Exercise 15.6

Chapter 15 Multiple Integrals 15.6 1E

The area of the surface with equation $z = f(x, y)$, $(x, y) \in D$, where f_x and f_y are continuous, is $A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$.

We have $z = 2 + 3x + 4y$. Then, $\frac{\partial z}{\partial x} = 3$ and $\frac{\partial z}{\partial y} = 4$.

$$A(S) = \iint_D \sqrt{1 + (3)^2 + (4)^2}\, dA$$

From the dimensions of the rectangle, we get the limits of x as $(0, 5)$ and the limits of y as $(1, 4)$.

On substituting the known values in the expression for area, we get

$$A(S) = \int_1^4 \int_0^5 \sqrt{1 + (3)^2 + (4)^2}\, dx\, dy.$$

Evaluate the iterated integral.

$$\begin{aligned} A(S) &= \int_1^4 \int_0^5 \sqrt{1 + (3)^2 + (4)^2}\, dx\, dy \\ &= \int_1^4 5\sqrt{26}\, dy \\ &= 15\sqrt{26} \end{aligned}$$

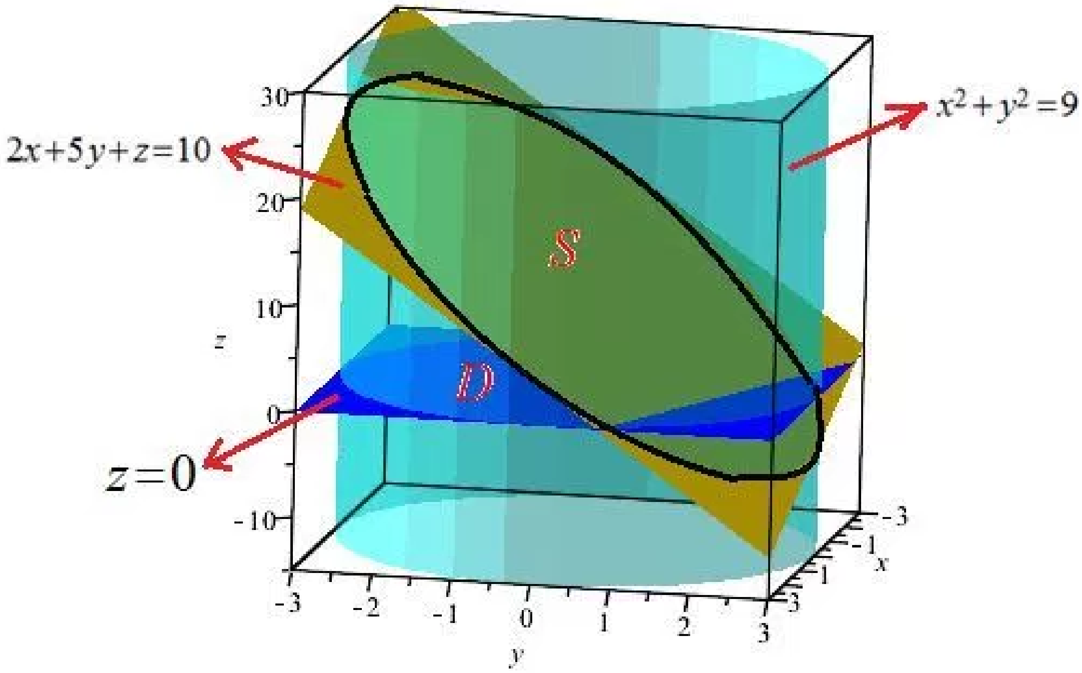
Therefore, the area of the surface is $\boxed{15\sqrt{26}}$.

Chapter 15 Multiple Integrals 15.6 2E

Consider the plane $2x+5y+z=10$.

Find the area of the surface of the given plane insider the cylinder $x^2+y^2=9$.

The surface S is shown in the below figure:



Area of the surface with equation $z=g(x,y)$ is given by the following formula:

$$A(S)=\iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\, dA$$

Here, D is the domain of integration.

The projection of the cylinder on xy- plane is a disk of radius 3.

So clearly, the surface S is lies above and below the disk D with center the origin and radius 3.

The region D can be written as $D=\left\{(x,y)\,|\,-3\le x\le 3, -\sqrt{9-x^2}\le y\le \sqrt{9-x^2}\right\}.$

Use polar coordinates, $x=r\cos\theta, y=r\sin\theta, dxdy=rdrd\theta.$

The region D can be written in polar coordinates as $D=\left\{(r,\theta)\,|\,0\le r\le 3, 0\le\theta\le 2\pi\right\}.$

Rewrite the plane equation as follows:

$z=10-2x-5y.$

Find the partial derivatives $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}.$

$$\begin{aligned}\frac{\partial z}{\partial x}&=\frac{\partial}{\partial x}(10-2x-5y)\\&=-2\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y}&=\frac{\partial}{\partial y}(10-2x-5y)\\&=-5\end{aligned}$$

$$\begin{aligned}\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}&=\sqrt{(-2)^2+(-5)^2+1}\\&=\sqrt{4+25+1}\\&=\sqrt{30}\end{aligned}$$

The surface area of the plane inside the cylinder is,

$$\begin{aligned}A&=\iint_D\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\;dA\\&=\iint_D\sqrt{30}\;dxdy\\&=\sqrt{30}\iint_D(1)\;dxdy\end{aligned}$$

$$\begin{aligned}&=\sqrt{30}\int_0^{2\pi}\int_0^3 r\;dr\;d\theta\\&=\sqrt{30}\left[\theta\right]_0^{2\pi}\left[\frac{r^2}{2}\right]_0^3\\&=\sqrt{30}\left[2\pi\right]\left[\frac{3^2}{2}\right]\\&=9\sqrt{30}\pi\end{aligned}$$

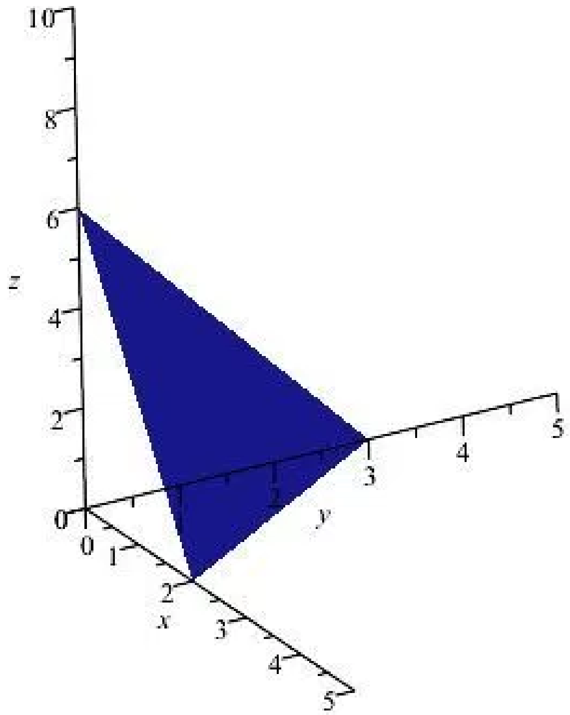
Therefore, the area of the plane that lies inside the cylinder is $\boxed{9\sqrt{30}\pi}.$

Chapter 15 Multiple Integrals 15.6 3E

Find the area of the surface of the plane $3x+2y+z=6$ that lies in the first octant.

Rewrite the plane equation $z=f(x,y)=6-3x-2y.$

At first sketch the plane $z=6-3x-2y$ in the first octant.



From the figure, it is obvious that, the variable *y* varies from 0 to 3,

Lower limit of *x* is obviously zero as the plane lies in the first quadrant, the upper limit of *x* is obtained by setting $z=0$ in $z=6-3x-2y$ and solving for *x*.

$$0 = 6-3x-2y$$
$$3x = 6-2y$$
$$x = \frac{6-2y}{3}$$

Hence, on the given surface the variable *x* varies from 0 to $\frac{6-2y}{3}$.

