Long Answer Questions-I (PYQ)

[4 Mark]

Q.1. Show that: $tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$

Ans.

$$\text{Let } \sin^{-1}\frac{3}{4}=\theta \quad \Rightarrow \quad \sin\theta=\frac{3}{4} \quad \left[\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$$

$$\Rightarrow \quad \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\frac{3}{4} \qquad \left[\because \ \sin 2x=\frac{2\tan x}{1+\tan^2 x}\right]$$

$$\Rightarrow \quad 3+3\tan^2\frac{\theta}{2}=8\tan\frac{\theta}{2} \quad \Rightarrow \quad 3\tan^2\frac{\theta}{2}-8\tan\frac{\theta}{2}+3=0$$

$$\Rightarrow \quad \tan\frac{\theta}{2}=\frac{8\pm\sqrt{64-36}}{6} \quad \Rightarrow \quad \tan\frac{\theta}{2}=\frac{8\pm\sqrt{28}}{6}$$

$$\Rightarrow \quad \tan\frac{\theta}{2}=\frac{8\pm2\sqrt{7}}{6} \quad \Rightarrow \quad \tan\frac{\theta}{2}=\frac{4\pm\sqrt{7}}{3}$$

$$\Rightarrow \quad \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3} \qquad \left[\because \ \theta=\sin^{-1}\frac{3}{4}\right]$$

Q.2. Find the value of $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$

Ans.

$$\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)=\tan^{-1}\left(\frac{\frac{x}{y}-\frac{x-y}{x+y}}{1+\frac{x}{y}\cdot\frac{x-y}{x+y}}\right) \qquad \left[\text{Here } \frac{x}{y}\cdot\frac{x-y}{x+y}>-1\right]$$

$$=\tan^{-1}\left(\frac{x^2+xy-xy+y^2}{y(x+y)}\times\frac{y(x+y)}{xy+y^2+x^2-xy}\right)$$

$$=\tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right)=\tan^{-1}(1)=\frac{\pi}{4}$$

Q.3. Evaluate: $tan\left\{2\tan^{-1}\left(\frac{1}{5}\right)+\frac{\pi}{4}\right\}$

Ans.

$$\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right)+\frac{\pi}{4}\right\}=\tan\left\{\tan^{-1}\frac{2\times\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}+\tan^{-1}1\right\}$$

$$=\tan\left\{\tan^{-1}\left(\frac{2}{5}\times\frac{25}{24}\right)+\tan^{-1}1\right\}=\tan\left\{\tan^{-1}\frac{5}{12}+\tan^{-1}1\right\}$$

$$=\tan\left\{\tan^{-1}\frac{\frac{5}{12}+1}{1-\frac{5}{12}\times1}\right\}=\tan\left\{\tan^{-1}\left(\frac{17}{12}\times\frac{12}{7}\right)\right\}=\tan\left\{\tan^{-1}\left(\frac{17}{7}\right)\right\}=\frac{17}{7}$$

Q.4. Prove that: $\cot^{-1}7+\cot^{-1}8+\cot^{-1}18=\cot^{-1}3$

Ans.

We have,

LHS $= \cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18$

$$= \left(\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8}\right) + \tan^{-1}\frac{1}{18}$$

$$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}\right) + \tan^{-1}\frac{1}{18} \qquad \left[\because \frac{1}{7}\times\frac{1}{8} < 1\right]$$

$$= \tan^{-1}\frac{3}{11} + \tan^{-1}\frac{1}{18} = \tan^{-1}\left(\frac{\frac{3}{11}+\frac{1}{18}}{1-\frac{3}{11}\times\frac{1}{18}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{65}{198}}{\frac{195}{198}}\right) = \tan^{-1}\left(\frac{65}{195}\right) = \tan^{-1}\frac{1}{3} \left[\because \frac{3}{11}\times\frac{1}{18} < 1\right]$$

$= \cot^{-1} 3 =$ RHS

Q.5. Prove that: $\sin^{-1}\left(\frac{63}{65}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$

Ans.

