Then $x = -at_1t_2$, [Art. 229].

$$\therefore RV^2 = a^2t_1^2t_2^2 = RM \,.\, RN.$$

7. Take O for origin, and

$$y^2 - 4ax = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$y^2 + 4ax = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

as the equations of the parabolas, the diameter and common tangent at O being the axes. Let the coordinates of the point P be $(at^2, 2at)$. Its polar with regard to (ii) is

$$y = -\frac{x}{t} - at,$$

which touches (i) at the point $(at^2, -2at)$. $\therefore$ PP' is parallel to the axis of y, the common tangent at O.

Coordinates of any point on the parabola expressed in terms of one variable.

227. It is often convenient to express the coordinates of any point on the curve in terms of one variable.

It is clear that the values

$$x = \frac{a}{m^2}, \quad y = \frac{2a}{m}$$

always satisfy the equation to the curve.

Hence, for all values of m, the point

$$\left(\frac{a}{m^2}, \ \frac{2a}{m}\right)$$

lies on the curve. By Art. 206, this m is equal to the tangent of the angle which the tangent at the point makes with the axis.

The equation to the tangent at this point is

$$y = mx + \frac{a}{m},$$

and the normal is, by Art. 207, found to be

$$my + x = 2a + \frac{a}{m^2}.$$

228. The coordinates of the point could also be expressed in terms of the m of the normal at the point; in this case its coordinates are am^2 and $-2am$.

The equation of the tangent at the point $(am^2, -2am)$ is, by Art. 205,

$$my + x + am^2 = 0,$$

and the equation to the normal is

$$y = mx - 2am - am^3.$$

229. The simplest substitution (avoiding both negative signs and fractions) is

$$\mathbf{x = at^2} \text{ and } \mathbf{y = 2at.}$$

These values satisfy the equation $y^2 = 4ax$.

The equations to the tangent and normal at the point $(at^2, 2at)$ are, by Arts. 205 and 207,

$$ty = x + at^2,$$

and $$y + tx = 2at + at^3.$$

The equation to the straight line joining

$$(at_1^2, 2at_1) \text{ and } (at_2^2, 2at_2)$$

is easily found to be

$$y(t_1 + t_2) = 2x + 2at_1t_2.$$

The tangents at the points

$$(at_1^2, 2at_1) \text{ and } (at_2^2, 2at_2)$$

are $$t_1y = x + at_1^2,$$

and $$t_2y = x + at_2^2.$$

The point of intersection of these two tangents is clearly

$$\{at_1t_2, a(t_1 + t_2)\}.$$

The point whose coordinates are $(at^2, 2at)$ may, for brevity, be called the point "t."

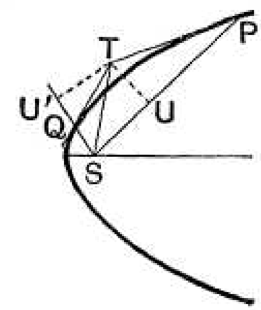
In the following articles we shall prove some important properties of the parabola making use of the above substitution.

230. *If the tangents at P and Q meet in T, prove that*

(1) *TP and TQ subtend equal angles at the focus S,*

(2) $ST^2 = SP \, . \, SQ$,

and (3) *the triangles SPT and STQ are similar.*



Let P be the point $(at_1^2, 2at_1)$, and Q be the point $(at_2^2, 2at_2)$, so that (Art. 229) T is the point $\{at_1t_2, a(t_1+t_2)\}$.

(1) The equation to SP is $y = \frac{2at_1}{at_1^2 - a}(x-a)$,

i.e. $(t_1^2 - 1)y - 2t_1x + 2at_1 = 0$.

The perpendicular, TU, from T on this straight line

$$= \frac{a(t_1^2-1)(t_1+t_2) - 2t_1 \, . \, at_1t_2 + 2at_1}{\sqrt{(t^2-1)^2 + 4t_1^2}} = a\frac{(t_1^3 - t_1^2t_2) + (t_1 - t_2)}{t_1^2+1}$$

$= a(t_1 - t_2)$.

Similarly TU' has the same numerical value.

The angles PST and QST are therefore equal.

(2) By Art. 202 we have $SP = a(1+t_1^2)$ and $SQ = a(1+t_2^2)$.

