2

Random Variable and Discrete Probability Distribution

Contents :

2.1 Random Variable

We have studied about random experiment, sample space and probability in the chapter of probability. In this chapter, we shall study random variable and discrete probability distribution.

First of all, we shall define random variable and then we shall understand it by illustration.

Random Variable : Let U be a sample space of a random experiment. Every element of U need not always be a number. However, we wish to assign a specific number to each outcome.

A function associating a real number with each outcome of U is called a random variable. It is denoted by X. That is, a random variable based on a sample space U is denoted by $X : U \rightarrow R$.

For example,

(i) The number of heads (H) in tossing an unbiased coin three times

(ii) The number of accidents during a week in a city

(iii) The weight of a person (in kilogram)

(iv) The maximum temperature of a day at a particular place (in Celsius)

Now, let us understand the concept of random variable by some illustrations.

(1) A balanced die is tossed once. If the number observed on the die is denoted by 'u' then the elements of the sample space U of this experiment can be shown in the notation of a set as follows :

$U = \{u \mid u = 1, 2, 3, 4, 5, 6\}$

That is $U = \{1, 2, 3, 4, 5, 6\}$

If we associate a real number X with element u of sample space by

$X(u)$ = the number obtained on the die then we can write

$X(u) = u, u = 1, 2, 3, 4, 5, 6$

Thus, variable X will be a random variable assuming values 1, 2, 3, 4, 5 and 6.

In the above illustration, the element of U are numeric. Now, we consider an illustration in which the elements of U are non-numeric.

(2) Suppose a box contains four balls : one red, one blue, one yellow and one white ball. We donote the red ball by R, the blue ball by B, the yellow ball by Y and the white ball by W. A person draws three balls at a time at random from the box. The sample space associated with this experiment is

$U = \{RBY, RBW, BYW, WYR\}$

Suppose for the element u of U,

$X(u) =$ the number of white ball in u then $X(RBY) = 0,\ X(RBW) = 1,\ X(BYW) = 1,\ X(WYR) = 1$.

Thus, random variable X assumes the values in the set $\{0, 1\}$. The outcomes of this sample space are not in numbers but we associate them with real numbers by a random variable.

(3) Suppose the heights of students in a class lie between 120 cm and 180 cm. If we measure the height of a student of this class then it will assume any value between 120 cm and 180 cm.

Here, the sample space is $U = \{u \mid 120 \le u \le 180\}$.

If we denote the height of a selected student by X then $X(u) = u =$ the height (in cm) of a selected student. Thus, X becomes a random variable which will be denoted as $X = x,\ 120 \le x \le 180$.

In the above example (1) and example (2), random variable X assumes particular countable values whereas in example (3), random variable X can assume any value in the interval [120, 180]. This random variable differs from the random variables in earlier two examples.

Now, we shall understand the difference between these random variables in the following section.

2.1.1 Discrete Random Variable

A random variable X which can assume a finite or countable infinite number of values in the set R of real numbers is called a discrete random variable.

For example (i) Birth year of a randomly selected student.

(ii) Number of broken eggs in a box of 6 eggs.

Now, we shall understand about the discrete random variable by some specific examples.

(1) Suppose there is one black and two white balls in a box. Suppose the black ball is denoted by B and two white balls by W_1 and W_2. A person can play the following game by paying ₹ 15.

The person playing a game is asked to select two balls randomly with replacement from the box. He is paid an amount according to the colour of the balls selected by him as per the following conditions :

If a white ball is selected then ₹ 5 are paid for each selected white ball and if a black ball is selected then ₹ 15 are paid per black ball.

If we denote the net amount earned (amount received – amount paid for the game) by the player corresponding to each outcome of the experiment by X then X becomes a discrete random variable. The values assumed by the variable X are denoted in the following table :

Outcome of the experiment (Event)	The amount received by the person by playing the game	The amount paid to play the game	The value of X (in ₹)
W_1W_1	$5 + 5 = 10$	15	$X(W_1W_1) = 10 - 15 = -5$
W_1W_2	$5 + 5 = 10$	15	$X(W_1W_2) = 10 - 15 = -5$
W_1B_1	$5 + 15 = 20$	15	$X(W_1B_1) = 20 - 15 = 5$
W_2W_1	$5 + 5 = 10$	15	$X(W_2W_1) = 10 - 15 = -5$
W_2W_2	$5 + 5 = 10$	15	$X(W_2W_2) = 10 - 15 = -5$
W_2B_1	$5 + 15 = 20$	15	$X(W_2B_1) = 20 - 15 = 5$
B_1W_1	$15 + 5 = 20$	15	$X(B_1W_1) = 20 - 15 = 5$
B_1W_2	$15 + 5 = 20$	15	$X(B_1W_2) = 20 - 15 = 5$
B_1B_1	$15 + 15 = 30$	15	$X(B_1B_1) = 30 - 15 = 15$

Thus, the random variable X assumes the values -5, 5 and 15 only. That is the total number of values of X is finite.

(2) Suppose a coin is tossed until either a tail (T) or four heads (H) occur. Let X denote the number of tosses required.

The sample space associated with this random experiment is

$U = \{T, HT, HHT, HHHT, HHHH\}$

The random variable X denotes the number of tosses required for the coin associated with the experiment and it assumes any one value out of 1, 2, 3 and 4 for the sample points of the sample space.

$$X(T) = 1,\ X(HT) = 2,\ X(HHT) = 3$$

$$X(HHHT) = 4,\ X(HHHH) = 4$$

The discrete random variable X assumes the finite number of values.

(3) Consider the random variable X denoting the number of tails before getting the first head in the experiment of tossing a coin till the first head is obtained.

In this experiment, head will appear either in the first trial or in the second trial or in the third trial and so on... Similarly, the first head may be obtained after tossing a coin infinite times. Hence, the sample space associated with random experiment becomes

$$U = \{H, TH, TTH, TTTH, TTTTH, ...\}$$

Thus, the number of tails before getting the first head will be $0, 1, 2, 3, 4....$

Thus, the random variable X assumes any one value from the countable infinite number of values $0, 1, 2, 3, 4....$

2.1.2 Continuous Random Variable

A random variable X which can assume any value in R, the set of real numbers or in any interval of R is called a continuous random variable.

For example (i) The actual amount of coffee in a coffee mug having a capacity of 250 millilitre.

(ii) Waiting time for a lift on any one floor of a high-rise office building.

Now, we shall understand more about the continuous random variable by the following examples.

(1) Denote the time taken by a student to finish a test of 3 hours duration by random variable X. The sample space here is

$$U = \{u | 0 \le u \le 3\}.$$

Since the time taken by any student for the exam takes any real value from 0 to 3 and the random variable X, the actual time taken by a student to complete the exam, will also be any real value from 0 to 3.

Thus,

$$X(u) = u, \ 0 \le u \le 3.$$

That means $X = x, 0 \le x \le 3$

The random variable X assumes any real value from 0 to 3, which is a subset of R and hence X is a continuous random variable.

(2) Suppose there are two stations A and B on an express highway. The distance of station B from station A is 200 km. Let us consider an experiment to know the place of accident between two stations A and B. For the sake of simplicity, let us fix the position of station A at 0 km and of station B at 200 km. The sample space of this experiment is any real value between 0 to 200. So, we can write the sample space for this experiment as

$$U = \{u \mid 0 \le u \le 200\}$$

Suppose the random variable X denotes the distance (in kilometer) of the place of the accident between two stations A and B from the station A. Then the random variable X is defined as below :

$X(u)$ = distance of the place of accident from the station A.

In short, we can define the random variable X as $X = x, 0 \le x \le 200$.

The random variable X assumes any real value from 0 to 200, which is subset of R, the set of real numbers. So, X is a continuous random variable.

2.2 Discrete Probability Distribution

Suppose $X : U \to R$ is a random variable which assumes all the values of the subset $\{x_1, x_2, ..., x_n\}$ of *R*. Further, suppose *X* assumes a value x_i with probability $P(X = x_i) = p(x_i)$. If $p(x_i) > 0, \ i = 1, 2, ..., n$ and $\Sigma\, p(x_i) = 1$ then the set of real values $\{x_1, x_2, ..., x_n\}$ and $\{p(x_1), p(x_2), ..., p(x_n)\}$ is called the discrete probability distribution of a random variable *X*. The discrete probability distribution of a random variable *X* is expressed in a tabular form as follow :

$X = x$	x_1	x_2	....	x_i	...	x_n
$p(x)$	$p(x_1)$	$p(x_2)$	....	$p(x_i)$	...	$p(x_n)$

Here, $0 < p(x_i) < 1, i = 1, 2, ..., n$ and $\Sigma\, p(x_i) = 1$

2.2.1 Illustrations for Probability Distribution of Discrete Variable

Illustration 1 : Determine whether the values given below are appropriate as the values of a probability distribution of a discrete random variable *X*, which assumes the values 1, 2, 3 and 4 only.