Let $\sin^{-1}\left(\frac{5}{13}\right) = \alpha, \cos^{-1}\left(\frac{3}{5}\right) = \beta$

$\Rightarrow \quad \sin\alpha = \frac{5}{13}, \ \cos\beta = \frac{3}{5}$

$\Rightarrow \quad \cos\alpha = \sqrt{1-\left(\frac{5}{13}\right)^2}, \ \sin\beta = \sqrt{1-\left(\frac{3}{5}\right)^2}$

$\Rightarrow \quad \cos\alpha = \frac{12}{13}, \quad \sin\beta = \frac{4}{5}$

Now, $\sin(\alpha+\beta) = \sin\alpha.\cos\beta + \cos\alpha.\sin\beta$

$= \frac{5}{13}.\frac{3}{5} + \frac{12}{13}.\frac{4}{5} = \frac{15}{65} + \frac{48}{65} = \frac{63}{65} \quad \Rightarrow \quad \alpha+\beta = \sin^{-1}\left(\frac{63}{65}\right)$

Putting the value of α and β we get

$$\sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5} = \sin^{-1}\left(\frac{63}{65}\right)$$

Q.6. Prove the following:

$$cos[\tan^{-1}\{sin(\cot^{-1}x)\}] = \sqrt{\frac{1+x^2}{2+x^2}}$$

Ans.

$$\text{LHS} = \cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1}x\right)\right\}\right]$$

Let $\cot^{-1}x = \theta \Rightarrow x = \cot\theta$

$$\text{Now LHS} = \cos/\tan^{-1}\{\sin\theta\}] = \cos\left[\tan^{-1}\left\{\frac{1}{\operatorname{cosec}\theta}\right\}\right]$$

$$= \cos\left[\tan^{-1}\frac{1}{\sqrt{1+\cot^2\theta}}\right] = \cos\left[\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right]$$

Let $\tan^{-1}\frac{1}{\sqrt{1+x^2}} = \alpha$

$$\Rightarrow \frac{1}{\sqrt{1+x^2}} = \tan\alpha \Rightarrow \frac{1}{1+x^2} = \tan^2\alpha$$

$$\Rightarrow \frac{1}{1+x^2} = \frac{\sin^2\alpha}{\cos^2\alpha} \Rightarrow \frac{1}{1+x^2}+1 = \frac{\sin^2\alpha}{\cos^2\alpha}+1$$

$$\Rightarrow \frac{2+x^2}{1+x^2} = \frac{1}{\cos^2\alpha} \Rightarrow \cos\alpha = \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}$$

$$\Rightarrow \alpha = \cos^{-1}\left(\sqrt{\frac{1+x^2}{2+x^2}}\right)$$

$$\therefore \text{LHS} = \cos\left(\cos^{-1}\sqrt{\frac{1+x^2}{2+x^2}}\right) = \sqrt{\frac{1+x^2}{2+x^2}}$$

Q.7. Prove that: $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

Ans.

$$\text{LHS} = \tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\times\frac{1}{5}}+\tan^{-1}\left(\frac{1}{8}\right) \qquad \left[\because \frac{1}{2}\times\frac{1}{5} = \frac{1}{10} < 1\right]$$

$$= \tan^{-1}\left(\frac{7}{9}\right)+\tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\times\frac{1}{8}}\right) = \tan^{-1}\left(\frac{65}{72}\times\frac{72}{65}\right)$$

$$= \tan^{-1}(1) = \frac{\pi}{4} = \text{RHS}$$

Q.8. Prove that: $2\tan^{-1}\left(\frac{1}{5}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

Ans.

$$\text{LHS} = 2\tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right)$$

$$= 2\left\{\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)\right\} + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$$

$$= 2\tan^{-1}\left\{\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\cdot\frac{1}{8}}\right\} + \tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2 - 1} \qquad [\because \sec^{-1}x = \tan^{-1}\sqrt{x^2-1}]$$

$$= 2\tan^{-1}\frac{\frac{13}{40}}{\frac{39}{40}} + \tan^{-1}\sqrt{\frac{50}{49}-1} = 2\tan^{-1}\frac{13}{40}\times\frac{40}{39} + \tan^{-1}\sqrt{\frac{1}{49}}$$

$$= 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right) + \tan^{-1}\left(\frac{1}{7}\right) \quad \left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right) = \tan^{-1}\left(\frac{25}{28}\times\frac{28}{25}\right) = \tan^{-1}(1) = \frac{\pi}{4} = \text{RHS}$$

Q.9. If $y = \cot^{-1}(\sqrt{\cos x}) - \tan^{-1}(\sqrt{\cos x})$, then prove that $\sin y = \tan^2\left(\frac{x}{2}\right)$.

Ans.