Also $$ST^2 = (at_1t_2 - a)^2 + a^2(t_1+t_2)^2$$
$$= a^2[t_1^2t_2^2 + t_1^2 + t_2^2 + 1] = a^2(1+t_1^2)(1+t_2^2).$$

Hence $$ST^2 = SP \, . \, SQ.$$

(3) Since $\frac{ST}{SP} = \frac{SQ}{ST}$ and the angles TSP and TSQ are equal, the triangles SPT and STQ are similar, so that

$$\angle SQT = \angle STP \text{ and } \angle STQ = \angle SPT.$$

231. *The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.*

Let the three points on the parabola be

$$(at_1^2, 2at_1), \ (at_2^2, 2at_2), \text{ and } (at_3^2, 2at_3).$$

The area of the triangle formed by these points, by Art. 25,

$$= \tfrac{1}{2}[at_1^2(2at_2 - 2at_3) + at_2^2(2at_3 - 2at_1) + at_3^2(2at_1 - 2at_2)]$$
$$= -a^2(t_2-t_3)(t_3-t_1)(t_1-t_2).$$

The intersections of the tangents at these points are (Art. 229) the points

$$\{at_2t_3, a(t_2+t_3)\}, \ \{at_3t_1, a(t_3+t_1)\}, \text{ and } \{at_1t_2, a(t_1+t_2)\}.$$

The area of the triangle formed by these three points

$$= \tfrac{1}{2}\{at_2t_3(at_3 - at_2) + at_3t_1(at_1 - at_3) + at_1t_2(at_2 - at_1)\}$$
$$= \tfrac{1}{2}a^2(t_2-t_3)(t_3-t_1)(t_1-t_2).$$

The first of these areas is double the second.

232. *The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.*

Let P, Q, and R be the points at which the tangents are drawn and let their coordinates be

$$(at_1^2,\ 2at_1),\ (at_2^2,\ 2at_2),\ \text{and}\ (at_3^2,\ 2at_3).$$

As in Art. 229, the tangents at Q and R intersect in the point

$$\{at_2t_3,\ a(t_2+t_3)\}.$$

Similarly, the other pairs of tangents meet at the points

$$\{at_3t_1,\ a(t_3+t_1)\}\ \text{and}\ \{at_1t_2,\ a(t_1+t_2)\}.$$

Let the equation to the circle be

$$x^2+y^2+2gx+2fy+c=0 \qquad \text{..........(1)}.$$

Since it passes through the above three points, we have

$$a^2t_2^2t_3^2+a^2(t_2+t_3)^2+2gat_2t_3+2fa(t_2+t_3)+c=0 \qquad \text{......(2)},$$

$$a^2t_3^2t_1^2+a^2(t_3+t_1)^2+2gat_3t_1+2fa(t_3+t_1)+c=0 \qquad \text{......(3)},$$

and

$$a^2t_1^2t_2^2+a^2(t_1+t_2)^2+2gat_1t_2+2fa(t_1+t_2)+c=0 \qquad \text{......(4)}.$$

Subtracting (3) from (2) and dividing by $a(t_2-t_1)$, we have

$$a\{t_3^2(t_1+t_2)+t_1+t_2+2t_3\}+2gt_3+2f=0.$$

Similarly, from (3) and (4), we have

$$a\{t_1^2(t_2+t_3)+t_2+t_3+2t_1\}+2gt_1+2f=0.$$

From these two equations we have

$$2g=-a(1+t_2t_3+t_3t_1+t_1t_2)\ \text{and}\ 2f=-a[t_1+t_2+t_3-t_1t_2t_3].$$

Substituting these values in (2), we obtain

$$c=a^2(t_2t_3+t_3t_1+t_1t_2).$$

The equation to the circle is therefore

$$x^2+y^2-ax(1+t_2t_3+t_3t_1+t_1t_2)-ay(t_1+t_2+t_3-t_1t_2t_3)$$
$$+a^2(t_2t_3+t_3t_1+t_1t_2)=0,$$

which clearly goes through the focus $(a, 0)$.

233. *If O be any point on the axis and POP' be any chord passing through O, and if PM and $P'M'$ be the ordinates of P and P', prove that $AM \,.\, AM'=AO^2$, and $PM \,.\, P'M'=-4a \,.\, AO$.*

Let O be the point $(h, 0)$, and let P and P' be the points

$$(at_1^2,\ 2at_1)\ \text{and}\ (at_2^2,\ 2at_2).$$

The equation to PP' is, by Art. 229,

$$(t_2+t_1)\,y-2x=2at_1t_2.$$

If this pass through the point $(h, 0)$, we have

$$-2h=2at_1t_2,$$

i.e.

$$t_1t_2=-\frac{h}{a}.$$

Hence $$AM \cdot AM' = at_1^2 \cdot at_2^2 = a^2 \cdot \frac{h^2}{a^2} = h^2 = AO^2,$$

and $$PM \cdot PM' = 2at_1 \cdot 2at_2 \cdot = 4a^2\left(-\frac{h}{a}\right) = -4a \cdot AO.$$

Cor. If O be the focus, $AO = a$, and we have

$$t_1 t_2 = -1, \text{ i.e. } t_2 = -\frac{1}{t_1}.$$

The points $(at_1^2, 2at_1)$ and $\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$ are therefore at the ends of a focal chord.