(i) $p(1) = 0.25, p(2) = 0.75, p(3) = 0.25, p(4) = -0.25$

(ii) $p(1) = 0.15, p(2) = 0.27, p(3) = 0.29, p(4) = 0.29$

(iii) $p(1) = \frac{1}{19}, \quad p(2) = \frac{9}{19}, \quad p(3) = \frac{3}{19}, \quad p(4) = \frac{4}{19}$

(i) The value of $P(4)$ is –0.25, which is negative. It does not satisfy the condition $p(x_i) > 0, i = 1, 2, 3, 4$ of discrete probability distribution. So, given values are not suitable for the probability distribution of a discrete variable. Thus, the given distribution cannot be called a probability distribution of a discrete variable.

(ii) For every value 1, 2, 3 and 4 of *X*, $p(x) > 0$, and $p(1) + p(2) + p(3) + p(4) = 1$. Thus, both the conditions of probability distribution of discrete variable are satisfied. So, the given values are appropriate and the given distribution is probability distribution of a discrete variable.

(iii) Here $p(x_i) > 0, i = 1, 2, 3, 4$ but, sum of probabilities

i.e. $p(1) + p(2) + p(3) + p(4) = \frac{17}{19}$, is not 1. So, the given values are not appropriate for the probability distribution. So, the given distribution cannot be called a probability distribution of discrete variable.

Illustration 2 : Determine when the following distribution is a probability distribution of discrete variable. Hence obtain the probability for $x = 2$:

$$p(x) = c\left(\frac{1}{4}\right)^x, \quad x = 1, 2, 3, 4$$

Here, $p(1) = c\left(\frac{1}{4}\right)$, $p(2) = c\left(\frac{1}{4}\right)^2 = c\left(\frac{1}{16}\right)$, $p(3) = c\left(\frac{1}{4}\right)^3 = c\left(\frac{1}{64}\right)$, $p(4) = c\left(\frac{1}{4}\right)^4 = c\left(\frac{1}{256}\right)$

Now, total probability should be 1 for a discrete probability distribution.

$p(1)+p(2)+p(3)+p(4)=1$

$\therefore\ c\left(\frac{1}{4}\right)+c\left(\frac{1}{16}\right)+c\left(\frac{1}{64}\right)+c\left(\frac{1}{256}\right)=1$

$\therefore\ c\left[\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}\right]=1$

$\therefore\ c\left[\frac{85}{256}\right]=1$

$\therefore\ c=\frac{256}{85}$

Thus, when $c=\frac{256}{85}$, the given distribution becomes probability distribution of a discrete variable.

Now, $P(X=2)=c\left(\frac{1}{4}\right)^2$

$=\frac{256}{85}\times\frac{1}{16}$

$=\frac{16}{85}$

$\therefore$ The probability of $X=2$ is $\frac{16}{85}$.

Illustration 3 : A random variable X denotes the number of accidents per year in a factory and the probability distribution of X is given below :

$X=x$	0	1	2	3	4
$p(x)$	$4K$	$15K$	$25K$	$5K$	K

(i) Find the constant K and rewrite the probability distribution.

(ii) Find the probability of the event that one or two accidents will occur in this factory during the year.

(iii) Find the probability that no accidents will take place during the year in the factory.

(i) By the definition of discrete probability distribution, we must have

$p(0)+p(1)+p(2)+p(3)+p(4)=1$

That is $4K+15K+25K+5K+K=1$

$\therefore 50K=1$

$\therefore K=\frac{1}{50}$

$=0.02$

Thus, when $K=0.02$, the given distribution becomes a probability distribution of a discrete variable, which is given below :

$X=x$	0	1	2	3	4	Total
$p(x)$	0.08	0.30	0.50	0.10	0.02	1

(ii) Probability of occurrence of one or two accidents

$= P(X=1)+P(X=2)$

$= 0.30 + 0.50$

$= 0.80$

(iii) Probability that accidents do not occur :

$= P(X=0)$

$= 0.08$

Illustration 4 : In a factory, packets of produced blades are prepared having 50 blades in each packet. A quality control engineer randomly selects a packet from these packets and examines all the blades of the selected packet. If 4 or more defective blades are observed in the selected packet then the packet is rejected. The probability distribution of the defective blades in the packet is given below :

Number of defective blades in the packet	0	1	2	3	4	5	6 or more
Probability	$9K$	$3K$	$3K$	$2K$	$2K$	$K-0.02$	0.02

From the given probability distribution,

(i) Find constant *K*.

(ii) Find the probability that the randomly selected packet is accepted by the quality control engineer.

(i) Let X = number of defective blades found during the inspection of the packet.

By definition of discrete probability distribution

$p(0)+p(1)+p(2)+p(3)+p(4)+p(5)+p(6 \text{ or more})=1.$

$\therefore 9K+3K+3K+2K+2K+K-0.02+0.02=1$

$\therefore 20K=1$

$\therefore K=\frac{1}{20}=0.05$

(ii) The randomly selected packet is accepted by the quality control engineer only when 3 or less defective blades are found in the packet.

$\therefore P(X \le 3)$

$=p(0)+p(1)+p(2)+p(3)$

$=9K+3K+3K+2K$

$=17K$

$=17(0.05)$

$=0.85 \quad (\because K=0.05)$

Illustration 5 : There are 4 red and 2 white balls in a box. 2 balls are drawn at random from the box without replacement. Obtain probability distribution of number of white balls in the selected balls.

Suppose X denotes the number of white balls in the selected two balls. X may assume the values 0, 1 and 2.

$X = 0$ means there will not be any white balls in the selected two balls that means both the selected balls are red.

$$\therefore P(X=0) = P\,(2 \text{ red balls}) = \frac{{}^4C_2}{{}^6C_2} = \frac{6}{15}$$

Now, $x = 1$ means there will be one white ball and one red ball in the two selected balls.

$$\therefore P(X=1) = P\,(1 \text{ White ball, } 1 \text{ Red ball})$$

$$= \frac{{}^2C_1 \times {}^4C_1}{{}^6C_2}$$

$$= \frac{2 \times 4}{15} = \frac{8}{15}$$

And $X = 2$ means both the selected ball will be white.

$$\therefore P(X=2) = P\,(2 \text{ White balls})$$

$$= \frac{{}^2C_2}{{}^6C_2}$$

$$= \frac{1}{15}$$

Thus, probability distribution of random variable X can be written as follows :

$X = x$	0	1	2
$p(x)$	$\frac{6}{15}$	$\frac{8}{15}$	$\frac{1}{15}$

$p(x) > 0$ and $\Sigma p(x) = 1$

2.2.2 Mean and Variance

Now, we will discuss two important results based on the probability distribution of discrete random variable. One of them is expected value (mean) of the random variable and the other is variance of the random variable.

Let X be a discrete random variable which assumes one of the values $x_1, x_2, ..., x_n$ only and its probability distribution is as follows :

$X = x$	x_1	x_2	....	x_i	...	x_n
$p(x)$	$p(x_1)$	$p(x_2)$	....	$p(x_i)$	...	$p(x_n)$

Where $0 < p(x_i) < 1, i = 1, 2, ..., n$ and $\Sigma\, p(x_i) = 1$

The mean of discrete random variable is denoted by μ or $E(X)$. It is defined as follows :

$$\mu = E(X) = \Sigma x_i \, p(x_i)$$

This value is also called expected value of discrete variable *X*.

The variance of discrete random variable *X* is denoted by σ^2 or $V(X)$, which is defined as follows :

$$\sigma^2 = V(X) = E(X-\mu)^2$$

$$= E(X^2) - (\mu)^2$$

$$= E(X^2) - (E(X))^2$$

Where $E(X^2) = \Sigma x_i^2 p(x_i)$

Note : (i) We will use the following notations for the sake of simplicity.

$\Sigma x \, p(x)$ instead of $\Sigma x_i \, p(x_i)$

and

$\Sigma x^2 p(x)$ instead of $\Sigma x_i^2 p(x_i)$

(ii) The mean and variance of variable *X* are also called mean and variance of the distribution of *X* respectively.

(iii) The value of the variance of variable *X* is always positive.