The area of the surface with equation $z=f(x,y)$, $(x,y)\in D$,

Where f_x and f_y are continuous functions, is

$$A(S) = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA \quad ......(1)$$

Use the formula (1) and also use limits of the variable *x* and *y*, to find the required

Surface area .

$$z = 6 - 3x - 2y \Rightarrow \frac{\partial z}{\partial x} = -3 \text{ and } \frac{\partial z}{\partial y} = -2$$

$$A(S) = \int_0^3 \int_0^{\frac{6-2y}{3}} \sqrt{1 + (-3)^2 + (-2)^2}\, dx\, dy$$

$$= \int_0^3 \left[\int_0^{\frac{6-2y}{3}} \sqrt{14}\, dx \right] dy$$

$$= \sqrt{14} \int_0^3 [x]_0^{\frac{6-2y}{3}} dy$$

$$= \sqrt{14} \int_0^3 \left(\frac{6-2y}{3} \right) dy$$

$$= \sqrt{14} \left[\frac{6y - y^2}{3} \right]_0^3$$

$$= \sqrt{14} \left[\frac{6(3) - 3^2}{3} - 0 \right]$$

$$= 3\sqrt{14}$$

Therefore, the area of the surface is $\boxed{3\sqrt{14} \text{ unit}^2}$.

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Consider the part of the surface $z = 1 + 3x + 2y^2$

The area of the surface with equation *z* = *f*(*x*, *y*), $(x, y) \in D$, where *fx* and *fy* are continuous,

is $A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$.

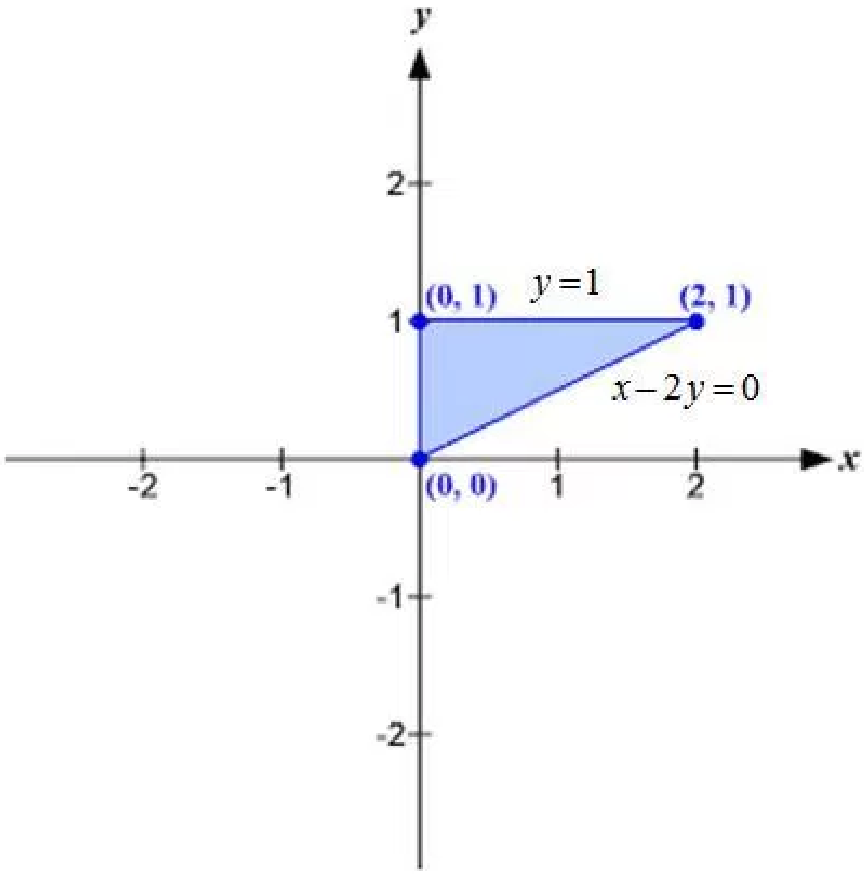
Since *z* = 1 + 3*x* + 2*y*2.

Then, $\frac{\partial z}{\partial x} = 3$ and $\frac{\partial z}{\partial y} = 4y$.

Therefore,

$$A(S) = \iint_D \sqrt{1 + (3)^2 + (4y)^2}\, dA$$

Now, let us sketch the region that lies above the triangle with vertices (0, 0), (0, 1), and (2, 1).



From the figure, see that the limits of *x* as (0, 2*y*) and the limits for *y* as (0, 1).

On substituting the known values in the expression for area, obtain that

$$A(S) = \int_0^1 \int_0^{2y} \sqrt{1 + (3)^2 + (4y)^2}\, dx\, dy.$$

Now evaluate the iterated integral.

$$\begin{aligned} A(S) &= \int_0^1 \int_0^{2y} \sqrt{1 + (3)^2 + (4y)^2}\, dx\, dy \\ &= \int_0^1 \sqrt{10 + 16y^2}\, (x)_0^{2y}\, dy \\ &= \int_0^1 2y\sqrt{10 + 16y^2}\, dy \end{aligned}$$

Suppose $10+16y^2=u$. Then $32y\,dy=du$. That is $2y\,dy=\frac{1}{16}du$

Limits: When $y=0 \Rightarrow u=10$

When $y=1 \Rightarrow u=26$

Therefore,

$$
\begin{aligned}
A(S) &= \int_0^1 2y\sqrt{10+16y^2}\ dy \\
&= \int_{10}^{26} \sqrt{u}\left(\frac{1}{16}du\right) \\
&= \frac{1}{16}\left(\frac{2}{3}u^{\frac{3}{2}}\right)_{10}^{26} \\
&= \frac{1}{24}\left(26^{\frac{3}{2}}-10^{\frac{3}{2}}\right)
\end{aligned}
$$

$$= \frac{1}{24}\left(26\sqrt{26}-10\sqrt{10}\right)$$

$$= \boxed{\frac{1}{12}\left(13\sqrt{26}-5\sqrt{10}\right)}$$

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Recall that, the surface area of an equation $z = g(x, y)$ is given by the following formula:

$$\iint_D f\left(x, y, g(x, y)\right)\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Here, D is the domain of integration.

The domain of integration is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the $xy-$ plane.

The domain is shown below:

$$D = \left\{(r, \theta) \mid 0 \le r \le 2, 0 \le \theta \le 2\pi\right\}.$$

Since the part of the sphere that lies above the xy-plane is

$$z = 4 - x^2 - y^2.$$

The partial derivatives of $z = 4 - x^2 - y^2$ are calculated as,

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}\left(4 - x^2 - y^2\right)\\ &= -2x\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}\left(4 - x^2 - y^2\right)\\ &= -2y\end{aligned}$$

Substitute all the values, and then the surface area can be calculated as,

$$\begin{aligned}S &= \iint_D f\left(x, y, g(x, y)\right)\sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}\, dA\\ &= \iint_D \sqrt{(-2x)^2 + (-2y)^2 + 1}\, dA\\ &= \iint_D \sqrt{4x^2 + 4y^2 + 1}\, dA\end{aligned}$$

Use polar coordinates, $x = r\cos\theta, y = r\sin\theta, r^2 = x^2 + y^2$ and $dA = rdrd\theta$, then the surface area of the surface is becomes,

$$S = \iint_D \sqrt{4x^2 + 4y^2 + 1}\, dA$$
$$= \int_0^{2\pi} \int_0^2 \sqrt{1+4r^2}\, rdrd\theta$$

Use substitution method: Let $1+4r^2 = t^2$, then derivative is

$$8rdr = 2tdt$$
$$rdr = \frac{1}{4}tdt$$

Limits of integration: If $r = 2$ then $t = \sqrt{1+4(2)^2} = \sqrt{17}$ and if $r = 0$ then

$$t = \sqrt{1+4(0)^2} = 1$$

Hence, the surface area of the paraboloid that lies above the xy-plane is,

$$S = \iint_D \sqrt{4x^2 + 4y^2 + 1}\, dA$$
$$= \int_0^{2\pi} \int_0^2 \sqrt{1+4r^2}\, rdrd\theta$$
$$= \int_0^{2\pi} \int_1^{\sqrt{17}} \sqrt{t^2}\, \frac{1}{4} tdtd\theta$$
$$= \frac{1}{4}\int_0^{2\pi} \int_1^{\sqrt{17}} t^2 dtd\theta$$
$$= \frac{1}{4}(\theta)_0^{2\pi} \left(\frac{t^3}{3}\right)_1^{\sqrt{17}}$$
$$= \frac{1}{12}(2\pi)\left[\left(\sqrt{17}\right)^3 - 1\right]$$
$$= \boxed{\frac{\pi}{6}\left(17\sqrt{17} - 1\right)}$$

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The formula for the surface area of a surface given by function $z = f(x, y)$ over region D is:

$$A(s) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Find the partial derivatives and limits of integration and use this formula.