234. *To prove that the orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.*

Let the equations to the three tangents be

$$y = m_1 x + \frac{a}{m_1} \quad \text{.............................. (1)},$$

$$y = m_2 x + \frac{a}{m_2} \quad \text{.............................. (2)},$$

and $$y = m_3 x + \frac{a}{m_3} \quad \text{.............................. (3)}.$$

The point of intersection of (2) and (3) is found, by solving them, to be

$$\left\{\frac{a}{m_2 m_3},\ a\left(\frac{1}{m_2} + \frac{1}{m_3}\right)\right\}.$$

The equation to the straight line through this point perpendicular to (1) is (Art. 69)

$$y - a\left(\frac{1}{m_2} + \frac{1}{m_3}\right) = -\frac{1}{m_1}\left[x - \frac{a}{m_2 m_3}\right],$$

i.e. $$y + \frac{x}{m_1} = a\left[\frac{1}{m_2} + \frac{1}{m_3} + \frac{a}{m_1 m_2 m_3}\right] \text{..................(4)}.$$

Similarly, the equation to the straight line through the intersection of (3) and (1) perpendicular to (2) is

$$y + \frac{x}{m_2} = a\left(\frac{1}{m_3} + \frac{1}{m_1} + \frac{a}{m_1 m_2 m_3}\right) \text{.................. (5)},$$

and the equation to the straight line through the intersection of (1) and (2) perpendicular to (3) is

$$y + \frac{x}{m_3} = a\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{a}{m_1 m_2 m_3}\right) \text{.................. (6)}.$$

The point which is common to the straight lines (4), (5), and (6), *i.e.* the orthocentre of the triangle, is easily seen to be the point whose coordinates are

$$x = -a, \quad y = a\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_1 m_2 m_3}\right),$$

and this point lies on the directrix.

EXAMPLES XXVIII

1. If ω be the angle which a focal chord of a parabola makes with the axis, prove that the length of the chord is $4a \operatorname{cosec}^2 \omega$ and that the perpendicular on it from the vertex is $a \sin \omega$.

2. A point on a parabola, the foot of the perpendicular from it upon the directrix, and the focus are the vertices of an equilateral triangle. Prove that the focal distance of the point is equal to the latus rectum.

3. Prove that the semi-latus-rectum is a harmonic mean between the segments of any focal chord.

4. If T be any point on the tangent at any point P of a parabola, and if TL be perpendicular to the focal radius SP and TN be perpendicular to the directrix, prove that $SL = TN$.

Hence obtain a geometrical construction for the pair of tangents drawn to the parabola from any point T.

5. Prove that on the axis of any parabola there is a certain point K which has the property that, if a chord PQ of the parabola be drawn through it, then

$$\frac{1}{PK^2} + \frac{1}{QK^2}$$

is the same for all positions of the chord.

6. The normal at the point $(at_1^2, 2at_1)$ meets the parabola again in the point $(at_2^2, 2at_2)$; prove that

$$t_2 = -t_1 - \frac{2}{t_1}.$$

7. A chord is a normal to a parabola and is inclined at an angle θ to the axis; prove that the area of the triangle formed by it and the tangents at its extremities is $4a^2 \sec^3 \theta \operatorname{cosec}^3 \theta$.

8. If PQ be a normal chord of the parabola and if S be the focus, prove that the locus of the centroid of the triangle SPQ is the curve

$$36ay^2(3x - 5a) - 81y^4 = 128a^4.$$

9. Prove that the length of the intercept on the normal at the point $(at^2, 2at)$ made by the circle which is described on the focal distance of the given point as diameter is $a\sqrt{1+t^2}$.

10. Prove that the area of the triangle formed by the normals to the parabola at the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$ is

$$\frac{a^2}{2}(t_2-t_3)(t_3-t_1)(t_1-t_2)(t_1+t_2+t_3)^2.$$

11. Prove that the normal chord at the point whose ordinate is equal to its abscissa subtends a right angle at the focus.

12. A chord of a parabola passes through a point on the axis (outside the parabola) whose distance from the vertex is half the latus rectum; prove that the normals at its extremities meet on the curve.

13. The normal at a point P of a parabola meets the curve again in Q, and T is the pole of PQ; shew that T lies on the diameter passing through the other end of the focal chord passing through P, and that PT is bisected by the directrix.

14. If from the vertex of a parabola a pair of chords be drawn at right angles to one another and with these chords as adjacent sides a rectangle be made, prove that the locus of the further angle of the rectangle is the parabola

$$y^2 = 4a(x-8a).$$

15. A series of chords is drawn so that their projections on a straight line which is inclined at an angle α to the axis are all of constant length c; prove that the locus of their middle point is the curve

$$(y^2-4ax)(y\cos\alpha + 2a\sin\alpha)^2 + a^2c^2 = 0.$$

16. Prove that the locus of the poles of chords which subtend a right angle at a fixed point (h, k) is

$$ax^2 - hy^2 + (4a^2+2ah)x - 2aky + a(h^2+k^2) = 0.$$

17. Prove that the locus of the middle points of all tangents drawn from points on the directrix to the parabola is

$$y^2(2x+a) = a(3x+a)^2.$$

18. Prove that the orthocentres of the triangles formed by three tangents and the corresponding three normals to a parabola are equidistant from the axis.