We consider the following examples to find mean and variance of the discrete probability distribution.

Illustration 6 : Find constant C for the following discrete probability distribution. Hence obtain mean and variance of this distribution.

$$\boldsymbol{p(x) = C \cdot {}^4P_x, \; x = 0, 1, 2, 3, 4}$$

From the property of discrete probability distribution we must have

$$p(0) + p(1) + p(2) + p(3) + p(4) = 1$$

$$\therefore \; C \cdot {}^4P_0 + C \cdot {}^4P_1 + C \cdot {}^4P_2 + C \cdot {}^4P_3 + C \cdot {}^4P_4 = 1$$

$$\therefore C\left[\frac{4!}{4!} + \frac{4!}{3!} + \frac{4!}{2!} + \frac{4!}{1!} + \frac{4!}{0!}\right] = 1$$

$$\therefore C[1+4+12+24+24] = 1$$

$$\therefore C[65] = 1$$

$$\therefore C = \frac{1}{65}$$

Thus, the probability distribution can be written in the tabular form as follow :

$X = x$	0	1	2	3	4	Total
$p(x)$	$\frac{1}{65}$	$\frac{4}{65}$	$\frac{12}{65}$	$\frac{24}{65}$	$\frac{24}{65}$	1

Now, mean of the distribution $= \mu = \Sigma xp(x)$

$$= 0\left(\frac{1}{65}\right) + 1\left(\frac{4}{65}\right) + 2\left(\frac{12}{65}\right) + 3\left(\frac{24}{65}\right) + 4\left(\frac{24}{65}\right)$$

$$= \frac{0+4+24+72+96}{65}$$

$$= \frac{196}{65}$$

Now, we obtain $E\left(X^2\right)$.

$$E\left(X^2\right) = \Sigma\, x^2 p(x)$$

$$= (0)^2\left(\frac{1}{65}\right) + (1)^2\left(\frac{4}{65}\right) + (2)^2\left(\frac{12}{65}\right) + (3)^2\left(\frac{24}{65}\right) + (4)^2\left(\frac{24}{65}\right)$$

$$= 0 + \frac{4}{65} + \frac{48}{65} + \frac{216}{65} + \frac{384}{65}$$

$$= \frac{652}{65}$$

Hence, variance of the distribution $= V(X)$

$$= E\left(X^2\right) - \left(E(X)\right)^2$$

$$= \frac{652}{65} - \left(\frac{196}{65}\right)^2$$

$$= \frac{42380 - 38416}{4225} = \frac{3964}{4225}$$

Illustration 7 : There are two red and one green balls in a box. Two balls are drawn at random with replacement from the box. Obtain probability distribution of number of red balls in the two balls drawn and find its mean and variance.

Let us denote the number of red balls in the selected two balls by X. Then we obtain the probability distribution of X as follow.

Let us denote the two red balls of the box by R_1 and R_2 and green ball by G.

The number of red balls in the selected balls and its probability can be obtained as in the following table.

Selected two balls (Event)	**Probability of the event**	$X = x$
R_1R_1	$\frac{1}{9}$	2
R_1R_2	$\frac{1}{9}$	2
R_1G	$\frac{1}{9}$	1
R_2R_1	$\frac{1}{9}$	2
R_2R_2	$\frac{1}{9}$	2
R_2G	$\frac{1}{9}$	1
GR_1	$\frac{1}{9}$	1
GR_2	$\frac{1}{9}$	1
GG	$\frac{1}{9}$	0

From the above table we can say that :

(i) Probability of getting 0 red ball

$= P(X = 0)$

$= \frac{1}{9}$

(ii) Probability of getting 1 red ball

$= P(X = 1)$

$= \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9}$

$= \frac{4}{9}$

(iii) Probability of getting 2 red balls

$= P(X = 2)$

$= \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9}$

$= \frac{4}{9}$

Thus, the probability distribution of X can be written in the tabular from as follow :

$X=x$	0	1	2	Total
$p(x)$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{4}{9}$	1

Now, mean of the distribution = $\mu = E(X)$

$$= \Sigma\, x\, p(x)$$

$$= 0\left(\frac{1}{9}\right)+1\left(\frac{4}{9}\right)+2\left(\frac{4}{9}\right)$$

$$= \frac{0+4+8}{9}$$

$$= \frac{12}{9}$$

Now, we first find $E\left(X^2\right)$ to obtain variance of the distribution.

$$E\left(X^2\right) = \Sigma\, x^2 p(x)$$

$$= 0^2\left(\frac{1}{9}\right)+1^2\left(\frac{4}{9}\right)+2^2\left(\frac{4}{9}\right)$$

$$= \frac{0+4+16}{9}$$

$$= \frac{20}{9}$$

So, using the formula $V(X) = E\left(X^2\right) - \left(E(X)\right)^2$,

$$V(X) = \frac{20}{9} - \left(\frac{12}{9}\right)^2$$

$$= \frac{20}{9} - \frac{144}{81}$$

$$= \frac{180-144}{81}$$

$$= \frac{36}{81}$$

Illustration 8 : There are 2 black and 2 white balls in a box. Two balls are drawn without replacement from it. Obtain probability distribution of the number of white balls in the selected balls. Hence find its mean and variance.

Suppose X = number of white balls in the selected two balls then by the formula of probability

(i) Probability of $X = 0$

$$= P(X=0) = P\ (0 \text{ white balls}) = \frac{{}^2C_0}{{}^4C_2} = \frac{1}{6}$$

(ii) Probability of $X = 1$

$= P(X=1) = P$ (1 white ball and 1 black ball)

$$= \frac{{}^2C_1 \times {}^2C_1}{{}^4C_2}$$

$$= \frac{2 \times 2}{6}$$

$$= \frac{4}{6}$$

(iii) Probability of $X = 2$

$= P(X=2) = P$ (2 white balls)

$$= \frac{{}^2C_2}{{}^4C_2}$$

$$= \frac{1}{6}$$

Thus, the probability distribution of random variable X can be written in the tabular form as,

$X = x$	0	1	2	Total
$p(x)$	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$	1

Now, mean of the probability distribution $= E(X)$

$$= \Sigma\, x\, p(x)$$

$$= 0\left(\frac{1}{6}\right) + 1\left(\frac{4}{6}\right) + 2\left(\frac{1}{6}\right)$$

$$= \frac{0+4+2}{6}$$

$$= 1$$

Now, to obtain variance of the probability distribution, we first find $E\left(X^2\right)$.

$$E\left(X^2\right) = \Sigma\, x^2 p(x)$$

$$= 0^2\left(\frac{1}{6}\right) + 1^2\left(\frac{4}{6}\right) + 2^2\left(\frac{1}{6}\right)$$

$$= \frac{0+4+4}{6}$$

$$= \frac{8}{6}$$

$$\therefore V(X) = E\left(X^2\right) - \left(E(X)\right)^2$$

$$= \frac{8}{6} - (1)^2 \qquad (\because E(X) = 1)$$

$$= \frac{8-6}{6}$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

Illustration 9 : Let X denote the maximum integer among the outcomes of tossing two dice simultaneously. Obtain the probability distribution of variable X and find its mean and variance.

By tossing two dice simultaneously, we have 36 events in the sample space U and the maximum integer of the outcomes will be one of the numbers 1, 2, 3, 4, 5 or 6. The following table gives the possible outcomes for variable X and the corresponding probability :

Event u of U	**Maximum integer** $X(u) = x$	$P(X = x)$
(1, 1)	1	$\frac{1}{36}$
(1, 2), (2, 1), (2, 2)	2	$\frac{3}{36}$
(1, 3), (2, 3), (3, 3) (3, 2), (3, 1)	3	$\frac{5}{36}$
(1, 4), (2, 4), (3, 4), (4, 4) (4, 3), (4, 2), (4, 1)	4	$\frac{7}{36}$
(1, 5), (2, 5), (3, 5), (4, 5), (5, 5) (5, 4), (5, 3), (5, 2), (5, 1)	5	$\frac{9}{36}$
(1, 6), (2, 6), (3, 6), (4, 6), (5, 6) (6, 6), (6, 5), (6, 4), (6, 3), (6, 2) (6, 1)	6	$\frac{11}{36}$
		Total 1

Now, mean of $X = E(X)$

$$= \Sigma x\, p(x)$$

$$= 1\left(\frac{1}{36}\right)+2\left(\frac{3}{36}\right)+3\left(\frac{5}{36}\right)+4\left(\frac{7}{36}\right)+5\left(\frac{9}{36}\right)+6\left(\frac{11}{36}\right)$$

$$= \frac{161}{36}$$

Now, $E\left(X^2\right) = \Sigma x^2 p(x)$

$$= 1^2\left(\frac{1}{36}\right)+2^2\left(\frac{3}{36}\right)+3^2\left(\frac{5}{36}\right)+4^2\left(\frac{7}{36}\right)+5^2\left(\frac{9}{36}\right)+6^2\left(\frac{11}{36}\right)$$

$$= \frac{791}{36}$$

Variance of $X = V(X)$

$$= E\left(X^2\right)-\left(E(X)\right)^2$$

$= \frac{791}{36} - \left(\frac{161}{36}\right)^2$

$= \frac{791}{36} - \frac{25921}{1296}$

$= \frac{791 \times 36 - 25921}{1296}$

$= \frac{28476 - 25921}{1296}$

$= \frac{2555}{1296}$

Illustration 10 : It is observed from the life table that the probability that a 40 years old man will live one more year is 0.95. Life insurance company wishes to sell one year life insurance policy of Rs. 10,000 to such a man. What should be the minimum premium of the policy so that expected gain of the company would be positive ?

