Complex Number

Q.1. Find the value of x and y , given that $(x + iy)(2 - 3i) = 4 + i$.

Solution : 1

We have $(x + iy)(2 - 3i) = 4 + i$

Or, $(2x + 3y) + i(2y - 3x) = 4 + i$

Comparing real and imaginary parts we have ,

$2x + 3y = 4$ ---------------- (1)

$-3x + 2y = 1$ --------------- (2)

solving these two equations , we get $x = 5/13$ and $y = 14/13$.

Q.2. If the ratio $(z - i)/(z - 1)$ is purely imaginary , prove that the point z lies on the circle whose centre is the point $1/2\ (1 + i)$ and radius is $1/\sqrt{2}$.

Solution : 2

$(z - i)/(z - 1) = (x + iy - i)/(x + iy - 1)$

$= \{x + i(y - 1)\}/\{(x - 1) + iy\}$

$= [\{x + i(y - 1)\}\{(x - 1) - iy\}]/[\{(x - 1) + iy\}\{(x - 1) - iy\}]$

$= [\{x(x - 1) + y(y - 1)\} + i\{(y - 1)(x - 1) - xy] / [(x - 1)2 + y2]$

As the given ratio is purely imaginary ,

Hence , $[x(x - 1) + y(y - 1)]/[(x - 1)^2 + y^2] = 0$

Or, $x^2 + y^2 - x - y = 0$.

This is the equation of the circle with centre $(1/2 , 1/2)$ i.e. $1/2\ (1 + i)$ and radius is $\sqrt{\{(1/2)^2 + (1/2)^2 - 0\}}$ i.e. $1/\sqrt{2}$.

The centre = $1/2\ (1 + i)$ and radius = $1/\sqrt{2}$.

Q.3. Find the modulus and amplitude of the complex number $(2 + 3i)/(3 + 2i)$.

Solution : 3

We have , $(2 + 3i)/(3 + 2i) = (2 + 3i)(3 - 2i)/(3 + 2i)(3 - 2i)$

$= (12 + 5i)/13$

$= (12/13) + (5/13)i$

Therefore , Modulus = $\sqrt{[(12/13)^2 + (5/13)^2]} = 1$.

And Amplitude = $\tan - 1\ [(5/13)/(12/13)] = \tan - 1\ (5/12)$.

Q.4. Express $(1 - 2i)/(2 + i) + (3 + i)/(2 - i)$ in the form of a + ib.

Solution : 4

$(1 - 2i)/(2 + i) + (3 + i)/(2 - i) = [(1 - 2i)(2 - i) + (3 + i)(2 + i)]/(2 + i)(2 - i)$

$= (2 - 4i - i + 2i^2 + 6 + i^2 + 2i + 3i)/(2^2 - i^2)$

$= (2 - 5i - 2 + 6 - 1 + 5i)/(4 + 1)$

$= 5/5 = 1 = 1 +0i$.

Q.5. Express $13i/(2 - 3\ i)$ in the form of A + iB.

Solution : 5

We have, $13i/(2 - 3i) = [13i\ (2 + 3i)]/[(2 - 3i)(2 + 3i)]$

$= (26i + 39i^2)/(4 - 9i^2)$

$= (26i - 39)/(4 + 9)$

$= (26i - 39)/13$

$= 2i - 3$

$= - 3 + 2i$ in the form of A + iB.

Q.6. Find the cube root of − 27 and show that the sum of the cube roots is equal to zero.

Solution : 6

Let the cube root of − 27 be x .

Therefore, $x = (-27)^{1/3}$

Or, $x^3 = -27 = 27(\cos π + i\sin π)$

Or, $x = 3 (\cos π + i\sin π)^{1/3}$

$= 3[\cos (2kπ + π)/3 + i\sin(2kπ + π)/3]$

[Where k = 0, 1, 2]

At k = 0 , $x = 3\{\cos π/3 + i\sin π/3\}$;

k = 1, $x = 3\{\cos 3π/3 + i\sin 3π/3\}$;

k = 2, $x = 3\{\cos 5π/3 + i\sin 5π/3\}$.

Therefore the roots are : x1 = 3(1/2 +i√3/2); $x_2 = 3 (-1 + 0) = -3$ and $x_3 = 3(1/2 - i√3/2)$.

Their sum = $x_1 + x_2 + x_3 = 3\{1/2 + i√3/2 - 1 + 1/2 - i√3/2\} = 0$.

Q.7. Find the amplitude of the complex number : sin 6π/5 + i(1 − cos 6π/5).

Solution : 7

We have sin 6π/5 + i(1 − cos 6π/5) = 2 sin 3π/5 cos 3π/5 + i(2 sin2 3π/5)

= 2 sin 3π/5 (cos 3π/5 + isin 3π/5)

Comparing with r(cosθ + isinθ) ,

Amplitude = θ = 3π/5 = 108º.