19. T is the pole of the chord PQ; prove that the perpendiculars from P, T, and Q upon any tangent to the parabola are in geometrical progression.

20. If r_1 and r_2 be the lengths of radii vectores of the parabola which are drawn at right angles to one another from the vertex, prove that

$$r_1^{\frac{4}{3}} r_2^{\frac{4}{3}} = 16a^2 \left(r_1^{\frac{2}{3}} + r_2^{\frac{2}{3}}\right).$$

21. A parabola touches the sides of a triangle ABC in the points D, E, and F respectively; if DE and DF cut the diameter through the point A in b and c respectively, prove that Bb and Cc are parallel.

22. Prove that all circles described on focal radii as diameters touch the directrix of the curve, and that all circles on focal radii as diameters touch the tangent at the vertex.

23. A circle is described on a focal chord as diameter; if m be the tangent of the inclination of the chord to the axis, prove that the equation to the circle is

$$x^2+y^2-2ax\left(1+\frac{2}{m^2}\right)-\frac{4ay}{m}-3a^2=0.$$

24. LOL' and MOM' are two chords of a parabola passing through a point O on its axis. Prove that the radical axis of the circles described on LL' and MM' as diameters passes through the vertex of the parabola.

25. A circle and a parabola intersect in four points; shew that the algebraic sum of the ordinates of the four points is zero.

Shew also that the line joining one pair of these four points and the line joining the other pair are equally inclined to the axis.

26. Circles are drawn through the vertex of the parabola to cut the parabola orthogonally at the other point of intersection. Prove that the locus of the centres of the circles is the curve

$$2y^2(2y^2+x^2-12ax)=ax(3x-4a)^2.$$

27. Prove that the equation to the circle passing through the points $(at_1^2,\ 2at_1)$ and $(at_2^2,\ 2at_2)$ and the intersection of the tangents to the parabola at these points is

$$x^2+y^2-ax[(t_1+t_2)^2+2]-ay(t_1+t_2)(1-t_1t_2)+a^2t_1t_2(2-t_1t_2)=0.$$

28. TP and TQ are tangents to the parabola and the normals at P and Q meet at a point R on the curve; prove that the centre of the circle circumscribing the triangle TPQ lies on the parabola

$$2y^2=a(x-a).$$

29. Through the vertex A of the parabola $y^2=4ax$ two chords AP and AQ are drawn, and the circles on AP and AQ as diameters intersect in R. Prove that, if θ_1, θ_2, and ϕ be the angles made with the axis by the tangents at P and Q and by AR, then

$$\cot\theta_1+\cot\theta_2+2\tan\phi=0.$$

30. A parabola is drawn such that each vertex of a given triangle is the pole of the opposite side; shew that the focus of the parabola lies on the nine-point circle of the triangle, and that the orthocentre of the triangle formed by joining the middle points of the sides lies on the directrix.

ANSWERS

25. Take the general equation to the circle and introduce the condition that the point $(at^2, 2at)$ lies on it; the sum of the roots of the resulting equation in t is then found to be zero.

28. It can be shewn that the normals at the points "t_1" and "t_2" meet on the parabola when $t_1t_2=2$; then use the previous example.

SOLUTIONS/HINTS

1. See Art. 225. From a figure it is clear that the perpendicular from the vertex on the focal chord $= a \sin \omega$.

2. Let PM be perpendicular to the directrix.

Then $SM^2 = SP^2$, *i.e.* $(2a)^2 + (2at)^2 = a^2(t^2+1)^2$,

i.e. $t^4 - 2t^2 - 3 = 0$. $\therefore t^2 = 3$. $\therefore SP = a(t^2+1) = 4a$.

3. $SP = a(1+t^2)$, $SP' = a\left(1+\frac{1}{t^2}\right)$. [Art. 233, Cor.]

$$\therefore \frac{1}{SP} + \frac{1}{SP'} = \frac{1}{a}\left\{\frac{1}{1+t^2} + \frac{t^2}{1+t^2}\right\} = \frac{2}{2a}.$$

4. Removing the origin to the focus, the tangent $y = x\tan\theta + a\cot\theta$ becomes $y\cos\theta - x\sin\theta = a\operatorname{cosec}\theta$;

and since $P\hat{S}X = 2\theta$ (Fig. p. 188), the equation to TL is $x\cos 2\theta + y\sin 2\theta = p$, where $SL = p$. Whence, solving,

$$x = p - 2a. \quad \therefore TN = p = SL.$$

5. If P and Q are the points t_1, t_2 then, by Art. 229, K is $(-at_1t_2, 0)$.

$$\therefore \frac{1}{PK^2} + \frac{1}{QK^2} = \frac{1}{a^2t_1^2(t_1+t_2)^2 + 4a^2t_1^2} + \frac{1}{a^2t_2^2(t_1+t_2)^2 + 4a^2t_2^2}$$

$$= \frac{1}{a^2}\cdot\frac{t_1^2+t_2^2}{t_1^2t_2^2}\cdot\frac{1}{t_1^2+t_2^2+2t_1t_2+4},$$

and this by the question $= \frac{1}{AK^2}$ (by taking P at A and hence Q at infinity)

$$= \frac{1}{a^2 t_1^2 t_2^2}; \text{ hence } t_1 t_2 = -2.$$

$\therefore$ the point $(2a, 0)$ has the given property.