Let X be the company's gain and yearly premium of the policy be ₹ K, $K > 0$. Then gain of the company is $X = K$ if 40 year old man will live for one year and gain of the company is $X = K - 10{,}000$ if 40 year old man will die within a year.

Thus, the probability distribution of the gain of the company is as follow :

$X = x$	K	$K - 10000$
$p(x)$	0.95	0.05

Hence, expected gain of the company

$= E(X)$

$= \Sigma x\, p(x)$

$= K(0.95) + (K - 10000)(0.05)$

$= K(0.95) + K(0.05) - 500$

$= K(0.95 + 0.05) - 500$

$= K - 500$

Now, for positive expected gain, we must have

$K - 500 > 0$

$\therefore K > 500$

So, the company should fix the premium more than ₹ 500 so that the expected gain of the company be will positive.

EXERCISE 2.1

1. Examine whether the following distribution is a probability distribution of a discrete random variable X :

$$p(x) = \frac{x+2}{25},\ \ x = 1, 2, 3, 4, 5$$

2. If the following distribution is a probability distribution of variable X then find constant K.

$$p(x) = \frac{6-|x-7|}{K},\ \ x = 4, 5, 6, 7, 8, 9, 10$$

3. The probability distribution of a random variable X is defined as follows :

$$p(x) = \frac{K}{(x+1)!},\ \ x = 1, 2, 3\,;\ K = \text{constant}$$

Hence find (i) constant K (ii) $P(1 < X < 4)$

4. The probability distribution of a random variable X is as follows :

$X = x$	-2	-1	0	1	2
$p(x)$	$\frac{K}{3}$	$\frac{K}{3}$	$\frac{K}{3}$	$2K$	$4K$

Then (i) determine acceptable value of constant K. (ii) Find the Mean of the distribution.

5. The probability distribution of a random variable X is $P(x)$. Variable X can assume the values $x_1 = -2,\ x_2 = -1,\ x_3 = 1$ and $x_4 = 2$ and if $4P(x_1) = 2P(x_2) = 3P(x_3) = 4P(x_4)$ then obtain mean and variance of this probability distribution.

6. A die is randomly tossed two times. Determine the probability distribution of the sum of the numbers appearing both the times on the die and obtain expected value of the sum.

7. A box contains 4 red and 2 blue balls. Three balls are simultaneously drawn at random. If X denotes the number of red balls in the selected balls, find the probability distribution of X and find the expected number of red balls in the selected balls.

8. A coin is tossed till either a head or 5 tails are obtained. If a random variable X denotes the necessary number of trials of tossing the coin then obtain probability distribution of the random variable X and calculate its mean and variance.

9. A shopkeeper has 6 tickets in a box. 2 tickets among them are worth a prize of ₹ 10 and the remaining tickets are worth a prize of ₹ 5. If a ticket is drawn at random from the box, find the expected value of the prize.

*

2.3 Binomial Probability Distribution

In the earlier sections we considered continuous and discrete random variable and probability distribution of a discrete random variable. Now, we shall study an important probability distribution of a discrete random variable.

In some random experiments, there are only two outcomes. We call such outcomes as success and failure. These outcomes are mutually exclusive. We call such experiments as dichotomous experiments. The illustrations of some of these situations are given in the table below :

Experiment		Possible outcomes	
		Success	**Failure**
(i)	To know the effect of advertisement given to increase the sale of produced units	sale increased	sale did not increase
(ii)	To find the error in a letter typed by a type-writer	Error observed	Error not observed
(iii)	To know the effect of a drug on blood pressure given to the patients of high blood pressure	Blood pressure decreased	Blood pressure did not decrease
(iv)	To inspect whether produced item is defective	Item is defective	Item is not defective

If we denote the success by S and failure by F for such types of dichotomous experiment and the probabilities of such outcomes by p and q respectively then

$P(S) = p$ and $P(F) = q,\ 0 < p < 1,\ 0 < q < 1,\ p + q = 1$

Since there are only two outcomes of such an experiment and both are mutually exclusive, we have $p + q = 1$ and hence $q = 1 - p$.

If it is possible to repeat such a dichotomous random experiment n times and each repetition is done under identical conditions then the probability of success p remains constant in each trial. We call such experiments as Bernoulli Trials. Its actual definition can be given as follows :

Bernoulli Trials : Suppose dichotomous random experiment has two outcomes, success (S) and failure (F). If this experiment is repeated n times under identical conditions and the probability $p(0 < p < 1)$ of getting a success at each trial is constant then such trials are called Bernoulli Trials.

Properties of Bernoulli Trials

(1) The probability of getting a success at each Bernoulli trial remains constant.

(2) Bernoulli trials are mutually independent. That means getting success or failure at any trial does not depend on getting success or failure at the previous trial.

(3) Success and failure are mutually exclusive and exhustive events. Therefore $q = 1 - p$.

Binomial Probability Distribution

Suppose X denotes the number of successes in a sequence of success *(S)* and failure (F) obtained in n Bernoulli trials, then X is called a binomial random variables and X assumes any value in the finite set $\{0, 1, 2, ..., n\}$. The probability distribution of the binomial random variable X is defined by the following formula :

$$P(X = x) = p(x) = {}^nC_x p^x q^{n-x},\ x = 0,\ 1, 2, ..., n, 0 < p < 1, q = 1 - p$$

This probability distribution is called binomial Probability Distribution. We shall call such a distribution in short as binomial distribution.

If positive integer n and probability of success p are known here, the whole probability distribution that means the probability of each possible value of X can be determined. Hence, n and p are called parameters of the binomial distribution. We denote binomial distribution having parameters n and p as $b(n, p)$.

Note : If we repeat an experiment having such Bernoulli trials N times and $p(x)$ is the probability of getting x successes in the experiment then expected frequency of number of successes in N repetitions $= N \cdot p(x)$

2.3.1 Properties of Binomial Distribution

(1) Binomial distribution is a discrete distribution.

(2) Its parameters are n and p.

(3) The mean of the distribution is np which denotes average (expected) number of successes in n Bernoulli trials.

(4) The variance of the distribution is npq and its standard deviation is $\sqrt{npq}$.

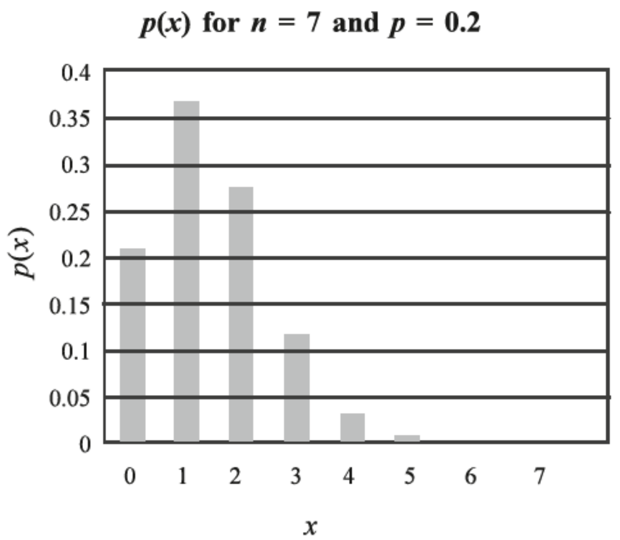
(5) For binomial distribution, mean is always greater than the variance and $\frac{\text{Variance}}{\text{Mean}} = q$ = probability of failure.