Given $y = \cot^{-1}(\sqrt{\cos x}) - \tan^{-1}(\sqrt{\cos x})$

$\Rightarrow \quad y = \frac{\pi}{2} - \tan^{-1}(\sqrt{\cos x}) - \tan^{-1}(\sqrt{\cos x})$

$\Rightarrow \quad y = \frac{\pi}{2} - 2\tan^{-1}(\sqrt{\cos x}) \qquad \Rightarrow \quad y = \frac{\pi}{2} - \cos^{-1}\left(\frac{1-\cos x}{1+\cos x}\right)$

$\Rightarrow \quad y = \sin^{-1}\left(\frac{1-\cos x}{1+\cos x}\right) \qquad \Rightarrow \quad \sin y = \frac{1-\cos x}{1+\cos x}$

$\Rightarrow \quad \sin y = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} \qquad \Rightarrow \quad \sin y = \tan^2\frac{x}{2}$

$$a\begin{bmatrix}\text{Note}: \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2},\ x \in R \\ \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, x \in [-1,1] \\ \text{and} \quad 2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}, x \geq 0\end{bmatrix}$$

Q.10. Prove that: $\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2}$

Ans.

$$\text{LHS} \quad = \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{4}{5}\sqrt{1-\frac{25}{169}} + \frac{5}{13}\sqrt{1-\frac{16}{25}}\right) + \sin^{-1}\frac{16}{65} \qquad \left[\because \left(\frac{4}{5}\right)^2 + \left(\frac{5}{13}\right)^2 \le 1\right]$$

$$= \sin^{-1}\left(\frac{4}{5}\times\frac{12}{13} + \frac{5}{13}\times\frac{3}{5}\right) + \sin^{-1}\frac{16}{65} = \sin^{-1}\frac{63}{65} + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left(\frac{63}{65}\sqrt{1-\frac{16^2}{65^2}} + \frac{16}{65}\sqrt{1-\left(\frac{63}{65}\right)^2}\right) \qquad \left[\because \left(\frac{63}{65}\right)^2 + \left(\frac{16}{65}\right)^2 \le 1\right]$$

$$= \sin^{-1}\left(\frac{63}{65}\times\frac{63}{65} + \frac{16}{65}\times\frac{16}{65}\right) = \sin^{-1}\left(\frac{63^2+16^2}{65^2}\right)$$

$$= \sin^{-1}\left(\frac{65^2}{65^2}\right) = \sin^{-1}(1) = \frac{\pi}{2} = \text{RHS}$$

Q.11. Prove that: $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

Ans.

$$\text{LSH} = \text{Let } \cos^{-1}\frac{4}{5} = x, \cos^{-1}\frac{12}{13} = y \qquad [x,y \in [0,\pi]]$$

$$\Rightarrow \quad \cos x = \frac{4}{5}, \cos y = \frac{12}{13}$$

$$\therefore \quad \sin x = \sqrt{1-\left(\frac{4}{5}\right)^2}, \sin y = \sqrt{1-\left(\frac{12}{13}\right)^2} \qquad [\because x,y \in [0,\pi] \Rightarrow \sin x \text{ and } \sin y \text{ are } +\text{ve}]$$

$$\Rightarrow \quad \sin x = \frac{3}{5}, \ \sin y = \frac{5}{13}$$

Now, $\cos(x+y) = \cos x . \cos y - \sin x . \sin y$

$$= \frac{4}{5}\times\frac{12}{13} - \frac{3}{5}\times\frac{5}{13} \qquad \Rightarrow \qquad \cos(x+y) = \frac{33}{65}$$

$$\Rightarrow \qquad x + y = \cos^{-1}\left(\frac{33}{65}\right) \qquad \left[\because \frac{33}{65} \in [-1,1]\right]$$

Putting the value of x and y we get

$$\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\left(\frac{33}{65}\right) = \text{RHS}$$

Q.12. Prove that: $\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}$

Ans.