6. The chord $y(t_1 + t_2) = 2x + 2at_1 t_2$ and the normal

$$y + t_1 x = 2at_1 + at_1^3$$

are coincident if $t_1 + t_2 = -\frac{2}{t_1}$. $\quad \therefore t_2 = -t_1 - \frac{2}{t_1}$.

7. In the result of Ex. XXVII. 2,

put $\qquad x_1 = at_1 t_2$ and $y_1 = a(t_1 + t_2)$.

$$\therefore \Delta = \frac{a^2}{2}(t_1 - t_2)^3. \quad \text{But } t_2 = -t_1 - \frac{2}{t_1} \quad \text{(Ex. 6)}.$$

$$\therefore t_1 - t_2 = 2t_1 + \frac{2}{t_1} = 2(\tan\theta + \cot\theta) = 2\sec\theta \,.\, \operatorname{cosec}\theta.$$

$$\therefore \Delta = 4a^2 \sec^3\theta \operatorname{cosec}^3\theta.$$

8. If (x, y) be the centroid,

$3x = a(1 + t_1^2 + t_2^2) \ldots \text{(i)}, \; 3y = 2a(t_1 + t_2) \ldots \text{(ii)}$ [p. 13, Ex. 2];

and $\qquad t_1 + t_2 = -\frac{2}{t_1} \ldots\ldots\ldots\ldots \text{(iii)} \qquad$ (Ex. 6).

From (ii) and (iii), $t_1 = -\frac{4a}{3y}$; $\therefore t_2 = \frac{4a}{3y} + \frac{3y}{2a}$.

Substitute in (i); $\therefore 3x = a\left(5 + \frac{32a^2}{9y^2} + \frac{9y^2}{4a^2}\right)$.

$$\therefore 36ay^2(3x - 5a) = 81y^4 + 128a^4.$$

9. In the figure on page 188, $S\hat{P}G = S\hat{G}P$.

$$\therefore \tan S\hat{G}P = \frac{2at}{2a} = t. \quad \therefore \cos S\hat{P}G = \frac{1}{\sqrt{1+t^2}}.$$

If the circle cut PG in H,

$$PH = SP \,.\, \cos S\hat{P}H = a(1+t^2)\frac{1}{\sqrt{1+t^2}} = a\sqrt{1+t^2}.$$

10. Solving $y + t_1 x = 2at_1 + at_1^3$, and $y + t_2 x = 2at_2 + at_2^3$, we obtain $x = a(2 + t_1^2 + t_2^2 + t_1 t_2)$, $y = -at_1 t_2 (t_1 + t_2)$, and similar expressions for the other points.

$$\Delta = \tfrac{1}{2}a^2 \Sigma \left[t_1 t_2 (t_1 + t_2)\{(2 + t_2^2 + t_3^2 + t_2 t_3) - (2 + t_3^2 + t_1^2 + t_3 t_1)\}\right]$$
$$= \tfrac{1}{2}a^2 (t_1 + t_2 + t_3) \Sigma t_1 t_2 (t_2 + t_1)(t_2 - t_1),$$
$$= \tfrac{1}{2}a^2 (t_1 + t_2 + t_3) \Sigma t_1 t_2 (t_2 - t_1)(t_1 + t_2 + t_3),$$

since $\Sigma t_1 t_2 t_3 (t_2 - t_1) = 0$, $= \dfrac{a^2}{2}(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)(t_1 + t_2 + t_3)^2$.

11. Let t_1, t_2 be the extremities of the chord.

Then $\quad 2at_1 = at_1^2. \quad \therefore\ t_1 = 2.$

Also $\quad t_2 = -t_1 - \dfrac{2}{t_1} = -2 - 1 = -3 \quad$ (Ex. 6).

The m's of the lines SP, SP' are $\dfrac{2t_1}{t_1^2 - 1}$, $\dfrac{2t_2}{t_2^2 - 1}$, and

$$1 + \frac{2t_1}{t_1^2 - 1} \cdot \frac{2t_2}{t_2^2 - 1} = 1 - \frac{4 \cdot 6}{3 \cdot 8} = 0.$$

$\therefore\ P\hat{S}P'$ is a right angle.

12. The chord joining t_1 and t_2 cuts the axis of x where $x = -at_1 t_2$. Hence $t_1 t_2 = 2$.

If the normal at t_1 cuts the curve again at t_3,

then $\quad t_3 = -t_1 - \dfrac{2}{t_1}$ (Ex. 6) $\quad = -\dfrac{2}{t_2} - t_2.$

Hence the normal at t_2 passes through t_3.

