08

In earlier classes, we have already learnt how to find the squares and cubes of expression of the form $a + b$ and $a - b$. But for higher powers like $(a+b)^8, (a-b)^{10}$, etc. the calculation become difficult. This difficulty can be overcome by using binomial theorem. It gives an easier way to expand $(a+b)^n$, where n is a positive integer.

BINOMIAL THEOREM

CHAPTER CHECKLIST

- Binomial Theorem for Positive Integral Index
- Applications of Binomial Theorem

|TOPIC 1|

Binomial Theorem for Positive Integral Index

BINOMIAL EXPRESSION

An algebraic expression consisting of two terms with +ve or −ve sign between them, is called **binomial expression**.

e.g. $(a+2b)$, $\left(\frac{p}{x^2} - \frac{q}{x^4}\right)$, $\left(3x - \frac{2}{y}\right)$

PASCAL'S TRIANGLE

In earlier classes, *we have already studied that*

(i) $(a+b)^0 = 1$
(ii) $(a+b)^1 = a+b$
(iii) $(a+b)^2 = a^2 + 2ab + b^2$
(iv) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
(v) $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

In these expansions, *we observe that*

(i) The total number of terms in the expansion is one more than the index.
(ii) Powers of first quantity 'a' is decreasing by 1, whereas powers of second quantity 'b' increase by 1, in the successive terms.
(iii) In each term of the expansion, the sum of indices of a and b is the same and is equal to the index of $(a+b)$.

Also, the coefficient of variables in the above expansions follow a particular pattern, *which is given below*

Index	Coefficients
0	1
1	1 ∇ 1
2	1 ∇ 2 ∇ 1
3	1 ∇ 3 ∇ 3 ∇ 1
4	1 ∇ 4 ∇ 6 ∇ 4 ∇ 1
5	1 ∇ 5 ∇ 10 ∇ 10 ∇ 5 ∇ 1

It can be seen that the addition of 1's in the row for index 1 gives rise to 2 in the row for index 2. The addition of 1, 2 and 2, 1 in the row for index 2 gives rise to 3 and 3 in the row for index 3 and so on.

Also each row is bounded by 1 on both sides. The structure looks like a triangle with 1 at the top vertex and running down the two slanting sides. This structure or array of numbers, is known as Pascal's triangle.

PASCAL'S TRIANGLE WITH THE HELP OF COMBINATION

To find the expansion of the binomial for any power without writing all the rows of the Pascal's triangle that come before the row of the desired index. We can use a rule which is based on combination.

We know that $^nC_r = \dfrac{n!}{r!\,(n-r)!} (0 \le r \le n)$ and n is non-negative integer. Here $^nC_0 = {^nC_n} = 1$,

Write the Pascal triangle as given below

Index	Coefficients
0	$^0C_0$ (=1)
1	$^1C_0$ (=1) $^1C_1$ (=1)
2	$^2C_0$ (=1) $^2C_1$ (=2) $^2C_2$ (=1)
3	$^3C_0$ (=1) $^3C_1$ (=3) $^3C_2$ (=3) $^3C_3$ (=1)
4	$^4C_0$ (=1) $^4C_1$ (=4) $^4C_2$ (=6) $^4C_3$ (=4) $^4C_4$ (=1)
5	$^5C_0$ (=1) $^5C_1$ (=5) $^5C_2$ (=10) $^5C_3$ (=10) $^5C_4$ (=5) $^5C_5$ (=1)

Clearly, with the help of this pattern, row of the Pascal's triangle for any index can be written without writing the earlier rows.

e.g. For $n = 7$, the row would be

$$^7C_0, {^7C_1}, {^7C_2}, {^7C_3}, {^7C_4}, {^7C_5}, {^7C_6}, {^7C_7}$$

Binomial Theorem for any Positive Integer

Theorem *If a and b are two real numbers, then for any positive integer n, we have*

$$(a+b)^n = {^nC_0}a^nb^0 + {^nC_1}a^{n-1}b^1 + {^nC_2}a^{n-2}b^2 + \dots + {^nC_n}a^0b^n$$

$$\mathbf{(a+b)^n = {^nC_0}a^n + {^nC_1}a^{n-1}b + {^nC_2}a^{n-2}b^2 + \dots + {^nC_n}b^n}$$

$$\Rightarrow \qquad (a+b)^n = \sum_{r=0}^{n} {^nC_r}\, a^{n-r} b^r$$

where, $^nC_0, {^nC_1}, \dots, {^nC_n}$ are called binomial coefficients. These binomial coefficients can also be written as $C_0, C_1, \dots, C_n$.

SOME IMPORTANT OBSERVATIONS

(i) The total number of terms in the expansion is one more than the index, i.e. the total number of terms in $(a+b)^n$ is $n+1$.

(ii) In the successive terms of the expansion, powers of the first quantity a go on decreasing by 1, i.e. it is n in the first term, $(n-1)$ in the second term and so on ending with zero in the last term whereas the powers of the second quantity b increase by 1, i.e. it is zero in first term, 1 in the second term and so on ending with n in the last term.

(iii) In each term of the expansion, the sum of the indices of a and b is the same and is equal to the index of $a+b$ i.e. n.

(iv) The binomial coefficients of terms equidistant from the beginning and end are equal.

Important Formulae

(Related to Binomial Coefficients)

(i) $^nC_r = \dfrac{n!}{r!\,(n-r)!} (0 \le r \le n)$

(ii) $^nC_0 = {^nC_n} = 1,\ {^nC_1} = n$

(iii) $^nC_r = {^nC_{n-r}}$

(iv) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

(v) $^nC_x = {^nC_y} \Leftrightarrow x = y$ or $x + y = n$

(vi) $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}$

(vii) $^nC_0 + {^nC_1} + {^nC_2} + \dots + {^nC_n} = 2^n$

(viii) $^nC_0 + {^nC_2} + {^nC_4} + \dots = {^nC_1} + {^nC_3} + {^nC_5} + \dots = 2^{n-1}$

EXAMPLE |1| Find the number of terms in the expansions of following expressions.

(i) $(x+3y)^2$ (ii) $(1-z)^4$

Sol. (i) Given expression is $(x+3y)^2$. Here, $n=2$

$\therefore$ The number of terms in expansion is $(n+1)$

i.e. $2+1=3$

(ii) Given expression is $(1-z)^4$. Here, $n=4$

$\therefore$ The number of terms in the expansion is $(n+1)$ i.e. $4+1=5$.

EXAMPLE |2| Using binomial theorem, expand $(x^2+2y)^5$.

Sol. By binomial theorem, we have

$$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}b^2 + \ldots + {}^nC_n (b)^n$$

Here, $a = x^2$, $b = 2y$ and $n=5$

$$\therefore (x^2+2y)^5 = {}^5C_0 (x^2)^5 + {}^5C_1 (x^2)^4 (2y) + {}^5C_2 (x^2)^3 (2y)^2 + {}^5C_3 (x^2)^2 (2y)^3 + {}^5C_4 (x^2)(2y)^4 + {}^5C_5 (2y)^5$$

$$= x^{10} + 5x^8 (2y) + 10x^6 (4y^2) + 10x^4 (8y^3) + 5x^2 (16y^4) + 32y^5$$

$$= x^{10} + 10x^8 y + 40x^6 y^2 + 80x^4 y^3 + 80x^2 y^4 + 32y^5$$

EXAMPLE |3| Prove that $\sum_{r=0}^{n} 3^r\, {}^nC_r = 4^n$. **[NCERT]**

Use $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots + {}^nC_n x^n$ and prove the result.

Sol. We have, $\sum_{r=0}^{n} {}^nC_r \times 3^r = {}^nC_0 3^0 + {}^nC_1 3 + {}^nC_2 3^2 + {}^nC_3 3^3 + \ldots + {}^nC_n 3^n$

[on putting $r = 0, 1, 2, \ldots, n$]

$$= {}^nC_0 + {}^nC_1 3 + {}^nC_2 3^2 + {}^nC_3 3^3 + \ldots + {}^nC_n 3^n$$

$$= (1+3)^n \quad [\because (1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n]$$

$= 4^n$ **Hence proved.**

Particular Cases of Binomial Theorem

(i) **Expansion of $(a-b)^n$** Replacing b by $(-b)$ in the expansion of $(a+b)^n$, we get $(a-b)^n = [a+(-b)]^n$ $= {}^nC_0 a^n - {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \ldots + (-1)^n\, {}^nC_n b^n$

(ii) **Expansion of $(1+x)^n$** Replacing a by 1 and b by x in the expansion of $(a+b)^n$, we get

$$(1+x)^n = {}^nC_0 1^n + {}^nC_1 1^{n-1} x + {}^nC_2 1^{n-2} x^2 + \ldots + {}^nC_n x^n$$

$$= 1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$$

$$= 1 + nx + \frac{n(n-1)}{2!} x^2 + \ldots + x^n$$

(iii) **Expansion of $(1-x)^n$** Replacing a by 1 and b by $(-x)$ in the expansion of $(a+b)^n$, we get

$$(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 + \ldots + (-1)^n\, {}^nC_n x^n$$

$$= 1 - nx + \frac{n(n-1)}{2!} x^2 - \ldots + (-1)^n x^n$$

(iv) $(x+a)^n + (x-a)^n = 2[{}^nC_0 x^n + {}^nC_2 x^{n-2} a^2 + \ldots]$

$= 2$ (sum of terms at odd places)

Note

(i) The last term will be ${}^nC_n a^n$ or ${}^nC_{n-1} x a^{n-1}$ according as n is even or odd, respectively.