Take the partial derivative in terms of x:

$z = y^2 - x^2$

$\frac{\partial z}{\partial x} = -2x$

Take the partial derivative in terms of y:

$z = y^2 - x^2$

$\frac{\partial z}{\partial y} = 2y$

Plugging into the formula for surface area, we now have

$$A(s) = \iint_D \sqrt{1+(-2x)^2+(2y)^2}\,dA$$

The region of integration is between a circle of radius 1 and a circle of radius 2. We convert to polar coordinates and let r range from 1 to 2. The limits in θ go all the way around the circles, or 0 to 2π. To rewrite the integrand in polar coordinates, use the conversion formulas

$x = r\cos\theta$

$y = r\sin\theta$

Also, in converting to polar coordinates we must add a factor of r into the integrand:

$$\begin{aligned} A(s) &= \int_1^2 \int_0^{2\pi} r\sqrt{1+(-2r\cos\theta)^2+(2r\sin\theta)^2}\,d\theta dr \\ &= \int_1^2 \int_0^{2\pi} r\sqrt{1+\left(4r^2\cos^2\theta\right)+\left(4r^2\sin^2\theta\right)}\,d\theta dr \\ &= \int_1^2 \int_0^{2\pi} r\sqrt{1+4r^2\left(\cos^2\theta+\sin^2\theta\right)}\,d\theta dr \end{aligned}$$

Use the Pythagorean identity, $\cos^2\theta+\sin^2\theta=1$, to simplify:

$$\begin{aligned} A(s) &= \int_1^2 \int_0^{2\pi} r\sqrt{1+4r^2(1)}\,d\theta dr \\ &= \int_1^2 \int_0^{2\pi} r\sqrt{1+4r^2}\,d\theta dr \end{aligned}$$

Integrate in terms of θ, leaving r constant:

$$\begin{aligned} A(s) &= \int_1^2 \int_0^{2\pi} r\sqrt{1+4r^2}\,d\theta dr \\ &= \int_1^2 \left(\theta r\sqrt{1+4r^2}\right)\Big|_0^{2\pi} dr \\ &= \int_1^2 \left(2\pi r\sqrt{1+4r^2}-0\right)dr \\ &= 2\pi\int_1^2 \left(r\sqrt{1+4r^2}\right)dr \end{aligned}$$

To integrate in terms of r, use a substitution:

$u = 1+4r^2$

$du = 8rdr$

$\frac{du}{8} = rdr$

Find the limits in terms of u:

$1+4(1)^2 = 5$

$1+4(2)^2 = 17$

Plug in u, du, and the new limits, and integrate:

$$A(s) = 2\pi\int_5^{17}\left(\sqrt{u}\right)\frac{du}{8}$$
$$= \frac{\pi}{4}\int_5^{17}\left(u^{1/2}\right)du$$
$$= \frac{\pi}{4}\left(\frac{2}{3}u^{3/2}\right)\Big|_5^{17}$$
$$= \frac{\pi}{6}\left(17^{3/2}-5^{3/2}\right)$$
$$= \boxed{\frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right)}$$

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Consider the surface $z = \frac{2}{3}\left(x^{\frac{3}{2}} + y^{\frac{3}{2}}\right)$ and the intervals $0 \le x \le 1$ and $0 \le y \le 1$.

The region T is described by $T = \left\{(x,y) \middle|\ 0 \le x \le 1,\ \ 0 \le y \le 1\right\}$.

The area of surface bounded by the surface is $A = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$.

The partial derivatives with respect to x are,

$$z = \frac{2}{3}\left(x^{\frac{3}{2}} + y^{\frac{3}{2}}\right)$$
$$\frac{\partial z}{\partial x} = \frac{2}{3}\cdot\frac{3}{2}x^{\frac{1}{2}}$$
$$= \sqrt{x}$$

The partial derivatives with respect to y are,

$$z = \frac{2}{3}\left(x^{\frac{3}{2}} + y^{\frac{3}{2}}\right)$$
$$\frac{\partial z}{\partial y} = \frac{2}{3}\cdot\frac{3}{2}y^{\frac{1}{2}}$$
$$= \sqrt{y}$$

The area of surface is,

$$A(S)=\int_0^1\int_0^1\sqrt{\left(\sqrt{x}\right)^2+\left(\sqrt{y}\right)^2+1}\,dydx$$
$$=\int_0^1\int_0^1\sqrt{x+y+1}\,dydx$$
$$=\int_0^1\frac{2}{3}\left[(1+x+y)^{\frac{3}{2}}\right]_0^1dx$$
$$=\frac{2}{3}\int_0^1\left[(1+x+1)^{\frac{3}{2}}-(1+x)^{\frac{3}{2}}\right]dx$$
$$=\frac{2}{3}\int_0^1\left[(2+x)^{\frac{3}{2}}-(1+x)^{\frac{3}{2}}\right]dx$$
$$=\frac{2}{3}\times\frac{2}{5}\int_0^1\left[(2+x)^{\frac{5}{2}}-(1+x)^{\frac{5}{2}}\right]_0^1$$
$$=\frac{4}{15}\left[(3)^{\frac{5}{2}}-(2)^{\frac{5}{2}}-(2)^{\frac{5}{2}}-(1)^{\frac{5}{2}}\right]$$
$$=\frac{4}{15}-\frac{32}{15}\sqrt{2}+\frac{12}{5}\sqrt{3}$$
$$=1.4068$$

Hence, the area of the surface is $\boxed{1.4068}$.

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Consider the part of the surface $z=xy$ that lies within the cylinder $x^2+y^2=1$.

The polar coordinates are $x=r\cos\theta,\ y=r\sin\theta$ and $r^2=x^2+y^2$

The region D is described by $D=\{(r,0)|\ 0\le r\le 1,\ \ 0\le\theta\le 2\pi\}$.

The area of surface is $A=\iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$.

The partial derivatives with respect to x are,

$z=xy$

$z_x=y$

The partial derivatives with respect to y are,

$z=xy$

$z_y=x$

The area of surface is,

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$A = \iint_D \sqrt{1 + y^2 + x^2}\, dA$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1 + r^2}\, r\ dr d\theta \qquad ......(1)$$

Let $1 + r^2 = t$

Then

$$2rdr = 1\ dt$$

$$rdr = \frac{1}{2}\ dt$$

Substitute the values into the equation (1),

$$A = \frac{1}{2}\int_0^{2\pi}\int_1^2 \sqrt{t}\, dt$$

$$= \frac{1}{2}\cdot\frac{2}{3}\int_0^{2\pi}\left[t^{3/2}\right]_0^2 d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\left(2\sqrt{2} - 1\right)d\theta$$

$$= \left(\frac{2\sqrt{2} - 1}{3}\right)\left[\theta\right]_0^{2\pi}$$

$$= \frac{2\pi\left(2\sqrt{2} - 1\right)}{3}$$

Hence, the area of the surface is $\boxed{\frac{2\pi\left(2\sqrt{2} - 1\right)}{3}}$.

