COORDINATE GEOMETRY

The coordination of algebra and geometry is called coordinate geometry. Historically, coordinates were introduced to help geometry. And so well did they do this job that the very identity of geometry was changed. The word 'geometry' today generally means coordinate geometry.

In coordinate geometry, all the properties of geometrical figures are studied with the help of algebraic equations. Students should note that the object of coordinate geometry is to use some known facts about a curve in order to obtain its equation and then deduce other properties of the curve from the equation so obtained. For this purpose, we require a coordinate system. There are various types of coordinate systems present in two dimensions e.g., rectangular, oblique, polar, triangular system, etc. Here, we will only discuss rectangular coordinate system in detail.

Cartesian Coordinates

Let $X'OX$ and $Y'OY$ be two fixed straight lines at right angles. $X'OX$ is called axis of x and $Y'OY$ is called axis of y, and O is named as origin. From any point 'P', a line is drawn parallel to OY. The directed line $OM = x$ and $MP = y$. Here, OM is abscissa and MP is ordinate of the point 'P'. The abscissa OM and the ordinate MP together written as (x, y) are called coordinates of point 'P'. Here, (x, y) is an ordered pair of real numbers x and y, which determine the position of point 'P'.

Since $X'OX$ is perpendicular to $Y'OY$, this system of representation is called rectangular (or orthogonal) coordinate system (Fig. 1.1(a)).

When the axes coordinates $X'OX$ and $Y'OY$ are not at right angles, they are said to be oblique axes.

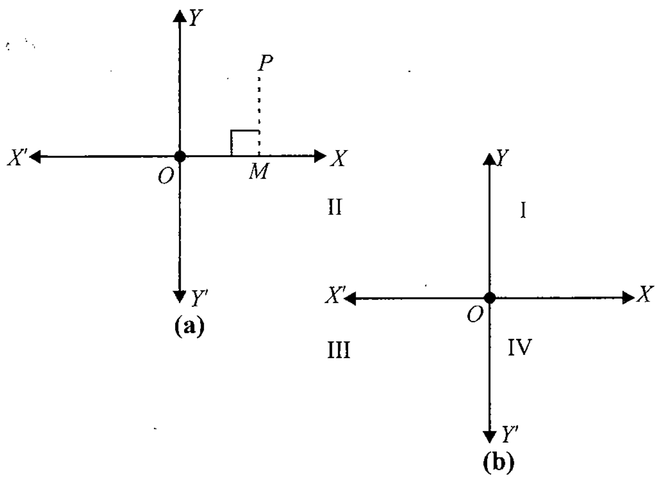


Fig. 1.1

The axes of rectangular coordinate system divide the plane into four infinite regions, called quadrants, each bounded by two half-axes. These are numbered from 1st to 4th and denoted by roman numerals (Fig. 1.1 (b)). The signs of the two coordinates are given below:

	x	y
First quadrant	+	+
Second quadrant	–	+
Third quadrant	–	–
Fourth quadrant	+	–

Lattice point (with respect is coordinate geometry): Lattice point is defined as a point whose abscissa and ordinate are integers.

DISTANCE FORMULA

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

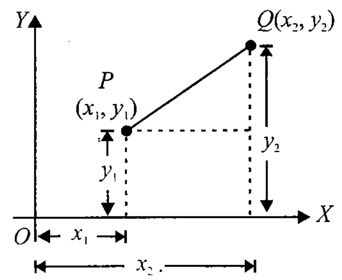


Fig. 1.2

Therefore, distance of (x_1, y_1) from origin $= \sqrt{x_1^2 + y_1^2}$.

Note:

- *If distance between two points is given, then use* $\pm$ *sign.*
- *Distance between* $(x_1, 0)$ *and* $(x_2, 0)$ is $|x_1 - x_2|$.
- *Distance between* $(0, y_1)$ *and* $(0, y_2)$ is $|y_1 - y_2|$.
- *Circumcentre* $P(x, y)$ *is a point which is equidistant from the vertices of triangle* $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$.
  Hence, $AP = BP = CP$, *which gives two equations in* x *and* y, *solving which we get circumcentre.*

Example 1.1 **In ΔABC, prove that $AB^2 + AC^2 = 2(AO^2 + BO^2)$, where O is the middle point of BC.**

Sol.

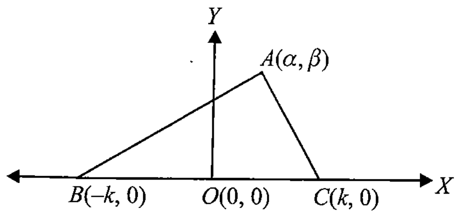


Fig. 1.3

We take O as the origin and OC and OY as the x-and y-axes, respectively.

Let $BC = 2k$, then $B \equiv (-k, 0)$, $C \equiv (k, 0)$.

Let $A \equiv (\alpha, \beta)$

Now L.H.S.

$$= AB^2 + AC^2$$
$$= (\alpha + k)^2 + (\beta - 0)^2 + (\alpha - k)^2 + (\beta - 0)^2$$
$$= \alpha^2 + k^2 + 2\alpha k + \beta^2 + \alpha^2 + k^2 - 2\alpha k + \beta^2$$
$$= 2\alpha^2 + 2\beta^2 + 2k^2$$
$$= 2(\alpha^2 + \beta^2 + k^2)$$

and R.H.S

$$= 2(AO^2 + BO^2)$$
$$= 2[(\alpha - 0)^2 + (\beta - 0)^2 + (-k - 0)^2 + (0 - 0)^2]$$
$$= 2(\alpha^2 + \beta^2 + k^2)$$

$\therefore$ L.H.S. = R.H.S.

Example 1.2 **Find the coordinates of the circumcentre of the triangle whose vertices are $A(5, -1)$, $B(-1, 5)$, and $C(6, 6)$. Find its radius also.**

Sol.

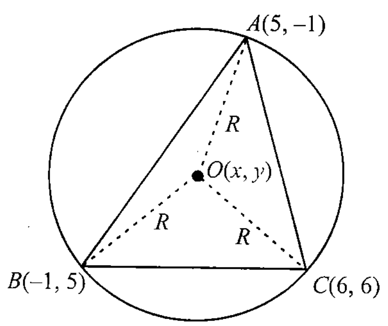


Fig. 1.4

Let circumcentre be $O(x, y)$, then

$$(OA)^2 = (OB)^2 = (OC)^2 = (\text{radius})^2 = R^2 \quad \text{(i)}$$

$$\Rightarrow (x - 5)^2 + (y + 1)^2 = (x + 1)^2 + (y - 5)^2$$
$$= (x - 6)^2 + (y - 6)^2$$

Taking first two relations, we get

$$(x - 5)^2 + (y + 1)^2 = (x + 1)^2 + (y - 5)^2$$

$$\Rightarrow \quad x = y \quad \text{(ii)}$$

Taking last two relations, we get

$$(x + 1)^2 + (y - 5)^2 = (x - 6)^2 + (y - 6)^2$$

$$\Rightarrow (x + 1)^2 + (x - 5)^2 = (x - 6)^2 + (x - 6)^2 \quad \text{[from (ii)]}$$

$$\Rightarrow \quad 2x^2 - 8x + 26 = 2x^2 - 24x + 72$$

$$x = 23/8$$

$$\Rightarrow \quad \text{Circumcentre} = (23/8, 23/8)$$

$$\Rightarrow \quad R^2 = (x - 5)^2 + (y + 1)^2 = (OA)^2$$

$$\Rightarrow \quad = [(23/8) - 5]^2 + [(23/8) + 1]^2$$

$$= (-17)^2/(64) + (31)^2/(64)$$

$$= 1250/64$$

$$\Rightarrow \quad \text{Radius} = 25\sqrt{2}/8 \text{ units}$$

Example 1.3 **Two points $O(0, 0)$ and $A(3, \sqrt{3})$ with another point P form an equilateral triangle. Find the coordinates of P.**

Sol. Let the coordinates of P be (h, k).

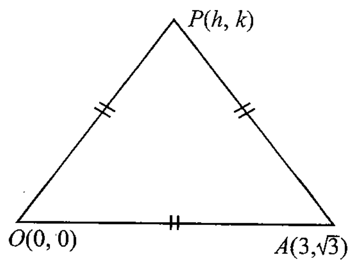


Fig. 1.5

$$\therefore \quad OA = OP = AP \text{ or } OA^2 = OP^2 = AP^2$$

$$\therefore \quad OA^2 = OP^2$$

$$\Rightarrow \quad 12 = h^2 + k^2 \quad \text{(i)}$$

and $\quad OP^2 = AP^2$

$$\Rightarrow \quad h^2 + k^2 = (h - 3)^2 + (k - \sqrt{3})^2$$

$$\Rightarrow \quad 3h + \sqrt{3}k = 6 \text{ or } h = 2 - k/\sqrt{3} \quad \text{(ii)}$$

Using Eq. (ii) in Eq. (i), we get

$$(2 - k/\sqrt{3})^2 + k^2 = 12 \text{ or } k^2 - \sqrt{3}k - 6 = 0$$

or $(k - 2\sqrt{3})(k + \sqrt{3}) = 0$

$$\therefore \quad k = 2\sqrt{3} \text{ or } -\sqrt{3}$$

From Eq. (ii), we get that

when $k = 2\sqrt{3}, h = 0$,

when $k = -\sqrt{3}, h = 3$

Hence, the coordinates of P are

$(0, 2\sqrt{3})$ or $(3, -\sqrt{3})$

Example 1.4 **If O is the origin and if coordinates of any two points Q_1 and Q_2 are (x_1, y_1) and (x_2, y_2), respectively, prove that $OQ_1 \cdot OQ_2 \cos \angle Q_1 OQ_2 = x_1x_2 + y_1y_2$.**

Sol.

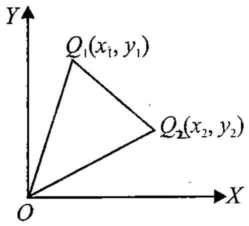


Fig. 1.6

In ΔQ_2OQ_1,

$$Q_1Q_2^2 = OQ_1^2 + OQ_2^2 - 2OQ_1OQ_2 \cos \angle Q_1OQ_2$$

$$\Rightarrow (x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2OQ_1\, OQ_2 \cos\angle Q_1OQ_2$$

(using cosine Rule)

$$\Rightarrow \quad x_1x_2 + y_1y_2 = OQ_1\, OQ_2 \cos Q_1OQ_2$$

Example 1.5 **Let $A = (3, 4)$ and B is a variable point on the lines $|x| = 6$. If $AB \le 4$, then the number of position of B with integral coordinates is**

a. 5 **b.** 4

c. 6 **d.** 10

Sol. $B = (\pm 6, y)$. So, $AB \le 4$

$$\Rightarrow \quad (3 \mp 6)^2 + (y - 4)^2 \le 16$$

$$\therefore \quad 9 + (y - 4)^2 \le 16,$$

$$(\because 81 + (y - 4)^2 \le 16 \text{ is absurd})$$

$$\Rightarrow \quad y^2 - 8y + 9 \le 0$$

$$\Rightarrow \quad 4 - \sqrt{7} \le y \le 4 + \sqrt{7}$$

But y is an integer.

$$\Rightarrow \quad y = 2, 3, 4, 5, 6$$

AREA OF A TRIANGLE

The area of a triangle, whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$$\frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

Proof:

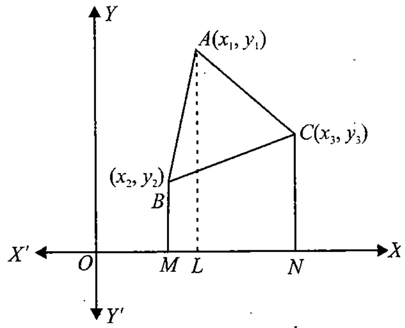


Fig. 1.7

Let ABC be a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$. Draw AL, BM, and CN as perpendiculars from A, B, and C on the x-axis. Clearly, $ABML$, $ALNC$ and $BMNC$ are all trapeziums.

We have,

Area of $\Delta\, ABC$ = Area of trapezium $ABML$ + Area of trapezium $ALNC$ – Area of trapezium $BMNC$

$$\Rightarrow \text{Area of } \Delta ABC = \frac{1}{2}(BM + AL)(ML) + \frac{1}{2}(AL + CN)(LN) - \frac{1}{2}(BM + CN)(MN)$$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right\|$$

Note:

- *Area of a triangle can also be found by easy method, i.e., Stair method.*

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

$$= \frac{1}{2}|\{(x_1y_2 + x_2y_3 + x_3y_1) - (y_1x_2 + y_2x_3 + y_3x_1)\}|$$

- *If three points A, B, and C are collinear, then area of triangle ABC is zero.*
- ***Sign of area:** If the points A, B, C are plotted in two-dimensional plane and taken on the anticlockwise sense, then the area calculated of the triangle ABC will be positive, while if the points are taken in clockwise sense, then the area calculated will be negative. But, if the points are taken arbitrarily, then the area calculated may be positive or negative, the numerical value being the same in both cases. In case, the area calculated is negative, we will consider it as positive.*

Area of Polygon

The area of polygon whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3), …, (x_n, y_n) is

$$= \frac{1}{2}|\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_ny_1 - x_1y_n)\}|$$

Stair Method

Repeat first coordinates one time in last for down arrow use + ve sign and for up arrow use – ve sign.

$$\text{Area of polygon} = \frac{1}{2}\begin{Vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{Vmatrix}$$

$$= \frac{1}{2}|\{x_1y_2 + x_2y_3 + \cdots + x_ny_1) - (y_1x_2 + y_2x_3 + \cdots + y_nx_1)\}|$$

Note:

Points should be taken in cyclic order in coordinate plane.

Example 1.6 **Find the area of a triangle whose vertices are $A(3, 2)$, $B(11, 8)$, and $C(8, 12)$.**

Sol. Let $A = (x_1, y_1) = (3, 2)$, $B(x_2, y_2) = (11, 8)$, and $C = (x_3, y_3) = (8, 12)$.

Then, area of ΔABC

$$= \frac{1}{2}|\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}|$$
$$= \frac{1}{2}|\{3(8 - 12) + 11(12 - 2) + 8(2 - 8)\}|$$
$$= \frac{1}{2}|\{-12 + 110 - 48\}|$$
$$= 25 \text{ sq. units}$$

Example 1.7 **Prove that the area of the triangle whose vertices are $(t, t - 2)$, $(t + 2, t + 2)$, and $(t + 3, t)$ is independent of t.**

Sol. Let $A = (x_1, y_1) = (t, t - 2)$, $B = (x_2, y_2) = (t + 2, t + 2)$, and $C = (x_3, y_3) = (t + 3, t)$ be the vertices of the given triangle.

Then, area of ΔABC

$$= \frac{1}{2}|\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}|$$
$$= \frac{1}{2}|\{t(t + 2 - t) + (t + 2)(t - t + 2) + (t + 3)(t - 2 - t - 2)\}|$$
$$= \frac{1}{2}|\{2t + 2t + 4 - 4t - 12\}| = |-4| = 4 \text{ sq. units.}$$

Clearly, area of ΔABC is independent of t.

Example 1.8 **Find the area of the quadrilateral $ABCD$ whose vertices are respectively $A(1, 1)$, $B(7, -3)$, $C(12, 2)$, and $D(7, 21)$.**

Sol. Here, given points are in cyclic order, then

$$\text{Area} = \frac{1}{2}\begin{Vmatrix} 1 & 1 \\ 7 & -3 \\ 12 & 2 \\ 7 & 21 \\ 1 & 1 \end{Vmatrix}$$

$$= \frac{1}{2}|(-3 - 7) + (14 + 36) + (252 - 14) + (7 - 21)|$$
$$= 132 \text{ sq. units}$$

Example 1.9 **For what value of k are the points $(k, 2 - 2k)$, $(-k + 1, 2k)$ and $(-4 - k, 6 - 2k)$ are collinear?**

Sol. Let three given points be $A = (x_1, y_1) = (k, 2 - 2k)$, $B = (x_2, y_2) = (-k + 1, 2k)$, and $C = (x_3, y_3) = (-4 - k, 6 - 2k)$.

If the given points are collinear, then $\Delta = 0$

$$\Rightarrow x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$
$$\Rightarrow k(2k - 6 + 2k) + (-k + 1)(6 - 2k - 2 + 2k) + (-4 - k)(2 - 2k - 2k) = 0$$
$$\Rightarrow k(4k - 6) - 4(k - 1) + (4 + k)(4k - 2) = 0$$
$$\Rightarrow 4k^2 - 6k - 4k + 4 + 4k^2 + 14k - 8 = 0$$
$$\Rightarrow 8k^2 + 4k - 4 = 0$$
$$\Rightarrow 2k^2 + k - 1 = 0$$
$$\Rightarrow (2k - 1)(k + 1) = 0$$
$$\Rightarrow k = 1/2 \text{ or } -1$$

Hence, the given points are collinear for $k = 1/2$ or -1.