(ii) If n is odd, then number of terms in $[(x+a)^n + (x-a)^n]$ is $\left(\frac{n+1}{2}\right)$ and if n is even, then number of terms is $\left(\frac{n}{2}+1\right)$.

(v) $(x+a)^n - (x-a)^n = 2[{}^nC_1 x^{n-1} a^1 + {}^nC_3 x^{n-3} a^3 + \ldots]$

$= 2$ (sum of terms at even places)

Note

(i) The last term will be ${}^nC_{n-1} x a^{n-1}$ or ${}^nC_n a^n$ as n is even or odd, respectively.

(ii) If n is odd, then number of terms in $[(x+a)^n - (x-a)^n]$ is $\left(\frac{n+1}{2}\right)$ and if n is even, then number of terms is $\left(\frac{n}{2}\right)$.

EXAMPLE |4| Using binomial theorem, expand the following expressions.

(i) $(2x-3)^6$ (ii) $(1-2x)^5$

Sol. (i) We have, $(2x-3)^6 = {}^6C_0 (2x)^6 - {}^6C_1 (2x)^5 \times 3 + {}^6C_2 (2x)^4 (3)^2 - {}^6C_3 (2x)^3 (3)^3 + {}^6C_4 (2x)^2 (3)^4 - {}^6C_5 (2x) 3^5 + {}^6C_6 3^6$

$$= {}^6C_0\, 64x^6 - {}^6C_1 32x^5 \times 3 + {}^6C_2 16x^4 \times 9 - {}^6C_3 8x^3 \times 27 + {}^6C_2\, 4x^2 \times 81 - {}^6C_1\, 2x \times 243 + {}^6C_0 729$$

$[\because {}^nC_r = {}^nC_{n-r}]$

$$= 64x^6 - 6 \times 32 \times 3 \times x^5 + \frac{6\times 5}{2} \times 16x^4 \times 9 - \frac{6\times5\times4\times 8x^3 \times 27}{6} + \frac{6\times 5}{2} \times 4x^2 \times 81 - 6 \times 2x \times 243 + 729$$

$$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$$

(ii) Consider, $(1-2x)^5 = [1+(-2x)]^5$

$$= {}^5C_0 + {}^5C_1 (-2x) + {}^5C_2 (-2x)^2 + {}^5C_3 (-2x)^3 + {}^5C_4 (-2x)^4 + {}^5C_5 (-2x)^5$$

$$= 1 + 5(-2x) + 10(4x^2) + 10(-8x^3) + 5(16x^4) + 1(-32x^5)$$

$$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$$

which is the required expansion.

EXAMPLE |5| Expand $\left(\frac{2x}{3}-\frac{3}{2x}\right)^4$.

Use $(a-b)^n = {}^nC_0a^n - {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \dots + (-1)^n\, {}^nC_nb^n$

Sol. We have, $\left(\frac{2x}{3}-\frac{3}{2x}\right)^4 = {}^4C_0\left(\frac{2x}{3}\right)^4 - {}^4C_1\left(\frac{2x}{3}\right)^3\left(\frac{3}{2x}\right)$

$$+ {}^4C_2\left(\frac{2x}{3}\right)^2\left(\frac{3}{2x}\right)^2 - {}^4C_3\left(\frac{2x}{3}\right)\left(\frac{3}{2x}\right)^3 + {}^4C_4\left(\frac{3}{2x}\right)^4$$

$$= 1\times\frac{16x^4}{81} - 4\times\frac{8x^3}{27}\left(\frac{3}{2x}\right) + 6\cdot\frac{4x^2}{9}\left(\frac{9}{4x^2}\right) - 4\left(\frac{2x}{3}\right)\left(\frac{27}{8x^3}\right) + 1\times\left(\frac{81}{16x^4}\right)$$

$$\left[\because {}^4C_0 = {}^4C_4 = 1,\ {}^4C_3 = {}^4C_1 = 4 \text{ and } {}^4C_2 = \frac{4!}{2!2!} = \frac{4\times3\times2!}{2\times1\times2!} = 6\right]$$

$$= \frac{16}{81}x^4 - \frac{16}{9}x^2 + 6 - \frac{9}{x^2} + \frac{81}{16x^4}$$

EXAMPLE |6| Find the number of terms in the expansions of the following expressions. **[NCERT Exemplar]**

(i) $(x+a)^{100}+(x-a)^{100}$ (ii) $(1+\sqrt{5}x)^7+(1-\sqrt{5}x)^7$

(iii) $(y+a)^{30}-(y-a)^{30}$

Sol. (i) Given expression is $(x+a)^{100}+(x-a)^{100}$.

Here, $n = 100$, which is even.

$\therefore$ Total number of terms $= \frac{n}{2}+1 = \frac{100}{2}+1 = 51$

(ii) Given expression is $(1+\sqrt{5}x)^7+(1-\sqrt{5}x)^7$.

Here, $n = 7$, which is odd.

$\therefore$ Total number of terms $= \frac{n+1}{2} = \frac{7+1}{2} = \frac{8}{2} = 4$

(iii) Given expression is $(y+a)^{30}-(y-a)^{30}$.

Here $n = 30$, which is even.

$\therefore$ Total number of terms $= \frac{n}{2} = \frac{30}{2} = 15$

EXAMPLE |7| Find $(x+1)^6+(x-1)^6$. Hence, evaluate $(\sqrt{2}+1)^6+(\sqrt{2}-1)^6$. **[NCERT]**

Sol. We have, $(x+1)^6 = {}^6C_0x^6 + {}^6C_1x^5\times1 + {}^6C_2x^4\times(1)^2 + {}^6C_3x^3\times(1)^3 + {}^6C_4x^2\times(1)^4 + {}^6C_5x\times(1)^5 + {}^6C_6(1)^6$

$$\Rightarrow (x+1)^6 = {}^6C_0x^6 + {}^6C_1x^5 + {}^6C_2x^4 + {}^6C_3x^3 + {}^6C_2x^2 + {}^6C_1x + {}^6C_0 \quad [\because {}^nC_r = {}^nC_{n-r}]$$

$$\Rightarrow (x+1)^6 = x^6 + 6x^5 + \frac{6\times5}{2}x^4 + \frac{6\times5\times4}{6}x^3 + \frac{6\times5}{2}x^2 + 6x + 1$$

$$\Rightarrow (x+1)^6 = x^6+6x^5+15x^4+20x^3+15x^2+6x+1 \quad \text{...(i)}$$

Similarly,

$$(x-1)^6 = x^6-6x^5+15x^4-20x^3+15x^2-6x+1 \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$(x+1)^6+(x-1)^6 = 2[x^6+15x^4+15x^2+1]$$

Now, on putting $x = \sqrt{2}$, we get

$$(\sqrt{2}+1)^6+(\sqrt{2}-1)^6 = 2[(\sqrt{2})^6+15(\sqrt{2})^4+15(\sqrt{2})^2+1]$$

$$= 2(2^3+15\times2^2+15\times2+1) = 2(8+15\times4+30+1)$$

$$= 2(8+60+30+1) = 2\times99 = 198$$

EXAMPLE |8| In the expansion of $(x+a)^n$, if the sum of odd terms is denoted by O and the sum of even terms by E. Then, prove that

(i) $O^2-E^2 = (x^2-a^2)^n$ (ii) $4OE = (x+a)^{2n}-(x-a)^{2n}$

Sol. (i) We know that, $(x+a)^n = {}^nC_0\,x^na^0 + {}^nC_1x^{n-1}a^1 + {}^nC_2x^{n-2}a^2 + {}^nC_3x^{n-3}a^3 + \dots + {}^nC_na^n$

Now, sum of odd terms

i.e. $O = {}^nC_0x^n + {}^nC_2x^{n-2}a^2 + \dots$

and sum of even terms

i.e. $E = {}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + \dots$

Thus $(x+a)^n = O+E$ ...(i)

Similarly, $(x-a)^n = O-E$ ...(ii)

Now, on multiplying Eqs. (i) and (ii), we get

$$(O+E)(O-E) = (x+a)^n(x-a)^n$$

$$\Rightarrow O^2-E^2 = (x^2-a^2)^n$$

(ii) Clearly, $4\,OE = (O+E)^2-(O-E)^2$

$$\Rightarrow 4OE = [(x+a)^n]^2-[(x-a)^n]^2$$

[from Eqs. (i) and (ii)]

$$= (x+a)^{2n}-(x-a)^{2n}$$ **Hence proved.**

EXAMPLE |9| Evaluate $(x^2-\sqrt{1-x^2})^4+(x^2+\sqrt{1-x^2})^4$. **[NCERT Exemplar]**

Sol. Let $E = (x^2-\sqrt{1-x^2})^4+(x^2+\sqrt{1-x^2})^4$

Put $\sqrt{1-x^2} = y$, we get

$$E = (x^2-y)^4+(x^2+y)^4$$

$$= 2[{}^4C_0(x^2)^4\,y^0 + {}^4C_2(x^2)^2\,y^2 + {}^4C_4(x^2)^0\,y^4]$$

$$\left[\begin{array}{l}\text{if } n \text{ is even, then } (x-a)^n+(x+a)^n \\ = 2\{{}^nC_0\,x^na^0 + {}^nC_2\,x^{n-2}a^2 + {}^nC_4x^{n-4}\,a^4+\dots\}\end{array}\right]$$

$$= 2[1\times x^8\times1 + 6\times x^4y^2 + 1\times1\times y^4]$$

$$= 2[x^8+6x^4(1-x^2)+(1-x^2)^2] \quad [\text{put } y = \sqrt{1-x^2}]$$

$$= 2(x^8+6x^4-6x^6+1+x^4-2x^2)$$

$$= 2x^8-12x^6+14x^4-4x^2+2$$

EXAMPLE |10| Using binomial theorem, $(x^2-2x+1)^3$ expand.