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Consider the surface $x^2+y^2+z^2=4$ that lies above the plane $z=1$.

The equation of surface is,

$$\begin{aligned} x^2+y^2+z^2&=4 \\ x^2+y^2+1&=4 \\ x^2+y^2&=3 \\ r&=\sqrt{3} \end{aligned}$$

The region D is described by $D=\left\{(r,\theta)\,\middle|\; 0\le r\le\sqrt{3},\;\; 0\le\theta\le 2\pi\right\}$.

The area of the surface is $A=\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$.

The partial derivatives of $z=\sqrt{4-x^2-y^2}$ with respect to x are,

$$\begin{aligned} z&=\sqrt{4-x^2-y^2} \\ z_x&=\frac{-x}{\sqrt{4-x^2-y^2}} \end{aligned}$$

The partial derivatives of $z=\sqrt{4-x^2-y^2}$ with respect to x are,

$$\begin{aligned} z&=\sqrt{4-x^2-y^2} \\ z_y&=\frac{-y}{\sqrt{4-x^2-y^2}} \end{aligned}$$

Thus, the area of the surface is,

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$A = \iint_D \sqrt{1 + \frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2}}\, dydx$$

$$= \iint \sqrt{\frac{4 - x^2 - y^2 + x^2 + y^2}{4 - x^2 - y^2}}\, dA$$

$$= \iint \frac{2}{\sqrt{4 - x^2 - y^2}}\, dA$$

Substitute the polar coordinates,

$$A = 2\int_0^{2\pi}\int_0^{\sqrt{3}} \frac{r}{\sqrt{4 - r^2}}\, drd\theta$$

Put

$$4 - r^2 = t$$

Then,

$$-2rdr = dt$$

The limits are,

$$\text{Lower Limit}: r = 0, \text{then } t = 4$$
$$\text{Upper Limit}: r = \sqrt{3}, \text{then } t = 1$$

The area of the surface becomes,

$$A = 2\int_0^{2\pi}\int_0^{\sqrt{3}} \frac{r}{\sqrt{4-r^2}}\,drd\theta$$
$$= -\int_0^{2\pi}\left[\int_4^1 \frac{1}{\sqrt{t}}\,dt\right]d\theta$$
$$= -\int_0^{2\pi} 2\left[\sqrt{t}\right]_4^1 d\theta$$
$$= -\int_0^{2\pi} (-2)\,d\theta$$
$$= 2[\theta]_0^{2\pi}$$
$$= 2(2\pi)$$
$$= \boxed{4\pi}$$

Therefore, the area of the surface is $\boxed{4\pi}$.

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The formula for the surface area of a surface given by function $z = f(x, y)$ over region D is:

$$A(s) = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

Find the partial derivatives and limits of integration and use this formula.
Since we are concerned with only the surface above the xy-plane, we can rewrite the surface
$x^2+y^2+z^2=a^2$
That we are working with as
$z=\sqrt{a^2-x^2-y^2}$.

Take the partial derivative in terms of x:
$$z=\sqrt{a^2-x^2-y^2}$$
$$\frac{\partial z}{\partial x}=\frac{1}{2}(a^2-x^2-y^2)^{-1/2}(-2x)$$
$$=-x(a^2-x^2-y^2)^{-1/2}$$

Take the partial derivative in terms of y:
$$z=\sqrt{a^2-x^2-y^2}$$
$$\frac{\partial z}{\partial y}=\frac{1}{2}(a^2-x^2-y^2)^{-1/2}(-2y)$$
$$=-y(a^2-x^2-y^2)^{-1/2}$$

Plugging into the formula for surface area, we now have

$$A(s) = \iint_D \sqrt{1+\left(-x(a^2-x^2-y^2)^{-1/2}\right)^2+\left(-y(a^2-x^2-y^2)^{-1/2}\right)^2}\,dA$$
$$= \iint_D \sqrt{1+\left(x^2(a^2-x^2-y^2)^{-1}\right)+\left(y^2(a^2-x^2-y^2)^{-1}\right)}dA$$
$$= \iint_D \sqrt{1+(a^2-x^2-y^2)^{-1}(x^2+y^2)}dA$$
$$= \iint_D \sqrt{1+\left(\frac{x^2+y^2}{a^2-(x^2+y^2)}\right)}dA$$

The region is within the cylinder $x^2+y^2=ax$. We are going to want to integrate in polar coordinates, so we convert this boundary equation using the conversion formulas

$r^2 = x^2+y^2$ ...... (1)

$x = r\cos\theta$

$y = r\sin\theta$

The equation for the cylinder then becomes

$r^2 = ar\cos\theta$

$r = a\cos\theta$

The cross-section of this cylinder is a disk, and ordinarily a function with $\cos\theta$ would take a period of 2π to traverse a full circle. However, once $\cos\theta$ becomes negative, the r values become negative and begin to repeat the same circle. We therefore want to traverse only the positive θ values, and we let the θ limits range from $-\pi/2$ to $\pi/2$. The r limits range from the origin to the function that borders r, or 0 to $a\cos\theta$.

We again apply (1) to convert the integrand to polar coordinates, and we also fill in the limits of integration. Converting to polar coordinates requires multiplying a factor of r into the integrand:

$$A(s) = \iint_D \sqrt{1+\left(\frac{x^2+y^2}{a^2-(x^2+y^2)}\right)}dA$$
$$= \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} r\sqrt{1+\left(\frac{r^2}{a^2-r^2}\right)}drd\theta$$

To integrate in terms of r, use a substitution:

$u = a^2-r^2$

$du = -2rdr$

$rdr = \frac{-du}{2}$

We can also write this substitution as

$r^2 = a^2 - u$

So that we can plug in for r^2.

Find the new limits of integration in terms of u:

$$a^2 - 0^2 = a^2$$

$$\begin{aligned} a^2 - (a\cos\theta)^2 &= a^2 - a^2\cos^2\theta \\ &= a^2(1-\cos^2\theta) \\ &= a^2\sin^2\theta \end{aligned}$$

The last step uses the fact that the Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$, can be rearranged to be $\sin^2\theta = 1 - \cos^2\theta$ and substituted in.

Plug in with u and du and simplify:

$$\begin{aligned} A(s) &= \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} r\sqrt{1+\left(\frac{r^2}{a^2-r^2}\right)}\,dr\,d\theta \\ &= \int_{-\pi/2}^{\pi/2}\int_{a^2}^{a^2\sin^2\theta} \sqrt{1+\left(\frac{a^2-u}{u}\right)}\left(\frac{-du}{2}\right)d\theta \\ &= -\frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_{a^2}^{a^2\sin^2\theta} \sqrt{1+\left(\frac{a^2}{u}\right)-\frac{u}{u}}\,du\,d\theta \end{aligned}$$

$$= -\frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_{a^2}^{a^2\sin^2\theta} \sqrt{1+a^2u^{-1}-1}\,du\,d\theta$$

$$= -\frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_{a^2}^{a^2\sin^2\theta} \sqrt{a^2u^{-1}}\,du\,d\theta$$

$$= -\frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_{a^2}^{a^2\sin^2\theta} au^{-1/2}\,du\,d\theta$$

Integrate in terms of u:

$$\begin{aligned} A(s) &= -\frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_{a^2}^{a^2\sin^2\theta} au^{-1/2}\,du\,d\theta \\ &= -\frac{a}{2}\int_{-\pi/2}^{\pi/2}\left(2u^{1/2}\right)\Big|_{a^2}^{a^2\sin^2\theta} d\theta \\ &= -a\int_{-\pi/2}^{\pi/2}\left(|a\sin\theta| - |a|\right)d\theta \end{aligned}$$

Notice that taking the square root of a square gives the absolute value of the quantity. Because of symmetry, whether a was originally positive or negative doesn't matter—the area of the surface in question will be the same. So we assume a positive. Between our limits of integration, $\sin\theta$ is positive from 0 to $\pi/2$ but negative from $-\pi/2$ to 0, and since its values are otherwise symmetric,