Example 1.10 **If the vertices of a triangle have rational coordinates, then prove that the triangle cannot be equilateral.**

Sol. Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ be the vertices of a triangle ABC, where x_i, y_i, $i = 1, 2, 3$ are rational. Then, the area of ΔABC is given by

$$\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$
$$= \text{a rational number} \qquad [\because x_i, y_i \text{ are rational}]$$

If possible, let the triangle ABC be an equilateral triangle, then its area is given by

$$\Delta = \frac{\sqrt{3}}{4}(\text{side})^2 = \frac{\sqrt{3}}{4}(AB)^2 \qquad (\because AB = BC = CA)$$

$= \frac{\sqrt{3}}{4}$ (a rational number) [$\because$ vertices are rational $\therefore AB^2$ is a rational]

= an irrational number

This is a contradication to the fact that the area is a rational number. Hence, the triangle cannot be equilateral.

Example 1.11 **If the coordinates of two points A and B are (3, 4) and (5, – 2), respectively. Find the coordinates of any point P if $PA = PB$ and area of $\Delta PAB = 10$ sq. units.**

Sol. Let the coordinates of P be (x, y). Then, $PA = PB$.

$\Rightarrow \quad PA^2 = PB^2$

$\Rightarrow \quad (x-3)^2 + (y-4)^2 = (x-5)^2 + (y+2)^2$

$\Rightarrow \quad x - 3y - 1 = 0 \quad$ (i)

Now, area of $\Delta PAB = 10$ sq. units

$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{vmatrix} = \pm 10$

$\Rightarrow \quad 6x + 2y - 26 = \pm 20$

$\Rightarrow \quad 6x + 2y - 46 = 0$

or $\quad 6x + 2y - 6 = 0$

$\Rightarrow \quad 3x + y - 23 = 0$ or $3x + y - 3 = 0 \quad$ (ii)

Solving $x - 3y - 1 = 0$ and $3x + y - 23 = 0$, we get $x = 7$, $y = 2$

Solving $x - 3y - 1 = 0$ and $3x + y - 3 = 0$, we get $x = 1$, $y = 0$

Thus, the coordinates of P are (7, 2) or (1, 0).

Example 1.12 **Given that $P(3, 1)$, $Q(6, 5)$, and $R(x, y)$ are three points such that the angle PRQ is a right angle and the area of $\Delta RQP = 7$, then find the number of such points R.**

Sol. Obviously, R lies on the circle with P and Q as end points of diameter.

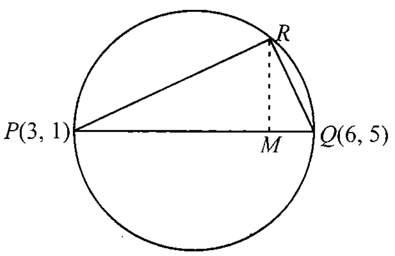


Fig. 1.8

Distance between points $P(3, 1)$ and $Q(6, 5)$ is 5 units. Hence, radius is 2.5 units

Area of triangle $PQR = \frac{1}{2} RM \times PQ = 7$ (given)

$\Rightarrow \quad RM = \frac{14}{5} = 2.8$

Now the value of RM cannot be greater than the radius.

Hence. no such triangle is possible.

SECTION FORMULA

Formula for Internal Division

Coordinates of the point that divides the line segment joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio $m : n$ are given by

$$x = \frac{mx_2 + nx_1}{m+n},$$

$$y = \frac{my_2 + ny_1}{m+n}$$

Proof:

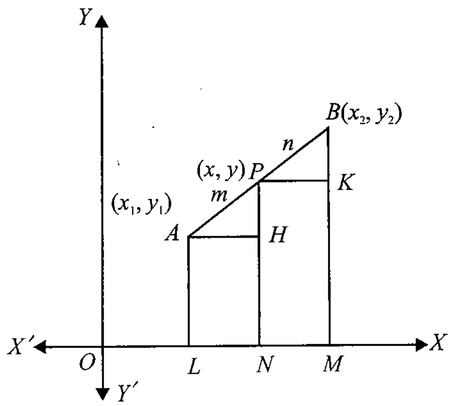


Fig. 1.9

From the figure,

Clearly, ΔAHP and ΔPKB are similar.

$\Rightarrow \quad \frac{AP}{BP} = \frac{AH}{PK} = \frac{PH}{BK}$

$\Rightarrow \quad \frac{m}{n} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$

Now,

$\frac{m}{n} = \frac{x - x_1}{x_2 - x}$

$\Rightarrow \quad mx_2 - mx = nx - nx_1$

$\Rightarrow \quad x = \frac{mx_2 + nx_1}{m+n}$

Similarly $\quad \frac{m}{n} = \frac{y - y_1}{y_2 - y}$

$$\Rightarrow \quad my_2 - my = ny - ny_1$$

$$\Rightarrow \quad y = \frac{my_2 + ny_1}{m+n}$$

Thus, the coordinates of P are $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$.

Formula for External Division

Coordinates of the point that divides the line segment joining the points (x_1, y_1) and (x_2, y_2) externally in the ratio m: n are given by

$$x = \frac{mx_2 - nx_1}{m-n}, y = \frac{my_2 - ny_1}{m-n}$$

Proof:

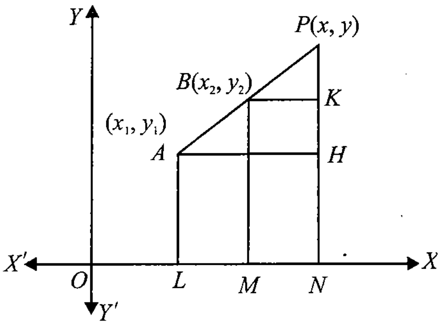


Fig. 1.10

From the figure,

Clearly, triangles PAH and PBK are similar. Therefore,

$$\frac{AP}{PB} = \frac{AH}{BK} = \frac{PH}{PK}$$

$$\Rightarrow \quad \frac{m}{n} = \frac{x - x_1}{x - x_2} = \frac{y - y_1}{y - y_2}$$

Now,

$$\frac{m}{n} = \frac{x - x_1}{x - x_2}$$

$$\Rightarrow \quad mx - mx_2 = nx - nx_1$$

$$\Rightarrow \quad x = \frac{mx_2 - nx_1}{m-n}$$

and

$$\frac{m}{n} = \frac{y - y_1}{y - y_2}$$

$$\Rightarrow \quad my - my_2 = ny - ny_1$$

$$\Rightarrow \quad y = \frac{my_2 - ny_1}{m-n}$$

Thus, the coordinates of P are $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$.

Note:

- *If the ratio, in which a given line segment is divided, is to be determined, then sometimes, for convenience (instead of taking the ratio m:n), we take the ratio* λ*: 1 and apply the formula for internal division* $\left(\frac{\lambda x_2 + x_1}{\lambda+1}, \frac{\lambda y_2 + y_1}{\lambda+1}\right)$.

  If the value of λ *turns out to be positive, it is an internal division, otherwise it is an external division.*
- *The midpoint of* (x_1, y_1) *and* (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- *To prove that A, B, C, D are vertices of*

Parallelogram	*Show that diagonals AC and BD bisect each other*
Rhombus	*Show that diagonals AC and BD bisect each other and adjacent sides AB and BC are equal*
Square	*Show that diagonals AC and BD are equal and bisect each other and adjacent sides AB and BC are equal*
Rectangle	*Show that diagonals AC and BD are equal and bisect each other*

Example 1.13 **Find the coordinates of the point which divides the line segments joining the points (6, 3) and (–4, 5) in the ratio 3 : 2 (i) internally and (ii) externally.**

Sol. Let $P(x, y)$ be the required point.

i. For internal division, we have

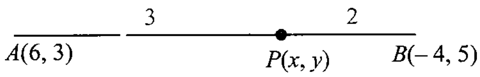


Fig. 1.11

$$x = \frac{3(-4) + 2(6)}{3+2}$$

and $$y = \frac{3(5) + 2(3)}{3+2}$$

$$\Rightarrow \quad x = 0 \text{ and } y = 21/5$$

So the coordinates of P are (0, 21/5).

ii. For external division, we have

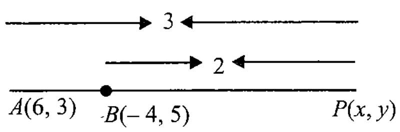


Fig. 1.12

$$x = \frac{3(-4) - 2(6)}{3 - 2}$$

and $$y = \frac{3(5) - 2(3)}{3 - 2}$$

$\Rightarrow$ $x = -24$ and $y = 9$

So the coordinates of P are $(-24, 9)$.

Example 1.14 **In what ratio does the x-axis divide the line segment joining the points (2, – 3) and (5, 6)?**

Sol. Let the required ratio be λ:1.

Then, the coordinates of the point of division are

$[(5\lambda + 2)/(\lambda - 1), (6\lambda - 3)/(\lambda + 1)]$

But, it is a point on x-axis on which y-coordinates of every point is zero.

$\Rightarrow$ $(6\lambda - 3)/(\lambda + 1) = 0$

$\Rightarrow$ $\lambda = \frac{1}{2}$

Thus, the required ratio is (1/2): 1 or 1 : 2

Example 1.15 **Given that A(1, 1) and B(2, – 3) are two points and D is a point on AB produced such that $AD = 3AB$. Find the coordinates of D.**

Sol. We have, $AD = 3AB$. Therefore, $BD = 2AB$.

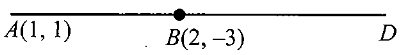


Fig. 1.13

Thus, D divides AB externally in the ratio $AD : BD = 3 : 2$

Hence, the coordinates of D are

$$\left(\frac{3(2) - 2(1)}{3 - 2}, \frac{3(-3) - 2(1)}{3 - 2}\right) = (4, -11)$$

Example 1.16 **Determine the ratio in which the line $3x + y - 9 = 0$ divides the segment joining the points (1, 3) and (2, 7).**

Sol. Suppose the line $3x + y - 9 = 0$ divides the line segment joining $A(1, 3)$ and $B(2, 7)$ in the ratio $k : 1$ at point C. Then, the coordinates of C are

$$\left(\frac{2k + 1}{k + 1}, \frac{7k + 3}{k + 1}\right)$$

But, C lies on $3x + y - 9 = 0$. Therefore,

$$3\left(\frac{2k + 1}{k + 1}\right) + \frac{7k + 1}{k + 1} - 9 = 0$$

$\Rightarrow 6k + 3 + 7k + 3 - 9k - 9 = 0$

$\Rightarrow$ $k = \frac{3}{4}$

So, the required ratio is 3 : 4 internally

Example 1.17 **Prove that the points (–2, – 1), (1, 0), (4, 3), and (1, 2) are the vertices of a parallelogram. Is it a rectangle?**

Sol. Let the given points be A, B, C, and D, respectively.

Then, the coordinates of the midpoint of AC are

$$\left(\frac{-2 + 4}{2}, \frac{-1 + 3}{2}\right) = (1, 1)$$

Coordinates of the midpoint of BD are

$$\left(\frac{1 + 1}{2}, \frac{0 + 2}{2}\right) = (1, 1)$$

Thus, AC and BD have the same midpoint.

Hence, $ABCD$ is a parallelogram.

Now, we shall see whether $ABCD$ is a rectangle or not.

We have

$$AC = \sqrt{(4 - (-2))^2 + (3 - (-1))^2}$$
$$= 2\sqrt{13},$$

and

$$BD = \sqrt{(1 - 1)^2 + (0 - 2)^2} = 2$$

Clearly, $AC \neq BD$. So, $ABCD$ is not a rectangle.

Example 1.18 **If P divides OA internally in the ratio $\lambda_1 : \lambda_2$ and Q divides OA externally in the ratio $\lambda_1 : \lambda_2$, then prove that OA is the harmonic mean of OP and OQ.**

Sol.

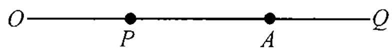


Fig. 1.14

We have, $$\frac{1}{OP} = \frac{\lambda_1 + \lambda_2}{\lambda_1 OA}$$

and $$\frac{1}{OP} = \frac{\lambda_1 - \lambda_2}{\lambda_1 OA}$$

$\Rightarrow$ $$\frac{1}{OP} + \frac{1}{OQ} = \frac{2}{OA}$$

$\Rightarrow OP$, OA and OQ are in H.P.

Example 1.19 **Given that $A_1, A_2, A_3, \ldots, A_n$ are n points in a plane whose coordinates are $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$, respectively. A_1A_2 is bisected at the point P_1, P_1A_3 is divided in the ratio 1 : 2 at P_2, P_2A_4 is divided in the ratio 1 : 3 at P_3, P_3A_5 is divided in the ratio 1 : 4 at P_4, and so on until all n points are exhausted. Find the final point so obtained.**

Sol. The coordinates of P_1(midpoint of A_1A_2) are

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

P_2 divides P_1A_3 in 1 : 2; therefore, coordinates of P_2 are

$$\left(\frac{2\left(\frac{x_1+x_2}{2}\right)+x_3}{2+1}, \frac{2\left(\frac{y_1+y_2}{2}\right)+y_3}{2+1}\right)$$

i.e., $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

Now P_3 divides P_1A_4 in 1 : 3, therefore,

$$P_3 = \left(\frac{3\frac{(x_1+x_2+x_3)}{3}+x_4}{3+1}, \frac{3\frac{(y_1+y_2+y_3)}{3}+y_4}{3+1}\right)$$

$$= \left[\frac{1}{4}(x_1+x_2+x_3+x_4), \frac{1}{4}(y_1+y_2+y_3+y_4)\right]$$

Proceeding in this manner, we can show that the coordinates of the final point are

$$[(x_1+x_2+\cdots+x_n)/n, (y_1+y_2+\cdots+y_n)/n]$$

COORDINATES OF THE CENTROID, INCENTRE, AND EX-CENTRES OF A TRIANGLE

Centroid of a Triangle

The point of concurrency of the medians of a triangle is called the centroid of the triangle. The coordinates of the centroid of the triangle with vertices as (x_1, y_1), (x_2, y_2), and (x_3, y_3) are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

Proof:

Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ be the vertices of ΔABC whose medians are AD, BE, and CF, respectively. So D, E, and F are respectively the midpoint of BC, CA, and AB.

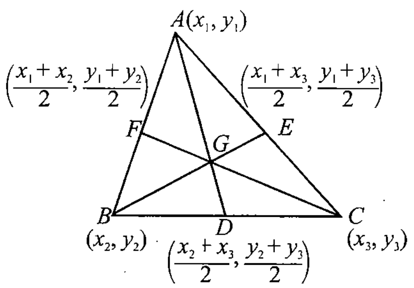


Fig. 1.15

Coordinates of D are

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$$

Coordinates of a points G dividing AD in the ratio 2:1 are

$$\left(\frac{1(x_1)+2\left(\frac{x_2+x_3}{2}\right)}{1+2}, \frac{1(y_1)+2\left(\frac{y_2+y_3}{2}\right)}{1+2}\right)$$

$$= \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

Incentre of a Triangle

The point of concurrency of the internal bisectors of the angles of a triangle is called the incentre of the triangle. The coordinates of the incentre of a triangle with vertices as $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

where a = BC, $b = AC$ and $C = AB$

Proof: Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle ABC, and let a, b, c be the lengths of the sides BC, CA, AB, respectively.

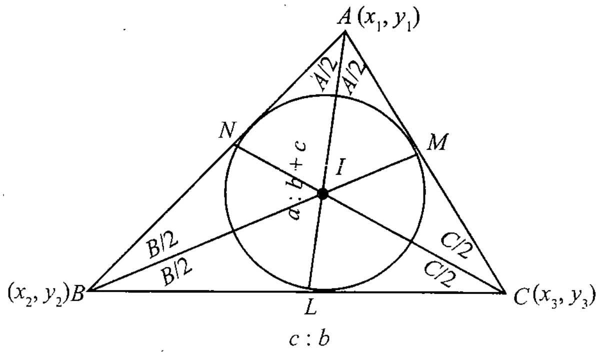


Fig. 1.16

The point at which the bisectors of the angles of a triangle intersect is called the incentre of the triangle.

Let AL, BM, and CN be respectively the internal bisectors of the angles A, B and C.

As AL bisects $\angle BAC$ internally, we have

$$\frac{BL}{LC} = \frac{BA}{AC} = \frac{c}{b} \quad \text{(i)}$$

$$\Rightarrow \frac{LC}{BL} = \frac{b}{c}$$

$$\Rightarrow \frac{LC}{BL}+1 = \frac{b}{c}+1$$

$$\Rightarrow \frac{LC+BL}{BL} = \frac{b+c}{c}$$

$$\Rightarrow \frac{BC}{BL} = \frac{b+c}{c}$$

$$\Rightarrow \frac{a}{BL} = \frac{b+c}{c}$$

$$\Rightarrow BL = \frac{ac}{b+c} \quad \text{(ii)}$$

Since BI is the bisector of $\angle B$, so it divides AIL in the ratio $AI : IL$

$$\therefore \quad \frac{AI}{IL} = \frac{AB}{BL} = \frac{c}{(ac)/(b+c)} = \frac{b+c}{a}$$

[Using (ii)]

$$\Rightarrow \quad AI : IL = (b+c) : a \qquad \text{(iii)}$$

From (i), L divides BC in the ratio $c : b$

$\Rightarrow$ Coordinates of L are

$$\left(\frac{bx_2 + cx_3}{b+c}, \frac{by_2 + cy_3}{b+c}\right)$$

From (iii), I divides AL in the ratio $(b + c) : a$. So the coordinates of I are

$$\left(\frac{ax_1 + (b+c)\left(\frac{bx_2 + cx_3}{b+c}\right)}{a+b+c}, \frac{ay_1 + (b+c)\left(\frac{by_2 + cy_3}{b+c}\right)}{a+b+c}\right)$$

$$\text{or} \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

Ex-centre of a Triangle

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle ABC, and let a, b, c be the lengths of the sides BC, CA, AB, respectively. The circle which touches the side BC and the other two sides AB and AC produced is called the escribed circle opposite to the angle A. The bisectors of the external angle B and C meet at a point I_1 which is the centre of the escribed circle opposite to the angle A.