13. Let PS cut the curve in P'. Draw the ordinates PN, $P'N'$ and TH perpendicular to the axis.

By Ex. 6 $\quad t_1 t_2 + t_1^2 = -2.$

$\therefore\ AN + AH = at_1^2 + at_1 t_2 = -2a = -2AX.$

$\therefore\ X$ is the middle point of NH, *i.e.* PT is bisected by the directrix.

Also $$TH = a(t_1 + t_2) = -\frac{2a}{t_1} = P'N'. \quad \text{[Art. 233.]}$$

14. The lines joining the origin to the common points of $y(t_1 + t_2) = 2x + 2at_1t_2$ and $y^2 = 4ax$ are

$$y^2 \cdot 2at_1t_2 = 4ax\{y \cdot (t_1 + t_2) - 2x\}.$$

They are at right angles if $t_1t_2 + 4 = 0$.

The coordinates of the further corner of the rectangle are then $\quad x = a(t_1^2 + t_2^2)$, and $y = 2a(t_1 + t_2)$.

$$y^2 = 4a^2(t_1^2 + t_2^2 + 2t_1t_2) = 4a^2\left(\frac{x}{a} - 8\right) = 4a(x - 8a).$$

15. Let $y(t_1 + t_2) = 2x + 2at_1t_2$ be the equation of the chord. Its middle point is given by

$$2x' = a(t_1^2 + t_2^2), \text{ and } y' = a(t_1 + t_2).$$

Whence $$4ax' - y'^2 = a^2(t_1 - t_2)^2.$$

If the chord is inclined at an angle ϕ to the given line,

$$\tan\phi = \frac{\tan\alpha - \dfrac{2}{t_1 + t_2}}{1 + \tan\alpha \cdot \dfrac{2}{t_1 + t_2}} = \frac{y'\sin\alpha - 2a\cos\alpha}{y'\cos\alpha + 2a\sin\alpha}$$

$$\therefore \sec^2\phi = 1 + \tan^2\phi = \frac{y'^2 + 4a^2}{(y'\cos\alpha + 2a\sin\alpha)^2}.$$

Again, $$c^2\sec^2\phi = a^2\{(t_1^2 - t_2^2)^2 + 4(t_1 - t_2)^2\}$$

$$= a^2(t_1 - t_2)^2\{(t_1 + t_2)^2 + 4\} = (4ax' - y'^2)\left(\frac{y'^2}{a^2} + 4\right).$$

Eliminating ϕ, the locus of (x', y') is

$$(y\cos\alpha + 2a\sin\alpha)^2(y^2 - 4ax) + a^2c^2 = 0.$$

16. The chord joining t_1, t_2 will subtend a right angle at (h, k) if $(k - 2at_1)(k - 2at_2) + (h - at_1^2)(h - at_2^2) = 0$,

i.e. if $$k^2 - 2ak(t_1 + t_2) + 4a^2t_1t_2 + h^2 - a(t_1^2 + t_2^2)h + a^2t_1^2t_2^2 = 0. \quad \ldots(1)$$

The coordinates of the pole are given by

$$x = at_1t_2, \text{ and } y = a(t_1 + t_2),$$

whence $$y^2 - 2ax = a^2(t_1^2 + t_2^2).$$

Substituting in (1), we have

$$k^2 - 2ky + 4ax + h^2 - h\frac{(y^2 - 2ax)}{a} + x^2 = 0,$$

or $$ax^2 - hy^2 + (4a^2 + 2ah)x - 2aky + a(h^2 + k^2) = 0.$$

17. The intersection of $x + a = 0$ with $ty = x + at^2$ is

$$\left(-a, \frac{at^2 - a}{t}\right).$$

If (x, y) be the middle point of the tangent

$$2x = at^2 - a, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and $$2y = \frac{at^2 - a}{t} + 2at = 3at - \frac{a}{t}.$$

$$\therefore\ 2ty = 3at^2 - a;\ \text{also}\ 6x = 3at^2 - 3a.$$

$$\therefore\ ty = 3x + a.\ \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Eliminate t between (1) and (2);

$$\therefore\ (2x + a)y^2 = a(3x + a)^2.$$

18. Solving for the normals at t_1, t_2 we obtain

$$x = a(t_1^2 + t_2^2 + t_1t_2 + 2),\ y = -at_1t_2(t_1 + t_2).$$

The line $x - t_3y = c$ (which is perpendicular to the normal at t_3) will pass through this point if

$$c = a\{t_1^2 + t_2^2 + t_1t_2 + 2 + t_1t_2t_3(t_1 + t_2)\}$$
$$= \{(t_1 + t_2)(t_1 + t_2 + t_3 + t_1t_2t_3) + 2 - \Sigma t_1t_2\}.$$

One perpendicular of the normal triangle is

$$x - t_3y = a\{(t_1 + t_2)(t_1 + t_2 + t_3 + t_1t_2t_3) + 2 - \Sigma t_1t_2\}.$$

Similarly, another is

$$x - t_1y = a\{(t_2 + t_3)(t_1 + t_2 + t_3 + t_1t_2t_3) + 2 - \Sigma t_1t_2\}.$$

Subtracting, $y = a(t_1 + t_2 + t_3 + t_1t_2t_3)$

= length of ordinate of orthocentre of tangent triangle. [Art. 234.]