Sol. We have, $(x^2-2x+1)^3=[(x-1)^2]^3=(x-1)^6$

$$= {}^6C_0(x)^6-{}^6C_1(x)^5(1)^1+{}^6C_2(x)^4(1)^2$$
$$-{}^6C_3(x)^3(1)^3+{}^6C_4(x)^2(1)^4-{}^6C_5(x)^1(1)^5+{}^6C_6(1)^6$$
$$=x^6-6x^5+15x^4-20x^3+15x^2-6x+1$$

Note

To solve such questions we consider any two terms as one term and then expand by using suitable theorem.

EXAMPLE |11| Expand, the following using binomial theorem.

(i) $(1-x+x^2)^4$

(ii) $\left(1+\dfrac{x}{2}-\dfrac{2}{x}\right)^4, x\neq 0$

Sol. (i) We have, $(1-x+x^2)^4=[(1-x)+x^2]^4$

$$= {}^4C_0(1-x)^4+{}^4C_1(1-x)^3(x^2)+{}^4C_2(1-x)^2(x^2)^2+{}^4C_3(1-x)(x^2)^3+{}^4C_4(x^2)^4$$

[on taking $(1-x)$ as a and x^2 as b]

$$=(1-x)^4+4x^2(1-x)^3+6x^4(1-x)^2+4x^6(1-x)+1\cdot x^8$$
$$=(1-4x+6x^2-4x^3+x^4)+4x^2(1-3x+3x^2-x^3)+6x^4(1-2x+x^2)+4(1-x)x^6+x^8$$
$$=1-4x+6x^2-4x^3+x^4+4x^2-12x^3+12x^4-4x^5+6x^4-12x^5+6x^6+4x^6-4x^7+x^8$$
$$=1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$$

(ii) We have, $\left(1+\dfrac{x}{2}-\dfrac{2}{x}\right)^4=\left[\left(1+\dfrac{x}{2}\right)-\dfrac{2}{x}\right]^4$

$$= {}^4C_0\left(1+\frac{x}{2}\right)^4-{}^4C_1\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right)+{}^4C_2\left(1+\frac{x}{2}\right)^2\left(\frac{2}{x}\right)^2-{}^4C_3\left(1+\frac{x}{2}\right)\left(\frac{2}{x}\right)^3+{}^4C_4\left(\frac{2}{x}\right)^4$$
$$=\left(1+\frac{x}{2}\right)^4-4\left(1+\frac{x}{2}\right)^3\frac{2}{x}+\frac{4\times 3}{2}\left(1+\frac{x}{2}\right)^2\frac{4}{x^2}-4\left(1+\frac{x}{2}\right)\times\frac{8}{x^3}+\frac{16}{x^4}$$
$$=\left(1+\frac{x}{2}\right)^4-\frac{8}{x}\left(1+\frac{x}{2}\right)^3+\frac{24}{x^2}\left(1+\frac{x}{2}\right)^2-\frac{32}{x^3}\left(1+\frac{x}{2}\right)+\frac{16}{x^4}$$

Now, on expanding $\left(1+\dfrac{x}{2}\right)^4,\left(1+\dfrac{x}{2}\right)^3,\left(1+\dfrac{x}{2}\right)^2$, we get

$$\left(1+\frac{x}{2}-\frac{2}{x}\right)^4=\left(1+4\cdot\frac{x}{2}+6\cdot\frac{x^2}{4}+4\cdot\frac{x^3}{8}+\frac{x^4}{16}\right)-8\cdot\frac{1}{x}\left(1+3\cdot\frac{x}{2}+3\cdot\frac{x^2}{4}+\frac{x^3}{8}\right)$$
$$+24\cdot\frac{1}{x^2}\left(1+x+\frac{x^2}{4}\right)-32\times\frac{1}{x^3}\left(1+\frac{x}{2}\right)+\frac{16}{x^4}$$
$$=\left(1+2x+\frac{3x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}\right)-\left(\frac{8}{x}+12+6x+x^2\right)+\left(\frac{24}{x^2}+\frac{24}{x}+6\right)-\left(\frac{32}{x^3}+\frac{16}{x^2}\right)+\frac{16}{x^4}$$
$$=\frac{x^4}{16}+\frac{x^3}{2}+x^2\left(\frac{3}{2}-1\right)+x(2-6)+(1-12+6)+(24-8)\frac{1}{x}+(24-16)\frac{1}{x^2}-\frac{32}{x^3}+\frac{16}{x^4}$$
$$=\frac{x^4}{16}+\frac{x^3}{2}+\frac{x^2}{2}-4x-5+\frac{16}{x}+\frac{8}{x^2}-\frac{32}{x^3}+\frac{16}{x^4}$$

TOPIC PRACTICE 1

OBJECTIVE TYPE QUESTIONS

1 The total number of terms in the expansion of $(a+x)^{51}-(a-x)^{51}$ after simplification is

(a) 51 (b) 26
(c) 52 (d) 102

2 The number of terms in the expansion $(1-3x+3x^2-x^3)^9$ is equal to

(a) 9 (b) 27
(c) 10 (d) 28

3 If $x\neq 0$, then $\left(x^2+\dfrac{3}{x}\right)^4$ is equal to

(a) $x^8+12x^5+\dfrac{108}{x}+\dfrac{81}{x^4}$

(b) $x^8+12x^5+54x^3+\dfrac{180}{x}+\dfrac{81}{x^3}$

(c) $x^8+12x^5+54x^2+\dfrac{108}{x}+\dfrac{81}{x^4}$

(d) None of the above

4 Which among the following is /are true?

I. $(1.1)^{10000}>100$
II. $(1.1)^{10000}<100$
III. $(1.1)^{10000}>1000$
IV. $(1.1)^{10000}<1000$

(a) I and IV are true (b) II and IV are true
(c) I and III are true (d) Only IV is true

5 The number of terms in the expansion of $(2x+3y-4z)^n$ is

(a) $n+1$ (b) $n+2$

(c) $\frac{(n+1)(n+2)}{2}$ (d) $(n+1)(n+2)$

VERY SHORT ANSWER Type Questions

6 Find the number of terms in the expansions of the following expressions.

(i) $(1+x^2)^4$ (ii) $(1-x)^5$

(iii) $\left(x+\frac{y}{x}\right)^7$ (iv) $\left(x-\frac{1}{3x}\right)^7$

(v) $(3x+2y)^4$ (vi) $[(a+2b)^2]^4$

(vii) $(4+x^2+4x)^3$ (viii) $(x^2+1-2x)^8$

7 Find the number of terms in the expansions of the following expressions.

(i) $(x+2a)^{10}+(x-2a)^{10}$

(ii) $(1+5\sqrt{2}x)^9+(1-5\sqrt{2}x)^9$

(iii) $(y+b)^{20}-(y-b)^{20}$

(iv) $(z+3a)^{13}-(z-3a)^{13}$

(v) $(3x+2y)^6+(3x-2y)^6$

(vi) $\left(2x+\frac{1}{y}\right)^7+\left(2x-\frac{1}{y}\right)^7$

(vii) $(z+3y)^8-(z-3y)^8$

(viii) $(2a+5b)^9-(2a-5b)^9$

8 Prove that $\sum_{r=0}^{n} 5^r \cdot {}^nC_r = 6^n$.

SHORT ANSWER Type Questions

9 Using binomial theorem, expand the following expressions.

(i) $(2x+3y)^5$ (ii) $(2x-3y)^4$

(iii) $(1-x)^6$ (iv) $(1-3x)^7$

(v) $(x^2+2a)^4$ (vi) $(\sqrt{x}+\sqrt{y})^{10}$

(vii) $\left(x+\frac{1}{x}\right)^5$ (viii) $\left(x^2+\frac{3}{x}\right)^4, x \neq 0.$

(ix) $\left(\frac{x}{3}+\frac{1}{x}\right)^5$ (x) $\left(ax-\frac{b}{x}\right)^6$