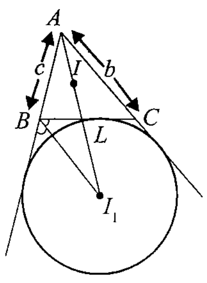


Fig. 1.17

$$\frac{BL}{LC} = \frac{c}{b}, \text{ also } \frac{AI_1}{I_1L} = -\frac{(b+c)}{a}$$

The coordinates of I_1 are given by

$$\left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

Similarly, coordinates of I_2 and I_3 (centres of escribed circles opposite to the angles B and C, respectively) are given by

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right)$$

$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

Circumcentre of a Triangle

Circumcentre $P(x, y)$ is a point which is equidistant from the vertices of triangle $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$.

Hence, $AP = BP = CP$, which gives two equations in x and y, solving which we get circumcentre.

Also circumcentre is given by

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$

Orthocentre

The point of concurrency of the altitudes of a triangle is called the orthocentre of the triangle.

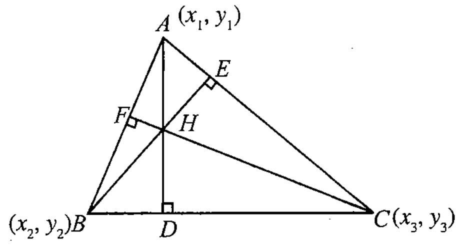


Fig. 1.18

In Fig. 1.18, point H is an orthocentre of ΔABC, and it is given by

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$$

Note:

- *Circumcentre O, Centroid G, and Orthocentre H of an acute ΔABC are collinear. G divides OH in the ratio* 1 : 2, *i.e.*, $OG : GH = 1 : 2$
- *In an isosceles triangle, centroid, orthocentre, incentre, and circumcentre lie on the same line. In an equilateral triangle, all these four points coincide.*

Example 1.20 **If a vertex of a triangle is (1, 1), and the middle points of two sides passing through it are (–2, 3) and (5, 2), then find the centroid and the incentre of the triangle.**

Sol.

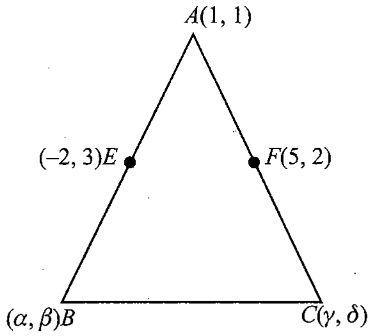


Fig. 1.19

Let E and F be the midpoints of AB and AC.

Let the coordinates of B and C be (α, β) and (γ, δ), respectively, then

$$-2 = \frac{1+\alpha}{2}, 3 = \frac{1+\beta}{2},$$

$$5 = \frac{1+\gamma}{2}, 2 = \frac{1+\delta}{2}$$

$$\therefore \quad \alpha = -5, \beta = 5, \gamma = 9, \delta = 3$$

Therefore, coordinates of B and C are $(-5, 5)$ and $(9, 3)$, respectively.

Then, centroid is

$$\left(\frac{1-5+9}{3}, \frac{1+5+3}{3}\right), \text{ i.e., } \left(\frac{5}{3}, 3\right)$$

and $\quad a = BC = \sqrt{(-5-9)^2 + (5-3)^2} = 10\sqrt{2}$

$$b = CA = \sqrt{(9-1)^2 + (3-1)^2} = 2\sqrt{17}$$

and $\quad c = AB = \sqrt{(1+5)^2 (1-5)^2} = 2\sqrt{13}$

Then incentre is

$$\left(\frac{10\sqrt{2}(1) + 2\sqrt{17}(-5) + 2\sqrt{13}(9)}{10\sqrt{2} + 2\sqrt{17} + 2\sqrt{13}}, \frac{10\sqrt{2}(1) + 2\sqrt{17}(5) + 2\sqrt{13}(3)}{10\sqrt{2} + 2\sqrt{17} + 2\sqrt{13}}\right)$$

$$\text{i.e., } \left(\frac{5\sqrt{2} - 5\sqrt{17} + 9\sqrt{13}}{5\sqrt{2} + \sqrt{17} + \sqrt{13}}, \frac{5\sqrt{2} + 5\sqrt{17} + 3\sqrt{13}}{5\sqrt{2} + \sqrt{17} + \sqrt{13}}\right)$$

Example 1.21 **Find the orthocentre of the triangle whose vertices are (0, 0), (3, 0), and (0, 4).**

Sol. This is a right-angled (at origin) triangle, therefore orthocentre = (0, 0).

Example 1.22 **If the vertices *P*, *Q*, *R* of a triangle *PQR* are rational points, which of the following points of the triangle *PQR* is (are) not necessarily rational?**

a. centroid **b. incentre**

c. circumcentre **d. orthocentre**

(A rational point is a point both of whose coordinates are rational numbers)

Sol. If $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$, where x_1, y_1 etc., are rational numbers, then Σx_1, Σy_1 are also rational.

So, the coordinates of the centroid $(\Sigma x_1/3\ \Sigma y_1/3)$ will be rational.

As $AB = c = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ may or may not be rational and it may be an irrational number of the form $\sqrt{p}$. Hence, the coordinates of the incentre $(\Sigma ax_1/\Sigma a, \Sigma ay_1/\Sigma a)$ may or may not be rational. If (α, β) is the circumcentre or orthocentre, α and β are found by solving two linear equations in α, β with rational coefficients. So α, β must be rational numbers.

Example 1.23 **If the circumcentre of an acute angled triangle lies at the origin and the centroid is the middle point of the line joining the points $(a^2 + 1, a^2 + 1)$ and $(2a, -2a)$, then find the orthocentre.**

Sol. We know from geometry that the circumcentre (O) centroid (G) and orthocentre (H) of a triangle lie on the line joining the circumcentre (0, 0) and the centroid $((a+1)^2/2, (a-1)^2/2)$.

Also $\quad \frac{HG}{GO} = \frac{2}{1} \Rightarrow$ H has coordinate

$$\left(\frac{3(a+1)^2}{2}, \frac{3(a-1)^2}{2}\right)$$

Example 1.24 **If a vertex, the circumcentre, and the centroid of a triangle are (0, 0), (3, 4), and (6, 8), respectively, then the triangle must be**

a. a right-angled triangle

b. an equilateral triangle

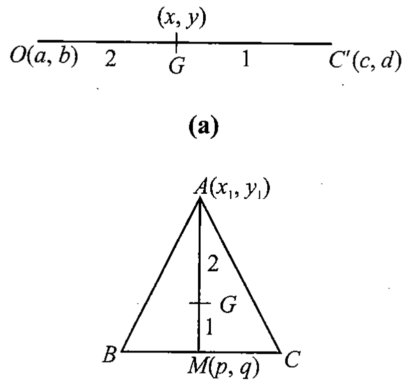
c. an isosceles triangle

d. a right-angled isosceles triangle

Sol. Clearly, (0, 0), (3, 4), and (6, 8) are collinear. So, the circumcentre M and the centroid G are on the median which is also the perpendicular bisector of the side. So, the Δ must be isosceles.

Example 1.25 **Orthocentre and circumcentre of a Δ *ABC* are (*a*, *b*) and (*c*, *d*), respectively. If the coordinates of the vertex *A* are (x_1, y_1), then find the coordinates of the middle point of *BC*.**

Sol.



(a)

(b)

Fig. 1.20

$$x = \frac{a+2c}{3}; y = \frac{b+2d}{3}$$

Now

$$x = \frac{x_1+2p}{3}; y = \frac{y_1+2q}{3}$$

$\therefore$

$$p = \frac{a+2c-x_1}{2}; q = \frac{b+2d-y_1}{2}$$

Concept Application Exercise 1.1

1. If the points (0, 0), $(2, 2\sqrt{3})$, and (p, q) are the vertices of an equilateral triangle, then (p, q) is
   a. (0, –4) **b.** (4, 4) **c.** (4, 0) **d.** (5, 0)
2. The distance between the points $(a \cos \alpha, a \sin \alpha)$ and $(a \cos \beta, a \sin \beta)$ is
   a. $a \cos \frac{\alpha-\beta}{2}$ **b.** $2a \cos \frac{\alpha-\beta}{2}$
   c. $2a \sin \frac{\alpha-\beta}{2}$ **d.** $a \sin \frac{\alpha-\beta}{2}$
3. Find the length of altitude through A of the triangle ABC, where $A \equiv (-3, 0)$, $B \equiv (4, -1)$, $C \equiv (5, 2)$.
4. If the point $(x, -1)$, $(3, y)$, $(-2, 3)$, and $(-3, -2)$ taken in order are the vertices of a parallelogram, then find the values of x and y.
5. If the midpoints of the sides of a triangle are (2, 1), (–1, –3), and (4, 5). Then find the coordinates of its vertices.
6. The three points (–2, 2), (8, –2), and (–4, –3) are the vertices of
   a. an isosceles triangle
   b. an equilateral triangle
   c. a right angled triangle
   d. none of these
7. The points (a, b), (c, d), and $\left(\frac{kc+la}{k+l}, \frac{kd+lb}{k+l}\right)$ are
   a. Vertices of an equilateral triangle
   b. Vertices of an isosceles triangle
   c. Vertices of a right-angled triangle
   d. Collinear
8. The points $(-a, -b)$, (a, b), (a^2, ab) are
   a. Vertices of an equilateral triangle
   b. Vertices of a right-angled triangle
   c. Vertices of an isosceles triangle
   d. Collinear
9. Circumcentre of the triangle formed by the line $y = x$, $y = 2x$, and $y = 3x + 4$ is
   a. (6, 8) **b.** (6, –8)
   c. (3, 4) **d.** (–3, –4)
10. Find the area of the pentagon whose vertices are $A(1, 1)$, $B(7, 21)$, $C(7, -3)$, $D(12, 2)$, and $E(0, -3)$.
11. If the middle points of the sides of a triangle are (–2, 3), (4, –3), and (4, 5), then find the centroid of the triangle.
12. The line joining $A(b \cos \alpha, b \sin \alpha)$ and $B(a \cos \beta, a \sin \beta)$ is produced to the point $M(x, y)$ so that AM and BM are in the ratio $b : a$. Then prove that $x + y \tan (\alpha + \beta/2) = 0$.
13. A point moves such that the area of the triangle formed by it with the points (1, 5) and (3, –7) is 21 sq. units. Then, find the locus of the point.
14. If (1, 4) is the centroid of a triangle and the coordinates of its any two vertices are (4, –8) and (–9, 7), find the area of the triangle.
15. A triangle with vertices (4, 0), (– 1, – 1), (3, 5) is
   a. isosceles and right-angled
   b. isosceles but not right-angled
   c. right-angled but not isosceles
   d. neither right-angled nor isosceles

LOCUS AND EQUATION TO A LOCUS

Locus

The curve described by a point which moves under given condition or conditions is called its locus. For example, suppose C is a point in the plane of the paper and P is a variable point in the plane of the paper such that its distance from C is always equal to a (say). It is clear that all the positions of the moving point P lie on the circumference of a circle whose centre is C and whose radius is a. The circumference of this circle is, therefore, the "Locus" of point P when it moves under the condition that its distance from the point C is always equal to constant a.

Let A and B be two fixed points in the plane of the paper, and P be a variable point in the plane of the paper which moves in such a way that its distance from A and B is always same. Thus, the "locus" of P is the perpendicular bisector of AB when it moves under the condition that its distance from A and B is always equal.

Equation to Locus of a Point

The equation to the locus of a point is the relation which is satisfied by the coordinates of every point on the locus of the point.

Steps to find locus of a points

Step I: Assume the coordinates of the point say (h, k) whose locus is to be found.

Step II: Write the given condition in mathematical form involving h, k.

Step III: Eliminate the variable(s), if any.

Step IV: Replace h by x and k by y in the result obtained in step III.

The equation so obtained is the locus of the point which moves under some stated condition(s).

Example 1.26 **The sum of the squares of the distances of a moving point from two fixed points $(a, 0)$ and $(-a, 0)$ is equal to a constant quantity $2c^2$. Find the equation to its locus.**

Sol. Let $P(h, k)$ be any position of the moving point and let $A(a, 0)$, $B(-a, 0)$ be the given points.

Then, we have

$$PA^2 + PB^2 = 2c^2 \text{ (given)}$$

$$\Rightarrow (h-a)^2 + (k-0)^2 + (h+a)^2 + (k-0)^2 = 2c^2$$

$$2h^2 + 2k^2 + 2a^2 = 2c^2$$

$$\Rightarrow \quad h^2 + k^2 = c^2 - a^2$$

Hence, equation to locus (h, k) is $x^2 + y^2 = c^2 - a^2$.

Example 1.27 **Find the locus of a point, so that the join of (–5, 1) and (3, 2) subtends a right angle at the moving point.**

Sol. Let $P(h, k)$ be a moving point and let $A(-5, 1)$ and $B(3, 2)$ be given points.

By the given condition, we have

$$\angle APB = 90°$$

$\Rightarrow \Delta APB$ is a right-angled triangle

$$\Rightarrow \quad AB^2 = AP^2 + PB^2$$

$$\Rightarrow (3+5)^2 + (2-1)^2 = (h+5)^2 + (k-1)^2 + (h-3)^2 + (k-2)^2$$

$$\Rightarrow \quad 65 = 2(h^2 + k^2 + 2h - 3k) + 39$$

$$\Rightarrow \quad h^2 + k^2 + 2h - 3k - 13 = 0$$

Hence, locus of (h, k) is $x^2 + y^2 + 2x - 3y - 13 = 0$.

Example 1.28 **A rod of length l slides with its ends on two perpendicular lines. Find the locus of its midpoint.**

Sol. Let the two perpendicular lines be the coordinates axes.

Let AB be a rod of length l.

Let the coordinates of A and B be $(a, 0)$ and $(0, b)$, respectively.

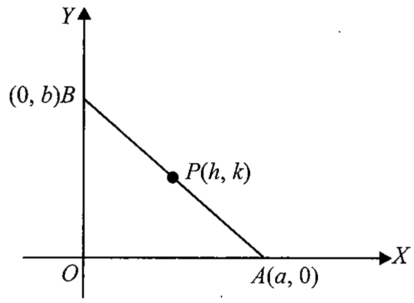


Fig. 1.21

As the rod slides, the values of a and b change.

So a and b are two variables.

Let $P(h, k)$ be the midpoint of the rod AB in one of the infinite positions it attains.

Then, $$h = \frac{a+0}{2} \text{ and } k = \frac{0+b}{2}$$

$$\Rightarrow \quad h = \frac{a}{2} \text{ and } k = \frac{b}{2} \qquad \text{(i)}$$

From ΔOAB, we have

$$AB^2 = OA^2 + OB^2$$

$$\Rightarrow \quad a^2 + b^2 = l^2$$

$$\Rightarrow \quad (2h)^2 + (2k)^2 = l^2 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 4h^2 + 4k^2 = l^2$$

Hence, the locus of (h, k) is $4x^2 + 4y^2 = l^2$

Example 1.29 ***AB* is a variable line sliding between the coordinate axes in such a way that *A* lies on *x*-axis and *B* lies on *y*-axis. If *P* is a variable point on *AB* such that $PA = b$, $PB = a$, and $AB = a + b$, find the equation of the locus of *P*.**

Sol. Let $P(h, k)$ be a variable point on AB such that $\angle OAB = \theta$.

Here θ is a variable.

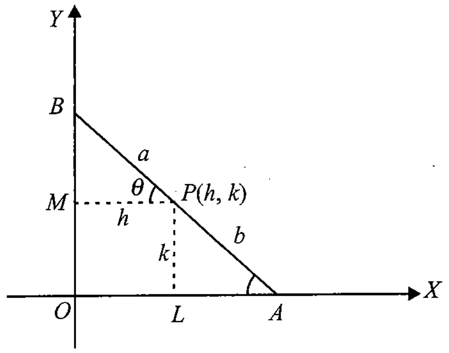


Fig. 1.22

From triangles ALP and PMB, we have

$$\sin\theta = \frac{k}{b} \quad \text{(i)}$$

$$\cos\theta = \frac{h}{a} \quad \text{(ii)}$$

Here θ is a variable. So, we have to eliminate θ.