Q.8. If i = √(−1), prove the following :

$(x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i) = x^4 + 4.$

Solution : 8

L.H.S. = $(x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)$

$= [(x + 1)^2 - i^2][(x - 1)^2 - i^2]$

$= [(x + 1)^2 + 1][(x - 1)^2 + 1]$

$= [x^2 + 2 + 2x][x^2 + 2 - 2x]$

$= (x^2 + 2)^2 - (2x)^2$

$= x^4 + 4 + 4x^2 - 4x^2$

$= x^4 + 4$ = R.H.S.

Q.9. If $z = x + iy$ and $|2z + 1| = |z - 2i|$, show that $3(x^2 + y^2) + 4(x + y) = 3$.

Solution : 9

We have, $|2z + 1| = |z - 2i|$

Or, $|2(x + iy) + 1| = |x + iy - 2i|$

Or, $|2x + 1 + 2iy| = |x + i(y - 2)|$

Or, $\sqrt{[(2x + 1)^2 + (2y)^2]} = \sqrt{[x^2 + (y - 2)^2]}$

Or, $4x^2 + 1 + 4x + 4y^2 = x^2 + y^2 + 4 - 4y$

Or, $3x^2 + 3y^2 + 4x + 4y - 3 = 0$

Or, $3(x^2 + y2) + 4(x + y) = 3$.

Q.10. Find the locus of the complex number $z = x + iy$ satisfying the relation $|2z + 3i| \geq |2z + 5|$. Illustrate the locus in the Argand plane.

Solution : 10

We have $|2z + 3i| \geq |2z + 5|$

Substituting $z = x + iy$, we get

$|2x + 2iy + 3i| \geq |2x + 2iy + 5|$

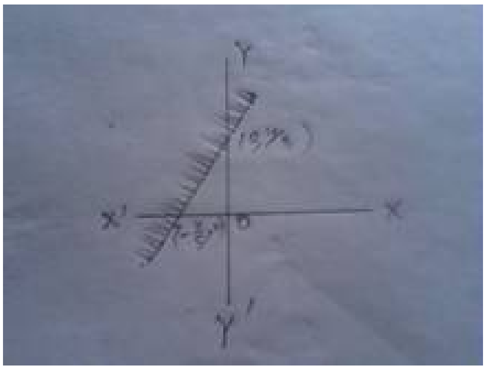
Or, $|2x + i(2y + 3)| \geq |(2x + 5) + i2y|$

Or, $\sqrt{\{(2x)^2 + (2y + 3)2\}} \geq \sqrt{\{(2x + 5)2 + (2y)^2\}}$

Or, $4x^2 + 4y^2 + 12y + 9 \geq 4x^2 + 4y^2 + 20x + 25$

Or, $12y - 20x \geq 16$

Or, $3y - 5x \geq 4$.



Q.11. Find the real values of x and y satisfying the equality : $[(x - 2) + (y - 3)i] /(1 + i) = 1 - 3i$.

Solution : 11

We have $[(x - 2) + (y - 3)i]/(1 + i) = 1 - 3i$.

Or, $(x - 2) + (y - 3)i = (1 + i)(1 - 3i)$

$= 1 + i - 3i - 3i2 = 1 - 2i + 3$

Or, $(x - 2) + (y - 3)i = 4 - 2i$.

Comparing real and imaginary parts, we get

$x - 2 = 4$ and $y - 3 = - 2$

Or, $x = 6$ and $y = 1$.

Q.12. If z = x + iy and [|z − 1 − i|+ 4]/[3|z − 1 − i|−2] = 1 , show that $x^2 + y^2 - 2x - 2y - 7 = 0$.

Solution : 12

We are given that z = x + iy and [|z − 1 − i |+ 4]/[3|z − 1 − i|−2] = 1

Or, |z − 1 − i|+ 4 = 3|z − 1 − i|−2

Or, 6 = 2|z − 1 − i|

Or, |z − 1 − i| = 3

Or, |x + iy − 1 − i|= 3

Or, $\sqrt{(x-1)^2 + (y-1)^2} = 3$

Or, $x^2 - 2x + 1 + y^2 - 2y + 1 = 9$

Or, $x^2 + y^2 - 2x - 2y - 7 = 0$.

Q.13. If (− 2 + √−3)(− 3 + 2√−3) = a + ib, find the real numbers a and b. With these values of a and b find the modulus of a + ib.

Solution : 13

We have,

(− 2 + √3i) (− 3 + 2√3i) = a + ib

Or, 6 − 6 − 7 √3i = a + ib

Or, a = 0 and b = − 7√3.