(xi) $\left(x-\frac{1}{2x}\right)^5$ (xii) $(1-2x+x^2)^3$

(xiii) $(\sqrt[3]{x}-\sqrt[3]{y})^6$

10 Using binomial theorem, expand the following expressions.

(i) $(2+\sqrt{3})^7+(2-\sqrt{3})^7$

(ii) $(\sqrt{2}+1)^6+(\sqrt{2}-1)^6$

(iii) $(3+\sqrt{2})^5-(3-\sqrt{2})^5$

(iv) $(\sqrt{3}+\sqrt{2})^6-(\sqrt{3}-\sqrt{2})^6$

11 Find the value of

(i) $(\sqrt{y+1}+\sqrt{y-1})^6+(\sqrt{y+1}-\sqrt{y-1})^6$

(ii) $(x+\sqrt{x^2-1})^6+(x-\sqrt{x^2-1})^6$

12 Using binomial theorem, expand $(x+y)^6-(x-y)^6$. Hence, find the value of $(\sqrt{2}+1)^6-(\sqrt{2}-1)^6$.

13 Using binomial theorem, expand the following expansions.

(i) $(1+x+x^2)^3$ (ii) $(1-x+x^2)^3$

(iii) $(x^2+3+2\sqrt{3}x)^3$

HINTS & ANSWERS

1. (b) We have $(a+x)^{51}-(a-x)^{51}$

Here, $n=51$, which is odd

$\therefore$ Total number of terms $=\frac{51+1}{2}=26$

2. (d) We have $(1-3x+3x^2-x^3)^9$

$=((1-x)^3)^9=(1-x)^{27}$

$\therefore$ Total number of terms $=27+1=28$

3. (c) By using binomial theorem, we have

$$\left(x^2+\frac{3}{x}\right)^4={}^4C_0(x^2)^4+{}^4C_1(x^2)^3\left(\frac{3}{x}\right)+{}^4C_2(x^2)^2\left(\frac{3}{x}\right)^2+{}^4C_3(x^2)\left(\frac{3}{x}\right)^3+{}^4C_4\left(\frac{3}{x}\right)^4$$

Ans. $x^8+12x^5+54x^2+\frac{108}{x}+\frac{81}{x^4}$.

4. (c) $(1\cdot1)^{10000}=(1+0\cdot1)^{10000}$

$={}^{10000}C_0\,(1)^{10000}+{}^{10000}C_1\,(1)^{9999}(0\cdot1)+\dots+$ Other terms

$=1+1000+$ Other terms

$=(1001+$ Other terms$)>1000$

$\Rightarrow (1\cdot1)^{10000}>1000$

5. (c) We have,

$$(2x+3y-4z)^n=\{2x+(3y-4z)\}^n$$
$$={}^nC_0(2x)^n(3y-4z)^0+{}^nC_1(2x)^{n-1}(3y-4z)^1+{}^nC_2(2x)^{n-2}(3y-4z)^2+\dots+{}^nC_{n-1}(2x)^1(3y-4z)^{n-1}+{}^nC_n(3y-4z)^n$$

Clearly, the first term in the above expansion gives one term, second term gives two terms, third terms gives three terms and so on.

So, total terms $=1+2+3+\dots+n+(n+1)$

$=\frac{(n+1)(n+2)}{2}$

6. (i) Given $n = 4$

$\therefore$ Number of terms $= (n+1) = 4 + 1 = 5$

(ii) Solve as part (i). **Ans.** 6

(iii) Solve as part (i). **Ans.** 8

(viii) Solve as part (i). **Ans.** 8

(v) Solve as part (i). **Ans.** 5

(vi) Solve as part (i). **Ans.** 9

(vii) $(4 + x^2 + 4x)^3 = \{(x+2)^2\}^3 = (x+2)^6$

Now solve as part (i). **Ans.** 7

(viii) $(x^2 + 1 - 2x)^8 = \{(x-1)^2\}^8 = (x-1)^{16}$

Now solve as part (i). **Ans.** 17

7. (i) $(x + 2a)^{10} + (x - 2a)^{10}$

$n = 10$, which is even

$\therefore$ Total number of terms $\frac{n}{2} + 1 = \frac{10}{2} + 1 = 6$

(ii) $n = 9$, which is odd

$\therefore$ Total number of terms $= \frac{n+1}{2} = \frac{9+1}{2} = 5$

(iii) Solve as part (i). **Ans.** 11

(iv) Solve as part (ii). **Ans.** 7

(v) Solve as part (i). **Ans.** 4

(vi) Solve as part (ii). **Ans.** 4

(vii) Solve as part (i). **Ans.** 5

(viii) Solve as part (ii). **Ans.** 5

8. $\sum_{r=0}^{n} 5^r \cdot {}^nC_r = {}^nC_0\, 5^0 + {}^nC_1\, 5^1 + {}^nC_2\, 5^2 + \ldots + {}^nC_n\, 5^n$

$= (1+5)^n = 6^n$

9. (i) Here, $a = 2x$, $b = 3y$ and $n = 5$

Given, $(2x + 3y)^5$

$= {}^5C_0\,(2x)^5 + {}^5C_1(2x)^4\,(3y)^1 + {}^5C_2\,(2x)^3\,(3y)^2 + {}^5C_3\,(2x)^2 \times (3y)^3 + {}^5C_4 \times (2x)^1 \times (3y)^4 + {}^5C_5 \times (2x)^0 \times (3y)^5$

$= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$

(ii) Here, $a = 2x$, $b = 3y$ and $n = 4$

Given, $(2x - 3y)^4$

$= {}^4C_0\,(2x)^4 - {}^4C_1\,(2x)^3 \times (3y)^1 + {}^4C_2\,(2x)^2 \times (3y)^2 - {}^4C_3\,(2x)^1 \times (3y)^3 + {}^4C_4\,(2x)^0\,(3y)^4$

Ans. $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$

(iii) Solve as part (ii).

Ans. $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$

(iv) Solve as part (ii).

Ans. $1 - 21x + 189\,x^2 - 945\,x^3 + 2835\,x^4 - 5103\,x^5 + 5103x^6 - 2187x^7$

(v) Solve as part (i).

Ans. $x^8 + 8x^6a + 24x^4a^2 + 32x^2a^3 + 16a^4$

(vi) Solve as part (i).

Ans. $x^5 + 10x^{9/2}y^{1/2} + 45x^4y + 120x^{7/2}y^{3/2} + 210x^3y^2 + 252x^{5/2}y^{5/2} + 210x^2y^3 + 120x^{3/2}y^{7/2} + 45xy^4 + 10x^{1/2}y^{9/2} + y^5$

(vii) Solve as part (i).

Ans. $x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$

(viii) Solve as part (i).

Ans. $x^8 + 12x^5 + 54\,x^2 + \frac{108}{x} + \frac{81}{x^4}$

(ix) Solve as part (i).

Ans. $\frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}$

(x) Solve as part (ii).

Ans. $a^6x^6 - 6a^5x^4\,b + 15a^4x^2b^2 - 20a^3b^3 + \frac{15a^2b^4}{x^2} - \frac{6ab^5}{x^4} + \frac{b^6}{x^6}$

(xi) Solve as part (ii) $x^5 - \frac{5x^3}{2} + \frac{5x}{2} - \frac{5}{4x} + \frac{5}{16x^3} - \frac{1}{32x^5}$

(xii) $(1 - 2x + x^2)^3 = \{(1-x)^2\}^3 = (1-x)^6$

Now solve as part (ii).

Ans. $1 + 4x + 6x^2 + 4x^3 + x^4$

(xiii) $(\sqrt[3]{x} - \sqrt[3]{y})^6 = (x^{1/3} - y^{1/3})^6$

Now, solve as part (i).

Ans. $x^2 - 6x^{5/3}y^{1/3} + 15x^{4/3}\,y^{2/3} - 20xy + 15x^{2/3}y^{4/3} - 6x^{1/3}y^{5/3} + y^2$

10. (i) $(2+\sqrt{3})^7 = {}^7C_0(2)^7 \times (\sqrt{3})^0 + {}^7C_1\,(2)^6 \times (\sqrt{3})^1 + {}^7C_2\,(2)^5\,(\sqrt{3})^2 + {}^7C_2\,(2)^4\,(\sqrt{3})^3 + {}^7C_4\,(2)^3\,(\sqrt{3})^4 + {}^7C_5\,(2)^2\,(\sqrt{3})^5 + {}^7C_6\,(2)^1\,(\sqrt{3})^6 + {}^7C_7 \times 2^0 \times (\sqrt{3})^7$ ...(i)

and $(2-\sqrt{3})^7 = {}^7C_0(2)^7\,(\sqrt{3})^0 - {}^7C_1\,(2)^6\,(\sqrt{3})^1 + {}^7C_2\,(2)^5\,(\sqrt{3})^2 - {}^7C_2\,(2)^4\,(\sqrt{3})^3 + {}^7C_4\,(2)^3\,(\sqrt{3})^4 - {}^7C_5\,(2)^2\,(\sqrt{3})^5 + {}^7C_6\,(2)^1\,(\sqrt{3})^6 - {}^7C_7\,2^0(\sqrt{3})^7$ ...(ii)

Now add Eq. (i) and (ii), we get

$(2+\sqrt{3})^7 + (2-\sqrt{3})^7$

$= 2\,[{}^7C_0\,(2)^7\,(\sqrt{3})^0 + {}^7C_2\,(2)^5\,(\sqrt{3})^2 + {}^7C_4\,(2)^3\,(\sqrt{3})^4 + {}^7C_6\,(2)^1\,(\sqrt{3})^6]$