Squaring (i) and (ii) and adding, we get

$$\frac{k^2}{b^2} + \frac{h^2}{a^2} = 1$$

Hence, the locus of (h, k) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Example 1.30 **Two points P and Q are given, R is a variable point on one side of the line PQ such that $\angle RPQ - \angle RQP$ is a positive constant 2α. Find the locus of the point R.**

Sol. Let the x-axis along QP and the middle point of PQ be origin and let $R \equiv (x_1, y_1)$.

Let $OP = OQ = a$ and $\angle RPM = \theta$ and $\angle RQM = \phi$

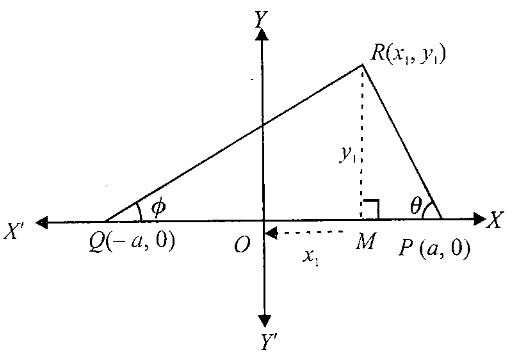


Fig. 1.23

In ΔRMP,

$$\tan\theta = \frac{RM}{MP} = \frac{y_1}{a - x_1} \quad \text{(i)}$$

In ΔRQM,

$$\tan\phi = \frac{RM}{QM} = \frac{y_1}{a + x_1} \quad \text{(ii)}$$

But given $\angle RPQ - \angle RQP = 2\alpha$ (constant)

$$\Rightarrow \quad \theta - \phi = 2\alpha$$

$$\Rightarrow \quad \tan(\theta - \phi) = \tan 2\alpha$$

$$\Rightarrow \quad \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \tan 2\alpha$$

$$\Rightarrow \quad \frac{\dfrac{y_1}{a - x_1} - \dfrac{y_1}{a + x_1}}{1 + \dfrac{y_1}{a - x_1}\dfrac{y_1}{a + x_1}} = \tan 2\alpha$$

$$\Rightarrow \quad \frac{2x_1y_1}{a^2 - x_1^2 + y_1^2} = \tan 2\alpha$$

$\Rightarrow a^2 - x_1^2 + y_1^2 = 2x_1y_1 \cot 2\alpha$ or $x_1^2 - y_1^2 + 2x_1y_1 \cot 2\alpha = a^2$

Hence, locus of the point $R(x_1, y_1)$ is $x^2 - y^2 + 2xy \cot 2\alpha = a^2$.

Example 1.31 **If the coordinates of a variable point P is $(a\cos\theta, b\sin\theta)$, where θ is a variable quantity, then find the locus of P.**

Sol. Let $P \equiv (x, y)$. According to the question

$$x = a\cos\theta \quad \text{(i)}$$

$$y = b\sin\theta \quad \text{(ii)}$$

Squaring and adding (i) and (ii), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Example 1.32 **Find the locus of a point such that the sum of its distance from the points (0, 2) and (0, – 2) is 6.**

Sol. Let $P(h, k)$ be any point on the locus and let $A(0, 2)$ and $B(0, -2)$ be the given points.

By the given condition, we get

$$PA + PB = 6$$

$$\Rightarrow \sqrt{(h-0)^2+(k-2)^2}+\sqrt{(h-0)^2+(k+2)^2}=6$$

$$\Rightarrow \quad \sqrt{h^2+(k-2)^2}=6-\sqrt{(h-0)^2+(k+2)^2}$$

$$\Rightarrow \quad h^2+(k-2)^2=36-12\sqrt{h^2+(k+2)^2}+h^2+(k+2)^2$$

$$\Rightarrow \quad -8k-36=-12\sqrt{h^2+(k+2)^2}$$

$$\Rightarrow \quad (2k+9)=3\sqrt{h^2+(k+2)^2}$$

$$\Rightarrow \quad (2k+9)^2=9(h^2+(k+2)^2)$$

$$\Rightarrow \quad 4k^2+36k+81=9h^2+9k^2+36k+36$$

$$\Rightarrow \quad 9h^2+5k^2=45$$

Hence, locus of (h, k) is $9x^2+5y^2=45$.

Concept Application Exercise 1.2

1. Find the locus of a point whose distance from $(a, 0)$ is equal to its distance from y-axis.
2. The coordinates of the points A and B are $(a, 0)$ and $(-a, 0)$, respectively. If a point P moves so that $PA^2 - PB^2 = 2k^2$, when k is constant, then find the equation to the locus of the point P.
3. If $A(\cos\alpha, \sin\alpha)$, $B(\sin\alpha, -\cos\alpha)$, $C(1, 2)$ are the vertices of a ΔABC, then as α varies, then find the locus of its centroid.
4. Let $A(2, -3)$ and $B(-2, 1)$ be vertices of a triangle ABC. If the centroid of the triangle moves on the line $2x+3y=1$, then find the locus of the vertex C.
5. Q is a variable point whose locus is $2x+3y+4=0$; corresponding to a particular position of Q, P is the point of section of OQ, O being the origin, such that $OP : PQ = 3 : 1$. Find the locus of P?
6. Find the locus of the middle point of the portion of the line $x\cos\alpha + y\sin\alpha = p$ which is intercepted between the axes, given that p remains constant.
7. Find the locus of the point of intersection of lines $x\cos\alpha + y\sin\alpha = a$ and $x\sin\alpha - y\cos\alpha = b$ (α is a variable).

SHIFTING OF ORIGIN

Let O be the origin and let $X'OX$ and $Y'OY$ be the axis of x and y, respectively. Let O' and P be two points in the plane having coordinates (h, k) and (x, y), respectively referred to $X'OX$ and $Y'OY$ as the coordinates axes. Let the origin be transferred to O' and let $X'O'X$ and $Y'O'Y$ be new rectangular axes. Let the coordinates of P referred to new axes as the coordinates axes be (X, Y).

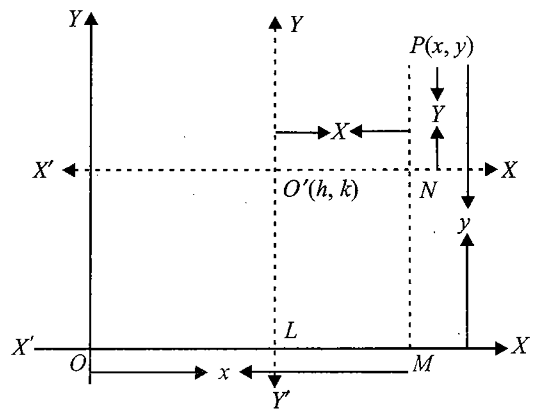


Fig. 1.24

Then,

$O'N = X$, $PN = Y$, $OM = x$, $PM = y$, $OL = h$, and $O'L = k$

Now,

$x = OM = OL + LM = OL + O'N = h + X$

and

$y = PM = PN + NM = PN + O'L = Y + k$

$\Rightarrow x = X + h$ and $y = Y + k$

Thus, if (x, y) are coordinates of a point referred to old axes and (X, Y) are the coordinates of the same point referred to new axes, then $x = X + h$ and $y = Y + k$. Therefore, the origin is shifted at a point (h, k), we must substitute $X + h$ and $Y + k$ for x and y, respectively.

The transformation formula from new axes to old axes is

$X = x - h$, $Y = y - k$

The coordinates of the old origin referred to the new axes are $(-h, -k)$.

Example 1.33 **If the origin is shifted to the point (1, –2) without rotation of axes what do the following equations become?**

i. $2x^2 + y^2 - 4x + 4y = 0$ and

ii. $y^2 - 4x + 4y + 8 = 0$.

Sol. i. Substituting $x = X + 1$, $y = Y + (-2) = Y - 2$ in the equation $2x^2 + y^2 - 4x + 4y = 0$, we get

$2(X+1)^2 + (Y-2)^2 - 4(X+1) + 4(Y-2) = 0$

or $2X^2 + Y^2 = 6$

ii. Substituting $x = X + 1$, $y = Y - 2$ in the equation $y^2 - 4x + 4y + 8 = 0$, we get

$(Y-2)^2 - 4(X+1) + 4(Y-2) + 8 = 0$

or $Y^2 = 4X$

Example 1.34 **At what point the origin be shifted, if the coordinates of a point (4, 5) become (– 3, 9)?**

Sol. Let (h, k) be the point to which the origin is shifted. Then,

$$x = 4, y = 5, X = -3, Y = 9$$

$$\therefore \quad x = X + h \text{ and } y = Y + k$$

$$\Rightarrow \quad 4 = -3 + h \text{ and } 5 = 9 + k$$

$$\Rightarrow \quad h = 7 \text{ and } k = -4$$

Hence, the origin must be shifted to (7, – 4).

Example 1.35 **Shift the origin to a suitable point so that the equation $y^2 + 4y + 8x - 2 = 0$ will not contain term in y and the constant term.**

Sol. Let the origin be shifted to (h, k). Then,

$$x = X + h \text{ and } y = Y + k$$

Substituting $x = X + h$, $y = Y + k$ in the equation $y^2 + 4y + 8x - 2 = 0$, we get

$$(Y + k)^2 + 4(Y + k) + 8(X + h) - 2 = 0$$

$$\Rightarrow \quad Y^2 + (4 + 2k)\,Y + 8X + (k^2 + 4k + 8h - 2) = 0$$

For this equation to be free from the term containing Y and the constant term, we must have

$$4 + 2k = 0 \text{ and } k^2 + 4k + 8h - 2 = 0$$

$$\Rightarrow \quad k = -2 \text{ and } h = 3/4$$

Hence, the origin is shifted at the point (3/4, – 2).

Example 1.36 **The equation of a curve referred to new axes, axes retaining their directions, and origin (4, 5) is $X^2 + Y^2 = 36$. Find the equation referred to the original axes.**

Sol. With the above notation, we have

$$x = X + 4, y = Y + 5$$

$$\Rightarrow \quad X = x - 4, Y = y - 5$$

$\therefore$ The required equation is

$$(x - 4)^2 + (y - 5)^2 = 36$$

$\Rightarrow$ $x^2 + y^2 - 8x - 10y + 5 = 0$ which is equation referred to the original axes.

Example 1.37 **Find the equation to which the equation**

$$x^2 + 7xy - 2y^2 + 17x - 26y - 60 = 0$$

is transformed if the origin is shifted to the point (2, – 3), the axes remaining parallel to the original axis.

Sol. Here the new origin is (2, - 3).

Then, $x = X + 2$, $y = Y - 3$.

and the given equation transforms to

$$(X + 2)^2 + 7(X + 2)(Y - 3) - 2(Y - 3)^2 + 17(X + 2) - 26(Y - 3) - 60 = 0$$

$$\Rightarrow \quad X^2 + 7XY - 2Y^2 - 4 = 0$$

ROTATION OF AXIS

Rotation of Axes without Changing the Origin

Let O be the origin. Let $P \equiv (x, y)$ with respect to axes OX and OY and let $P \equiv (x', y')$ with respect to axes OX' and OY' where $\angle X'OX = \angle YOY' = \theta$

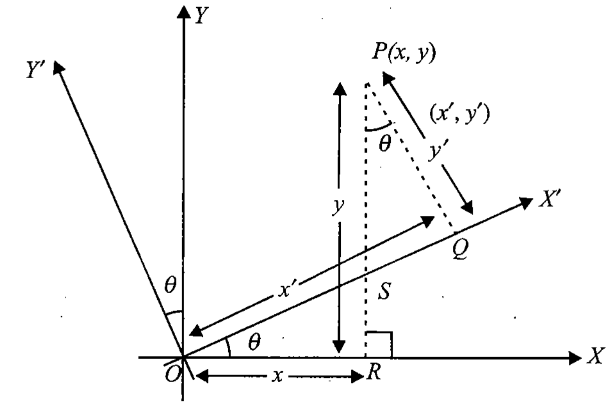


Fig. 1.25

In Fig. 1.25, we have

$$SR = x\tan\theta,\ OS = x\sec\theta,$$
$$PS = y - x\tan\theta$$

Now in triangle PQS,

$$\sin\theta = \frac{SQ}{PS} = \frac{x' - x\sec\theta}{y - x\tan\theta}$$

$$\Rightarrow \quad x' = y\sin\theta - x\frac{\sin^2\theta}{\cos\theta} + \frac{x}{\cos\theta}$$

$$= y\sin\theta + x\frac{1 - \sin^2\theta}{\cos\theta}$$

$$\Rightarrow \quad x' = x\cos\theta + y\sin\theta$$

Also $$\cos\theta = \frac{PQ}{PS} = \frac{y'}{y - x\tan\theta}$$

$$\Rightarrow \quad y' = -x\sin\theta + y\cos\theta$$

$$\Rightarrow \quad x = x'\cos\theta - y'\sin\theta,$$
$$y = x'\sin\theta + y'\cos\theta$$

	x	y
x'	$\cos\theta$	$\sin\theta$
y'	$-\sin\theta$	$\cos\theta$

Note:

Compare real and imaginary parts of the equation $(x + iy) = (x' + iy')(\cos\theta + i\sin\theta)$ to remember the formula

Example 1.38 **The equation of a curve referred to a given system of axes is $3x^2 + 2xy + 3y^2 = 10$. Find its equation if the axes are rotated through an angle 45°, the origin remaining unchanged.**

Sol. With the above notation, we have

$$x = x' \cos 45° - y' \sin 45° = \frac{x' - y'}{\sqrt{2}}$$

and $y = x' \sin 45° + y' \cos 45° = \dfrac{x' + y'}{\sqrt{2}}$

Thus, $3x^2 + 2xy + 3y^2 = 10$ transforms to

$$3\left(\frac{x'-y'}{\sqrt{2}}\right)^2 + 2\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) + 3\left(\frac{x'+y'}{\sqrt{2}}\right)^2 = 10$$

$\Rightarrow\ 2x'^2 + y'^2 = 5.$

Removal of the term *xy*, from $f(x, y) = ax^2 + 2hxy + by^2$ without changing the origin

Clearly, $h \neq 0$.

Rotating the axes through an angle θ, we have

$x = x' \cos \theta - y' \sin \theta$

$y = x' \sin \theta + y' \cos \theta.$

$\therefore f(x, y) = ax^2 + 2hxy + by^2$

$= a(x' \cos \theta - y' \sin \theta)^2 + 2h(x' \cos \theta - \theta y' \sin \theta).(x' \sin \theta + y' \cos \theta) + b(x' \sin \theta + y' \cos \theta)^2$

$= (a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta) x'^2 + 2[(b - a)\cos \theta \sin \theta + h(\cos^2 \theta - \sin^2 \theta)] x'y' + (a \sin^2 \theta - 2h \cos \theta \sin \theta + b \cos^2 \theta) y'^2$

$= F(x', y')$, (*say*).

In $F(x', y')$, we require that the coefficient of the XY-term to be zero.

$\therefore 2[(b - a) \cos \theta \sin \theta + h(\cos^2 \theta - \sin^2 \theta)] = 0.$

$\Rightarrow (a - b) \sin 2\theta = 2h \cos 2\theta.$

$$\Rightarrow \tan 2\theta = \frac{2h}{a-b} \text{ or } \cot 2\theta = \frac{a-b}{2h}$$

We use the former or the later equation according as $a \neq b$ or $a = b$. These yield θ, the angle through which the axes are to be rotated (the origin remaining unchanged) in order to remove the xy-term from $f(x, y)$.

Example 1.39 **Given the equation $4x^2 + 2\sqrt{3}\,xy + 2y^2 = 1$. Through what angle should the axes be rotated so that the term xy is removed from the transformed equation.**

Sol. Comparing the given equation, with

$ax^2 + 2hxy + by^2$, we get $a = 4$, $h = \sqrt{3}$, $b = 2$.

Let θ be the angle through which the axes are to be rotated.