Q.14. Given z = x + iy and z1 = 1 + 2i, determine the region in the complex plane represented by 1 < |z − z1| ≤ 3. Represent it on the Argand plane.

Solution : 14

Fig. Solution – 15(a)/Page – 423[10 yrs]

We have, $z = x + iy$ and $z^1 = 1 + 2i$,

Then $z - z^1 = (x + iy) - (1 + 2i) = (x - 1) + i(y - 2)$

Therefore, $|z - z^1| = \sqrt{[(x - 1)2 + (y - 2)^2]}$

But $1 < |z - z^1| \leq 3$

Or, $1 < \sqrt{[(x - 1)^2 + (y - 2)2]} \leq 3$

Or, $1 < (x - 1)^2 + (y - 2)^2 \leq 9$

Now, $(x - 1)^2 + (y - 2)^2 > 1$

The points satisfying this inequality will lie outside the circle drawn on Argand plane with centre (1, 2) and radius unity.

And in $(x - 1)^2 + (y - 2)^2 \leq 9$

Points satisfying it will lie outside/on the boundary of circle with centre (1, 2) and radius thrice the unity.

Q.15. Sketch in the complex plane the set of points z satisfying : $| z - 3 |/| z + 1| = 3$.

Solution : 15

Fig. soln 14(a)/page – 453 [10 yrs]

Let $z = x + iy$,

Then, $z - 3 = x + iy - 3 = (x - 3) + iy$ and $z + 1 = x + iy + 1 = (x + 1) + iy$.

Now $| z - 3 | = \sqrt{[(x - 3)2 + y2]}$ and $| z + 1| = \sqrt{[(x + 1)^2 + y^2]}$

Therefore, $|z - 3 |/|z + 1| = \sqrt{[(x - 3)^2 + y2]}/\sqrt{[(x + 1)^2 + y^2]} = 3$

Squaring we get,

$[(x - 3)^2 + y^2]/[(x + 1)^2 + y^2] = 9$

Or, $(x - 3)^2 + y^2 = 9 [(x + 1)^2 + y^2]$

Or, $x^2 + y^2 - 6x + 9 = 9 [x^2 + y^2 + 2x + 1]$

Or, $8x^2 + 8y^2 + 24x = 0$

Or, $x^2 + y^2 + 3x = 0$

Or, $[x^2 + 2. (3/2) . x + 9/4] + y^2 = 9/4$

Or, $(x + 3/2)^2 + y^2 = (3/2)^2$

Which is a circle with centre (– 3/2, 0) and radius 3/2 units.

Q.16. Given that : [2√3(cos 30º) – 2i(sin 30º)]/ [√2(cos 45º + i sin 45º)] = A + i B. Find the value of A and B.

Solution : 16

We have,
[2√3(cos 30º) – 2 i (sin 30º)]/[√2(cos 45º + i sin 45º)] = A + i B.

Or, [2√3 ×√3/2 – 2 i × 1/2]/[√2(1/√2 + i ×1/√2)] = A + i B.

Or, (3 – i)/(1 + i) = A + i B.

Rationalizing the denominator we get,

[(3 – i)(1 – i)]/[(1 + i)(1 – i)] = A + i B.

Or, $(3 + i^2 - 3i - i)/(1 - i^2)$ = A + i B.

Or, [3 + (–1) – 4i]/[1 – (–1)] = A + i B.

Or, (2 – 4 i)/2 = A + i B.

Or, 1 – 2 i = A + i B.

Equating real and imaginary parts we get,

A = 1 and B = –2.

Q.17. If z_1, z_2 ε C, prove that : $| z_1 - z_2| \le | z_1 | + | z_2 |$.

Solution : 17

Let z1 = $x_1 + i\, y_1$ and $z_2 = x_2 + i\, y_2$,

Then, $z_1 - z_2 = (x_1 + i\, y_1) - (x_2 + i\, y_2) = (x1 - x_2) + i\, (y1 - y_2)$

$| z1 – z_2 | = \sqrt{[(x_1 – x_2)2 + (y_1– y_2)^2]} = \sqrt{[x_1^2 + x_2^2– 2x_1x_2 + y_1^2 + y_2^2 – 2y_1y_2]}$

$| z1| = \sqrt{[x_1^2 + y_1^2]}$, $| z_2| = \sqrt{[x_2^2 + y_2^2]}$ and $| z_1 | + | z_2| = \sqrt{[x_1^2 + y_1^2]} + \sqrt{[x_2^2 + y_2^2]}$

Therefore, $[| z_1 | + | z_2 |]2 = \{\sqrt{[x_1^2 + y_1^2]}\}^2 + \{\sqrt{[x_2^2 + y_2^2]}\}^2 + 2\sqrt{[x_1^2 + y_1^2]}.\sqrt{[x_2^2 + y_2^2]}$