Ans. 10084

(ii) Solve as part (i). **Ans** 198

(iii) $(3+\sqrt{2})^5-(3-\sqrt{2})^5$

$$(3+\sqrt{2})^5 = {}^5C_0\,(3)^5\times(\sqrt{2})^0 + {}^5C_1\,(3)^4\times(\sqrt{2}) + {}^5C_2\,(3)^3\times(\sqrt{2})^2 + {}^5C_3\,(3)^2\times(\sqrt{2})^3. + {}^5C_4\,(3)^1\times(\sqrt{2})^4 + {}^5C_5\,(3)^0\times(\sqrt{2})^5 \quad \ldots\text{(i)}$$

$$(3-\sqrt{2})^5 = {}^5C_0\,(3)^5\times(\sqrt{2})^0 - {}^5C_1\,(3)^4\times(\sqrt{2})^1 + {}^5C_2\,(3)^3\times(\sqrt{2})^2 - {}^5C_3\,(3)^2\times(\sqrt{2})^3 + {}^5C_4\,(3)^1\times(\sqrt{2})^4 - {}^5C_5\,(3)^0\times(\sqrt{2})^5$$

Now subtract Eq. (ii) from (ii) Eq. (i),

$$(3+\sqrt{2})^5-(3-\sqrt{2})^5 = 1178\sqrt{2}$$

(iv) Solve as part (iii). **Ans.** $396\sqrt{6}$

11. (i) Solve as Example 9. **Ans.** $16y\,(4y^2-3)$

(ii) Solve as Example 9. **Ans.** $64x^6-96x^4+36x^2-2$

12. $(x+y)^6-(x-y)^6$

$$= {}^6C_0x^6 + {}^6C_1x^5y + {}^6C_2x^4y^2 + {}^6C_3x^3y^3 + {}^6C_4x^2y^4 + {}^6C_5xy^5 + {}^6C_6y^6 - [{}^6C_0x^6 + {}^6C_1x^5(-y)^1 + {}^6C_2x^4(-y)^2 + {}^6C_3x^3(-y)^3 + {}^6C_4x^2(-y)^4 + {}^6C_5x(-y)^5 + {}^6C_6(-y)^6]$$

$$= 2\,(6x^5y + 20x^3y^3 + 6xy^5)$$

$$= 4xy\,(3x^4 + 10x^2y^2 + 3y^4)$$

On substituting $x=\sqrt{2}$ and $y=1$, we get the answer.

Ans. $140\sqrt{2}$

13. (i) $(1+x+x^2)^3 = [(1+x)+x^2]^3$

$$= {}^3C_0\,(1+x)^3 + {}^3C_1\,(1+x)^2\,(x^2)^1 + {}^3C_2\,(1+x)\,(x^2)^2 + {}^3C_3(x^2)^3$$

$$= x^6+3x^5+6x^4+7x^3+6x^2+3x+1$$

(ii) $(1-x+x^2)^3 = [(1-x)+x^2]^3$

$$= {}^3C_0\,(1-x)^3 - {}^3C_1\,(1-x)^2(x^2)^1 + {}^3C_2\,(1-x)^1\,(x^2)^2 - {}^3C_3\,(x^2)^3$$

$$= -x^6-3x^5+5x^3-3x+1$$

(iii) $(x^2+3+2\sqrt{3}x)^3 = (x+\sqrt{3})^6$

$$= {}^6C_0\,(x)^6\times(\sqrt{3})^0 + {}^6C_1\,x^5\,(\sqrt{3})^1 + {}^6C_2\,x^4\,(\sqrt{3})^2 + {}^6C_3x^3\,(\sqrt{3})^3 + {}^6C_4\,x^2\,(\sqrt{3})^4 + {}^6C_5\,x^1\,(\sqrt{3})^5 + {}^6C_6\,(\sqrt{3})^6$$

$$= x^6+6\sqrt{3}x^5+45x^4+60\sqrt{3}x^3+135x^2+54\sqrt{3}x+27$$

|TOPIC 2|
Applications of Binomial Theorem

Sometimes, the calculations for powers of higher numbers like $(98)^5$, $(101)^6$ etc., become difficult by using repeated multiplication. Also, division of an expression, involving n (a rational number) as index by a number is difficult to calculate by division method. Such problems can be solved easily by using the binomial theorem which *are given below*

PROBLEM BASED ON EXPONENT OF A NUMBER

If exponent of a number is integer and the given number without index is less than or greater than 100 then write it as difference of two numbers or sum of two numbers. Out of these two numbers by to take one number as multiple of 10. Also, if the given number without index is decimal, then write it as the difference/sum of two numbers out of which one number is 1 and then use suitable binomial expansion.

EXAMPLE |1| Compute the value of $(96)^3$.

Sol. Here, given number without index is 96. Since, it is less than 100, so it can be written as $100-4$.

Now, $(96)^3 = (100-4)^3$

$$= {}^3C_0(100)^3 - {}^3C_1\,(100)^2\,4 + {}^3C_2(100)\,4^2 - {}^3C_3 4^3$$

[by binomial theorem]

$$= 1000000 - 3(10000)4 + 3(100)(16) - 64$$

$$= 1000000 - 120000 + 4800 - 64 = 884736$$

Note If given number without index is less than 100, then write it as difference of two numbers and if number is greater than 100, then write it as sum of two numbers.

EXAMPLE |2| Expand $(102)^5$.

Sol. We have, $(102)^5 = (100+2)^5$

$$= {}^5C_0(100)^5 + {}^5C_1(100)^4(2)^1 + {}^5C_2(100)^3(2)^2 + {}^5C_3(100)^2(2)^3 + {}^5C_4(100)(2)^4 + {}^5C_5(2)^5$$

$$= {}^5C_0(10)^{10} + {}^5C_1(10)^8\,2 + {}^5C_2(10)^6\times 4 + {}^5C_2(10)^4\times 8 + {}^5C_1(10)^2\times 16 + ({}^5C_0)\times 32 \quad [\because {}^nC_r = {}^nC_{n-r}]$$

$$= 1\times10^{10} + 5\times10^8\times 2 + \frac{5\times4}{2}\times10^6\times4 + \frac{5\times4}{2}\times10^4\times8 + 5\times100\times16 + 32$$

$$= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32$$

$$= 11040808032$$

EXAMPLE |3| Find an approximation of $(0.99)^5$, using the first three terms of its expansion. **[NCERT]**

Write $0.99 = (1 - 0.01)$. Then, use $(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - {}^nC_3x^3 + {}^nC_4x^4 - \dots + (-1)^n\,{}^nC_nx^n$ to expand $(0.99)^5$.
Note that there will be six terms but here we have to take only first three terms.

Sol. We have, $(0.99)^5 = (1-0.01)^5$

$$= 1 - {}^5C_1 \times (0.01) + {}^5C_2 \times (0.01)^2 - \dots$$
$$= 1 - 0.05 + 10 \times 0.0001 - \dots$$
$$= 1.001 - 0.05 = 0.951$$

EXAMPLE |4| Which number is larger, $(1.1)^{10000}$ or 1000? **[NCERT]**

(i) Firstly, write the number $(1.1)^{10000}$ in the form $(1+0.1)^{10000}$ and then use binomial theorem.
(ii) Secondly, use the inequality.

Sol. We have, $(1.1)^{10000} = [1 + (0.1)]^{10000}$

$$= {}^{10000}C_0 + {}^{10000}C_1(0.1) + {}^{10000}C_2(0.1)^2 + \dots$$

[expanding by binomial theorem]

$$= 1 + 10000(0.1) + \text{Other positive terms}$$
$$= 1 + 1000 + \text{Other positive terms}$$
$$= 1001 + \text{Other positive terms} > 1000$$

Hence, $(1.1)^{10000} > 1000$

PROBLEM BASED ON DIVISION

Firstly, write the term having exponent, as the sum or difference of two terms or numbers in such a way that after expanding, terms will become a multiple of divisor. In the expansion $(1+\alpha)^n = 1 + {}^nC_1\alpha + {}^nC_2\alpha^2 + \dots + {}^nC_n\alpha^n$

We can conclude that

$$(1+\alpha)^n - 1 = {}^nC_1\alpha + {}^nC_2\alpha^2 + \dots + {}^nC_n\alpha^n$$

is divisible by α i.e. it is a multiple of α.

Note While writing the term having exponent as sum or difference of two numbers, number should be a factor of given divisor.

EXAMPLE |5| If 25^{15} is divided by 13, then find the remainder. **[NCERT Exemplar]**

Sol. We have, $25^{15} = (26-1)^{15}$

$$= {}^{15}C_0(26)^{15} - {}^{15}C_1(26)^{14}(1)^1 + {}^{15}C_2(26)^{13}(1)^2 - \dots - {}^{15}C_{15}$$
$$= {}^{15}C_0(26)^{15} - {}^{15}C_1(26)^{14} + \dots - 1 - 12 + 12$$
$$= ({}^{15}C_0 2 \times 13 \times (26)^{14} - {}^{15}C_1 2 \times 13 \times (26)^{13} + \dots\dots - 13) + 12$$
$$= 13({}^{15}C_0 \times 2 \times (26)^{14} - {}^{15}C_1 \times 2 \times (26)^{13} + \dots - 1) + 12$$

It is clear that when right hand side is divided by 13, we get the remainder 12.