Then $$\tan 2\theta = \frac{2h}{a-b}$$

$$\Rightarrow \quad \tan 2\theta = \frac{2\sqrt{3}}{4-2} = \sqrt{3} = \tan\frac{\pi}{3}$$

$$\Rightarrow \quad 2\theta = \frac{\pi}{3}, \pi + \frac{\pi}{3}$$

$$\Rightarrow \quad \theta = \frac{\pi}{6}, \frac{2\pi}{3}$$

Change of Origin and Rotation of Axes

If origin is changed to $O'(\alpha, \beta)$ and axes are rotated about the new origin O' by an angle θ in the anticlockwise sense such that the new coordiantes of $P(x, y)$ becomes (x', y'), then the equations of transformation will be

$$x = \alpha + x' \cos \theta - y' \sin \theta$$

and

$$y = \beta + x' \sin \theta + y' \cos \theta$$

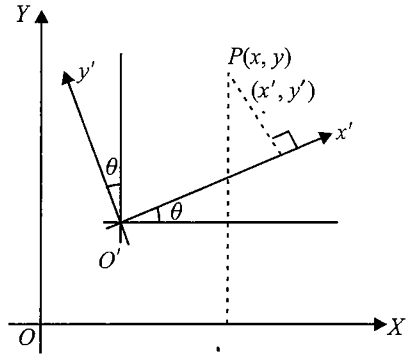


Fig. 1.26

Example 1.40 **What does the equation $2x^2 + 4xy - 5y^2 + 20x - 22y - 14 = 0$ become when referred to rectangular axes through the point (–2, – 3), the new axes being inclined at an angle of 45° with the old axes?**

Sol. Let O' be (–2, –3). Since the axes are rotated about O' by an angle 45° in anticlockwise direction, let (x', y') be the new coordinates with respect to new axes and (x, y) be the coordinates with respect to old axes. Then, we have

$$x = -2 + x' \cos 45° - y' \sin 45° = -2 + \left(\frac{x'-y'}{\sqrt{2}}\right)$$

$$y = -3 + x' \sin 45° + y' \cos 45° = -3 + \left(\frac{x'+y'}{\sqrt{2}}\right)$$

The new equation will be

$$2\left\{-2 + \left(\frac{x'-y'}{\sqrt{2}}\right)\right\}^2 + 4\left(-2 + \left(\frac{x'-y'}{\sqrt{2}}\right)\right)\left(-3 + \left(\frac{x'+y'}{\sqrt{2}}\right)\right)$$

$$-5\left(-3 + \left(\frac{x'+y'}{\sqrt{2}}\right)\right)^2 + 20\left(-2 + \left(\frac{x'-y'}{\sqrt{2}}\right)\right)$$

$$-22\left(-3 + \left(\frac{x'+y'}{\sqrt{2}}\right)\right) - 14 = 0$$

$\Rightarrow \quad x'^2 - 14x'y' - 7y'^2 - 2 = 0$

Hence, new equation of curve is

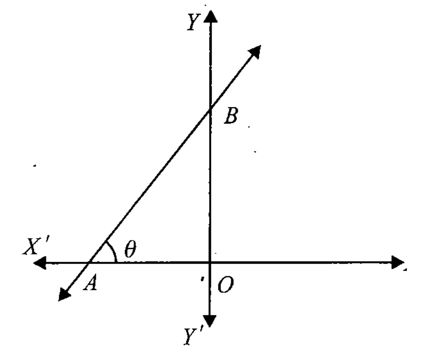
$$x^2 - 14xy - 7y^2 - 2 = 0$$

STRAIGHT LINE

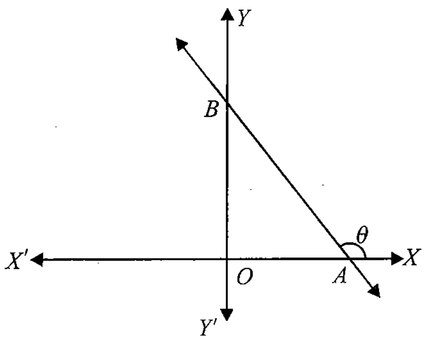
A straight line is a curve such that every point on the line segment joining any two points on it lies on it.

Slope (Gradient) of a Line

The trigonometrical tangent of an angle that a line makes with the positive direction of the x-axis in anticlockwise sense is called the slope or gradient of the line.



(a)



(b)

Fig. 1.27

Note:

- *The slope of a line is generally denoted by m. Thus, $m = \tan\theta$.*
- *Since a line parallel to x-axis makes an angle of* 0° *with x-axis; therefore, its slope is tan* 0° = 0.
- A line parallel to y-axis, i.e., perpendicular to x-axis makes an angle of 90° with x-axis, so its slope is $\tan \pi/2 = \infty$.
- *The slope of a line equally inclined with the axis is* 1 *or* – 1, *as it makes* 45° *or* 135° *angle with x-axis. Slope of a line in terms of coordinates of any two points on it is given as shown below:*

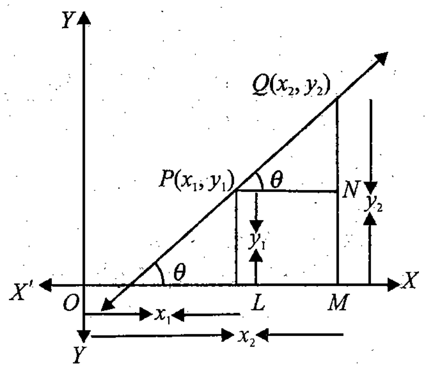


Fig. 1.28

From the figure, slope is

$$\tan\theta = \frac{QN}{PN} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{\textit{Difference of ordinates}}{\textit{Difference of abscissae}}$$

Angle between Two Lines

The angle θ between the lines having slope m_1 and m_2 is given by

$$\tan\theta = \pm\,\frac{m_2 - m_1}{1 + m_1 m_2}$$

Proof:

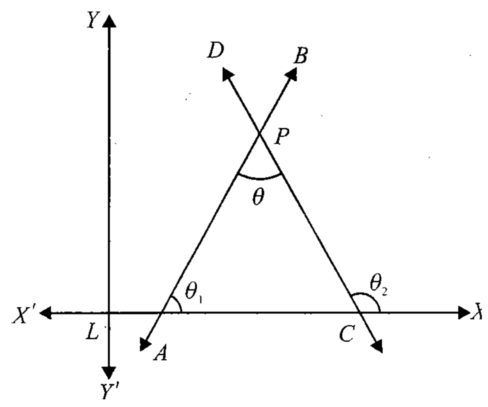


Fig. 1.29

Let m_1 and m_2 be the slopes of two given lines AB and CD which intersect at a point P and make angles θ_1 and θ_2, respectively with the positive direction of x-axis.

Then,

$$m_1 = \tan\theta_1 \text{ and } m_2 = \tan\theta_2$$

Let $\angle APC = \theta$ be the angle between the given lines.

Then,

$$\theta_2 = \theta + \theta_1$$

$$\Rightarrow \quad \theta = \theta_2 - \theta_1$$

$$\Rightarrow \quad \tan\theta = \tan(\theta_2 - \theta_1)$$

$$\Rightarrow \quad \tan\theta = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}.$$

$$\Rightarrow \quad \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} \qquad \text{(i)}$$

Since $\angle APD = \pi - \theta$ is also the angle between AB and CD.

Therefore,

$$\tan\angle APD = \tan(\pi - \theta) = -\tan\theta$$

$$= -\frac{m_2 - m_1}{1 + m_1 m_2} \qquad \text{(ii) [Using (i)]}$$

From (i) and (ii), we find that the angle between two lines of slopes m_1 and m_2 is given by

$$\tan\theta = \pm\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\pm\frac{m_2 - m_1}{1 + m_1 m_2}\right)$$

The acute angle between the lines is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

Example 1.41 **If A (−2, 1), B(2, 3), and C(−2, −4) are three points, then find the angle between BA and BC.**

Sol. Let m_1 and m_2 be the slopes of BA and BC, respectively. Then,

$$m_1 = \frac{3 - 1}{2 - (-2)} = \frac{2}{4} = \frac{1}{2}$$

and

$$m_2 = \frac{-4 - 3}{-2 - 2} = \frac{7}{4}$$

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4} \times \frac{1}{2}}\right|$$

$$= \left|\frac{\frac{10}{8}}{\frac{15}{8}}\right| = \frac{2}{3}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{2}{3}\right)$$

Example 1.42 **Determine x so that the line passing through (3, 4) and (x, 5) makes 135° angle with the positive direction of x-axis**

Sol. Since the line passing through (3, 4) and (x, 5) makes an angle of 135° with x-axis; therefore, its slope is

$$\tan 135° = -1.$$

But, the slope of the line is also equal to

$$\frac{5 - 4}{x - 3}$$

$$\Rightarrow \quad -1 = \frac{5 - 4}{x - 3}$$

$$\Rightarrow \quad -x + 3 = 1$$

$$\Rightarrow \quad x = 2$$

Condition for Parallelism of Lines

It two lines of slopes m_1 and m_2 are parallel, then the angle θ between is 0°.

$$\therefore \quad \tan\theta = \tan 0° = 0$$

$$\Rightarrow \quad \frac{m_2 - m_1}{1 + m_1 m_2} = 0$$

$$\Rightarrow \quad m_2 = m_1$$

Thus, when two lines are parallel, their slopes are equal.

Condition for Perpendicularity of Two Lines

If two lines of slopes m_1 and m_2 are perpendicular, then the angle θ between them is 90°.

$$\therefore \quad \cot\theta = 0$$

$$\Rightarrow \quad \frac{1 + m_1 \cdot m_2}{m_2 - m_1} = 0$$

$$\Rightarrow \quad m_1 m_2 = -1$$

Thus, when lines are perpendicular, the product of their slope is − 1.

If m is the slope of a line, then the slope of a line perpendicular to it is $-(1/m)$.

Example 1.43 **Let A(6, 4) and B(2, 12) be two given points. Find the slope of a line perpendicular to AB.**

Sol. Let m be the slope of AB, than

$$m = \frac{12 - 4}{2 - 6} = \frac{8}{-4} = -2$$

So, the slope of a line ⊥ to AB

$$= -\frac{1}{m} = \frac{1}{2}$$

Intercepts of a Line on the Axes

If a straight line cuts x-axis at A and the y – axis at B then OA and OB are known as the intercepts of the line on x-axis and y-axis respectively.

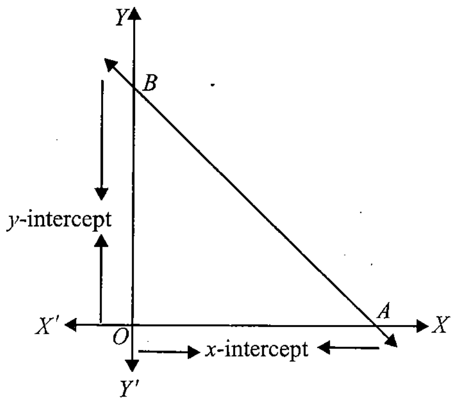


Fig. 1.30

The intercepts are positive or negative according as the line meets with positive or negative directions of the coordinates axes.

In figure OA = x-intercept, OB = y-intercept

OA is positive or negative according as A lies on OX or OX' respectively.

Similarly OB is positive or negative according as B lies on OY or OY' respectively.

Note:

- *If line has equal intercept on axes, then its slope is* –1.

Equation of a Line Parallel to x-Axis

Equation of a line parallel to x-axis at a distance b from it.

Then, clearly the ordinates of each point on AB is b.

Thus, AB can be considered as the locus of a point at a distance b from x-axis.

Thus, if $P(x, y)$ is any point on AB, then $y = b$.

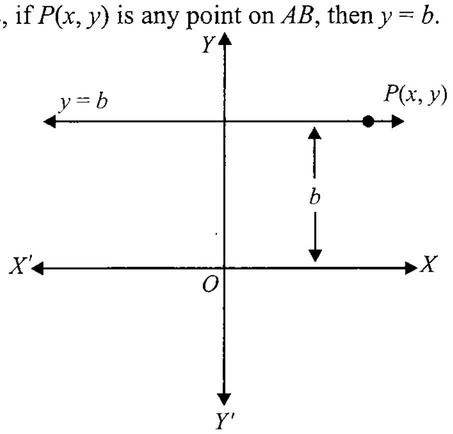


Fig. 1.31

Hence, the equation of a line parallel to x-axis at a distance b from it is $y = b$.

Since x-axis is parallel to itself at a distance 0 from it; therefore, the equation of x-axis is $y = 0$.

Equation of a Line Parallel to y-Axis

Let AB be a line parallel to y-axis and at a distance a from it. Then the abscissa of every point on AB is a. So it can be treated as the locus of a point at a distance a from y-axis.

Thus, if $P(x, y)$ is any point in AB, then $x = a$.

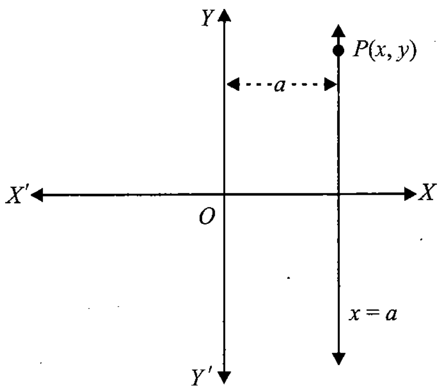


Fig. 1.32

DIFFERENT FORMS OF LINE

Slope Intercept Form of a Line

The equation of a line with slope m that makes an intercept c on y-axis is

$$y = mx + c$$

Proof: Let the given line intersects y-axis at Q and makes an angle θ with x-axis. Then $m = \tan\theta$. Let $P(x, y)$ be any point on the line as shown in the figure.

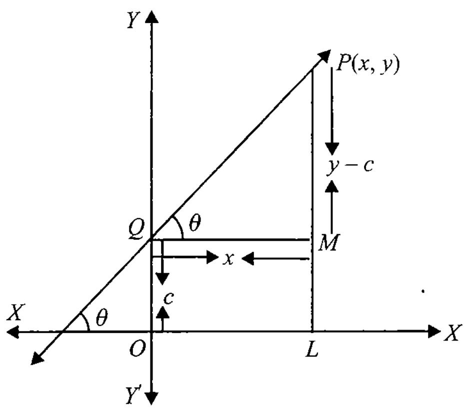


Fig. 1.33

From ΔPMQ, we have

$$\tan\theta = \frac{PM}{QM} = \frac{y-c}{x}$$

$$\Rightarrow \quad m = \frac{y-c}{x}$$

$$\Rightarrow \quad y = mx + c$$

which is the required equation of the line.

Point–Slope Form of a Line

The equation of a line which passes through the point (x_1, y_1) and has the slope 'm' is

$$y - y_1 = m(x - x_1)$$

Proof: Let $Q(x_1, y_1)$ be the point through which the line passes and let $P(x, y)$ be any point on the line. Then, the slope of the line is

$$\frac{y-y_1}{x-x_1}$$

But m is the slope of the line. Therefore,

$$m = \frac{y-y_1}{x-x_1} \Rightarrow y - y_1 = m(x - x_1)$$

Thus, $y - y_1 = m(x - x_1)$ is the required equation of the line.

Example 1.44 **Find the equation of a straight line which cuts-off an intercept of 5 units on negative direction of y-axis and makes an angle of 120° with the positive direction of x-axis.**

Sol. Here, $m = \tan 120° = \tan(90 + 30°) = -\cot 30° = -\sqrt{3}$ and $c = -5$. So, the equation of the line is

$$y = -\sqrt{3}x - 5 \Rightarrow \sqrt{3}x + y + 5 = 0$$

Example 1.45 **Find the equation of a straight line cutting off an intercept – 1 from y-axis and being equally inclined to the axes.**

Sol. Since the required line is equally inclined with coordinates axes; therefore, it makes an angle of either 45° or 135° with the x-axes.

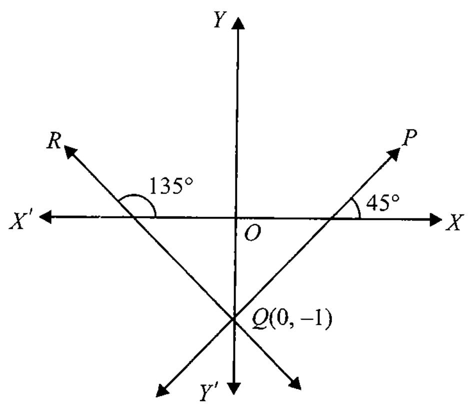


Fig. 1.34

So, its slope is either $m = \tan 45°$ or $m = \tan 135°$, i.e., $m = 1$ or -1. It is given that $c = -1$. Hence, the equations of the lines are

$$y = x - 1 \text{ and } y = -x - 1$$

Example 1.46 **Find the equation of a line that has y- intercept 4 and is perpendicular to the line joining (2, – 3) and (4, 2).**

Sol. Let m be the slope of the required line.

Since the required line is perpendicular to the line joining $A(2, -3)$ and $B(4, 2)$. Therefore,

$$m \times \text{slope of } AB = -1$$

$$\Rightarrow \quad m \times \frac{2+3}{4-2} = -1$$

$$\Rightarrow \quad m = -\frac{2}{5}$$

The required line cuts-off an intercept 4 on y-axis, so $c = 4$.

Hence, the equation of the required line is

$$y = -\frac{2}{5}x + 4$$

$$\Rightarrow \quad 2x + 5y - 20 = 0$$

Two-Point Form of a Line

The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

Proof:

Let m be the slope of the passing through (x_1, y_1) and (x_2, y_2), then

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

So, the equation of the line is

$$y - y_1 = m(x - x_1) \text{ (Using point – slope form)}$$

Substituting the value of m, we obtain

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

This is the required equation of the line in two-point form.

Example 1.47 **Find the equation of the perpendicular bisector of the line segment joining the points $A(2, 3)$ and $B(6, -5)$.**

Sol. The slope of AB is given by m

$$= \frac{-5-3}{6-2} = -2$$

$\Rightarrow$ The slope of a line $\perp$ to AB

$$= -\frac{1}{m} = \frac{1}{2}$$

Let P be the midpoint of AB, then the coordinates of P are

$\left(\frac{2+6}{2}, \frac{3-5}{2}\right)$, i.e., $(4, -1)$

Thus, the required line passes through $P(4, -1)$ and has slope 1/2.