(6) If $p < \frac{1}{2}$ then the skewness of the distribution is positive for any value of n.

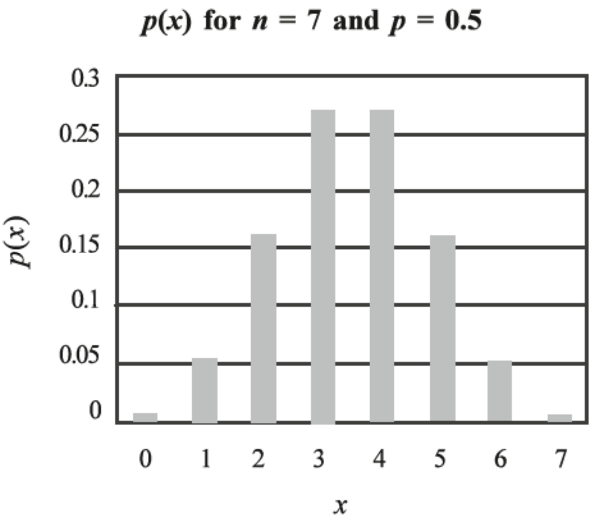
(7) If $p = \frac{1}{2}$ then the distribution becomes symmetric that means the skewness of the distribution is zero for any value of n.

(8) If $p > \frac{1}{2}$ then the skewness of the distribution is negative for any value of n.

The properties (6), (7) and (8) can be clearly seen from the following graphs :

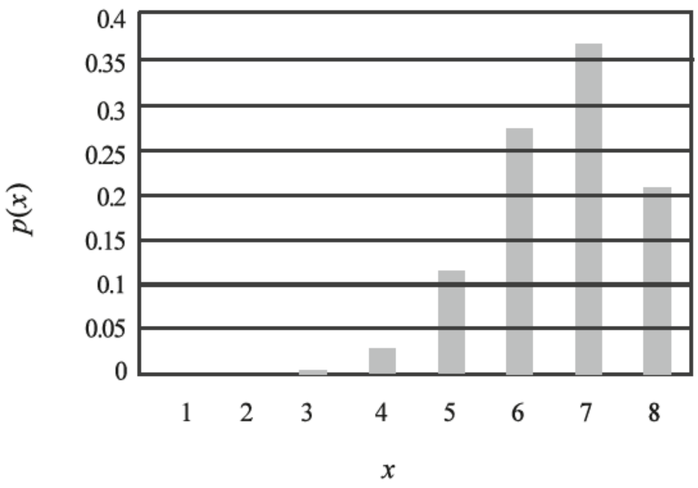


Positively Skewed Binomial Distribution



Symmetric Binomial Distribution

$p(x)$ for n = 7 and p = 0.8



Negatively Skewed Binomial Distribution

2.3.2 Illustrations of Binomial Distribution

Illustration 11 : There are 3 % defective items in the items produced by a factory. 4 items are selected at random from the items produced. What is the probability that there will not be any detective item ?

If the event that the selected items is defective is considered as success then the probability of success $p = 0.03$ and $n = 4$. None of the selected items is defective means $X = 0$.

Now,

$$p(x) = {}^nC_x p^x q^{n-x}, \ x = 0, 1, 2, ..., n$$

Putting the values of n, p, $q = 1-p$ and x in the formula,

$$P(X = 0) = {}^4c_0 (0.03)^0 (0.97)^{4-0}$$

$$= (0.97)^4$$

$$= 0.8853$$

Thus, the probability of getting no defective item in the selected 4 items is 0.8853.

Illustration 12 : The probability that a person living in a city is a non-vegetarian is 0.20. Find the probability of at the most two persons out of 6 persons randomly selected from the city is non-vegetarian.

If we consider the event that a person is non-vegetarian as success then we are given the probability of success $p = 0.20$ and $n = 6$.

If we take $X =$ number of non-vegetarians among the selected persons then the probability of $X \leq 2$

is obtained by putting the values of *n, p* and *x* in the formula of binomial probability distribution

$$p(x) = {}^{n}C_{x} p^{x} q^{n-x},\ x = 0, 1, 2, \ldots, n$$

$$p(X \le 2) = p(X = 0 \ \text{ or } \ X = 1 \ \text{ or } \ X = 2)$$

$$= p(0) + p(1) + p(2)$$

$$= {}^{6}C_{0}(0.20)^{0}(0.80)^{6} + {}^{6}C_{1}(0.20)^{1}(0.80)^{6-1} + {}^{6}C_{2}(0.20)^{2}(0.80)^{6-2}$$

$$= 0.2621 + 6(0.20)(0.3277) + 15(0.04)(0.4096)$$

$$= 0.2621 + 0.3932 + 0.2458$$

$$= 0.9011$$

Illustration 13 : The mean and variance of a binomial distribution are 3.9 and 2.73 respectively. Find the number of Bernoulli trials conducted in this distribution and write $p(x)$.

Here, variance $= npq = 2.73$ and mean $= np = 3.9$.

$$\therefore q = \frac{\text{Variance}}{\text{Mean}} = \frac{2.73}{3.9} = 0.7 \ \text{ and } \ p = 1 - q = 0.3$$

$$\text{Now } \ n = \frac{np}{p} = \frac{\text{Mean}}{p} = \frac{3.9}{0.3} = 13$$

Thus, the number of Bernoulli trials conducted in this distribution is 13. Since $n = 13,\ p = 0.3$ and $q = 0.7$ in the distribution, its $p(x)$ can be written as follow :

$$p(x) = {}^{13}C_{x}(0.3)^{x}(0.7)^{13-x},\ x = 0, 1, 2, \ldots, 13.$$

Illustration 14 : During a war, on an average one ship out of 9 got sunk in a certain voyage. Find the probability that exactly 5 out of a convoy of 6 ships would arrive safely.

Suppose $X =$ the number of ships that arrive safely out of a convoy of 6 ships during a war.

$n =$ total number of ships in a convoy $= 6$

$p =$ probability that a ship arrives safely in a certain voyage $= \frac{8}{9}$

$\therefore$ The probability that exactly 5 out of a convoy of 6 ships would arrive safely can be obtained by putting corresponding values in the formula

$$p(x) = {}^{n}C_{x} p^{x} q^{n-x},\ x = 0, 1, 2, \ldots, n,$$

$$p(5) = {}^{6}C_{5}\left(\frac{8}{9}\right)^{5}\left(\frac{1}{9}\right)^{1}$$

$$= 6\left(\frac{32{,}768}{59{,}049}\right)\left(\frac{1}{9}\right)$$

$$= \frac{196608}{531441}$$

$$= 0.3700$$

Illustration 15 : Assume that on an average one line out of 4 telephone lines remains busy between 2 pm and 3 pm on week days. Find the probability that out of 6 randomly selected telephone lines (i) not more than 3 (ii) at least three of them will be busy.

Suppose $p=$ the probability of the event that the selected telephone line remains busy between 2 pm to 3 pm = $\frac{1}{4}$

and $X=$ the number of busy telephone lines out of 6 telephone lines between 2 pm to 3 pm.

It is given here that $n = 6$.

(i) The event that not more than 3 lines out of 6 randomlly selected telephone lines will be busy is the event that 3 or less telephone lines will be busy.

That is $X \le 3$.

$\therefore$ To find probability of this event we use the formula of binomial probability distribution

$$p(x) = {}^nC_x p^x q^{n-x},\ x = 0, 1, 2, ..., n$$

$\therefore$ $P(X \le 3)$

$= P\ (X = 0 \text{ or } 1 \text{ or } 2 \text{ or } 3)$

$= 1 - P\ (X = 4 \text{ or } 5 \text{ or } 6)$

$= 1 - \left[p\ (4) + p(5) + p(6)\right]$

$$= 1 - \left[{}^6C_4 \left(\tfrac{1}{4}\right)^4 \left(\tfrac{3}{4}\right)^2 + {}^6C_5 \left(\tfrac{1}{4}\right)^5 \left(\tfrac{3}{4}\right)^1 + {}^6C_6 \left(\tfrac{1}{4}\right)^6 \left(\tfrac{3}{4}\right)^0\right]$$

$$= 1 - \left[15\left(\tfrac{1}{256}\right)\left(\tfrac{9}{16}\right) + 6\left(\tfrac{1}{1024}\right)\left(\tfrac{3}{4}\right) + \left(\tfrac{1}{4096}\right)\right]$$

$$= 1 - \left[\tfrac{135}{4096} + \tfrac{18}{4096} + \tfrac{1}{4096}\right]$$

$$= 1 - \tfrac{154}{4096} = \tfrac{3942}{4096} = 0.9624$$

(ii) The probability that at least 3 telephone lines will be busy

$= P(X \ge 3)$

$= P\ (X = 3 \text{ or } 4 \text{ or } 5 \text{ or } 6)$

$= p(3) + p\ (4) + p(5) + p(6)$

Now, from the above calculations we will get the values of $p\ (4)$, $p(5)$ and $p(6)$. So, we first find the value of $p(3)$.