19. Let $(at_1^2, 2at_1)$; $(at_2^2, 2at_2)$; $\{at_1t_2, a(t_1 + t_2)\}$ be the coordinates of P, Q, T respectively, and

$x - ty + at^2 = 0$ the equation of any tangent.

$\therefore$ (perpendicular from $T)^2 \times (1+t^2) = a^2\{t_1t_2 - t(t_1+t_2)+t^2\}^2,$

$$= a^2(t-t_1)^2(t-t_2)^2.$$

Also (perp. from P) (perp. from Q) $\times (1+t^2)$

$$= a^2(t_1^2 - 2tt_1 + t^2)(t_2^2 - 2tt_2 + t^2) = a^2(t-t_1)^2(t-t_2)^2.$$

Hence the required result.

20. Let r_1 be inclined at θ to the axis of x, and let $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ be the coordinates of the ends of the radii vectores.

Then $\tan\theta = \dfrac{2}{t_1}$; $\therefore\ t_1 = 2\cot\theta$. Hence $t_2 = -2\tan\theta$.

$$r_1^2 = a^2(t_1^4 + 4t_1^2) = 16a^2\cot^2\theta(1+\cot^2\theta)$$

$$= 16a^2\left(\frac{1+\tan^2\theta}{\tan^4\theta}\right).$$

$$r_2^2 = a^2(t_2^4 + 4t_2^2) = 16a^2\tan^2\theta(1+\tan^2\theta).$$

$$\therefore\ r_2^2/r_1^2 = \tan^6\theta,\ \tan^2\theta = r_2^{\frac{2}{3}}/r_1^{\frac{2}{3}}.$$

$$\therefore\ r_2^2 = 16a^2\frac{r_2^{\frac{2}{3}}}{r_1^{\frac{2}{3}}}\left(1+\frac{r_2^{\frac{2}{3}}}{r_1^{\frac{2}{3}}}\right). \quad \therefore\ r_1^{\frac{4}{3}}r_2^{\frac{4}{3}} = 16a^2(r_1^{\frac{2}{3}} + r_2^{\frac{2}{3}}).$$

21. Let D, E, F be the points t_1, t_2, t_3.

The point b is the intersection of chord DE and the diameter through A, that is

$$y(t_1+t_2) - 2ax - 2at_1t_2 = 0, \text{ and } y - a(t_2+t_3) = 0.$$

Let $y(t_1+t_2) - 2x - 2at_1t_2 - \lambda\{y - a(t_2+t_3)\} = 0$ be the equation of Bb.

Since it passes through B, whose coordinates are

$$[at_1t_3,\ a(t_1+t_3)],$$

$$\therefore\ (t_1+t_3)(t_1+t_2) - 2t_1t_3 - 2t_1t_2 - \lambda(t_1+t_3-t_2-t_3) = 0.$$

$\therefore\ \lambda = t_1 - t_3$. Hence the "$m$" of $Bb = \dfrac{2}{t_2+t_3}$.

$\therefore\ Bb$ is parallel to FE. Similarly Cc is parallel to FE.

22. (i) The equation of the circle on a focal chord as diameter is $(x-at^2)\left(x-\frac{a}{t^2}\right)+(y-2at)\left(y+\frac{2a}{t}\right)=0.$

[Arts. 145 and 233.]

This meets $x+a=0$ where $\left\{y-a\left(t-\frac{1}{t}\right)\right\}^2=0.$

Hence the directrix touches the circle.

(ii) The equation of the circle on a focal radius as diameter is $(x-a)(x-at^2)+y(y-2at)=0.$ [Art. 145.]

Putting $x=0$ we have $(y-at)^2=0.$

Hence the tangent at the vertex touches the circle.

23. By last Ex., the equation of the circle is

$$x^2+y^2-ax\left(t^2+\frac{1}{t^2}\right)-2ay\left(t-\frac{1}{t}\right)-3a^2=0.$$

Also $\quad m=\tan 2P\hat{T}S=-\frac{2t}{1-t^2}. \quad \therefore \frac{2}{m}=t-\frac{1}{t}.$

$\therefore \frac{4}{m^2}+2=t^2+\frac{1}{t^2}.$ On substituting, we obtain the required result.

24. Let L, L', M, M' be the points t_1, t_2, t_3, t_4. In the equations of the circles on LL' and MM' as diameters the constant terms will be

$a^2t_1^2t_2^2+4a^2t_1t_2$ and $a^2t_3^2t_4^2+4a^2t_3t_4.$ [Art. 145.]

But, since the chords intersect on the axis,

$\therefore t_1t_2=t_3t_4.$ [Arts. 229 and 233.]