So its equation is

$$y + 1 = \frac{1}{2}(x-4) \text{ [(Using } y - y_1 = m(x - x_1)]$$

or $\quad x - 2y - 6 = 0$

Example 1.48 **Find the equations of the medians of the triangle ABC whose vertices are $A(2, 5)$, $B(-4, 9)$, and $C(-2, -1)$.**

Sol. Let D, E, F be the midpoints of BC, CA and AB, respectively. Then the coordinates of these points are $D(-3, 4)$, $E(0, 2)$, and $F(-1, 7)$, respectively.

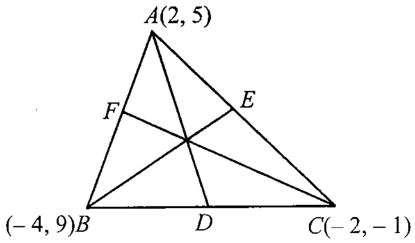


Fig. 1.35

The median AD passes through points $A(2, 5)$ and $D(-3, 4)$.

So, the equation of AD is

$$y - 5 = \frac{4-5}{-3-2}(x-2)$$

$$\Rightarrow \quad y - 5 = \frac{1}{5}(x-2)$$

$$\Rightarrow \quad x - 5y + 23 = 0$$

The median BE passes through points $B(-4, 9)$ and $E(0, 2)$

So, the equation of median BE is

$$(y - 9) = \left(\frac{2-9}{0+4}\right)(x+4)$$

$$\Rightarrow \quad 7x + 4y - 8 = 0$$

Similarly, the equation of the median CF is

$$(y + 1) = \frac{7+1}{-1+2}(x+2)$$

$$\Rightarrow \quad 8x - y + 15 = 0$$

Example 1.49 **In what ratio does the line joining the points (2, 3) and (4, 1) divide the segment joining the points (1, 2) and (4, 3)?**

Sol. The equation of the line joining the points (2, 3) and (4, 1) is

$$y - 3 = \frac{1-3}{4-2}(x-2)$$

$$\Rightarrow \quad y - 3 = -x + 2$$

$$\Rightarrow \quad x + y - 5 = 0 \qquad \text{(i)}$$

Suppose the line joining (2, 3) and (4, 1) divides the segment joining (1, 2) and (4, 3) at point P in the ratio $\lambda : 1$.

Then the coordinates of P are

$$\left(\frac{4\lambda+1}{\lambda+1}, \frac{3\lambda+2}{\lambda+1}\right)$$

Clearly, P lies on (i)

$$\Rightarrow \frac{4\lambda+1}{\lambda+1} + \frac{3\lambda+2}{\lambda+1} - 5 = 0$$

$$\Rightarrow \quad \lambda = 1$$

Hence, the required ratio is $\lambda : 1$, i.e., $1 : 1$.

Example 1.50 **Find the equations of the altitudes of the triangle whose vertices are $A(7, -1)$, $B(-2, 8)$, and $C(1, 2)$ and hence orthocentre of triangle.**

Sol.

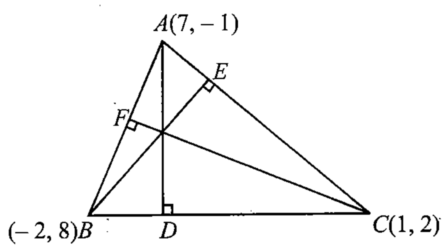


Fig. 1.36

Let AD, BE, and CF be three altitudes of triangle ABC.

Let m_1, m_2, and m_3 be the slopes of AD, BE, and CF, respectively.

Then, $AD \perp BC$

$\Rightarrow$ Slope of AD × Slope of $BC = -1$

$$\Rightarrow \quad m_1 \times \left(\frac{2-8}{1+2}\right) = -1$$

$$\Rightarrow \quad m_1 = \frac{1}{2}$$

$BE \perp AC$

$\Rightarrow$ Slope of BE × Slope of $AC = -1$

$$\Rightarrow \quad m_2 \times \left(\frac{-1-2}{7-1}\right) = -1$$

$$\Rightarrow \quad m_2 = 2$$

And, $CF \perp AB$

$\Rightarrow$ Slope of CF × Slope of $AB = -1$

$$\Rightarrow \quad m_3 \times \frac{-1-8}{7+2} = -1$$

$$\Rightarrow \quad m_3 = 1$$

Since AD passes through $A(7, -1)$ and has slope

$$m_1 = 1/2.$$

So, its equation is

$$y + 1 = \frac{1}{2}(x - 7)$$

$$\Rightarrow \quad x - 2y - 9 = 0$$

Similarly, equation of BE is

$$y - 8 = 2(x + 2)$$

$$\Rightarrow \quad 2x - y + 12 = 0$$

Equation of CF is

$$y - 2 = 1\,(x - 1)$$

$$\Rightarrow \quad x - y + 1 = 0$$

Intercept Form of a Line

The equation of a line which cuts-off intercepts a and b, respectively from the x and y-axes is

$$\frac{x}{a} + \frac{y}{b} = 1$$

Proof: Let AB be the line which cuts-off intercepts $OA = a$ and $OB = b$ on the x and y-axes respectively. Let $P(x, y)$ be any point on the line.

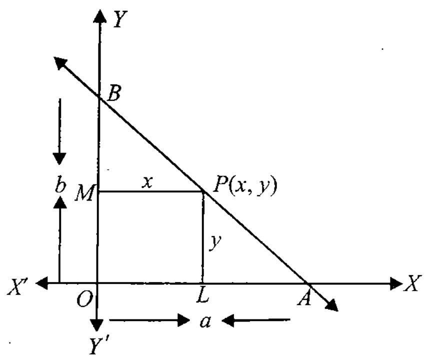


Fig. 1.37

From the diagram, we get that

Area of ΔOAB = Area of ΔOPA + Area of ΔOPB

$$\Rightarrow \quad \frac{1}{2}OA \times OB = \frac{1}{2}OA \times PL + \frac{1}{2}OB \times PM$$

$$\Rightarrow \quad \frac{1}{2}ab = \frac{1}{2}ay + \frac{1}{2}bx$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1$$

This is the equation of the line in the intercept form.

Example 1.51 **Find the equation of the line which passes through the point (3, 4) and the sum of its intercepts on the axes is 14.**

Sol. Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{(i)}$$

This passes through (3, 4), therefore

$$\frac{3}{a} + \frac{4}{b} = 1 \qquad \text{(ii)}$$

It is given that $a + b = 14$

$$\Rightarrow \quad b = 14 - a$$

Putting $b = 14 - a$ in (ii), we get

$$\frac{3}{a} + \frac{4}{14 - a} = 1$$

$$\Rightarrow \quad a^2 - 13a + 42 = 0$$

$$\Rightarrow \quad (a - 7)(a - 6) = 0$$

$$\Rightarrow \quad a = 7, 6$$

For $a = 7$, $b = 14 - 7 = 7$

and for $a = 6$, $b = 14 - 6 = 8$

Putting the values of a and b in (i), we get the equations of the lines

$$\frac{x}{7} + \frac{y}{7} = 1 \text{ and } \frac{x}{6} + \frac{y}{8} = 1$$

or $\quad x + y = 7$ and $4x + 3y = 24$

Example 1.52 **Find the equation of the straight line that**

i. makes equal intercepts on the axes and passes through the point (2, 3),

ii. passes through the point (−5, 4) and is such that the portion intercepted between the axes is divided by the point in the ratio 1 : 2.

Sol. i. Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1$$

Since it makes equal intercepts on the coordinates axes, therefore $a = b$.

So, the equation of the line is

$$\frac{x}{a} + \frac{y}{a} = 1 \text{ or } x + y = a$$

The line passes through the point (2, 3).

Therefore, $\quad 2 + 3 = a$

$$\Rightarrow \quad a = 5$$

Thus, the equation of the required line is $x + y = 5$.

ii. Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1$$

Clearly, this line meets the coordinate axes at $A(a, 0)$ and $B(0, b)$, respectively.

The coordinates of the point that divides the line joining $A(a, 0)$ and $B(0, b)$ in the ratio 1 : 2 are

$$\left(\frac{1(0) + 2(a)}{1+2}, \frac{1(b) + 2(0)}{1+2}\right) = \left(\frac{2a}{3}, \frac{b}{3}\right)$$

It is given that the point (–5, 4) divides AB in the ratio 1 : 2.

Therefore, $2a/3 = -5$ and $b/3 = 4$

$\Rightarrow$ $a = -15/2$ and $b = 12$

Hence, the equation of the required line is

$$-\frac{x}{\frac{15}{2}} + \frac{y}{12} = 1$$

$\Rightarrow$ $8x - 5y + 60 = 0$

Normal Form or Perpendicular Form of a Line

The equation of the straight line upon which the length of the perpendicular from the origin is p and this perpendicular makes an angle α with +ve direction of x-axis is

$$x \cos\alpha + y \sin\alpha = p$$

Proof: Let the line AB be such that the length of the perpendicular OQ from the origin O to the line be p and $\angle XOQ = \alpha$.

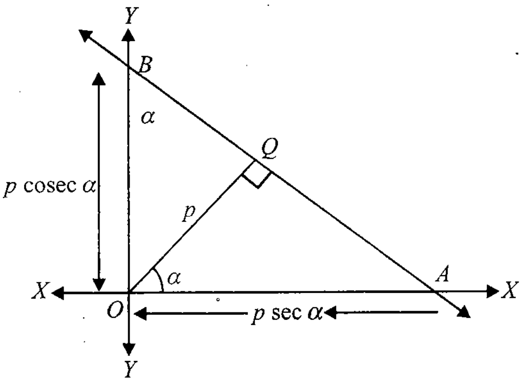


Fig. 1.38

From the diagram, using the intercept form, we get

Equation of line AB is

$$\frac{x}{p\sec\alpha} + \frac{y}{p\,\text{cosec}\,\alpha} = 1$$

or $\quad x\cos\alpha + y\sin\alpha = p$

Example 1.53 **The length of the perpendicular from the origin to a line is 7 and the line makes an angle of 150° with the positive direction of *y*-axis. Find the equation of the line.**

Sol. Here $p = 7$ and $\alpha = 30°$

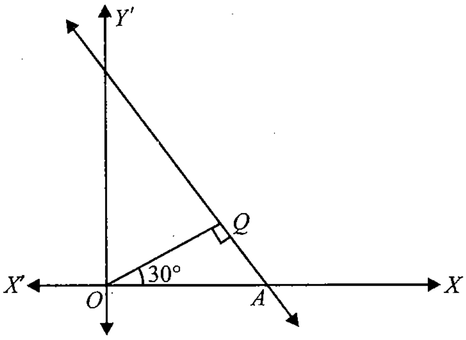


Fig. 1.39

Equation of the required line is

$$x\cos 30° + y \sin 30° = 7$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2}x + \frac{1}{2}y = 7$$

$$\Rightarrow \quad \sqrt{3}x + y = 14$$

ANGLE BETWEEN TWO STRAIGHT LINES WHEN THEIR EQUATIONS ARE GIVEN

Let the angle θ between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by

$$\tan\theta = \left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$$

Proof: Let m_1 and m_2 be the slopes of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, respectively.

Then,

$$m_1 = -a_1/b_1 \text{ and } m_2 = -a_2/b_2$$

Now,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

$$\Rightarrow \quad \tan\theta = \left|\frac{-\frac{a_1}{b_1} + \frac{a_2}{b_2}}{1 + \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right)}\right|$$

$$\Rightarrow \quad \tan\theta = \left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$$

$$\Rightarrow \quad \theta = \tan^{-1}\left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$$

Condition for the Lines to be Parallel

If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel, then

$$m_1 = m_2$$

$$\Rightarrow \quad -\frac{a_1}{b_1} = -\frac{a_2}{b_2}$$

$$\Rightarrow \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

Condition for the Lines to be Perpendicular

If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are perpendicular, then

$$m_1m_2 = -1 \Rightarrow \left(-\frac{a_1}{b_1}\right) \times \left(-\frac{a_2}{b_2}\right) = -1$$

$$\Rightarrow \quad a_1a_2 + b_1b_2 = 0$$

It follows from the above discussion that the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

i. Coincident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

ii. Parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

iii. Intersecting, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

iv. Perpendicular, if $a_1a_2 + b_1b_2 = 0$

Example 1.54 **Find the angle between the pairs of straight lines**

i. $x - y\sqrt{3} - 5 = 0$ and $\sqrt{3}x + y - 7 = 0$

ii. $y = (2 - \sqrt{3})x + 5$ and $y = (2 + \sqrt{3})x - 7$

Sol. i. The equations of two straight lines are

$$x - y\sqrt{3} - 5 = 0 \quad \text{(i)}$$

and

$$\sqrt{3}x + y - 7 = 0 \quad \text{(ii)}$$

Let m_1 and m_2 be the slopes of these two lines. Then,

$$m_1 = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$$

and $$m_2 = -\frac{\sqrt{3}}{1} = -\sqrt{3}$$

Clearly, $m_1m_2 = -1$. Thus, the two lines are at right angle.

ii. Let m_1 and m_2 be the slopes of the straight lines $y = (2 - \sqrt{3})x + 5$ and $y = (2 + \sqrt{3})x - 7$, respectively. Then,

$$m_1 = 2 - \sqrt{3} \text{ and } m_2 = 2 + \sqrt{3}$$

Let θ be the angle between the lines. Then,

$$\tan\theta = \pm\left(\frac{(m_1 - m_2)}{1 + m_1m_2}\right)$$

$$= \pm\left(\frac{(2 - \sqrt{3}) - (2 + \sqrt{3})}{1 + (2 - \sqrt{3})(2 + \sqrt{3})}\right)$$

$$= \pm\left(-\frac{2\sqrt{3}}{1 + 4 - 3}\right) = \pm\sqrt{3}$$

So, the acute angle between the lines is given by

$$\tan\theta = |\pm\sqrt{3}| = \sqrt{3}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

Example 1.55 **A straight canal is $4\frac{1}{2}$ miles from a place and the shortest route from this place to the canal is exactly north-east. A village is 3 miles north and four east from the place. Does it lie by the nearest edge of the canal?**

Sol. Let the given place be O. Take this as the origin and the east and north directions through O as the x-and y-axes, respectively.

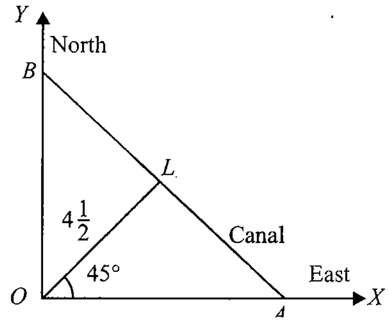


Fig. 1.40

Let AB be the nearest edge of the canal. From the question, OL is perpendicular to AB such that $OL = 4\frac{1}{2}$ miles and $\angle LOA = 45°$

So, the equation of the canal is

$$x\cos 45° + y\sin 45° = 4\frac{1}{2}$$

$$\Rightarrow \quad \sqrt{2}(x + y) = 9 \quad \text{(i)}$$

The position of the village is (4, 3). The village will lie on the edge of the canal if (4, 3) satisfies the Eq. (i).

Clearly, (4, 3) does not satisfy (i). Hence, the village does not lie by the nearer edge of the canal.

Example 1.56 **Reduce the line $2x - 3y + 5 = 0$ in slope-intercept, intercept, and normal forms.**

Sol. $y = \frac{2x}{3} + \frac{5}{3}$, $\tan\theta = m = 2/3$, $c = \frac{5}{3}$

Intercept form:

$$\frac{x}{\left(-\frac{5}{2}\right)} + \frac{y}{\left(\frac{5}{3}\right)} = 1, a = -\frac{5}{2}, b = \frac{5}{3}$$

Normal form:

$$-\frac{2x}{\sqrt{13}} + \frac{3y}{\sqrt{13}} = \frac{5}{\sqrt{13}}$$

$$\sin\alpha = \frac{3}{\sqrt{13}}, \cos\alpha = -\frac{2}{\sqrt{13}}, p = \frac{5}{\sqrt{13}}$$

Example 1.57 **A rectangle has two opposite vertices at the points (1, 2) and (5, 5). If the other vertices lie on the line $x = 3$, find the other vertices of the rectangle.**

Sol. Let $A \equiv (1, 2)$, $C \equiv (5, 5)$

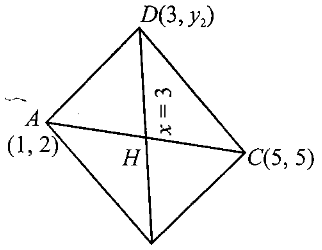


Fig. 1.41

Since vertices B and D lie on line $x = 3$; therefore, let $B \equiv (3, y_1)$ and $D \equiv (3, y_2)$.

Now since AC and BD bisect each other, therefore, middle points of AC and BD will be same

$$\frac{y_1 + y_2}{2} = \frac{2+5}{2} \text{ or } y_1 + y_2 = 7 \quad \text{(i)}$$

Also $\quad BD^2 = AC^2$

$\therefore \quad (y_1 - y_2)^2 = (1-5)^2 + (2-5)^2 = 25$

or $\quad y_1 - y_2 = \pm 5 \quad$ (ii)

Solving (i) and (ii), we get

$$y_1 = 6, y_2 = 1 \text{ or } y_1 = 1, y_2 = 6$$

Hence, other vertices of the rectangle are (3, 1) and (3, 6).