$$\text{Here LHS} = \cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$$

$$\text{Let } \sin^{-1}\frac{3}{5} = \theta \text{ and } \cot^{-1}\frac{3}{2} = \varphi \quad \Rightarrow \quad \sin\theta = \frac{3}{5} \text{ and } \cot\varphi = \frac{3}{2}$$

$$\Rightarrow \qquad \cos\theta = \frac{4}{5} \quad \text{and} \quad \sin\varphi = \frac{2}{\sqrt{13}}, \cos\varphi = \frac{3}{\sqrt{13}}$$

$$\text{Now}, \quad \cos(\theta+\varphi) = \cos\theta . \cos\varphi - \sin\theta . \sin\varphi$$

$$= \frac{4}{5}.\frac{3}{\sqrt{13}} - \frac{3}{5}.\frac{2}{\sqrt{13}} = \frac{12}{5\sqrt{13}} - \frac{6}{5\sqrt{13}} = \frac{6}{5\sqrt{13}}$$

Q.13. Prove that: $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{\frac{1}{2}\cos^{-1}a}{b}\right)=\frac{2b}{a}$

Ans.

$$\text{LHS}=\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right)$$

$$=\tan\left(\frac{\pi}{4}+x\right)+\tan\left(\frac{\pi}{4}-x\right)\qquad\left[\text{where } x=\frac{1}{2}\cos^{-1}\frac{a}{b}\right]$$

$$=\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}.\tan x}+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}.\tan x}=\frac{1+\tan x}{1-\tan x}+\frac{1-\tan x}{1+\tan x}$$

$$=\frac{(1+\tan x)^2+(1-\tan x)^2}{1-\tan^2 x}=\frac{1+\tan^2 x+2\tan x+1+\tan^2 x-2\tan x}{1-\tan^2 x}$$

$$=\frac{2(1+\tan^2 x)}{1-\tan^2 x}=\frac{2}{\cos 2x}$$

$$=\frac{2}{\cos 2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}=\frac{2}{\cos\left(\cos^{-1}\frac{a}{b}\right)}\qquad\left[\begin{array}{l}By Property \cos(\cos^{-1}x)=x \text{ if } x\in[-1,1]\\ Here \frac{a}{b}\in[-1,1]\end{array}\right]$$

$$=\frac{2}{\frac{a}{b}}=\frac{2b}{a}=\text{RHS}$$

Q.14. Prove the following:

$$\cot^{-1}\left(\frac{xy+1}{x-y}\right)+\cot^{-1}\left(\frac{yz+1}{y-z}\right)+\cot^{-1}\left(\frac{zx+1}{z-x}\right)=0\qquad(0<xy,\ yx,\ zx<1)$$

Ans.

$$\text{LHS}=\cot^{-1}\left(\frac{xy+1}{x-y}\right)+\cot^{-1}\left(\frac{yz+1}{y-z}\right)+\cot^{-1}\left(\frac{zx+1}{z-x}\right)$$

$$=\tan^{-1}\left(\frac{x-y}{1+xy}\right)+\tan^{-1}\left(\frac{y-z}{1+yz}\right)+\tan^{-1}\left(\frac{z-x}{1+zx}\right)$$

$$=\tan^{-1}x-\tan^{-1}y+\tan^{-1}y-\tan^{-1}z+\tan^{-1}z+\tan^{-1}x$$

$$=0=\text{RHS}$$

Q.15. Prove that: $\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)=12cos\text{–}1(35)$

Ans.

$$\text{LHS} = \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)$$

$$= \tan^{-1}\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\cdot\frac{2}{9}} \qquad \left[\because \quad \frac{1}{4}\times\frac{2}{9}<1\right]$$

$$= \tan^{-1}\left(\frac{17}{36}\times\frac{36}{34}\right) = \tan^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}.2\tan^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\cos^{-1}\frac{1-\left(\frac{1}{2}\right)^2}{1+\left(\frac{1}{2}\right)^2} \qquad \left[\begin{array}{c}\because \quad \frac{1}{2}\geq 0 \\ \text{and } 2\ \tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}, x \geq 0\end{array}\right]$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{3}{4}\times\frac{4}{5}\right) = \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right) = \text{RHS}$$

Q.16. Prove that: $\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$

Ans.

$$\text{LHS} \begin{array}{l} = \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\sqrt{1-\left(\frac{12}{13}\right)^2} + \sin^{-1}\frac{3}{5} \\ = \sin^{-1}\sqrt{1-\frac{144}{169}} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{3}{5}\end{array}$$

$$= \sin^{-1}\left[\frac{5}{13}\sqrt{1-\left(\frac{3}{5}\right)^2} + \frac{3}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}\right] \qquad \left[\because \left(\frac{5}{13}\right)^2 + \left(\frac{3}{5}\right)^2 \leq 1\right]$$

$$= \sin^{-1}\left[\frac{5}{13}\sqrt{1-\frac{9}{25}} + \frac{3}{5}\sqrt{1-\frac{25}{169}}\right]$$

$$= \sin^{-1}\left[\frac{5}{13}\times\frac{4}{5} + \frac{3}{5}\times\frac{12}{13}\right] = \sin^{-1}\left[\frac{20}{65}+\frac{36}{65}\right] = \sin^{-1}\left[\frac{56}{65}\right] = \text{RHS}$$