$p(3) = {}^6C_3 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^3 = 20\left(\frac{1}{64}\right)\left(\frac{27}{64}\right)$

$= \frac{540}{4096}$

Now, from the values of $p(4)$, $p(5)$ and $p(6)$ in question (i) and the value of $p(3)$ we obtained,

$$P(X \ge 3) = \frac{540}{4096} + \frac{135}{4096} + \frac{18}{4096} + \frac{1}{4096}$$

$$= \frac{694}{4096} = 0.1694$$

Illustration 16 : The parameters of binomial distribution of a random variable X are $n = 4$ and $p = \frac{1}{3}$. State the probability distribution of X in a tabular form and hence find the value of $P(X \le 2)$.

Here, parameters are $n = 4$ and $p = \frac{1}{3}$ $\therefore q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

Substituting the values of the parameters in the formula of binomial distribution,

$p(x) = {}^nC_x\, p^x\, q^{n-x}$, $x = 0, 1, 2, ..., n$ we have $p(x) = {}^4C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}$, $x = 0, 1, 2, 3, 4$.

Now, we calculate the values of $p(x)$ by putting the different values of x as 0, 1, 2, 3 and 4.

$$p(0) = {}^4C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{4-0} = \frac{16}{81}$$

$$p(1) = {}^4C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{4-1} = 4\left(\frac{1}{3}\right)\left(\frac{8}{27}\right) = \frac{32}{81}$$

$$p(2) = {}^4C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{4-2} = 6\left(\frac{1}{9}\right)\left(\frac{4}{9}\right) = \frac{24}{81}$$

$$p(3) = {}^4C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{4-3} = 4\left(\frac{1}{27}\right)\left(\frac{2}{3}\right) = \frac{8}{81}$$

$$p(4) = {}^4C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^{4-4} = 1\left(\frac{1}{81}\right)1 = \frac{1}{81}$$

These can be put in the tabular form as follows :

$X = x$	0	1	2	3	4	Total
$p(x)$	$\frac{16}{81}$	$\frac{32}{81}$	$\frac{24}{81}$	$\frac{8}{81}$	$\frac{1}{81}$	1

Now, $P(X \le 2)$

$= p(X=0)+p(X=1)+p(X=2)$

$= \frac{16}{81}+\frac{32}{81}+\frac{24}{81}$

$=\frac{72}{81}$

$=\frac{8}{9}$

Illustration 17 : In a binomial distribution, for $P(X=x)=p(x)$, $n=8$ and $2p(4)=5p(3)$. Find the probability of getting success in all the trials for this distribution.

Here, we have $2p(4)=5p(3)$ and $n=8$

$\therefore$ Putting $n=8$ in the formula of binomial distribution, we get,

$p(x)={}^8C_x p^x q^{8-x}$, $x = 0, 1, 2, ..., 8$

Putting the values of $p(4)$ and $p(3)$ from this formula in the given condition

$2p(4)=3p(3)$

$2 \times {}^8C_4\, p^4\, q^{8-4} = 5 \times {}^8C_3 p^3 q^{8-3}$

$\therefore\ 2 \times (70)\, p^4\, q^4 = 5 \times (56)\, p^3 q^5$

$\therefore\ 140\, p^4\, q^4 = 280 p^3 q^5$

$\therefore\ p = 2q$

$\therefore\ p = 2(1-p)$

$\therefore\ p = 2-2p$

$\therefore 3p = 2$

$\therefore\ p=\frac{2}{3}$ and $q=1-p=1-\frac{2}{3}=\frac{1}{3}$.

Now, getting success in all the trials means the event of getting 8 successes since we have total 8 trials..

The probability of this events is $p(8)$.

$\therefore\ p(8)={}^8C_8 \left(\frac{2}{3}\right)^8 \left(\frac{1}{3}\right)^{8-8}$

$=1\times\left(\frac{2}{3}\right)^8 \times 1$

$=\frac{256}{6561}$

Thus, the probability of getting success in all the trials is $\frac{256}{6561}$.

Illustration 18 : For a binomial distribution, mean = 18 and variance = 4.5. Determine whether the skewness of this distribution is positive or negative.

Here, mean $= np = 18$ and variance $= npq = 4.5$

$\therefore q = \frac{\text{Variance}}{\text{Mean}} = \frac{4.5}{18} = 0.25 = \frac{1}{4}$

$\therefore p = 1 - \frac{1}{4} = \frac{3}{4}$

Since the value of p is greater than $\frac{1}{2}$, the skewness of binomial distribution will be negative.

Illustration 19 : A balacned die is tossed 7 times. If the event of getting a number 5 or more is called success and X denotes the number of success in 7 trials then (i) Write the probability distribution of X. (ii) Find the probability of getting 4 successes. (iii) Find the probability of getting at the most 6 successes.

The sample space associated with tossing of a balanced die once is $U = \{1, 2, 3, 4, 5, 6\}$ and probability of getting each number is $\frac{1}{6}$.

If the event of getting a number 5 or more is called success then probability of success

$p =$ probability of getting 5 or 6 on the die

$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

$\therefore q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

Here, total number of trials is 7. $\therefore n = 7$

(i) Using the probability distribution of X $p(x) = {}^nC_x p^x q^{n-x}$, $x = 0, 1, 2, ..., n$,

$$p(x) = {}^7C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{7-x}, \quad x = 0, 1, 2, 3, 4, 5, 6, 7$$

(ii) Probability of getting 4 successes

$$p(4) = {}^7C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^{7-4}$$

$$= 35\left(\frac{1}{81}\right)\left(\frac{8}{27}\right)$$

$$= \frac{280}{2187}$$

(iii) Probability of getting at the most 6 successes

$= p(X \le 6)$

$= 1 - p(X > 6)$

$= 1 - p(X = 7) \quad \because x = 0, 1, 2, ..., 7$

$= 1 - {}^7C_7 \left(\frac{1}{3}\right)^7 \left(\frac{2}{3}\right)^{7-7}$

$= 1 - \frac{1}{2187} = \frac{2186}{2187}$

Illustration 20 : A social worker claims that 10 % of the young children in a city have vision problem. A sample survey agency takes a random sample of 10 young children from the city to test the claim. If at the most one young child is affected by the vision problem, the claim of the social worker is rejected. Find (i) the probability that the claim of the social worker is rejected (ii) the expected number of young children having vision problem in the randomly selected 10 young children.

Suppose p = probability that a young child has eye problem

$= 0.10$ (by accepting the claim of social worker)

And X = the number of young childrens having eye problem in the randomly selected 10 young children

Here, putting $n = 10$ and $p = 0.10$ in the formula

$p(x) = {}^nC_x \ p^x q^{n-x}, \quad x = 0, 1, 2, ..., n$

of binomial distribution,

$p(x) = {}^{10}C_x (0.10)^x (0.90)^{10-x}, \ x = 0, 1, 2, ..., 10$

(i) Probability that at the most one young child has eye problem

$= p(0) + p(1)$

$= {}^{10}C_0 (0.10)^0 (0.90)^{10-0} + {}^{10}C_1 (0.10)^1 (0.90)^{10-1}$

$= 0.3487 + 10 (0.10)(0.3874)$

$= 0.3487 + 0.3874$

$= 0.7361$

Now, sample survey agency rejects the claim of the social worker if at the most one young child has eye problem.

$\therefore$ Probability of rejecting the claim of social worker by the sample survey agency $= 0.7361$.

(ii) The expected number of young children having eye problem in the randomly selected 10 young child

$= E(X) = np$

$= 10 \times$ probability that the selected young child has eye problem

$= 10 \times 0.10$

$= 1$

Illustration 21 : An experiment is conducted to toss five balanced coins simultaneously. If we consider occurrence of head (H) on the coin as success then obtain probability distribution of the number of successes. If such an experiment is repeated 3200 times then obtain expected frequency distribution of the number of successes. For this distribution, obtain expected value of the number successes and also obtain its standard deviation.

Since the coins are balanced, probability of getting head will be $\frac{1}{2}$.