Example 1.58 **A vertex of an equilateral triangle is (2, 3) and the equation of the opposite side is $x + y = 2$, find the equation of the other sides of the triangle.**

Sol. Given line is

$$x + y - 2 = 0 \quad \text{(i)}$$

Its slope $\quad m_1 = -1.$

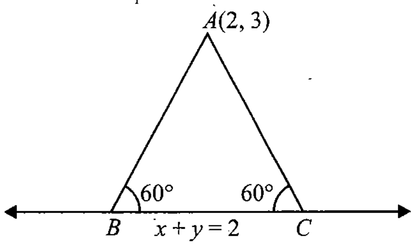


Fig. 1.42

Let the slope of the line be m which makes an angle of 60° with line in Eq. (i), then

$$\tan 60° = \left|\frac{m_1 - m}{1 + m_1 m}\right| \text{ or } \sqrt{3} = \left|\frac{-1-m}{1-m}\right|$$

or $\quad \sqrt{3} = \left|\frac{1+m}{m-1}\right| \text{ or } \frac{1+m}{m-1} = \pm\sqrt{3}$

or $\quad 1 + m = \pm\sqrt{3}(m-1)$

$\Rightarrow \quad m = \frac{\sqrt{3}+1}{\sqrt{3}-1}, \frac{\sqrt{3}-1}{\sqrt{3}+1}$

$$= 2 + \sqrt{3}, 2 - \sqrt{3}$$

Equation of other two sides of the triangle are

$$y - 3 = (2 + \sqrt{3})(x - 2)$$

and $\quad y - 3 = (2 - \sqrt{3})(x - 2)$

Example 1.59 **Two consecutive sides of a parallelogram are $4x + 5y = 0$ and $7x + 2y = 0$. If the equation to one diagonal is $11x + 7y = 9$, find the equation of the other diagonal.**

Sol.

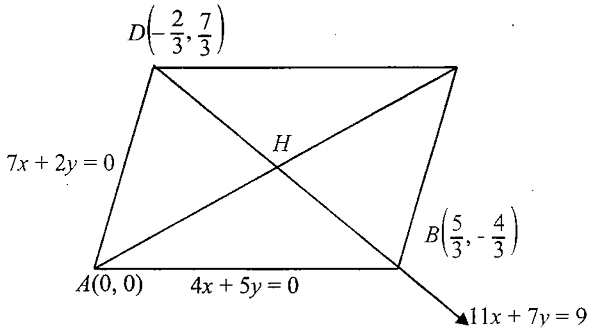


Fig. 1.43

Let the equation of sides AB and AD of the parallelogram $ABCD$ be as given in Eqs. (i) and (ii), respectively, i.e.,

$$4x + 5y = 0 \quad \text{(i)}$$

and $\quad 7x + 2y = 0 \quad$ (ii)

Solving (i) and (ii), we have

$$x = 0, y = 0$$

$$\therefore \quad A \equiv (0, 0)$$

Equation of one diagonal of the parallelogram is

$$11x + 7y = 9 \quad \text{(iii)}$$

Clearly, $A(0, 0)$ does not lie on diagonal as shown in Eq. (iii), therefore Eq. (iii) is the equation of diagonal BD.

Solving (i) and (iii), we get $B \equiv \left(\frac{5}{3}, -\frac{4}{3}\right)$

Solving (ii) and (iii), we get $D \equiv \left(-\frac{2}{3}, \frac{7}{3}\right)$

Since H is the middle point of BD

$$\therefore \quad H \equiv \left(\frac{1}{2}, \frac{1}{2}\right)$$

Now, equation of diagonal AC which passes through $A(0, 0)$ and $H\left(\frac{1}{2}, \frac{1}{2}\right)$ is

$$y - 0 = \frac{0 - \frac{1}{2}}{0 - \frac{1}{2}}(x - 0) \text{ or } y - x = 0$$

Example 1.60 **A line $4x + y = 1$ through the point $A(2, -7)$ meets the line BC whose equation is $3x - 4y + 1 = 0$ at the point B. Find the equation of the line AC, so that $AB = AC$.**

Sol.

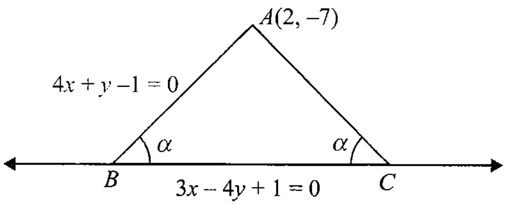


Fig. 1.44

Let the equation of BC be

$$3x - 4y + 1 = 0 \quad \text{(i)}$$

and the equation of AB be

$$4x + y - 1 = 0 \quad \text{(ii)}$$

Since $AB = AC$ $\therefore \angle ABC = \angle ACB = \alpha$ (say)

Slope of line BC = 4/3 and slope of $AB = -4$.

Let slope of $AC = m$, equating the two values of $\tan \alpha$, we get

$$\left|\frac{-4 - \frac{3}{4}}{1 - 4 \times \frac{3}{4}}\right| = \left|\frac{\frac{3}{4} - m}{1 + \frac{3}{4}m}\right|$$

$$\Rightarrow \quad \pm\frac{19}{8} = \frac{3 - 4m}{4 + 3m}$$

$$\Rightarrow \quad m = 52/89 \text{ or } m = -4$$

Therefore, equation of AC is

$y + 7 = -(52/89)(x - 2)$ or $52x + 89y + 519 = 0$

Example 1.61 **A variable straight line is drawn through the point of intersection of the straight lines $x/a + y/b = 1$ and $x/b + y/a = 1$ and meets the coordinate axes at A and B. Show that the locus of the midpoint of AB is the curve $2xy(a + b) = ab(x + y)$.**

Sol.

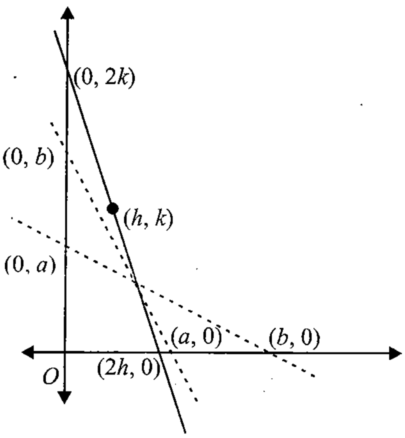


Fig. 1.45

Given lines are

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{(i)}$$

and

$$\frac{x}{b} + \frac{y}{a} = 1 \quad \text{(ii)}$$

Solving (i) and (ii), we get

$$x = \frac{ab}{a + b} \text{ and } y = \frac{ab}{a + b}$$

Now a variable line passing through these points meets the axis at points A and B. Let the midpoint of AB be (h, k) whose locus is to be found.

Then coordinates of A and B are $(2h, 0)$ and $(0, 2k)$.

Now points A, B and $[ab/(a + b), ab/(a + b)]$ are coliinear.

Then

$$\Delta = \frac{1}{2}\begin{vmatrix} 2h & 0 & 1 \\ 0 & 2k & 1 \\ \frac{ab}{a+b} & \frac{ab}{a+b} & 1 \end{vmatrix} = 0$$

$$\Rightarrow 4hk - 2h\frac{ab}{a + b} - 2k\frac{ab}{a + b} = 0$$

$$\Rightarrow \quad 2xy(a + b) = ab(x + y)$$

Example 1.62 **If the line $(x/a) + (y/b) = 1$ moves in such a way that $(1/a^2) + (1/b^2) = (1/c^2)$ where c is a constant, prove that the foot of the perpendicular from the origin on the straight line describes the circle $x^2 + y^2 = c^2$.**

Sol. Variable line is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{(i)}$$

Any line perpendicular to Eq. (i) and passing through the origin will be

$$\frac{x}{b} - \frac{y}{a} = 0 \quad \text{(ii)}$$

Now the foot of the perpendicular from the origin to the line Eq. (i) is the point of intersection of Eq. (i) and (ii).

Let be $P(\alpha, \beta)$, then $\frac{\alpha}{a} + \frac{\beta}{b} = 1$ (iii)

and

$$\frac{\alpha}{b} - \frac{\beta}{a} = 0 \quad \text{(iv)}$$

Squaring and adding Eqs. (iii) and (iv), we get

$$\alpha^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) + \beta^2\left(\frac{1}{b^2} + \frac{1}{a^2}\right) = 1$$

But

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2} \text{ (given)}$$

Hence c is a constant and a, b are parameters (variables). Therefore,

$$(\alpha^2 + \beta^2)\frac{1}{c^2} = 1.$$

Hence, the locus of $P(\alpha, \beta)$ is

$$x^2 + y^2 = c^2$$

Equation of a Line Parallel to a Given Line

The equation of a line parallel to a given line $ax + by + c = 0$ is

$$ax + by + \lambda = 0$$

where λ is a constant. Find λ by using given condition.

Equation of a Line Perpendicular to a Given Line

The equation of a line perpendicular to a given line $ax + by + c = 0$ is

$$bx - ay + \lambda = 0$$

where λ is a constant. Find λ by using given condition.

Example 1.63 **Find the equation of the line which is parallel to $3x - 2y + 5 = 0$ and passes through the point (5, – 6).**

Sol. The equation of any line parallel to the line $3x - 2y + 5 = 0$ is

$$3x - 2y + \lambda = 0 \quad \text{(i)}$$

This passes through (5, – 6), therefore we get

$$3 \times (5) - 2 \times (-6) + \lambda = 0$$

$$\Rightarrow \quad \lambda = -27$$

Putting $\lambda = -27$ in (i), we get

$$3x - 2y - 27 = 0$$

which is the required equation.

Example 1.64 **Find the equation of the straight line that passes through the point (3, 4) and perpendicular to the line $3x + 2y + 5 = 0$.**

Sol. The equation of a line perpendicular to $3x + 2y + 5 = 0$ is

$$2x - 3y + \lambda = 0 \quad \text{(i)}$$

This passes through the point (3, 4), therefore we get

$$2 \times (3) - 3 \times (4) + \lambda = 0$$

$$\Rightarrow \quad \lambda = 6$$

Putting $\lambda = 6$ in (i), we get

$$2x - 3y + 6 = 0$$

which is the required equation.

Example 1.65 **Find the coordinates of the foot of the perpendicular drawn from the point (1, – 2) on the line $y = 2x + 1$.**

Sol. Let M be the foot of the perpendicular drawn from $P(1, -2)$ on the line $y = 2x + 1$.

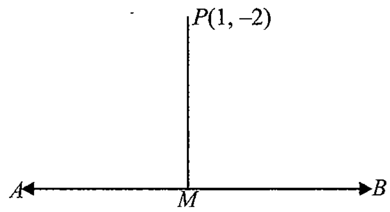


Fig. 1.46

Then M is the point of intersection of $y = 2x + 1$ and a line passing through $P(1, -2)$ and perpendicular to $y = 2x + 1$.

The equation of a line perpendicular to $y = 2x + 1$ or $2x - y + 1 = 0$ is

$$x + 2y + \lambda = 0 \quad \text{(i)}$$

This passes through $P(1, -2)$, therefore we get

$$\Rightarrow \quad 1 - 4 + \lambda = 0$$

$$\Rightarrow \quad \lambda = 3$$

Putting $\lambda = 3$ in (i), we get

$$x + 2y + 3 = 0$$

Point M is the point of intersection of $2x - y + 1 = 0$ and $x + 2y + 3 = 0$

Solving these equations by cross-multiplication, we get

$$\frac{x}{-5} = \frac{y}{-5} = \frac{1}{5}$$

$$\Rightarrow \quad x = -1 \text{ and } y = -1$$

Hence, the coordinates of the foot of the perpendicular are (–1, –1).

Example 1.66 **Find the image of the point (–8, 12) with respect to the line mirror $4x + 7y + 13 = 0$.**

Sol. Let the image of the point $P(-8, 12)$ in the line mirror AB be $Q(\alpha, \beta)$.

Then, PQ is perpendicularly bisected at R.

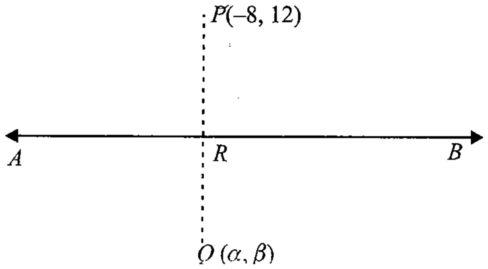


Fig. 1.47

The coordinates of R are $\left(\frac{\alpha - 8}{2}, \frac{\beta + 12}{2}\right)$

Since R lies on $4x + 7y + 13 = 0$, we get

$2\alpha - 16 + (7\beta + 84)/2 + 13 = 0$

$$\Rightarrow \quad 4\alpha + 7\beta + 78 = 0 \qquad \text{(i)}$$

Since $PQ \perp AB$, therefore (slope of AB) × (slope of PQ) $= -1$

$$\Rightarrow \quad -\frac{4}{7} \times \frac{\beta - 12}{\alpha + 8} = -1$$

$$\Rightarrow \quad 7\alpha - 4\beta + 104 = 0 \qquad \text{(ii)}$$

Solving (i) and (ii), we get

$$\alpha = -16, \beta = -2$$

Hence, the image of (–8, 12) in the line mirror $4x + 7y + 13 = 0$ is (–16, – 2).

Example 1.67 **A ray of the light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line $3x - 2y - 5 = 0$, the ray is reflected. Find the equation of the line containing the reflected ray.**

Sol. Solving the equations of LM and PA coordinates of A can be obtained.

If slope of AQ is determined, then the equation of AQ can be determined.

If slope of AQ is m, then equating the two values of $\tan\theta$ (considering the angles between AL and AP and between AM and AQ, m can be found.

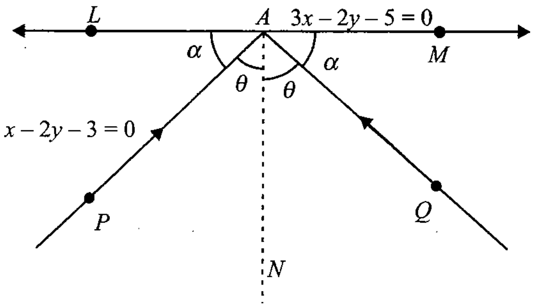


Fig. 1.48

Equation of line LM is

$$3x - 2y - 5 = 0 \qquad \text{(i)}$$

Equation of PA is

$$x - 2y - 3 = 0 \qquad \text{(ii)}$$

Solving (i) and (ii), we get

$$x = 1, y = -1$$

$$\therefore \quad A \equiv (1, -1)$$

Let slope of $AQ = m$, slope of $LM = 3/2$, slope of $PA = 1/2$

Let $\angle LAP = \angle QAM = \alpha$

As $\angle LAP = \alpha$

$$\Rightarrow \quad \tan\alpha = \left|\frac{\frac{3}{2} - \frac{1}{2}}{1 + \frac{3}{2} \times \frac{1}{2}}\right| = \frac{4}{7} \qquad \text{(iii)}$$

Again $\angle QAM = \alpha$

$$\therefore \quad \tan\alpha = \left|\frac{m - \frac{3}{2}}{1 + \frac{3}{2}m}\right| = \left|\frac{2m - 3}{2 + 3m}\right| \qquad \text{(iv)}$$

From Eq (iii) and (iv), we have

$$\left|\frac{2m - 3}{2 + 3m}\right| = \frac{4}{7}$$

or

$$\frac{2m - 3}{2 + 3m} \pm \frac{4}{7}$$

$$\therefore \quad m = \frac{1}{2}, \frac{29}{2}$$

But slope of $AP = \frac{1}{2}$

$\therefore$ Slope of $AQ = \frac{29}{2}$

Now, the equation of AQ will be

$$y + 1 = \frac{29}{2}(x - 1)$$

or

$$29x - 2y - 31 = 0$$

Example 1.68 **A ray of light is sent along the line $2x - 3y = 5$. After refracting across the line $x + y = 1$ it enters**

the opposite side after turning by 15° away from the line $x + y = 1$. Find the equation of the line along which the refracted ray travels.

Sol.