We integrate from 0 to $\pi/2$ only and double it:

$$A(s) = -2a\int_0^{\pi/2}(a\sin\theta - a)d\theta$$

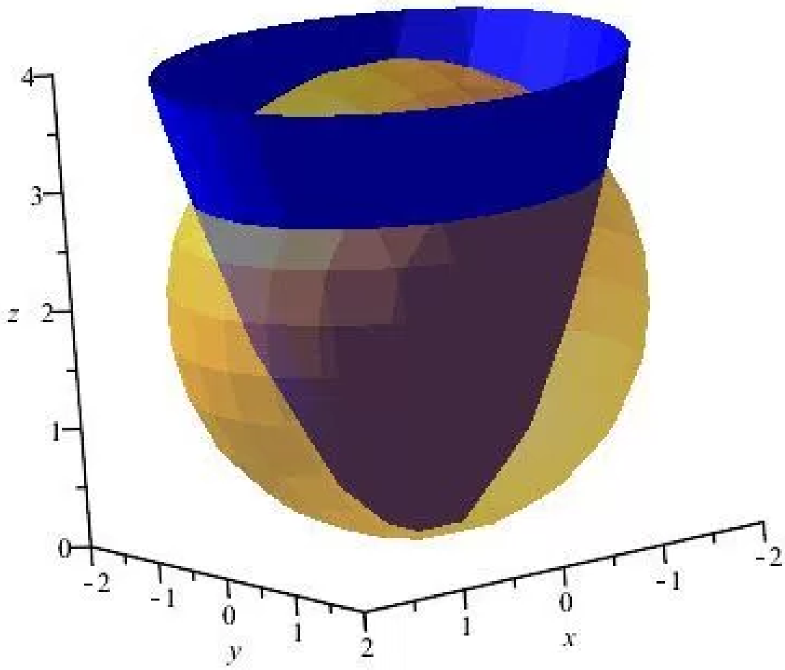
Integrate in terms of θ:

$$\begin{aligned}
A(s) &= -2a\int_0^{\pi/2}(a\sin\theta - a)d\theta \\
&= -2a\,(-a\cos\theta - a\theta)\Big|_0^{\pi/2} \\
&= -2a(-a\cos(\pi/2) - a(\pi/2) - (-a\cos(0) - a(0))) \\
&= -2a\left(-a(0) - \frac{a\pi}{2} + a(1) + 0\right) \\
&= -2a\left(-\frac{a\pi}{2} + a\right) \\
&= 2a^2\left(\frac{\pi}{2} - 1\right) \\
&= \boxed{a^2(\pi - 2)}
\end{aligned}$$

Chapter 15 Multiple Integrals 15.6 12E

Consider the sphere $x^2+y^2+z^2=4z$ and paraboloid $z=x^2+y^2$.

The objective is to find the surface area of the solid obtained from the part of the sphere that lies inside the paraboloid.



The surface area of the surface is given by

$$S=\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA.$$

Use cylindrical coordinates to find the surface area of the solid as follows:

$$x=r\cos\theta, y=r\sin\theta, \text{ and } r^2=x^2+y^2$$

Substitute these values in the above sphere and paraboloid.

$$r^2+z^2=4z, z=r^2$$

From the two equations,

$$\begin{aligned} z+z^2&=4z\\ z^2-3z&=0\\ z(z-3)&=0\\ z&=0 \text{ or } z=3 \end{aligned}$$

Substitute $z=0$ and $z=3$ in the equations $r^2+z^2=4z, z=r^2$ to get the values of r as follows:

$r=0, r=\sqrt{3}$.

Since the base of the part of the sphere is a circle upon *xy*-plane, so that $0 \le \theta \le 2\pi$.

Thus, the region of the integration is, $D=\left\{(r,\theta) \mid 0 \le r \le \sqrt{3}, 0 \le \theta \le 2\pi\right\}$.

Find the value of *z* from the sphere equation $x^2+y^2+z^2=4z$.

$$\begin{aligned} x^2+y^2+z^2 &= 4z \\ z^2-4z+\left(x^2+y^2\right) &= 0 \end{aligned}$$

It is in the form the quadratic equation.

Apply the quadratic formula to find the value of *z*.

$$\begin{aligned} z &= \frac{-(-4) \pm \sqrt{(-4)^2-4(1)\left(x^2+y^2\right)}}{2} \\ &= \frac{4 \pm \sqrt{16-4\left(x^2+y^2\right)}}{2} \\ &= \frac{4 \pm 2\sqrt{4-\left(x^2+y^2\right)}}{2} \\ &= 2 \pm \sqrt{4-x^2-y^2} \end{aligned}$$

Since here the objective is to find the surface of the part of the sphere, so take $z=2+\sqrt{4-x^2-y^2}$.

Now calculate the partial derivatives of the function *z*.

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}\left(2+\sqrt{4-x^2-y^2}\right) \\ &= \frac{-2x}{2\sqrt{4-x^2-y^2}} \\ &= \frac{-x}{\sqrt{4-x^2-y^2}} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}\left(2+\sqrt{4-x^2-y^2}\right) \\ &= \frac{-2y}{2\sqrt{4-x^2-y^2}} \\ &= \frac{-y}{\sqrt{4-x^2-y^2}} \end{aligned}$$

Substitute all the values into the surface area formula $S = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$.

$$\begin{aligned}
S &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \\
&= \iint_D \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{4 - x^2 - y^2}}\right)^2}\, dA \\
&= \iint_D \sqrt{1 + \frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2}}\, dA \\
&= \iint_D \sqrt{\frac{4 - x^2 - y^2 + x^2 + y^2}{4 - x^2 - y^2}}\, dA
\end{aligned}$$

$$= \iint_D \frac{2}{\sqrt{4 - x^2 - y^2}}\, dA$$

Use polar coordinates to simplify the integral as follows:

Substitute the polar coordinates $x = r\cos\theta, y = r\sin\theta, dA = rdrd\theta$ in the above integral.

$$S = \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{2r}{\sqrt{4-r^2}} drd\theta$$

Use substitution method: Let $4 - r^2 = t^2$

Its derivative is,

$$-2rdr = 2tdt$$
$$rdr = -tdt$$

Lower limit: If $r = 0$ then $t = 2$.

Upper limit: If $r = \sqrt{3}$ then $t = 1$.

So the surface area integral can be written as follows:

$$S = 2\int_0^{2\pi}\int_2^1 \frac{-tdt}{\sqrt{t^2}} d\theta$$
$$= 2\int_0^{2\pi}\int_2^1 (-dt) d\theta$$
$$= 2\int_0^{2\pi}\int_1^2 -(-dt) d\theta \qquad \left(\text{since } \int_a^b dx = \int_b^a (-dx)\right)$$
$$= 2\int_0^{2\pi} [t]_1^2 d\theta$$
$$= 2\int_0^{2\pi} [2-1] d\theta$$
$$= 2(\theta)_0^{2\pi}$$
$$= 2(2\pi)$$
$$= 4\pi$$

Hence, the required surface area is, $S = \boxed{4\pi}$.

Chapter 15 Multiple Integrals 15.6 13E

The formula for the surface area of a surface given by function $z = f(x, y)$ over region D is:

$$A(s) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Find the partial derivatives and limits of integration and use this formula.

Take the partial derivative in terms of x:

$z = e^{-x^2-y^2}$

$\frac{\partial z}{\partial x} = -2xe^{-x^2-y^2}$

Take the partial derivative in terms of y:

$z = e^{-x^2-y^2}$

$\frac{\partial z}{\partial y} = -2ye^{-x^2-y^2}$

Plugging into the formula for surface area, we now have

$$A(s) = \iint_D \sqrt{1+\left(-2xe^{-x^2-y^2}\right)^2+\left(-2ye^{-x^2-y^2}\right)^2}\,dA$$

The region is a disk of radius 2. We are going to want to convert to polar coordinates, so for limits of integration we let r range from 0 to 2, and θ must encompass the whole circle, going from 0 to 2π.

