

Integrals

✓ **Integration (Anti differentiation) :** Integration is the inverse process of differentiation. Instead of differentiating a function. we are given the derivative of a function and asked to find its primitive, i.e., the original function. Such a process is called integration or anti differentiation. Ex : $y = \int f(x) dx.$

Derivatives	Integrals (Antiderivatives)
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$
$\frac{d}{dx}(x) = 1$	$\int dx = x + c$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x dx = \sin x + c$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x dx = -\cos x + c$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x dx = \tan x + c$
$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$	$\int \operatorname{cosec}^2 x dx = -\cot x + c$
$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$	$\int \sec x \tan x dx = \sec x + c$
$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$	$\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + c$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$
$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + c$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \tan^{-1} x + c$
$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$	$\int \frac{dx}{1+x^2} = -\cot^{-1} x + c$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$
$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + c$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + c$
$\frac{d}{dx}(\log x) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log x + c$
$\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x$	$\int a^x dx = \frac{a^x}{\log a} + c$

✓ Integration by substitution method

$$\int \tan x \, dx = \log |\sec x| + c$$

$$\int \cot x \, dx = \log |\sin x| + c$$

$$\int \sec x \, dx = \log |\sec x + \tan x| + c$$

$$\int \operatorname{cosec} x \, dx = \log |\operatorname{cosec} x - \cot x| + c$$

✓ Integrals of some particular functions

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log |x + \sqrt{x^2 - a^2}| + c$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log |x + \sqrt{x^2 + a^2}| + c$$

✓ To find the integral $\int \frac{dx}{ax^2 + bx + c}$

we write, $ax^2 + bx + c = a \left[x^2 + \frac{b}{a}x + \frac{c}{a} \right] = a \left[\left(x + \frac{b}{2a} \right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2} \right) \right]$

Now; put $x + \frac{b}{2a} = t \Rightarrow dx = dt$ and $\frac{c}{a} - \frac{b^2}{4a^2} = \pm K^2$

The integral becomes $\frac{1}{a} \int \frac{dt}{t^2 \pm K^2}$

✓ To find the integrated of the type: $\int \frac{px + q}{ax^2 + bx + c} \, dx$

where p, q, a, b, c are constants.

To find the real numbers A, B such that,

$$px + q = A \frac{d}{dx} (ax^2 + bx + c) + B = A(2ax + b) + B$$

✓ Integration by partial fraction

Form of Rational function	Form of partial function
$\frac{px + q}{(x-a)(x-b)} ; a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px + q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
$\frac{px + qx + n}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px + qx + n}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
$\frac{px + qx + n}{(x-a)(x^2 + bx + c)}$	$\frac{A}{(x-a)} + \frac{Bx + C}{(x^2 + bx + c)}$

where $x^2 + bx + c$ cannot be factorised further.

✓ Integration by parts $\int f(x)g(x)dx = f(x)\int g(x)dx - \int [f'(x)\int g(x)dx]dx$

✓ Integral of the type $\int e^x [f(x) + f'(x)] dx = \int e^x f(x) dx$

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + c$$

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + c$$

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$$

✓ Fundamental theorem of Calculus

Area function : $A(x) = \int_a^x f(x) dx$

First fundamental theorem of integral calculus :

Theorem 1. Let f be a continuous function on the closed interval $[a, b]$ and let $A(x)$ be the area function. Then $A'(x) = f(x)$, for all $x \in [a, b]$

Second fundamental theorem of integral calculus :

Theorem 2. f be continuous function defined on the closed interval $[a, b]$ and F be an antiderivative of f .

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$$

✓ Definite Integral If $F(x)$ is the integral of $f(x)$ over the interval $[a, b]$, i.e. $\int f(x) dx = F(x)$ then the definite integral of $f(x)$ over the interval $[a, b]$ is denoted by $\int_a^b f(x)$ is defined as

$$\int_a^b f(x) dx = F(b) - F(a)$$

upper limit $\swarrow$
lower limit $\nwarrow$

✓ Definite integral as the limit of the sum

$$\int_a^b f(x) dx = \lim_{h \rightarrow 0} h [f(a) + f(a+h) + \dots + f(a+(n-1)h)]$$

OR

$$\int_a^b f(x) dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} [f(a) + f(a+h) + \dots + f(a+(n-1)h)]$$

where $h = \frac{b-a}{n} \rightarrow 0$ as $n \rightarrow \infty$

✓ Some properties of Definite Integrals :

$$P_0 : \int_a^b f(x) dx = \int_a^b f(t) dt$$

$$P_1 : \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$P_2 : \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$P_3 : \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$P_4 : \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$P_5 : \int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a-x) dx$$

$$P_6 : \int_0^{2a} f(x) dx = \begin{cases} 2 \int_0^a f(x) dx & : \text{if } f(2a-x) = f(x) \\ 0 & : f(2a-x) = -f(x) \end{cases}$$

$$P_7 : (i) \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx : f(x) \text{ is even function, i.e. } f(-x) = f(x)$$

$$(ii) \int_{-a}^a f(x) dx = 0 \text{ if } f(x) \text{ is odd function, i.e. } f(-x) = -f(x)$$