$p =$ probability of success

$=$ probability of getting head $= \frac{1}{2}$.

$\therefore q = 1 - p = \frac{1}{2}$

Here, $n =$ number of coins $= 5$, $x =$ the number of successes in tossing of five coins.

Putting the values of n, p and q in the formula

$p(x) = {}^nC_x \, p^x q^{n-x}, \ x = 0, 1, 2, ..., n$

of the binomial distribution

$$p(x) = {}^5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}, \quad x = 0, 1, 2, ..., 5$$

$$= {}^5C_x \left(\frac{1}{2}\right)^{x+5-x}$$

$$= {}^5C_x \left(\frac{1}{2}\right)^5$$

$$= \frac{{}^5C_x}{32}, \quad x = 0, 1, 2, ..., 5$$

Now, using the above formula, we calculate the probability for each x and the frequency for the number of successes in 3200 repetition of the experiment $= 3200 \times p(x)$, $x = 0, 1, 2, ...,5$.

We present the calculations in the following table :

x	$p(x)$	**Expected Frequency** $= \mathbf{N} \times p(x)$
0	$\frac{^5C_0}{32} = \frac{1}{32}$	$3200 \times \frac{1}{32} = 100$
1	$\frac{^5C_1}{32} = \frac{5}{32}$	$3200 \times \frac{5}{32} = 500$
2	$\frac{^5C_2}{32} = \frac{10}{32}$	$3200 \times \frac{10}{32} = 1000$
3	$\frac{^5C_3}{32} = \frac{10}{32}$	$3200 \times \frac{10}{32} = 1000$
4	$\frac{^5C_4}{32} = \frac{5}{32}$	$3200 \times \frac{5}{32} = 500$
5	$\frac{^5C_5}{32} = \frac{1}{32}$	$3200 \times \frac{1}{32} = 100$

(ii) Expected value of the number of successes

$= np$

$= 5\left(\frac{1}{2}\right) = 2.5$

(iii) Standard deviation of the number of successes

$= \sqrt{npq}$

$= \sqrt{5 \times \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right)} = \sqrt{\frac{5}{4}} = \sqrt{1.25}$

$= 1.118$

Illustration 22 : An advertisement company claims that 4 out of 5 house wives do not identify the difference between two different brands of butter. To check the claim, 5000 house wives are divided in groups, each group of 5 house wives. If the claim is true, in how many groups among these groups (i) at the most one house wife (ii) only two house wives can identify the difference between two different brands of butter ?

As per the claim made by an advertising company, 4 out of 5 house wives do not identify the difference between two different brands of butter.

$\therefore$ Its probability is $\frac{4}{5}$

That means the probability of identifying the difference between two different brands of butter by a house wife $= \frac{1}{5}$.

Let p = probability that the selected house wife can identify the difference between two different brands of butter $= \frac{1}{5}$.

To test the claim, selected 5000 housewives are divided in groups randomly, with each group having 5 house wives. So, there will be 1000 such groups.

If we take $X =$ the number of house wives who identify the difference between two different brands of butter in a group, $x = 0, 1, ..., 5$.

Thus, we have $n = 5$, $p = \frac{1}{5}$, $q = \frac{4}{5}$.

Putting the above values in the formula of binomial distribution

$$p(x) = {}^{n}C_{x}\ p^{x} q^{n-x},\ x = 0, 1, 2, ..., n$$

We get the following $p(x)$

$$p(x) = {}^{5}C_{x} \left(\frac{1}{5}\right)^{x} \left(\frac{4}{5}\right)^{5-x},\quad x = 0, 1, 2, ..., 5$$

Using this formula, we calculate the probabilities for different values of x and multiplying such probabilities by 1000 we get the number of groups out of 1000 groups in which 0, 1, 2, 3, 4 or 5 house wives can identify the difference between two different brands of the butter..

(i) The number of groups out of 1000 groups in which at the most one house wife can identify the difference between two different brands of butter

$$= 1000 \times \left[p(0) + p(1) \right]$$

$$= 1000 \times \left[{}^{5}C_{0} \left(\frac{1}{5}\right)^{0} \left(\frac{4}{5}\right)^{5-0} + {}^{5}C_{1} \left(\frac{1}{5}\right)^{1} \left(\frac{4}{5}\right)^{5-1} \right]$$

$$= 1000 \times \left[\frac{1024}{3125} + 5 \times \left(\frac{1}{5}\right) \times \left(\frac{256}{625}\right) \right]$$

$$= 1000 \times \left[\frac{1024}{3125} + \frac{256}{625} \right]$$

$= 1000 \times [0.32768 + 0.4096]$

$= 1000 \times [0.73728]$

$= 737.28$

$\simeq 737$ groups

(ii) The number of groups out of 1000 groups in which only two house wives can identify the difference between the two brands of butter.

$= 1000 \times p(2)$

$= 1000 \times {}^5C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^{5-2}$

$= 1000 \times 10 \times \left(\frac{1}{25}\right) \times \left(\frac{64}{125}\right)$

$= 1000 \times \frac{640}{3125}$

$= 1000 \times 0.2048$

$= 204.8$

$\simeq 205$ groups

EXERCISE 2.2

1. For a symmetrical binomial distribution with $n = 8$, find $p(X \leq 1)$.

2. Mean of a binomial distribution is 5 and its variance is equal to the probability of success. Find the parameters of this distribution and hence find the probability of the event of getting none of the failures for this distribution.

3. A person has kept 4 cars to run on rent. The probability that any car is rented during the day is 0.6. Find the probability that more than one but less than 4 cars are rented during a day.

4. There are 200 farms in a Taluka. Among the bore wells made in these 200 farms of the Taluka, salted water is found in 20 farms. Find the probability of the event of not getting salted water in 3 out of 5 randomly selected farms from the Taluka.

5. An example is given to 6 students to solve. The probability of getting correct solution of the problem by any student is 0.6. Students are trying to solve the problem independently. Find the probability of getting the correct solution by only 2 out of the 6 students.

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Summary

- **Random Variable :** A function associating a real number with each outcome of the sample space of a random experiment is called random variable.
- **Discrete Random Variable :** A random variable X which can assume a finite or countable infinite number of values in the set R of real numbers is called a discrete random variable.
- **Continuous Random Variable :** A random variable X which can assume any value in R, the set of real numbers or in any interval of R is called continuous random variable.
- **Discrete Probability Distribution :** Suppose $X : U \to R$ is a random variable which assumes all values of a finite set $\{x_1, x_2, ..., x_n\}$ of R. Also suppose X assumes a value x_i with probability $p(x_i)$. If $p(x_i) > 0$ for $i = 1, 2, ..., n$ and $\Sigma\, p(x_i) = 1$ then the set of real values $\{x_1, x_2, ..., x_n\}$ and $\{p(x_1), p(x_2), ..., p(x_n)\}$ is called the discrete probability distribution of a random variable X which is expressed in a tabular form as follows :

$X = x$	x_1	x_2	....	x_i	...	x_n	Total
$p(x)$	$p(x_1)$	$p(x_2)$	....	$p(x_i)$	...	$p(x_n)$	1

Here $0 < p(x_i) < 1, i = 1, 2, ..., n$

- **Bernoulli Trials :** Suppose dichotomous random experiment has two outcomes success (S) and failure (F). If this experiment is repeated under identical conditions and the probability $p(0 < p < 1)$ of getting success at each trial is constant then such trials are called Bernoulli Trials.
- **Binomial Random Variable :** Suppose X denotes the number of successes in the sequence of success (S) and failure (F) obtained in n Bernoulli trials then X is called a binomial random variable.
- **Binomial Probability Distribution :** The probability distribution of a binomial random variable X is called binomial probability distribution.

List of Formulae

(1) Mean of discrete probability distribution $= \mu$

$= E(X)$

$= \Sigma\, x\, p(x)$

(2) Variance of discrete probability distribution $= \sigma^2$

$= V(X)$

$= E(X^2) - (E(X))^2$

where $E(X^2) = \Sigma\, x^2\, p(x)$

(3) Binomial Probability Distribution

$$P(X = x) = p(x) = {}^nC_x\, p^x q^{n-x},\ x = 0,\ 1,\ 2,\ \ldots,n.$$

$$0 < p < 1,\ q = 1 - p$$

(4) Mean of binomial probability distribution $= np$

(5) Variance of binomial probability distribution $= npq$

(6) Standard deviation of binomial probability distribution $= \sqrt{npq}$

(7) If an experiment having Bernoulli trials repeats N times and $p(x)$ is the probability of getting x successes in the experiment then expected frequency of the number of successes in N repetitions $= N \cdot p(x)$

EXERCISE 2

Section A

Find the correct option for the following multiple choice questions :

1. Which variable of the following will be an illustration of discrete variable ?

(a) Height of a student (b) Weight of a student

(c) Blood Pressure of a student (d) Birth year of a student

2. Which variable of the following will be an illustration of continuous variable ?

(a) Number of accidents occuring at any place

(b) Number of rainy days during a year

(c) Maximum temperature during a day

(d) Number of children in a family

3. A random variable X assume the values -1, 0 and 1 with respective probability $\frac{1}{5}$, K and $\frac{1}{3}$, where $0 < K < 1$ and X does not assume any value other than these values. What will be the value of $E(X)$?