To convert the integrand to polar coordinates, use the conversion formulas

$r^2 = x^2 + y^2$

$x = r\cos\theta$

$y = r\sin\theta$ ...... (1)

We apply the formulas in (1) to convert the integrand to polar coordinates, and we also fill in the limits of integration. Converting to polar coordinates requires multiplying a factor of r into the integrand:

$$A(s) = \iint_D \sqrt{1+\left(-2xe^{-x^2-y^2}\right)^2+\left(-2ye^{-x^2-y^2}\right)^2}\,dA$$

$$= \int_0^2\int_0^{2\pi} r\sqrt{1+\left(-2r\cos\theta e^{-r^2}\right)^2+\left(-2r\sin\theta e^{-r^2}\right)^2}\,d\theta dr$$

$$= \int_0^2\int_0^{2\pi} r\sqrt{1+4r^2\cos^2\theta e^{-2r^2}+4r^2\sin^2\theta e^{-2r^2}}\,d\theta dr$$

$$= \int_0^2\int_0^{2\pi} r\sqrt{1+4r^2e^{-2r^2}(\cos^2\theta+\sin^2\theta)}\,d\theta dr$$

Use the Pythagorean identity, $\cos^2\theta+\sin^2\theta=1$, to simplify:

$$A(s)=\int_0^2\int_0^{2\pi} r\sqrt{1+4r^2e^{-2r^2}(1)}\,d\theta dr$$

$$A(s)=\int_0^2\int_0^{2\pi} r\sqrt{1+4r^2e^{-2r^2}}\,d\theta dr$$

Integrate in terms of θ:

$$A(s)=\int_0^2\int_0^{2\pi} r\sqrt{1+4r^2e^{-2r^2}}\,d\theta dr$$

$$=\int_0^2\left(\theta r\sqrt{1+4r^2e^{-2r^2}}\right)\Big|_0^{2\pi} dr$$

$$=\int_0^2\left(2\pi r\sqrt{1+4r^2e^{-2r^2}}-0\right)dr$$

$$=\boxed{2\pi\int_0^2 r\sqrt{1+4r^2e^{-2r^2}}\,dr}$$

We now have the surface area in a single integral, as the problem asks for. As specified, we use a calculator to solve, estimating the surface area to be approximately $\boxed{13.9783}$.

Chapter 15 Multiple Integrals 15.6 14E

The formula for the surface area of a surface given by function $z=f(x,y)$ over region D is:

$$A(s)=\iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

Find the partial derivatives and limits of integration and use this formula.

Take the partial derivative in terms of x:

$$z=\cos(x^2+y^2)$$

$$\frac{\partial z}{\partial x}=-2x\sin(x^2+y^2)$$

Take the partial derivative in terms of y:

$$z=\cos(x^2+y^2)$$

$$\frac{\partial z}{\partial y}=-2y\sin(x^2+y^2)$$

Plugging into the formula for surface area, we now have

$$A(s)=\iint_D\sqrt{1+\left(-2x\sin(x^2+y^2)\right)^2+\left(-2y\sin(x^2+y^2)\right)^2}\,dA$$

The region is inside a cylinder of radius 1, meaning the region of integration is a disk of radius 1. We are going to want to convert to polar coordinates, so for limits of integration we let r range from 0 to 1, and θ must encompass the whole circle, going from 0 to 2π.

To convert the integrand to polar coordinates, use the conversion formulas

$r^2 = x^2 + y^2$

$x = r\cos\theta$

$y = r\sin\theta$ ...... (1)

We apply the formulas in (1) to convert the integrand to polar coordinates, and we also fill in the limits of integration. Converting to polar coordinates requires multiplying a factor of r into the integrand:

$$A(s) = \iint_D \sqrt{1+\left(-2x\sin(x^2+y^2)\right)^2+\left(-2y\sin(x^2+y^2)\right)^2}\,dA$$

$$= \int_0^1\int_0^{2\pi} r\sqrt{1+\left(-2r\cos\theta\sin(r^2)\right)^2+\left(-2r\sin\theta\sin(r^2)\right)^2}\,d\theta dr$$

$$= \int_0^1\int_0^{2\pi} r\sqrt{1+\left(4r^2\cos^2\theta\sin^2(r^2)\right)+\left(4r^2\sin^2\theta\sin^2(r^2)\right)}d\theta dr$$

$$= \int_0^1\int_0^{2\pi} r\sqrt{1+4r^2\sin^2(r^2)\left(\cos^2\theta+\sin^2\theta\right)}d\theta dr$$

Use the Pythagorean identity, $\cos^2\theta+\sin^2\theta = 1$, to simplify:

$$A(s) = \int_0^1\int_0^{2\pi} r\sqrt{1+4r^2\sin^2(r^2)(1)}d\theta dr$$

$$A(s) = \int_0^1\int_0^{2\pi} r\sqrt{1+4r^2\sin^2(r^2)}d\theta dr$$

Integrate in terms of θ:

$$A(s) = \int_0^1\left(r\theta\sqrt{1+4r^2\sin^2(r^2)}\right)\Big|_0^{2\pi} dr$$

$$= \int_0^1\left(2\pi r\sqrt{1+4r^2\sin^2(r^2)}-0\right)dr$$

$$= \boxed{2\pi\int_0^1\left(r\sqrt{1+4r^2\sin^2(r^2)}\right)dr}$$

We now have the surface area in a single integral, as the problem asks for. As specified, we use a calculator to solve, estimating the surface area to be approximately $\boxed{4.1073}$.

Chapter 15 Multiple Integrals 15.6 15E

The formula for the surface area of a surface given by function $z = f(x,y)$ over region D is:

$$A(s) = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA \quad \text{......} \quad (1)$$

Find the partial derivatives and limits of integration and plug into this formula. Then compare using the Midpoint Rule versus a calculator.

Take the partial derivative in terms of x:

$$z = x^2 + y^2$$

$$\frac{\partial z}{\partial x} = 2x$$

Take the partial derivative in terms of y:

$$z = x^2 + y^2$$

$$\frac{\partial z}{\partial y} = 2y$$

We have been given the limits of integration; both x and y range from 0 to 1. Plug the limits of integration and partial derivatives into the surface integral formula, (1):

$$\begin{aligned} A(s) &= \int_0^1\int_0^1 \sqrt{1+(2x)^2+(2y)^2}\,dydx \\ &= \int_0^1\int_0^1 \sqrt{1+(4x^2)+(4y^2)}\,dydx \end{aligned} \qquad \ldots\ldots (2)$$

(a) Use the Midpoint Rule to estimate this double integral. The Midpoint Rule states that the double integral of a function $f(x., y)$ over a region R can be approximated as follows:

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\Delta A$$

Where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Here, if $[0,1]\times[0,1]$ is to be estimated using the Midpoint Rule with four squares, we must divide each side of the region into two intervals:

$$\begin{aligned} A(s) &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\Delta A \\ &= \sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\Delta A \\ &= f(\overline{x}_1,\overline{y}_1)\Delta A + f(\overline{x}_1,\overline{y}_2)\Delta A + f(\overline{x}_2,\overline{y}_1)\Delta A + f(\overline{x}_2,\overline{y}_2)\Delta A \end{aligned}$$

Since the x and y dimensions are each divided in half, both the x- and y-intervals are [0, 1/2] and [1/2, 1]. The midpoints of the four regions occur at (1/4, 1/4), (1/4, 3/4), (3/4, 1/4), and (3/4, 3/4). The area of each square is (1/2) (1/2) = 1/4. Plug in these values and the integrand of (2) for the function f:

$$A(s) \approx f(\bar{x}_1, \bar{y}_1)\Delta A + f(\bar{x}_1, \bar{y}_2)\Delta A + f(\bar{x}_2, \bar{y}_1)\Delta A + f(\bar{x}_2, \bar{y}_2)\Delta A$$

$$= \sqrt{1+\left(4(1/4)^2\right)+\left(4(1/4)^2\right)}(1/4) + \sqrt{1+\left(4(3/4)^2\right)+\left(4(1/4)^2\right)}(1/4)$$

$$+\sqrt{1+\left(4(1/4)^2\right)+\left(4(3/4)^2\right)}(1/4) + \sqrt{1+\left(4(3/4)^2\right)+\left(3(1/4)^2\right)}(1/4)$$

$$\approx \boxed{1.8279}$$

(b) Use a calculator to evaluate (2):

$$A(s) = \int_0^1\int_0^1 \sqrt{1+\left(4x^2\right)+\left(4y^2\right)}\,dydx$$

$$\approx \boxed{1.8616}$$

This is fairly close to the answer obtained using the Midpoint Rule in part (a), showing that the Midpoint Rule gives a good estimation for this integral.