Q.17. Prove that: $\tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Ans.

$$\text{LHS} = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) \qquad \left[\begin{array}{l}\because \ x.\frac{2x}{1-x^2} < 1 \\ \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-\text{xy}}, \text{xy} < 1\end{array}\right]$$

$$= \tan^{-1}\left(\frac{x+\frac{2x}{1-x^2}}{1-x\left(\frac{2x}{1-x^2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{x-x^3+2x}{1-x^2-2x^2}\right) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = \text{RHS}$$

Q.18. If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$ prove that $\frac{x^2}{a^2} - 2\frac{xy}{ab}\cos a + \frac{y^2}{b^2} = \sin^2$.

Ans.

Given, $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$

$$\Rightarrow \quad \cos^{-1}\left\{\frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right\}=\alpha \quad [\because \cos^{-1}x+\cos^{-1}y=\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-$$

$$\Rightarrow \quad \frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}=\cos\alpha$$

$$\Rightarrow \quad \frac{xy}{ab}-\sqrt{1-\frac{y^2}{b^2}-\frac{x^2}{a^2}+\frac{x^2y^2}{a^2b^2}}=\cos\alpha$$

$$\Rightarrow \quad \frac{xy}{ab}-\cos\alpha=\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}}$$

$$\Rightarrow \quad \left(\frac{xy}{ab}-\cos\alpha\right)^2=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \quad \frac{x^2y^2}{a^2b^2}+\cos^2\alpha-2\frac{xy}{ab}.\cos\alpha=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \quad \frac{x^2}{a^2}-2\frac{xy}{ab}\cos\alpha+\frac{y^2}{b^2}=1-\cos^2\alpha$$

$$\Rightarrow \quad \frac{x^2}{a^2}-2\frac{xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha.$$

Hence proved

Q.19. Prove that: $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\frac{\pi}{4}-\frac{x}{2},\ x\in\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$

Ans.

Now,
$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\tan^{-1}\left(\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\cos\frac{x}{2}.\sin\frac{x}{2}}\right)$$

$$=\tan^{-1}\left[\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}\right]$$

$$=\tan^{-1}\left(\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right)=\tan^{-1}\left(\frac{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}}-\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}}+\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}\right) \quad \begin{bmatrix} Divideeach \\ termbycos\frac{x}{2}\end{bmatrix}$$

$$=\tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)=\tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\frac{x}{2}}{1+\tan\frac{\pi}{4}\tan\frac{x}{2}}\right)$$

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]$$

$$=\frac{\pi}{4}-\frac{x}{2} \qquad \left[\because x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$$

$$\Rightarrow \quad -\frac{\pi}{2}<x<\frac{\pi}{2} \qquad \Rightarrow \quad -\frac{\pi}{4}<\frac{x}{2}<\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{\pi}{4}>-\frac{x}{2}>-\frac{\pi}{4} \qquad \Rightarrow \quad \frac{\pi}{4}+\frac{\pi}{4}>\frac{\pi}{4}-\frac{x}{2}>-\frac{\pi}{4}+\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{\pi}{2}>\frac{\pi}{4}-\frac{x}{2}>0 \qquad \Rightarrow \quad \left(\frac{\pi}{4}-\frac{x}{2}\right)\in\left(0,\frac{\pi}{2}\right)\subset\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

Q.20. Prove that: $\sin^{-1}\left(\frac{8}{17}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\cos^{-1}\left(\frac{36}{85}\right)$

Ans.