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) $\frac{2}{15}$ (d) $\frac{3}{15}$

4. A random variable X assumes the values -2, 0 and 2 only with respective probabilities $\frac{1}{5}, \frac{3}{5}$ and K. If $0 < K < 1$, what will be the value of K ?

(a) $\frac{1}{5}$ (b) $\frac{4}{5}$ (c) $\frac{2}{5}$ (d) $\frac{3}{5}$

5. Mean and variance of a discrete probability distribution are 3 and 7 respectively. What will be $E\left(X^2\right)$ for this distribution ?

(a) 10 (b) 4 (c) 40 (d) 16

6. For the probability distribution of a discrete random variable, $E(X)=5$ and $E\left(X^2\right)=35$. What will be the variance of this distribution ?

(a) 40 (b) 30 (c) 20 (d) 10

7. For a positively skewed binomial distribution with $n = 10$, which of the following values might be the value of mean ?

(a) 5 (b) 3 (c) 9 (d) 7

8. For which value of x, the value of $p(x)$ of binomial distribution with parameters $n = 4$ and $p = \frac{1}{2}$ becomes maximum ?

(a) 0 (b) 2 (c) 3 (d) 4

9. The binomial distribution has mean 5 and variance $\frac{10}{7}$. What will be the type of this distribution ?

(a) Positively skewed (b) Negatively skewed

(c) Symmetric (d) Nothing can be said about the distribution

10. Which of the following is the formula of probability of an event of not getting a success in the binomial distribution with parameters n and p ?

(a) ${}^nC_0 p^n q^0$ (b) ${}^nC_0 p^0 q^n$ (c) ${}^nC_0 pq^n$ (d) ${}^nC_0 p^n q$

Section B

Answer the following questions in one sentence :

1. Define discrete random variable.

2. Define continuous random variable.

3. Define discrete probability distribution.

4. State the formula to find mean of discrete variable.

5. State the formula to find variance of discrete variable.

6. Mean of a symmetrical binomial distribution is 7. Find the value of its parameter n.

7. The parameters of a binomial distribution are 10 and $\frac{2}{5}$. Calculate its variance.

8. State the relation between the probability of success and failure in Bernoulli trials.

9. State the relation between mean and variance of binomial distribution.

10. The probability of failure in a binomial distribution is 0.6 and the number of trials in it is 5. Find the probability of success.

Section C

Answer the following questions :

1. The probability distribution of a random variable X is as follows :

X	2	3	4	5
$p(x)$	0.2	0.3	$4C$	C

Determine the value of constant C.

2. Calculate mean of the discrete probability distribution $p(x) = \begin{cases} \frac{x-1}{6}; & x = 2, 3 \\ \frac{1}{2}; & x = 4 \end{cases}$

3. The probability distribution of a random variable is as follows :

$p(x) = \frac{x+3}{10}, \quad x = -2, 1, 2$

Hence calculate $E(X^2)$.

4. If $n=4$ for a symmetrical binomial distribution then find $p(4)$.

5. Define Bernoulli trials.

6. For a binomial distribution, if probability of success is double the probability of failure and $n=4$ then find variance of the distribution.

7. Find the standard deviation of the binomial distribution having $n=8$ and probability of failure $\frac{2}{3}$.

8. Find parameters of the binomial distribution where mean = 4 and variance = 2.

9. For a binomial distribution with $n=10$ and $q-p=0.6$, find mean of this distribution.

10. For a binomial distribution, standard deviation is 0.8 and probability of failure is $\frac{2}{3}$, find the mean of this distribution.

Section D

Answer the following questions :

1. The probability distribution of a random variable X is as follows :

$$p(x)=\begin{cases} K(x-1); & x=2, 3 \\ K; & x=4 \\ K(6-x); & x=5 \end{cases}$$

Find the value of constant K and the probability of the event that variable X assumes even numbers.

2. The probability distribution of a random variable X is as follows :

$$p(x)=C\left(x^2+x\right), x=-2, 1, 2$$

Find the value of C and show that $p(2)=3p(-2)$.

3. The distribution of a random variable X is $p(x)=K\cdot {}^5P_x,\ x=0, 1, 2, 3, 4, 5$

Find constant K and mean of this distribution.

4. What is discrete probability distribution ? State its properties.

5. State properties of binomial distribution.

6. In a game of hitting a target, the probability that Ramesh will fail in hitting the target is $\frac{2}{5}$. If he is given 3 trials to hit the target, find the probability of the event he hits the target successfully in 2 trials. State mean of this distribution.

7. A person is asked to select a number from positive integers 1 to 7. If the number selected by him is odd then he is entitled to get the prize. If he is asked to take 5 trials then find the probability of the event that he will be entitled to get a prize in only one trial.

8. The mean and variance of the binomial distribution are 2 and $\frac{6}{5}$ respectively. Find $p(1)$ and $p(2)$ for this binomial distribution.

9. 10 % apples are rotten in a box of apples. Find the probability that half of the 6 apples selected from the box with replacement will be rotten and find the variance of the number of rotten apples.

Section E

Solve the following :

1. The probability distribution of the monthly demand of laptop in a store is as follows :

Demand of laptop	1	2	3	4	5	6
Probability	0.10	0.15	0.20	0.25	0.18	0.12

Determine the expected monthly demand of laptop and find variance of the demand.

2. Two dice are thrown simultaneously once. Obtain the discrete probability distribution of the number of dice for which the number '6' comes up.

3. If the probability that any 50 year old person will die within a year is 0.01, find the probability that out of a group of 5 such persons

(i) none of them will die within a year

(ii) at least one of them will die within a year.

4. The probability that a student studying in 12th standard of science stream will get admission to engineering branch is 0.3. 5 students are selected from the students who studied in this stream. Find the probability of the event that the number of students admitted to engineering branch is more than the number of students who did not get admission to the engineering branch.

5. The probability that a bomb dropped from a plane over a bridge will hit the bridge is $\frac{1}{5}$. Two bombs are enough to destroy the bridge. If 6 bombs are dropped on the bridge, find the probability that the bridge will be destroyed.

6. Normally, 40 % students fail in one examination. Find the probability that at least 4 students in a group of 6 students pass in this examination.

7. There are 3 red and 4 white balls in a box. Four balls are selected at random with replacement from the box. Find the probability of the event of getting (i) 2 red balls and 2 white balls (ii) all four white balls among the selected balls using binomial distribution.

Section F

Solve the following :

1. There are one dozen mangoes in a box of which 3 mangoes are rotten. 3 mangoes are randomly selected from the box without replacement. If X denotes the number of rotten mangoes in the selected mangoes, obtain the probability distribution of X and hence find expected value and variance of the rotten mangoes in the selected mangoes.

2. It is known that 50 % of the students studying in the 10th standard have a habit of eating chocolate. To examine the information, 1024 investigators are appointed. Every investigator randomly selects 10 students from the population of such students and examines them for the habit of eating chocolate. Find the number of investigators who inform that less than 30 percent of the students have a habit of eating chocolate.

James Bernoulli
(1654 –1705)

James (Jacob) Bernoulli was born in Basel, Switzerland. He was one of the many prominent mathematicians in the Bernoulli family. Following his father's wish, he studied theology (divinity) and entered the ministry. But contrary to the desires of his parents, he also studied mathematics and astronomy. He travelled throughout Europe from 1676 to 1682; learning about the latest discoveries in mathematics and the sciences under leading figures of the time. He was an early proponent of Leibnizian calculus and had sided with Leibniz during the Leibniz-Newton calculus controversy. He is known for his numerous contributions to calculus, and along with his brother Johann, was one of the founders of the calculus of variations. However, his most important contribution was in the field of probability, where he derived the first version of the law of large numbers. He was appointed as professor of mathematics at the University of Basel in 1687, remained in this position for the rest of his life.

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