EXAMPLE |6| Find the remainder, when 5^{99} is divided by 13.

Sol. We know that, $5^4 = 625 = 13 \times 48 + 1$

$\Rightarrow$ $5^4 = 13\lambda + 1$, where λ is a positive integer.

$$\Rightarrow (5^4)^{24} = (13\lambda + 1)^{24}$$
$$= {}^{24}C_0(13\lambda)^{24} + {}^{24}C_1(13\lambda)^{23} + {}^{24}C_2(13\lambda)^{22} + \dots + {}^{24}C_{23}(13\lambda) + {}^{24}C_{24}$$

[by binomial theorem]

$$\Rightarrow 5^{96} = 13[{}^{24}C_0 13^{23}\lambda^{24} + {}^{24}C_1 13^{23}\lambda^{22} + \dots + {}^{24}C_{23}\lambda] + 1$$

= (a multiple of 13) + 1

On multiplying both sides by 5^3, we get

$$5^{96} \cdot 5^3 = 5^3 \cdot (\text{a multiple of } 13) + 5^3$$
$$\Rightarrow 5^{99} = [(\text{a multiple of } 13) + [13 \times 9 + 8]$$

$[\because 5^3 = 125 = 13 \times 9 + 8]$

Hence, the required remainder is 8.

EXAMPLE |7| Find the remainder when $6^n - 5n$ is divided by 25.

Write $6^n = (1+5)^n$ and expand by binomial theorem and then write in such a way that terms become multiple of 25. The remaining term gives the remainder.

Sol. We have, $6^n - 5n = (1+5)^n - 5n$

$$= {}^nC_0 + {}^nC_1(5) + {}^nC_2(5)^2 + \dots + {}^nC_n5^n - 5n$$
$$= 1 + n5 + \frac{n(n-1)}{2} \times (25) + \dots + 1(5^n) - 5n$$
$$= 1 + 25\left[\frac{n(n-1)}{2} + \dots + 5^{n-2}\right]$$
$$= 25 \times \text{an integer} + 1$$

Hence, $6^n - 5n$ leaves the remainder 1, when divided by 25.

EXAMPLE |8| Using binomial theorem, prove that $2^{3n} - 7n - 1$ is divisible by 49, where $n \in N$.

Sol. We have, $(2^3)^n - 7n - 1 = (8)^n - 7n - 1$

Here, 8 can be written as $1 + 7$. $[\because 7$ is a factor of 49]

$$\therefore \quad 2^{3n} - 7n - 1 = (1+7)^n - 7n - 1$$

By using binomial theorem, we get

$$2^{3n} - 7n - 1 = ({}^nC_0 + {}^nC_17 + {}^nC_27^2 + \dots + {}^nC_n7^n) - 7n - 1$$
$$= \left(1 + \frac{n!}{1!(n-1)!} \cdot 7 + \frac{n!}{2!(n-2)!}7^2 + \dots + 7^n\right) - 7n - 1$$
$$= 1 + 7n + \frac{n(n-1)}{2}7^2 + \dots + 7^n - 7n - 1$$
$$= \frac{n(n-1)}{2}7^2 + \dots + 7^n$$

Take common number outside, we get

$$2^{3n} - 7n - 1 = 7^2\left[\frac{n(n-1)}{2} + \dots + 7^{n-2}\right]$$
$$= 49\left[\frac{n(n-1)}{2} + \dots + 7^{n-2}\right]$$

Here, we see that above series is a multiple of 49.

Therefore, $2^{3n} - 7n - 1$ is divisible by 49. **Hence proved.**

EXAMPLE |9| Using binomial theorem, show that the expression $7^9 + 9^7$ is divisible by 64.

Sol. We have, $7^9 + 9^7 = (1+8)^7 - (1-8)^9$

$$= {}^7C_0 + {}^7C_1(8) + {}^7C_2(8)^2 + {}^7C_3(8)^3 + \dots + {}^7C_7(8)^7]$$
$$- [{}^9C_0 - {}^9C_1(8)^1 + {}^9C_2(8)^2 - \dots - {}^9C_9(8)^9]$$
$$= [1 + 7\times(8)^1 + 21\times(8)^2 + 35(8)^3 \dots + (8)^7]$$
$$-[1 - 9\times(8)^1 + 36(8)^2 - \dots - (8)^9]$$
$$= (7+9)\times(8)^1 + (21-36)\times(8)^2 + \dots$$
$$= 2\times 64 - 15\times 64 + \dots = 64(2 - 15 + \dots)$$

Hence, it is clear that it is divisible by 64. **Hence proved.**

EXAMPLE |10| If a and b are distinct integers, then prove that $(a-b)$ is a factor of $a^n - b^n$, whenever n is a positive integer. **[NCERT]**

Use the expansion of $(x+y)^n$.
i.e. $(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + \dots + {}^nC_ny^n$

Sol. We can write, $a^n = (a - b + b)^n$, then using expansion of $(x+y)^n$

we get $a^n = {}^nC_0(a-b)^n + {}^nC_1(a-b)^{n-1}b$
$$+ {}^nC_2(a-b)^{n-2}b^2 + \dots + {}^nC_nb^n$$
[taking $(a-b)$ as x and b as y]

$$\Rightarrow \quad a^n = (a-b)[{}^nC_0(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b$$
$$+ {}^nC_2(a-b)^{n-3}b^2 + \dots] + b^n$$
$$\therefore \quad a^n - b^n = (a-b)[{}^nC_0(a-b)^{n-1} + \dots]$$

Hence, $(a-b)$ is a factor of $a^n - b^n$. **Hence proved.**

TOPIC PRACTICE 2

OBJECTIVE TYPE QUESTIONS

1 If in the binomial expansion of $(a+b)^n$, the coefficients of 4th and 13th terms are equal to each other, then n equals

(a) 14 (b) 15
(c) 16 (d) 17

2 The value of $(99)^5$ is

(a) 9509900490
(b) 9509900495
(c) 9509900499
(d) None of the above

3 The greatest integer which divides the number $(101)^{100} - 1$, is

(a) 100 (b) 1000
(c) 10000 (d) 100000

4 The remainder left out when $8^{2n} - (62)^{2n+1}$ is divided by 9, is

(a) 0 (b) 2 (c) 7 (d) 8

SHORT ANSWER Type I Questions

5 Using binomial theorem, evaluate each of the following.

(i) $(96)^3$ (ii) $(101)^4$ (iii) $(90)^5$

6 Using binomial theorem, evaluate each of the following expansions.

(i) $(100.3)^4$ (ii) $(9.97)^5$ (iii) $(0.98)^4$

SHORT ANSWER Type II Questions

7 Which is larger $(1.01)^{10000}$ or 100?

8 Using binomial theorem, evaluate each of the following.

(i) $(12)^5 + (8)^5$
(ii) $(53)^7 - (47)^7$
(iii) $(13)^5 + (7)^5$
(iv) $(33)^6 - (27)^6$

9 Using binomial theorem, evaluate each of the following.

(i) $(10.3)^5 + (9.7)^5$
(ii) $(100.5)^4 - (99.5)^4$
(iii) $(10.4)^4 + (9.6)^4$
(iv) $(20.5)^5 - (19.5)^5$

10 Using binomial theorem, evaluate the following.

(i) $(9999)^4$
(ii) $(1001)^5$
(iii) $(105)^3$
(iv) $(98)^5$
(v) $(103)^3$
(vi) $(0.998)^8$

11 Show that $2^{4n} - 15n - 1$ is divisible by 225.

HINTS & ANSWERS

1. (b) The coefficients of the fourth and thirteenth terms in the binomial expansion of $(a+b)^n$ are ${}^nC_3$ and ${}^nC_{12}$, respectively.