Let $\sin^{-1}\left(\frac{8}{17}\right)=\alpha$ and $\sin^{-1}\left(\frac{3}{5}\right)=\beta$

$$\Rightarrow \quad \sin\alpha=\frac{8}{17} \quad \text{and} \quad \sin\beta=\frac{3}{5}$$

$$\Rightarrow \quad \cos\alpha=\sqrt{1-\sin^2\alpha} \quad \text{and} \quad \cos\beta=\sqrt{1-\sin^2\beta}$$

$$\Rightarrow \quad \cos\alpha=\sqrt{1-\frac{64}{289}} \quad \text{and} \quad \cos\beta=\sqrt{1-\frac{9}{25}}$$

$$\Rightarrow \quad \cos\alpha=\sqrt{\frac{225}{289}} \quad \text{and} \quad \cos\beta=\sqrt{\frac{16}{25}}$$

$$\Rightarrow \quad \cos\alpha=\frac{15}{17} \quad \text{and} \quad \cos\beta=\frac{4}{5}$$

Now, $\cos(\alpha+\beta)=\cos\alpha\,.\,\cos\beta-\sin\alpha\,.\,\sin\beta$

$$\Rightarrow \quad \cos(\alpha+\beta)=\frac{15}{17}\times\frac{4}{5}-\frac{8}{17}\times\frac{3}{5} \quad \Rightarrow \quad \cos(\alpha+\beta)=\frac{60}{85}-\frac{24}{85}$$

$$\Rightarrow \quad \cos(\alpha+\beta)=\frac{36}{85} \quad \Rightarrow \quad \alpha+\beta=\cos^{-1}\left(\frac{36}{85}\right)$$

$$\Rightarrow \quad \sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\cos^{-1}\left(\frac{36}{85}\right) \quad \text{[Putting the value of } \alpha,\beta]$$

Q.21. If $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$**, then find the value of *x*.**

Ans.

Given $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\times\frac{x+1}{x+2}}\right]=\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)}=\tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}=1 \quad \Rightarrow \quad \frac{2(x^2-2)}{-3}=1 \quad \Rightarrow \quad 2x^2-4=-3$$

$$\Rightarrow \quad 2x^2=1 \quad \Rightarrow \quad x^2=\frac{1}{2}$$

$$\therefore \quad x=\pm\frac{1}{\sqrt{2}}$$

Q.22. Solve: $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$

Ans.

Given: $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$

$\Rightarrow \quad \tan^{-1}(x-1)+\tan^{-1}(x+1)=\tan^{-1}3x-\tan^{-1}x$

$\Rightarrow \quad \tan^{-1}\left[\frac{(x-1)+(x+1)}{1-(x-1)(x+1)}\right]=\tan^{-1}\left[\frac{3x-x}{1+3x^2}\right] \quad \left[\text{Using } \tan^{-1}x\pm\tan^{-1}y=\tan^{-1}\frac{x\pm y}{1\mp xy}\right]$

$\Rightarrow \quad \tan^{-1}\frac{2x}{1-(x^2-1)}=\tan^{-1}\frac{2x}{1+3x^2} \quad \Rightarrow \quad \frac{2x}{2-x^2}=\frac{2x}{1+3x^2}$

Either $x=0$ or $2-x^2=1+3x^2 \quad \Rightarrow \quad 4x^2=1$

$\Rightarrow \quad x^2=\frac{1}{4} \qquad \therefore \quad x=\pm\frac{1}{2},\ 0$

Q.23. If 0 < *x* < 1, then solve the following for *x* :

$\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\left(\frac{8}{31}\right)$

Ans.

Given $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31} \qquad [\because\ 0<x<1 \quad \Rightarrow \quad (x+1)(x-1)<1]$

$\Rightarrow \quad \tan^{-1}\frac{x+1+x-1}{1-(x+1)(x-1)}=\tan^{-1}\frac{8}{31}$

$\Rightarrow \quad \tan^{-1}\frac{2x}{1-x^2+1}=\tan^{-1}\frac{8}{31} \quad \Rightarrow \quad \tan^{-1}\frac{2x}{2-x^2}=\tan^{-1}\frac{8}{31}$

$\Rightarrow \quad \frac{2x}{2-x^2}=\frac{8}{31} \qquad \Rightarrow \quad 16-8x^2=62x$

$\Rightarrow \quad 4x^2+31x-8=0$

$\Rightarrow \quad 4x^2+32x-x-8=0 \quad \Rightarrow \quad 4x(x+8)-1(x+8)=0$

$\Rightarrow \quad (x+8)(4x-1)=0 \quad \Rightarrow \quad x=-8 \quad \text{or} \quad x=\frac{1}{4}$

$\Rightarrow \quad x=\frac{1}{4} \qquad [x=-8 \text{ is not acceptable}]$

Q.24. Solve: $\cos(tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$.