Chapter 15 Multiple Integrals 15.6 17E

Consider the surface $z = 1+2x+3y+4y^2$ and the intervals are $1 \le x \le 4$ and $0 \le y \le 1$.

The region D is described by $D = \{(x,y) | 1 \le x \le 4,\ 0 \le y \le 1\}$.

The surface area is $A = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$.

The partial derivatives of z with respect to x are,

$$z = 1+2x+3y+4y^2$$
$$z_x = 2$$

The partial derivatives of z with respect to y are,

$$z = 1+2x+3y+4y^2$$
$$z_y = 3+8y$$

Use Maple software to find the area of the surface.

The key strokes and its results are shown below.

> $with(Student[MultivariateCalculus]):$

$SurfaceArea\left(1+2x+3y+4y^2, x=1..4, y=0..1, output=integral\right)$

$$\int_1^4\int_0^1 \sqrt{5+(-8y-3)^2}\,dy\,dx$$

$SurfaceArea\left(1+2x+3y+4y^2, x=1..4, y=0..1\right)$

$$\frac{45}{8}\sqrt{14}-\frac{15}{16}\ln\left(\sqrt{5}\sqrt{14}+3\sqrt{5}\right)+\frac{15}{16}\ln\left(3\sqrt{5}\sqrt{14}+11\sqrt{5}\right)$$

$\xrightarrow{\text{at 5 digits}}$

22.166

Therefore; the exact area is $\boxed{\frac{45}{8}\sqrt{14}+\frac{15}{16}\log\left(\frac{3\sqrt{70}+11\sqrt{5}}{\sqrt{70}+3\sqrt{5}}\right)\approx 22.166}$.

Chapter 15 Multiple Integrals 15.6 18E

Consider the surface $z=1+x+y+x^2$ and intervals $-2\le x\le 1$ and $-1\le y\le 1$.

The region D is described by $D=\left\{(x,y)\middle|-2\le x\le 1,\ -1\le y\le 1\right\}$.

The area of the surface is $A=\iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$.

The partial derivatives of z with respect to x are,

$$z=1+x+y+x^2$$
$$z_x=1+2x$$

The partial derivatives of z with respect to y are,

$$z=1+x+y+x^2$$
$$z_y=1$$

Use Maple software to find the area of the surface.

The key strokes and its results are shown below.

> $with(Student[MultivariateCalculus]):$

$SurfaceArea(1 + x + y + x^2, x = -2..1, y = -1..1, output = integral)$

$$\int_{-2}^{1}\int_{-1}^{1}\sqrt{2 + (-2x-1)^2}\,dy\,dx$$

$SurfaceArea(1 + x + y + x^2, x = -2..1, y = -1..1)$

$$3\sqrt{11} - \ln(\sqrt{2}\sqrt{11} - 3\sqrt{2}) + \ln(\sqrt{2}\sqrt{11} + 3\sqrt{2})$$

$\xrightarrow{\text{at 5 digits}}$

12.943

Hence, the area of the surface is $\boxed{12.943}$.

Chapter 15 Multiple Integrals 15.6 19E

The formula for the surface area of a surface given by function $z = f(x, y)$ over region D is:

$$A(s) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \quad \ldots\ldots (1)$$

Find the partial derivatives and limits of integration and plug into this formula. Then use a calculator to evaluate.

Take the partial derivative in terms of x:

$$z = 1 + x^2 y^2$$

$$\frac{\partial z}{\partial x} = 2xy^2$$

Take the partial derivative in terms of y:

$$z = 1 + x^2 y^2$$

$$\frac{\partial z}{\partial y} = 2yx^2$$

Plug into the surface integral formula, (1):

$$A(s)=\iint_D \sqrt{1+\left(2xy^2\right)^2+\left(2yx^2\right)^2}\,dA \quad \ldots\ldots (2)$$

The region of integration is a disk of radius 1. We are going to want to convert to polar coordinates, so for limits of integration we let r range from 0 to 1, and θ must encompass the whole circle, going from 0 to 2π.

To convert the integrand to polar coordinates, use the conversion formulas

$$r^2=x^2+y^2$$
$$x=r\cos\theta$$
$$y=r\sin\theta \quad \ldots\ldots (3)$$

Plug the conversion formulas (3) into equation (2), adding in the limits of integration. Converting to polar coordinates requires multiplying a factor of r into the integrand:

$$\begin{aligned}
A(s)&=\iint_D \sqrt{1+\left(2xy^2\right)^2+\left(2yx^2\right)^2}\,dA\\
&=\int_0^1\int_0^{2\pi} r\sqrt{1+\left(2r\cos\theta r^2\sin^2\theta\right)^2+\left(2r\sin\theta r^2\cos^2\theta\right)^2}\,d\theta dr\\
&=\int_0^1\int_0^{2\pi} r\sqrt{1+\left(4r^2\cos^2\theta r^4\sin^4\theta\right)+\left(4r^2\sin^2\theta r^4\cos^4\theta\right)}\,d\theta dr\\
&=\int_0^1\int_0^{2\pi} r\sqrt{1+4r^6\cos^2\theta\sin^2\theta\left(\sin^2\theta+\cos^2\theta\right)}\,d\theta dr
\end{aligned}$$

Use the Pythagorean identity, $\cos^2\theta+\sin^2\theta=1$, to simplify:

$$\begin{aligned}
A(s)&=\int_0^1\int_0^{2\pi} r\sqrt{1+4r^6\cos^2\theta\sin^2\theta(1)}\,d\theta dr\\
&=\int_0^1\int_0^{2\pi} r\sqrt{1+4r^6\cos^2\theta\sin^2\theta}\,d\theta dr
\end{aligned}$$

As specified by the problem, solve this integral using a calculator to estimate the surface area as $\boxed{3.3213}$.

Chapter 15 Multiple Integrals 15.6 21E

Consider the surface $z=ax+by+c$

The region D is described by in the $xy-$ plane is $D=\{(x,y)|0\le x\le 1, 0\le y\le 1\}$.

The area of the surface is $A=\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$.

The partial derivatives of z with respect to x are,

$$z=ax+by+c$$
$$z_x=a$$

The partial derivatives of z with respect to y are,

$$z=ax+by+c$$
$$z_x=b$$

The area of the region is,

$$A(D)=\int_0^1\int_0^1 dydx$$
$$=1$$

The surface area of the region is,

$$A=\iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$A(S)=\sqrt{1+a^2+b^2}\int_0^1\int_0^1 dydx$$
$$=\sqrt{1+a^2+b^2}\int_0^1 (y)_0^1\,dx$$
$$=\sqrt{1+a^2+b^2}\,(x)_0^1$$
$$=\sqrt{1+a^2+b^2}\,(1)$$
$$=\sqrt{1+a^2+b^2}\,A(D)$$

Therefore, the area of the surface is $\boxed{\sqrt{1+a^2+b^2}\,A(D)}$.