$= x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2\sqrt{[x_1^2 x_2^2 + x_1 2y_2^2 + x_2^2 y_1^2 + y_1^2 y_2^2]}$

$= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2\sqrt{[(x_1x^2 + y_1y_2)^2 + (x_1y_2 – x_2y_1)^2]}$

If $x_1y_2 – x_2y_1 = 0 \Rightarrow x_1y_2 = x_2y_1 \Rightarrow x_1/x_2 = y_1/y_2$,

Then, $[| z_1 | + | z_2 |]^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 = 2\sqrt{(x_1x^2 + y_1y_2)^2}$

$= x_1^2 + x_2^2 + y_1^2 + y_2^2 + 2(x_1x_2 + y_1y_2)$

$= (x_1 + x_2)^2 + (y_1 + y_2)^2 = |z_1 + z_2|^2$

Therefore, $| z_1| + | z_2| = | z_1 + z_2 | = \sqrt{[(x_1 + x_2)^2 + (y_1 + y_2)^2]}$

[When $x_1y_2 – x_2y_1 = 0$]

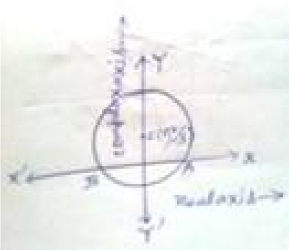
But, $|z_1– z_2| < |z_1 + z_2|$

$|z_1 + z_2| \le | z_1| + | z_2|$

Therefore, $|z_1 – z_2| \le |z_1| + |z_2|$

Q.18. If $z = x + i y$, $\omega = (2 – i z)/(2z – i)$, and $| \omega | = 1$, find the locus and illustrate it in the complex plane.

Solution : 18



Fig

We have, $z = x + i y$;

$\omega = (2 - i z)/(2z - i)$

$= [2 - i (x + i y)]/[2(x + i y) - i]$

$= [(2 + y) - i x]/[2x + i (2y - 1)]$

Now, $| \omega | = 1$; It means

$\sqrt{[(2 + y)^2 + x^2]} = \sqrt{[(2x)^2 + (2y - 1)^2]}$

Or, $4 + y^2 + 4y + x^2 = 4x^2 + 4y^2 + 1 - 4y$

Or, $3x^2 + 3y^2 - 8y - 3 = 0$

Or, $x^2 + y^2 - 8/3\ y - 1 = 0$

Which is a circle, $x^2 + (y - 4/3)^2 = (5/3)^2$

Whose centre is (0, 4/3) and radius = 5/3.

Q.19. Illustrate in the complex plane the set of points z satisfying : $| z + i - 2 | \leq 2$.

Solution : 19

Let $z = x + iy$, then

$| z + i - 2 | \leq 2. \Rightarrow |x + iy + i - 2 | \leq 2.$

Or, $\sqrt{[(x - 2)^2 + (y + 1)^2]} \leq 2$.

Or, $(x - 2)^2 + (y + 1)^2 \leq 4$.

Or, $x^2 + 4 - 4x + y^2 + 1 + 2y \leq 4$

Or, $x^2 + y^2 - 4x + 2y + 1 \leq 0$

Now , $x^2 + y^2 - 4x + 2y + 1 = 0$

Or, $(x - 2)^2 + (y + 1)^2 = 4$, is a circle with centre (2, − 1) and radius = $\sqrt{4} = 2$. **[Ans.]**

The circle cuts x-axis at $y = 0$,

Therefore, $x^2 - 4x + 1 = 0$ & $x = [4 \pm \sqrt{(16 - 4)}]/2$

$= [4 \pm 2\sqrt{3}]/2$

$= 2 \pm \sqrt{3} = 2 \pm 1.732 = 3.732, 0.276$.

It cuts x-axis at x = 0 ,

Therefore, $y^2 + 2y + 1 = 0 \Rightarrow (y + 1)^2 = 0 \Rightarrow y = -1, -1$.

Thus the circle cuts y-axis at (0, – 1).

Q.20. Solve the equation : $2z = | z | + 2i$.

Solution : 20

We have, $2z = | z | + 2i$.

Let $z = x + iy$, then $| z | = \sqrt{[x^2 + y^2]}$

Therefore, $| z | + 2i = \sqrt{[x^2 + y^2]} + 2i$, and $2z = 2x + 2iy$.

Comparing real and imaginary parts on both sides, we get

$2x = \sqrt{[x^2 + y^2]}$,

$2y = 2 \Rightarrow y = 1$.

Hence, $2x = \sqrt{[x^2 + 1]} \Rightarrow 4x^2 = x^2 + 1 \Rightarrow 3x^2 = 1 \Rightarrow x = \pm 1/\sqrt{3}$.

Thus, $z = \pm 1/\sqrt{3} + i$.