It is given that, coefficient of 4th term in $(a+b)^n$ = coefficient of 13th term in $(a+b)^n \Rightarrow {}^nC_3 = {}^nC_{12}$

Ans. 15 $[\because {}^nC_x = {}^nC_y \Rightarrow x = y \text{ or } x + y = n]$

2. (c) We have, $(99)^5 = (100-1)^5$

$$= {}^5C_0\,(100)^5 - {}^5C_1\,(100)^4 + {}^5C_2\,(100)^3 - {}^5C_3\,(100)^2 + {}^5C_4(100) - {}^5C_5(100)^0$$

$$= (100)^5 - 5(100)^4 + 10(100)^3 - 10(100)^2 + 5\,(100) - 1$$

Ans. 9509900499

3. (c) $(1+100)^{100} = 1 + 100\cdot 100 + \dfrac{100\cdot 99}{1\cdot 2}\cdot(100)^2 + ..$

$$\Rightarrow\ (101)^{100} - 1 = 10{,}000\left[1 + \frac{100\cdot 99}{1\cdot 2}(1) + ...\right]$$

Ans. 10000

4. (b) $8^{2n} - (62)^{2n+1} = (1+63)^n - (63-1)^{2n+1}$

$$= (1+63)^n + (1-63)^{2n+1}$$

$$= 2 + 63[{}^nC_1 + {}^nC_2\,(63) +$$

$$..+ (63)^{n-1} - {}^{(2n+1)}C_1 + {}^{(2n+1)}C_2(63) - ... + (-1)(63)^{(2n)}]$$

5. (i) $(96)^3 = (100-4)^3 = {}^3C_0\,(100)^3 - {}^3C_1\,(1000)^2\,(4)^1 + {}^3C_2\,(100)^1\,4^2 + {}^3C_3\,4^3$

$$= 884736$$

(ii) $(101)^4 = (100+1)^4$

$$= {}^4C_0\,(100)^4 + {}^4C_1\,(100)^3\,(1)^1 + {}^4C_2\,(100)^2\,(1)^2 + {}^4C_3\,(100)^1\,(1)^3 + {}^4C_4\,(1)^4$$

$= 104060401$ **Ans.** 104060401

(iii) $(90)^5 = (100-10)^5$

Now, solve as part (i). **Ans.** 5904900000

6. (i) $(100.3)^4 = (100+0.3)^4$

$$= {}^4C_0\,(100)^4 + {}^4C_1\,(100)^3\,(0.3)^1 + {}^4C_2\,(100)^2\,(0.3)^2 + {}^4C_3\,(100)^1\,(0.3)^3 + {}^4C_4\,(0.3)^4$$

$$= 101205410.8$$

(ii) $(9.97)^3 = (10-0.03)^3$

Now solve as part (i). **Ans.** 98508.97

(iii) $(0.98)^4 = (1-0.2)^4$ Now solve as part (i).

Ans. 0.922

7. Solve as Example 4. $(1.01)^{10000}$ is greater than 100.

8. (i) $(12)^5 + (8)^5 = (10+2)^5 + (10-2)^5$

$$(10+2)^5 = {}^5C_0 \times 10^5 + {}^5C_1 \times 10^4 \times 2 + {}^5C_2 \times 10^3 \times 2^2 + ...\quad ...(i)$$

$$(10-2)^5 = {}^{10}C_5 \times 5 - {}^5C_1 \times 10^4 \times 2 + {}^5C_2 \times 10^3 \times 2^2 ...(ii)$$

Now, adding Eq. (i) and Eq. (ii), we get

$$(12)^5 + (8)^5 = 2\,[\,{}^5C_0\,(10)^5 + {}^5C_2\,(10)^3\,(2)^2 + {}^5C_4\,(10)^1\,(2)^4]$$

$$= 281600$$

(ii) $(53)^7 - (47)^7 = (50+3)^7 - (50-3)^7$

$$(50+3)^7 = {}^7C_0\,(50)^7 + {}^7C_1\,(5)^6\,(3)^1 + {}^7C_2\,(50)^5\,(2)^2 \quad ...(i)$$

$$(50-3)^7 = {}^7C_0\,(50)^7 - {}^7C_1\,(50)^6\,(3)^1 + {}^7C_2\,(3)^2 \quad ...(ii)$$

Subtract Eq. (ii) from Eq. (i), we get

$$(50+3)^7 - (50-3)^7$$

$$= 2\,[\,{}^7C_1\,(50)^6\,(3)^1 + {}^7C_3\,(50)^4\,(3)^3 + {}^7C_5\,(50)^2\,(3)^5 + {}^7C_7\,(3)^7]$$

$$= 668088019374$$

(iii) Solve as part (i). **Ans.** 388100

(iv) Solve as part (ii). **Ans.** 904047480

9. (i) $(10.3)^5 + (9.7)^5 = (10+0.3)^5 + (10-0.3)^5$

$$(10+0.3)^5 = {}^5C_0\,10^5 + {}^5C_1\,(10)^4\,(0.3)^1 + {}^5C_2\,(10)^3\,(0.3)^2 + ... \quad ...(i)$$

$$(10-0.3)^5 = {}^5C_0\,10^5 - {}^5C_1\,(10)^4\,(0.3)^1 + {}^5C_2\,(10)^3\,(0.3)^2 \quad ...(ii)$$

On adding Eq. (i) and (ii), we get

$$(10+0.3)^5 + (10-0.3)^5 = 2\,[{}^5C_0\,(10)^5 + {}^5C_2\,(10)^3\,(0.3)^2 + {}^5C_4\,(10)^1\,(0.3)^4]$$

Ans. 201800.81

(ii) Solve as part (i). **Ans.** 4000100

(iii) Solve as part (i). **Ans.** 20192.0512

(iv) Solve as part (i). **Ans.** 801000.0625

10. (i) $(9999)^4 = (10000-1)^4$

Now solve as Q.5 (i). **Ans.** 996005996001

(ii) $(1001)^5 = (1000+1)^5$

Now solve as Q.5 (ii) **Ans.** 1005010010005001

(iii) $(105)^5 = (100+5)^5$

Now solve as Q.5 (ii) **Ans.** 1157625

(iv) $(98)^5 = (100-2)^5$

Now solve as Q.5 (i) **Ans.** 9039207968

(v) $(103)^3 = (100+3)^3$

Now solve as Q.5 (ii) **Ans.** 1092727

(vi) $(0.998)^8 = (1-0.002)^8$

Now solve as Q.6 (iii) **Ans.** 0.9841114

11. Solve as Example 8.

SUMMARY

- An algebraic expression consisting of two terms with +ve or –ve sign between them, is a binomial expression.
- **Bionomial Theorem for any Positive Integer** If a and b are two real numbers, then for any positive integer n, we have $(a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_nb^n = \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r$

  where, ${}^nC_0, {}^nC_1, \ldots\ldots, {}^nC_n$ are **binomial coefficients**.
- The coefficients of the expansions are arranged in an array. This array is called Pascal's triangle.
- **Important Points**

(i) The total number of terms in $(a + b)^n$ is $n + 1$.

(ii) The binomial coefficients of terms equidistant from the beginning and end are equal.

(iii) $(x + a)^n + (x - a)^n = 2[{}^nC_0x^n + {}^nC_2x^{n-2}a^2 + \ldots]$

If n is odd, the number of terms is $\left(\frac{n+1}{2}\right)$ and if n is even, then number of terms is $\left(\frac{n}{2} + 1\right)$.

(iv) $(x + a)^n - (x - a)^n = 2[{}^nC_1x^{n-1}a^1 + {}^nC_3x^{n-3}a^3 + \ldots]$

if n is odd, the number of terms is $\left(\frac{n+1}{2}\right)$ and if n is even, the number of terms is $\left(\frac{n}{2}\right)$.

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

1. In every term, the sum of indices of a and b in the expansion of $(a+b)^n$ is

(a) n (b) $n+1$ (c) $n+2$ (d) $n-1$

2. $(100)^{50}+(99)^{50}$ [NCERT Exemplar]

(a) $<(101)^{50}$ (b) $=101$

(c) $>(101)^{50}$ (d) >101

3. The coefficient of x^n in the expansion of $(1+x)(1-x)^n$ is

(a) $(1-n)$ (b) $(-1)^n(1-n)$

(c) $n-1$ (d) $(-1)^n(n-1)$

4. If the coefficients of 2nd, 3rd and the 4th terms in the expansion of $(1+x)^n$ are in AP then value of n is

(a) 2 (b) 7 (c) 11 (d) 14

5. If 6^n-5n is divided by 25, then remainder is

(a) 0 (b) 2 (c) 4 (d) 1

6. The coefficient of x^{50} after simplifying and collecting the like terms in the expansion of $(1+x)^{1000}+x(1+x)^{999}+x^2(1+x)^{998}+\ldots+x^{1000}$ is [NCERT Exemplar]

(a) $\dfrac{(1001)!}{(50)!(951)!}$ (b) $\dfrac{(101)!}{(50)!(49)!}$

(c) $\dfrac{(1001)!}{(150)!(851)!}$ (d) None of these

7. The last digit in 7^{300} is

(a) 7 (b) 3 (c) 9 (d) 1

SHORT ANSWER Type I Questions

8. Find $(a+b)^4-(a-b)^4$.

Hence, evaluate $(\sqrt{3}+\sqrt{2})^4-(\sqrt{3}-\sqrt{2})^4$. [NCERT]

9. Simplify the following expression

(i) $\left(y+\dfrac{1}{y}\right)^6-\left(y-\dfrac{1}{y}\right)^6$

(ii) $(2+\sqrt{y})^4+(2-\sqrt{y})^4$

SHORT ANSWER Type II Questions

10. If P and Q are the sum of odd and even terms in the expansion $(y+b)^n$, then prove that $(y+b)^{2n}+(y-b)^{2n}=2(P^2+Q^2)$.

11. Find the coefficient of x^5 in the expansion of $(1+x)^3(1-x)^6$.

LONG ANSWER Type Questions

12. The first three terms in the expansion of $(x+y)^n$ are 1, 56 and 1372 respectively. Find the values of x and y.

13. Using binomial theorem, determine which number is larger $(1.2)^{4000}$ or 800? [NCERT Exemplar]

14. In a survey, it was found that $(1.1)^{10000}$ number of people move on cycle (eco-friendly) and 1000 people use cars (need to be eco- friendly). Which is larger of the two? Which value system we should inculcate in public?