$\therefore$ the constant terms are equal.

Hence in the equation of the radical axis, there will be no constant term : $\therefore$ it passes through the origin.

25. On solving $y^2=4ax$, and $x^2+y^2+2gx+2fy+c=0$, we obtain a quartic for y which has no term in y^3.

$$\therefore\ y_1+y_2+y_3+y_4=0. \quad [\text{Art. } 2.]$$

Hence $\frac{4a}{y_1+y_2}=\frac{-4a}{y_3+y_4}$, *i.e.* the line joining one pair of points is equally inclined to the line joining the other pair. [See Art. 205.]

26. The centre is the intersection of the tangent at "t," viz. $\quad ty-x=at^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$

and the line bisecting AP at right angles, viz.

$$t\left(x-\frac{at^2}{2}\right)+2(y-at)=0,$$

or

$$tx+2y=\frac{at^3}{2}+2at. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (1), $\quad \frac{t^2y}{2}-\frac{tx}{2}=\frac{at^3}{2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$

Hence from (2) and (3), $t^2y+t(4a-3x)-4y=0$.

Also (1) is $\quad t^2a-ty+x=0$.

Whence $\frac{t^2}{4y^2+x(3x-4a)}=\frac{t}{y(x+4a)}=\frac{1}{y^2-x(3x-4a)}$,

and the required result follows by eliminating t.

27. Let $x^2+y^2+2gx+2fy+c=0$ be the equation of the circle through the three points.

Then

$$a^2t_1^2t_2^2+a^2(t_1+t_2)^2+2agt_1t_2+2af(t_1+t_2)+c=0, \ldots(\text{i})$$

$$a^2t_1^4+4a^2t_1^2+2agt_1^2+4aft_1+c=0, \ \ldots\ldots\ldots(\text{ii})$$

and

$$a^2t_2^4+4a^2t_2^2+2agt_2^2+4aft_2+c=0. \ldots\ldots\ldots(\text{iii})$$

From the sum of (ii) and (iii) subtract twice equation (i);

$$\therefore\ a(t_1+t_2)^2+2a+2g=0.$$

Subtracting (iii) from (ii),

$$a(t_1+t_2)(t_1^2+t_2^2)+4a(t_1+t_2)+2g(t_1+t_2)=-4f.$$

Substitute for g; $\therefore\ a(t_1+t_2)(1-t_1t_2)=-2f.$

Substitute for g and f in (ii), and we have

$$a^2t_1^4+4a^2t_1^2-a^2t_1^2(t_1^2+t_2^2+2t_1t_2)$$
$$-2a^2t_1^2-2a^2t_1(1-t_1t_2)(t_1+t_2)+c=0,$$

whence $c=a^2t_1t_2(2-t_1t_2)$. Hence, etc.

28. Let P and Q be the points t_1 and t_2.

Since the normals at P and Q meet at $R\,(t_3)$,

$$\therefore\ t_1+\frac{2}{t_1}=t_2+\frac{2}{t_2}=(-t_3),\ [\text{Ex. }6].\quad \therefore\ t_1t_2=2.$$

Also, from Ex. 27, if (x, y) be the coordinates of the centre,

$$2x=a\{(t_1+t_2)^2+2\},\text{ and }2y=a(t_1+t_2)(1-t_1t_2)=-a(t_1+t_2).$$

$$\therefore\ 2x=a\left\{\frac{4y^2}{a^2}+2\right\}.\quad \therefore\ 2y^2=a(x-a).$$

29. Let P and Q be the points t_1 and t_2. The equations of circles on AP, AQ as diameters are

$$x(x-at_1^2)+y(y-2at_1)=0,\text{ and }x(x-at_2^2)+y(y-2at_2)=0.$$

Whence the equation of AR (their common chord) is

$$x(t_1+t_2)+2y=0.$$

$$\therefore\ 2\tan\phi+t_1+t_2=0,\text{ that is, }2\tan\phi+\cot\theta_1+\cot\theta_2=0.$$

30. From Arts. 232 and 234 we have only to shew that the lines joining the middle points of the sides of the triangle are tangents to the parabola. Let O be one vertex of the triangle and OP, OQ the tangents from it;

then we are given that P and Q lie on the opposite sides of the triangle, so that the line joining the middle points of OP, OQ bisects the sides of the given triangle which pass through O. If P be $(at_1^2, 2at_1)$ and Q be $(at_2^2, 2at_2)$, then O is $[at_1t_2, a(t_1+t_2)]$.

$\therefore$ the middle points of OP, OQ are

$$\left[\frac{at_1(t_1+t_2)}{2}, \frac{a(3t_1+t_2)}{2}\right] \text{ and } \left[\frac{at_2(t_1+t_2)}{2}, \frac{a(t_1+3t_2)}{2}\right].$$

The equation to the straight line joining them is easily seen to be $y = \frac{2}{t_1+t_2}x + a.\frac{t_1+t_2}{2}$, which clearly touches the parabola.