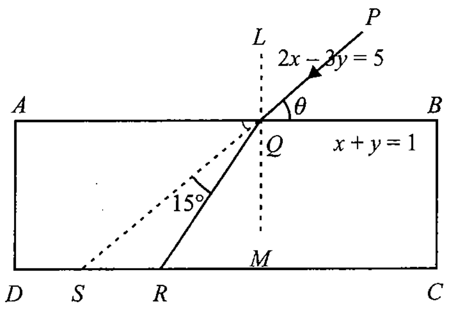


Fig. 1.49

Equation of line AB is

$$x + y = 1 \quad \text{(i)}$$

Equation of line QP is

$$2x - 3y = 5 \quad \text{(ii)}$$

QR is the refracted ray. According to question $\angle SQR = 15°$

Solving Eqs. (i) and (ii), we get

$$x = \frac{8}{5} \text{ and } y = -\frac{3}{5}$$

$$\Rightarrow \quad Q \equiv \left(\frac{8}{5}, -\frac{3}{5}\right)$$

$$\text{Slope of } QP = \frac{2}{3}$$

$$\Rightarrow \quad \text{Slope of } QS = \frac{2}{3}$$

Let slope of $\quad QR = m$

But $\quad \angle SQR = 15°$

$$\Rightarrow \quad \tan 15° = \left|\frac{\frac{2}{3} - m}{1 + \frac{2}{3}m}\right|$$

$$\Rightarrow \quad 2 - \sqrt{3} = \left|\frac{2 - 3m}{3 + 2m}\right|$$

$$\Rightarrow \quad \frac{2 - 3m}{3 + 2m} = \pm (2 - \sqrt{3})$$

$$\Rightarrow \quad 2 - 3m = \pm [6 - 3\sqrt{3} + (4 - 2\sqrt{3})m]$$

$$\text{or} \quad 2 - 3m = \begin{cases} 6 - 3\sqrt{3} + (4 - 2\sqrt{3})m \\ -6 + 3\sqrt{3} - (4 - 2\sqrt{3})m \end{cases}$$

$$\Rightarrow \quad m = \frac{3\sqrt{3} - 4}{7 - 2\sqrt{3}}, \frac{3\sqrt{3} - 8}{1 - 2\sqrt{3}}$$

For required line $m = \dfrac{3\sqrt{3} - 8}{1 - 2\sqrt{3}}$

Concept Application Exercise 1.3

1. Find the angle between lines $x = 2$ and $x - 3y = 6$.
2. If the coordinates of the points A, B, C, and D, be (a, b), (a', b'), $(-a, b)$, and $(a', -b')$, respectively, then find the equation of the line bisecting the line segments AB and CD.
3. If the coordinates of the vertices of the triangle ABC are $(-1, 6)$, $(-3, -9)$, and $(5, -8)$, respectively, then find the equation of the median through C.
4. Find the equation of the line perpendicular to the line $\frac{x}{a} - \frac{y}{b} = 1$ and passing through a point at which it cuts x-axis.
5. If the middle points of the sides BC, CA, and AB of the triangle ABC are $(1, 3)$, $(5, 7)$ and $(-5, 7)$, respectively, then find the equation of the side AB.
6. Find the equations of the lines which pass through the origin and are inclined at an angle $\tan^{-1} m$ to the line $y = mx + c$.
7. If $(-2, 6)$ is the image of the point $(4, 2)$ with respect to line $L = 0$, then find the equation of line L.
8. If the lines $x + (a - 1)y + 1 = 0$ and $2x + a^2y - 1 = 0$ are perpendicular, then find the values of a.
9. Find the area bounded by the curves $x + 2|y| = 1$ and $x = 0$.
10. Find the equation of the straight line passing through the intersection of the lines $x - 2y = 1$ and $x + 3y = 2$ and parallel to $3x + 4y = 0$.
11. A straight line through the point $(2, 2)$ intersects the lines $\sqrt{3}x + y = 0$ and $\sqrt{3}x - y = 0$ at the points A and B. Then find the equation to the line AB so that the triangle OAB is equilateral.
12. Find the equation of the straight line passing through the point $(4, 3)$ and making intercepts on the coordinate axes whose sum is -1.
13. A straight line through the point $A(3, 4)$ is such that its intercept between the axis is bisected at A. Find its equation.
14. The diagonals AC and BD of a rhombus intersect at $(5, 6)$. If $A \equiv (3, 2)$, then find the equation of diagonal BD.
15. If the foot of the perpendicular from the origin to a straight line is at the point $(3, -4)$. Then find the equation of the line.
16. If we reduce $3x + 3y + 7 = 0$ to the form $x \cos \alpha + y \sin \alpha = p$, then find the value of p.
17. Find the equation of the straight line which passes through the origin and makes angle 60° with the line $x + \sqrt{3}\, y + 3\sqrt{3} = 0$.

18. A line intersects the straight lines $5x - y - 4 = 0$ and $3x - 4y - 4 = 0$ at A and B, respectively. If a point $P(1, 5)$ on the line AB is such that $AP : PB = 2 : 1$ (internally), find point A.

19. In the given figure PQR is an equilateral triangle and $OSPT$ is a square. If $OT = 2\sqrt{2}$ units, find the equation of lines OT, OS, SP, QR, PR, and PQ.

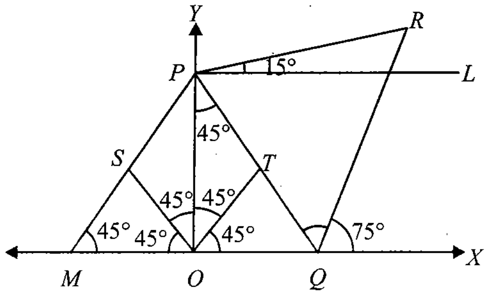


Fig. 1.50

20. Find the obtuse angle between the lines $x - 2y + 3 = 0$ and $3x + y - 1 = 0$.

21. Two fixed points A and B are taken on the coordinates axes such that $OA = a$ and $OB = b$. Two variable point A' and B' are taken on the same axes such that $OA' + OB' = OA + OB$. Find the locus of the point of intersection of AB' and $A'B$.

DISTANCE FORM OF A LINE (PARAMETRIC FORM)

The equation of the straight line passing through (x_1, y_1) and making an angle θ with the positive direction of x-axis is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

where r is the distance of the point (x, y) on the line from point (x_1, y_1).

Proof: Let the given line meets x-axis at A, y-axis at B and passes through the point $Q(x_1, y_1)$.

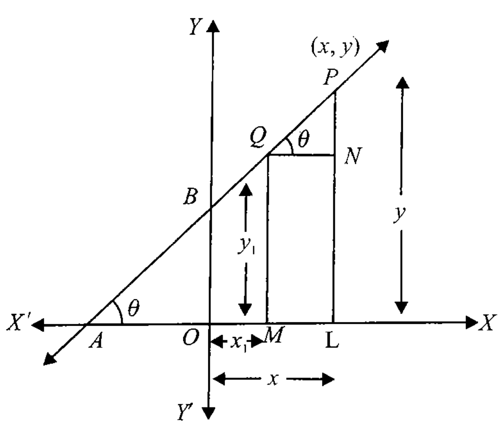


Fig. 1.51

Let $P(x, y)$ be any point on the line at a distance r from $Q(x_1, y_1)$ i.e., $PQ = r$.

In ΔPQN, we have

$$\cos\theta = \frac{QN}{PQ} = \frac{x - x_1}{r}$$

$$\Rightarrow \quad \cos\theta = \frac{x - x_1}{r} \qquad \text{(i)}$$

and

$$\sin\theta = \frac{PN}{PQ}$$

$$\Rightarrow \quad \sin\theta = \frac{y - y_1}{r} \qquad \text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

This is the required equation of the line in the distance form.

$\Rightarrow \quad x - x_1 = r\cos\theta$ and $y - y_1 = r\sin\theta$

$\Rightarrow \quad x = x_1 + r\cos\theta$ and $y = y_1 + r\sin\theta$.

Thus, the coordinates of any point on the line at a distance r from the given point (x_1, y_1) are $(x_1 + r\cos\theta, y_1 + r\sin\theta)$. If P is on the right side of (x_1, y_1), then r is positive and if P is on the left side of (x_1, y_1), then r is negative. Since different values of r determine different points on the line, therefore the above form of the line is also called parametric form or symmetric form of a line.

In the parametric form, one can determine the coordinates of any point on the line at a given distance from the given point through which it passes. At a given distance r from the point (x_1, y_1) on the line $(x - x_1)/\cos\theta = (y - y_1)/\sin\theta$ there are two points viz., $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ and $(x_1 - r\cos\theta, y_1 - r\sin\theta)$.

Example 1.69 **A straight line is drawn through the point $P(2, 3)$ and is inclined at an angle of 30° with the x-axis. Find the coordinates of two points on it at a distance 4 from P.**

Sol. Here $(x_1, y_1) = (2, 3)$, $\theta = 30°$, the equation of the line is

$$\frac{x - 2}{\cos 30°} = \frac{y - 3}{\sin 30°}$$

$$\Rightarrow \quad \frac{x - 2}{\frac{\sqrt{3}}{2}} = \frac{y - 3}{\frac{1}{2}}$$

$$\Rightarrow \quad x - 2 = \sqrt{3}(y - 3)$$

$$\Rightarrow \quad x - \sqrt{3}y = 2 - 3\sqrt{3}$$

Points on the line at a distance 4 from $P(2, 3)$ are

$(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$

or $(2 \pm 4\cos 30°, 3 \pm 4\sin 30°)$

or $(2 \pm 2\sqrt{3}, 3 \pm 2)$ or $(2 + 2\sqrt{3}, 5)$ and $(2 - 2\sqrt{3}, 1)$

Example 1.70 **Find the equation of the line passing through the point $A(2,3)$ and making an angle of 45° with the x-axis. Also determine the length of intercept on it between A and the line $x + y + 1 = 0$.**

Sol.

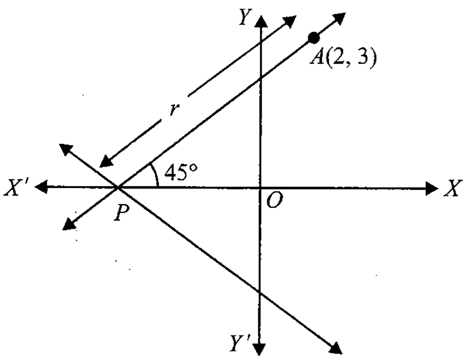


Fig. 1.52

The equation of a line passing through A and making an angle of 45° with the x-axis is

$$\frac{x-2}{\cos 45°} = \frac{y-3}{\sin 45°}$$

$$\frac{x-2}{\frac{1}{\sqrt{2}}} = \frac{y-3}{\frac{1}{\sqrt{2}}}$$

$$\Rightarrow \quad x - y + 1 = 0$$

Suppose this line meets the line $x + y + 1 = 0$ at P such that $AP = r$.

Then, the coordinates of P are given by

$$\frac{x-2}{\cos 45°} = \frac{y-3}{\sin 45°} = r$$

$$\Rightarrow \quad x = 2 + r\cos 45°, y = 3 + r\sin 45°$$

$$\Rightarrow \quad x = 2 + \frac{r}{\sqrt{2}}, y = 3 + \frac{r}{\sqrt{2}}$$

Thus, the coordinates of P are $\left(2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$

Since P lies on $x + y + 1 = 0$. Therefore,

$$2 + \frac{r}{\sqrt{2}} + 3 + \frac{r}{\sqrt{2}} + 1 = 0$$

$$\Rightarrow \quad \sqrt{2}\,r = -6$$

$$\Rightarrow \quad r = -3\sqrt{2}$$

Therefore, length $AP = |r| = 3\sqrt{2}$

Thus, the length of the intercept $= 3\sqrt{2}$

Example 1.71 **The line joining two points $A(2, 0)$, $B(3, 1)$ is rotated about A in anticlockwise direction through an angle of 15°. Find the equation of the line in the new position. If B goes to C in the new position, what will be the coordinates of C?**

Sol. The slope m of the line AB is given by

$$m = \frac{1-0}{3-2} = 1$$

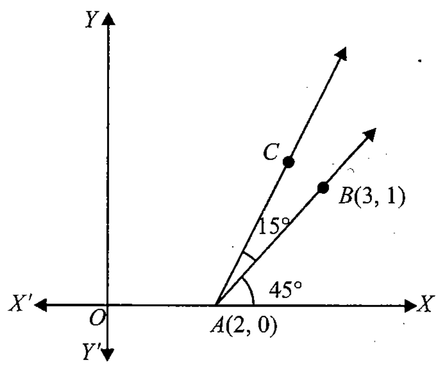


Fig. 1.53

So, AB makes an angle of 45° with x-axis. Now AB is rotated through 15° in anticlockwise direction and so it makes an angle of 60° with x-axis in its new position AC.

Clearly AC passes through $A(2, 0)$ and makes an angle of 60° with x-axis, therefore the equation of AC is

$$\frac{x-2}{\cos 60°} = \frac{y-0}{\sin 60°}$$

$$\frac{x-2}{\frac{1}{2}} = \frac{y-0}{\frac{\sqrt{3}}{2}}$$

We have

$$AB = \sqrt{(3-2)^2 + (1-0)^2} = \sqrt{2}.$$

So, the coordinates of C are given by

$$\frac{x-2}{\frac{1}{2}} = \frac{y-0}{\frac{\sqrt{3}}{2}} = \sqrt{2}$$

$$\Rightarrow \quad x = 2 + \frac{1}{2}\sqrt{2} = 2 + \frac{1}{\sqrt{2}} \text{ and } y = \frac{\sqrt{3}}{2}\sqrt{2} = \frac{\sqrt{6}}{2}$$

Hence, the coordinates of C are $\left(2 + \frac{1}{\sqrt{2}}, \frac{\sqrt{6}}{2}\right)$.

Example 1.72 **Find the distance of the point (1, 3) from the line $2x - 3y + 9 = 0$ measured along a line $x - y + 1 = 0$.**

Sol. The slope of the line $x - y + 1 = 0$ is 1. So it makes an angle of 45° with x-axis.

The equation of a line passing through (1, 3) and making an angle of 45° is

$$\frac{x-1}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} = r$$

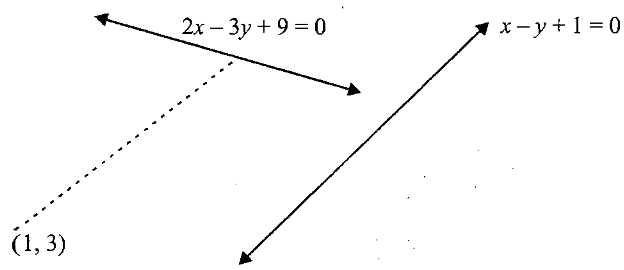


Fig. 1.54

Coordinates of any point on this line are $(1 + r\cos 45^\circ, 3 + r\sin 45^\circ) = \left(1 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$

If this point lies on the line $2x - 3y + 9 = 0$, then

$$2 + r\sqrt{2} - 9 - \frac{3r}{\sqrt{2}} + 9 = 0$$

$$\Rightarrow \quad r = 2\sqrt{2}$$

Hence, the required distance $= 2\sqrt{2}$

Concept Application Exercise 1.4

1. Two particles start from the point (2, − 1), one moves 2 units along the line $x + y = 1$ and the other 5 units along the line $x - 2y = 4$. If the particles move towards increasing, then find their new positions.
2. Find the distance between $A(2, 3)$ on the line of gradient 3/4 and the point of intersection P of this line with $5x + 7y + 40 = 0$.
3. The centre of a square is at the origin and one vertex is $A(2, 1)$. Find the coordinates of other vertices of the square.

CONCURRENCY OF THREE LINES

Three lines are said to be concurrent if they pass through a common point, i.e., they meet at a point.

Thus, if three line are concurrent the point of intersection of two lines lies on the third line. Let the three concurrent lines be

$$a_1x + b_1y + c_1 = 0 \quad \text{(i)}$$
$$a_2x + b_2y + c_2 = 0 \quad \text{(ii)}$$
$$a_3x + b_3y + c_3 = 0 \quad \text{(iii)}$$

Then the point of intersection of Eqs. (i) and (ii) must lie on the third.

The coordinates of the point of intersection of Eqs. (i) and (ii) are

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

This point lies on line (iii). Therefore, we get

$$\Rightarrow \quad a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

$$\Rightarrow \quad a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$$

$$\Rightarrow \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition of concurrency of three lines.

Alternative Method:

Three lines $L_1 \equiv a_1x + b_1y + c_1 = 0$; $L_2 \equiv a_2x + b_2y + c_2 = 0$; $L_3 \equiv a_3x + b_3y + c_3 = 0$ are concurrent iff there exist constants $\lambda_1, \lambda_2, \lambda_3$ not all zero at the same time so that $\lambda_1L_1 + \lambda_2L_2 + \lambda_3L_3 = 0$, i.e., $\lambda_1(a_1x + b_1y + c_1) + \lambda_2(a_2x + b_2y + c_2) + \lambda_3(a_3x + b_3y + c_3) = 0$.

Example 1.73 **Find the value of λ, if the lines $3x - 4y - 13 = 0$, $8x - 11y - 33$ and $2x - 3y + \lambda = 0$ are concurrent.**

Sol. The given lines are concurrent if

$$\begin{vmatrix} 3 & -4 & -13 \\ 8 & -11 & -33 \\ 2 & -3 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad 3(-11\lambda - 99) + 4(8\lambda + 66) - 13(-24 + 22) = 0$$

$$\Rightarrow \quad -\lambda - 7 = 0 \Rightarrow \lambda = -7$$

Alternative Method: The given equations are

$$3x - 4y - 13 = 0 \quad \text{(i)}$$
$$8x - 11y - 33 = 0 \quad \text{(ii)}$$
$$\text{and} \quad 2x - 3y + \lambda = 0 \quad \text{(iii)}$$

Solving Eqs. (i) and (ii), we get

$$x = 11 \text{ and } y = 5$$

Thus, (11, 5) is the point of intersection of Eqs. (i) and (ii).

The given lines will be concurrent if they pass through the common point, i.e., the point of intersection of any two lines lies on the third.