15. Show that $2^{4n+4}-15n-16$, where $n\in N$, is divisible by 225. [NCERT Exemplar]

HINTS & ANSWERS

1. (a) In the expansion of $(a+b)^n$, the sum of the indices of a and b is $n+0=n$ in the first term, $(n-1)+1=n$ in the second term and so on.

2. (a) Since, $(101)^{50}=(100+1)^{50}$

$$=100^{50}+{}^{50}C_1100^{49}+{}^{50}C_2100^{48}+\ldots+1 \quad \ldots\text{(i)}$$

and $(99)^{50}=(100-1)^{50}$

$$=100^{50}-{}^{50}C_1100^{49}+{}^{50}C_2100^{48}-\ldots+1 \quad \ldots\text{(ii)}$$

On subtracting Eq. (ii) from Eq. (i) we get $(101)^{50}-(99)^{50}$

$$=2\times{}^{50}C_1100^{49}+(2\times{}^{50}C_3\times100^{47}+\ldots)$$

$$=(100\times100^{49}+\text{a positive number})>100^{50}$$

$\Rightarrow$ $(101)^{50}-(99)^{50}>(100)^{50}$

$\Rightarrow$ $(101)^{50}>(100)^{50}+(99)^{50}$

3. (b) Coefficient of x^n in $(1+x)(1-x)^n$

$=$ Coefficent of x^n in $(1-x)^n$ + Coefficient of x^{n-1} in $(1-x)^n$

$= (-1)^n\ {}^nC_n + (-1)^{n-1}\ {}^nC_{n-1} = (-1)^n \cdot 1 + (-1)^{n-1} \cdot n$

Ans. $(-1)^n (1-n)$

4. (b) **Hint** $2\ {}^nC_2 = {}^nC_1 + {}^nC_3$

5. (d) For two numbers a and b, if we can find numbers q and r such that $a = bq + r$, then we say that b divides a with q as quotient and r as remainder. Thus, in order to show that $6^n - 5n$ leaves remainder 1 when divided by 25, we prove that $6^n - 5n = 25k + 1$, where k is some natural number.

We have,

$$(1+a)^n = {}^nC_0 + {}^nC_1 a + {}^nC_2 a^2 + \ldots + {}^nC_n a^n$$

For $a = 5$, we get

$$(1+5)^n = {}^nC_0 + {}^nC_1 5 + {}^nC_2 5^2 + \ldots + {}^nC_n 5^n$$

i.e., $(6)^n = 1 + 5n + 5^2 \cdot {}^nC_2 + 5^3 \cdot {}^nC_3 + \ldots + 5^n$

or $6^n - 5n = 25k + 1$, where $k = {}^nC_2 + 5 \cdot {}^nC_3 + \ldots + 5^{n-2}$

Ans. 1

6. (a) Since, the given series is a geometric series with the common ratio $\frac{x}{1+x}$, its sum $= \dfrac{(1+x)^{1000}\left[1-\left(\frac{x}{1+x}\right)^{1001}\right]}{\left[1-\left(\frac{x}{1+x}\right)\right]}$

$$= (1+x)^{1001} - x^{1001}$$

Hence, coefficient of x^{50} is given by 100, 50

Ans. $\dfrac{1001!}{50!\,951!}$

7. (d) $7^{300} = (7^2)^{150} = (50-1)^{150}$

$$= {}^{150}C_0 (50)^{150}(-1)^0 + {}^{150}C_1 (50)^{149}(-1)^1 + \ldots + {}^{150}C_{150}(50)^0(-1)^{150}$$

Thus, the last digit of 7^{300} is ${}^{150}C_{150} 1 \cdot 1 = 1$

8. **Hint** $(a+b)^4 - (a-b)^4 = 2[{}^4C_1\ a^3 b^1 + {}^4C_3\ ab^3]$

$$= 2\,[4\,a^3 b + 4ab^3] = 8\,(a^3 b + ab^3)$$

Put $a = \sqrt{3}$, $b = \sqrt{2}$ to get the required answer.

9. (i) $\left(y + \frac{1}{y}\right)^6 - \left(y - \frac{1}{y}\right)^6 = 2\left[{}^6C_1\ y^{6-1}\left(\frac{1}{y}\right)^1 + {}^6C_3\ y^{6-3}\left(\frac{1}{y}\right)^3 + {}^6C_5 y^{6-5}\left(\frac{1}{y}\right)^5\right]$

$$= 2\left[6\,y^5 \cdot \frac{1}{y} + 20y^3 \cdot \frac{1}{y^3} + 6y \cdot \frac{1}{y^5}\right] = 12\left(y^4 + \frac{1}{y^4} + \frac{10}{3}\right)$$

(ii) Use

$(x+a)^n + (x-a)^n = 2\,[{}^nC_0\ x^n + {}^nC_2\ x^{n-2}\ a^2 + \ldots]$

Here, $x = 2$, $a = \sqrt{y}$

10. $(y+b)^n = {}^nC_0\ y^n + {}^nC_1\ y^{n-1}\ b + {}^nC_2 y^{n-2}\ b^2 + \ldots + {}^nC_n\ b^n$

$= P + Q$ ...(i)

$(y-b)^n = {}^nC_0\ y^n - {}^nC_1 y^{n-1}\ b + {}^nC_2 y^{n-2}\ b^2 - \ldots + (-1)^n\ {}^nC_n\ b^n$

$= P - Q$ ...(ii)

Squaring Eqs. (i) and (ii) and then adding, we get

$(y+b)^{2n} + (y-b)^{2n} = (P+Q)^2 + (P-Q)^2$

$= P^2 + Q^2 + 2PQ + P^2 + Q^2 - 2PQ = 2\,(P^2 + Q^2)$

11. $(1+x)^3\,(1-x)^6$

$= (1 + x^3 + 3x + 3x^2)\ (1 + {}^6C_1 x + {}^6C_2\ x^2 - {}^6C_3 x^3 + {}^6C_4\ x^4 - {}^6C_5 x^5 + {}^6C_6\ x^6)$

Coefficient of $x^5 = -{}^6C_5 + {}^6C_2 + 3\,{}^6C_4 - 3\,{}^6C_3$

$= -6 + 15 + 45 - 60 = -6$

12. **Hint** $(x+y)^n = {}^nC_0\ x^n y^0 + {}^nC_1\ x^{n-1}\ y + {}^nC_2\ x^{n-2}\ y^2 + {}^nC_3\ x^{n-3}\ y^3 + \ldots$

Given, ${}^nC_0\ x^n y^0 = 1 \Rightarrow x^n = 1$ ...(i)

${}^nC_1\ x^{n-1} y = 56 \Rightarrow nx^{n-1}\ y = 56$ ...(ii)

${}^nC_2\ x^{n-2}\ y^2 = 1372 \Rightarrow \dfrac{n\,(n-1)}{2} \cdot x^{n-2}\ y^2 = 1372$ ...(iii)

Divide Eq. (i) by Eq. (ii), we get $\dfrac{x^n}{nx^{n-1}y} = \dfrac{1}{56}$

$\Rightarrow \dfrac{x}{ny} = \dfrac{1}{56} \Rightarrow 56\ x = ny$ ...(iv)

Divide Eq. (ii) by Eq. (iii), we get

$\dfrac{nx^{n-1}\ y}{\frac{n\,(n-1)}{2} x^{n-2}\ y^2} = \dfrac{56}{1372} \Rightarrow \dfrac{2x}{(n-1)y} = \dfrac{56}{1372}$

$\Rightarrow 2744\ x = 56\,(n-1)y$ ...(v)

Divide Eq. (i) by Eq. (iii), we get

$\dfrac{x^n}{\frac{n\,(n-1)}{2} x^{n-2}\ y^2} = \dfrac{1}{1372}$

$\Rightarrow \dfrac{2x^2}{n(n-1)y^2} = \dfrac{1}{1372} \Rightarrow 2744x^2 = n\,(n-1)\,y^2$ ...(vi)

Solve Eqs. (iv), (v) and (vi).

13. $(1.2)^{4000} = (1 + 0.2)^{4000}$

$= {}^{4000}C_0 + {}^{4000}C_1\ (0.2) +$ sum of positive terms

$= 1 + 4000\,(0.2) +$ a positive number

$= 1 + 800 +$ positive number

$= 800$

14. Solve as Q. 13.

15. $2^{4n+4} - 15n - 16 = (2)^{4n+1} - 15\,(n+1) - 1$

$= 16^{n+1} - 15\,(n+1) - 1$

$= (1 + 15)^{n+1} - 15\,(n+1) - 1$

$= \{{}^{n+1}C_0 + {}^{n+1}C_1 (15) + {}^{n+1}C_2\ (15)^2 + {}^{n+1}C_3\ (15)^3 + \ldots + {}^{n+1}C_{n+1}\ (15)^{n+1}\} - 15\,(n+1) - 1$

$= 1 + 15\,(n+1) + {}^{n+1}C_2 (15)^2 + {}^{n+1}C_3\ (15)^3 + \ldots + {}^{n+1}C_{n+1}\ (15)^{n+1} - 15\,(n+1) - 1$

$= 225\ \{{}^{n+1}C_2 + {}^{n+1}C_3 (15) + \ldots + {}^{n+1}C_{n+1}\ (15)^{n-1}\}$

$= 225 \times$ a natural number.