Chapter 15 Multiple Integrals 15.6 22E

The top of the half sphere is $x^2+y^2+z^2=a^2$ bounded by a circle $x^2+y^2=a^2$

The area of the surface is $A=\iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$.

The partial derivatives of z with respect to x are,

$$x^2+y^2+z^2=a^2$$
$$z=\sqrt{a^2-x^2-y^2}$$
$$z_x=\frac{-x}{\sqrt{a^2-x^2-y^2}}$$

The partial derivatives of z with respect to y are,

$$x^2+y^2+z^2=a^2$$
$$z=\sqrt{a^2-x^2-y^2}$$
$$z_y=\frac{-y}{\sqrt{a^2-x^2-y^2}}$$

The surface area of the sphere is,

$$A(S)=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\frac{a}{\sqrt{a^2-x^2-y^2}}\,dydx$$

$$=4\int_{0}^{a}\int_{0}^{\sqrt{a^2-x^2}}\frac{a}{\sqrt{a^2-x^2-y^2}}\,dydx$$

$$=4\lim_{t\to a-}\int_{0}^{t}\int_{0}^{\sqrt{t^2-x^2}}\frac{a}{\sqrt{a^2-x^2-y^2}}\,dydx$$

$$=4a\lim_{t\to a-}\int_{0}^{t}\left[\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_{0}^{\sqrt{t^2-x^2}}dx$$

$$=4a\lim_{t\to a-}\int_{0}^{t}\left[\sin^{-1}\left(\frac{\sqrt{t^2-x^2}}{\sqrt{a^2-x^2}}\right)\right]dx$$

$$=4a\int_{0}^{a}\sin^{-1}(1)\,dx$$

$$=4a\int_{0}^{a}\frac{\pi}{2}\,dx$$

$$=2\pi a^2$$

The surface area of the top half of the sphere is obtained as $2\pi a^2$.

Thus, the surface area of the sphere is $2(2\pi a^2)=\boxed{4\pi a^2}$.

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Consider the parabolic $y = x^2 + z^2$ cut off by the plane $y = 25$.

The area of the surface area is $A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$.

The partial derivatives of z with respect to x are,

$$y = x^2 + z^2$$
$$y_x = 2x$$

The partial derivatives of z with respect to y are,

$$y = x^2 + z^2$$
$$y_y = 2z$$

In order to convert the integral to polar coordinates, replace x by $r\cos\theta$ and y with $r\sin\theta$.

From the given inequality, the region D is described by $D = \{(r,\theta) \mid 0 \le r \le 5, 0 \le \theta \le 2\pi\}$.

The area of the surface is,

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + 4(r\cos\theta)^2 + 4(r\sin\theta)^2}\, dA \\ &= \int_0^{2\pi}\int_0^5 r\sqrt{1 + 4(r\cos\theta)^2 + 4(r\sin\theta)^2}\, dr\, d\theta \\ &= \frac{2\pi}{8} \cdot \frac{2}{3}\left[(1 + 4r^2)^{\frac{3}{2}}\right]_0^5 \\ &= \frac{\pi}{6}\left(101^{\frac{3}{2}} - 1\right) \end{aligned}$$

Hence, the area of the surface is $\boxed{A(S) = \frac{\pi}{6}\left(101^{\frac{3}{2}} - 1\right)}$.

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Area of surface with equation $z = f(x, y)$ where $(x, y) \in D$

$$AS = \iint_D \sqrt{\left[f_x(x,y)\right]^2 + \left[f_y(x,y)\right]^2 + 1}\, dA$$

Where f_x, f_y are continuous function. This shape has a lot of symmetry. We find the surface area over one eighth of it one quarter of the top half and then multiply by 8.

To find the surface area of that eighth, we use one cylinder as the surface equation and the curve of intersection between the two to find the limits of integration.

Rearrange the two equations to be

$$z^2 = 1 - y^2$$
$$z^2 = 1 - x^2$$

Set them equal to each other to find the (x, y) values of the curve of intersection;

$$1 - y^2 = 1 - x^2$$
$$y^2 = x^2$$
$$\pm y = \pm x$$

In the xy-plane, this forms two lines through the origin of slopes 1 and -1, bisecting all the quadrants. This is the projection of the cylinders' curve of intersection in the xy-plane. These lines slice the coordinate plane into four quarters; any one of these can give us our limits of integration. We choose the two lines $y = x$ and $y = -x$ for positive x. By choosing this area, we must use as the surface of the cylinder parallel to the y-axis, as that is the cylinder these boundaries will cut off in the way we want. So the surface we will be integrating is that of the $z^2 + x^2 = 1$ cylinder.

It remains to find the limits of integration in x. They are the most extreme values of x, so will be the origin, where the region of integration begins, and the boundary of our cylinder's surface as it intersects the xy-plane, which happens at $x = 1$ since the cylinder has a radius of 1.

To summarize, use the surface $z^2 + x^2 = 1$, which, since we are looking at only the positive half, we can rewrite as

$$z = \sqrt{1 - x^2}$$

With limits of integration in y between $-x$ and x and in x between 0 and 1.

Now we find the surface area. The formula for the surface area of a surface given by function $z = f(x, y)$ over region D is:

$$A(s) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Find the partial derivatives and limits of integration and use this formula.

Take the partial derivative in terms of x;

$$z = \sqrt{1 - x^2}$$
$$\frac{\partial z}{\partial x} = \frac{1}{2}(1 - x^2)^{-1/2}(-2x)$$
$$= -x(1 - x^2)^{-1/2}$$

Take the partial derivative in terms of y;

$$z=\sqrt{1-x^2}$$
$$\frac{\partial z}{\partial y}=0$$

Plugging these into the formula for surface area along with our limits of integration, we now have

$$\begin{aligned}A(s)&=\int_0^1\int_{-x}^{x}\sqrt{1+(-x(1-x^2)^{-1/2})^2+0^2}\,dydx\\&=\int_0^1\int_{-x}^{x}\sqrt{1+x^2(1-x^2)^{-1}}\,dydx\end{aligned}$$

Integrate in terms of y;

$$\begin{aligned}A(s)&=\int_0^1\int_{-x}^{x}\sqrt{1+x^2(1-x^2)^{-1}}\,dydx\\&=\int_0^1\left(y\sqrt{1+x^2(1-x^2)^{-1}}\right)\Big|_{-x}^{x}dx\\&=\int_0^1\left(x\sqrt{1+x^2(1-x^2)^{-1}}-(-x)\sqrt{1+x^2(1-x^2)^{-1}}\right)dx\\&=\int_0^1\left(2x\sqrt{1+x^2(1-x^2)^{-1}}\right)dx\end{aligned}$$

$$A(s)=2\int_0^1\left(x\sqrt{1+\frac{x^2}{1-x^2}}\right)dx$$

To integrate in terms of x, use a substitution;

$$u=1-x^2$$
$$du=-2xdx$$
$$xdx=\frac{-du}{2}$$

The substitution can also be written this way;

$$x^2=1-u$$

To substitute in for the extra x^2

Find the limits of integration in u;

$1-0^2=1$

$1-1^2=0$

Plug in u, du and the new limits of integration;

$$A(s)=2\int_1^0\left(\sqrt{1+\frac{1-u}{u}}\right)\left(\frac{-du}{2}\right)$$
$$=-\int_1^0\left(\sqrt{1+\frac{1}{u}-\frac{u}{u}}\right)du$$
$$=-\int_1^0\left(\sqrt{1+\frac{1}{u}-1}\right)du$$
$$=-\int_1^0\left(u^{-1/2}\right)du$$

Integrate in terms of u;

$$A(s)=-\int_1^0\left(u^{-1/2}\right)du$$
$$=-(2u^{1/2})\Big|_1^0 \quad \text{t}$$
$$=-(2(0)-2(1))$$
$$=2$$

The surface area of 1/8 of the solid is 2. Therefore, the surface area of the entire solid is

$(2)(8)=\boxed{16}$.