Ans.

Given $\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$

$\Rightarrow \quad \cos(\tan^{-1}x)=\cos\left(\frac{\pi}{2}-\cot^{-1}\frac{3}{4}\right) \quad \Rightarrow \quad \tan^{-1}x=\frac{\pi}{2}-\cot^{-1}\frac{3}{4}$

$\Rightarrow \quad \frac{\pi}{2}-\cot^{-1}x=\frac{\pi}{2}-\cot^{-1}\frac{3}{4} \qquad \Rightarrow \quad \cot^{-1}x=\cot^{-1}\frac{3}{4}$

$\Rightarrow \quad x=\frac{3}{4} \qquad \left[\begin{array}{l}\text{Note: } \sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)\\ \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\end{array}\right]$

Q.25. Solve for x : 2 tan^{-1} (cos x) = tan^{-1} (2 cosec x)

Ans.

Given $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right) \qquad \left[\because\ 2\tan^{-1}A = \tan^{-1}\left(\frac{2A}{1-A^2}\right)\right]$

$\Rightarrow \quad \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x} \quad \Rightarrow \quad \cot x = 1$

$\therefore \quad x = \frac{\pi}{4}$

Q.26. Solve for x : $2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x),\ x \neq \frac{\pi}{2}$

Ans.

Given, $2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x)$

$\Rightarrow \quad \tan^{-1}\left(\frac{2\sin x}{1-\sin^2 x}\right) = \tan^{-1}(2\sec x)$

$\Rightarrow \quad \frac{2\sin x}{1-\sin^2 x} = 2\sec x \qquad \left[\because x \neq \frac{\pi}{2} \Rightarrow 1-\sin^2 x \neq 0\right]$

$\Rightarrow \quad \frac{2\sin x}{\cos^2 x} = 2\sec x \qquad \Rightarrow \sin x = \sec x.\cos^2 x$

$\Rightarrow \quad \sin x = \frac{1}{\cos x}\cdot\cos^2 x \qquad \Rightarrow \quad \sin x = \cos x$

$\Rightarrow \quad \tan x = 1 \qquad \Rightarrow \quad x = \frac{\pi}{4}$

Q.27. Solve the following for x: $\tan^{-1}\left(\frac{x-2}{x-3}\right) + \tan^{-1}\left(\frac{x+2}{x+3}\right) = \frac{\pi}{4}, |x| < 1$

Ans.

Given: $\tan^{-1}\left(\frac{x-2}{x-3}\right)+\tan^{-1}\left(\frac{x+2}{x+3}\right)=\frac{\pi}{4}, |x|<1$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{x-2}{x-3}+\frac{x+2}{x+3}}{1-\left(\frac{x-2}{x-3}\right)\left(\frac{x+2}{x+3}\right)}\right)=\frac{\pi}{4} \qquad \left[\because \frac{x-2}{x-3}.\frac{x+2}{x+3}=\frac{x^2-4}{x^2-9}<1 \text{ for } |x|<1\right]$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{(x-2)(x+3)+(x+2)(x-3)}{(x-3)(x+3)-(x-2)(x+2)}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{x^2+3x-2x-6+x^2-3x+2x-6}{x^2-9-x^2+4}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{2x^2-12}{-5}\right\}=\frac{\pi}{4} \qquad \Rightarrow \qquad \frac{2x^2-12}{-5}=\tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{2x^2-12}{-5}=1 \qquad \Rightarrow \qquad 2x^2-12=-5$$

$$\Rightarrow \quad 2x^2-7=0 \qquad \Rightarrow \qquad x^2=\frac{7}{2}$$

$$\Rightarrow \quad x=\pm\sqrt{\frac{7}{2}} \text{ Not acceptable as } |x|<1.$$

Hence, there is no solution.

Q.28. Solve the following for $x:\ \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)+\tan^{-1}\left(\frac{2x}{x^2-1}\right)=\frac{2\pi}{3}$

Ans.

$$\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)+\tan^{-1}\left(\frac{2x}{x^2-1}\right)=\frac{2\pi}{3}$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{-(1-x^2)}{1+x^2}\right)+\tan^{-1}\left[\frac{2x}{-(1-x^2)}\right]=\frac{2\pi}{3} \qquad \begin{bmatrix}\cos^{-1}(-x)=\pi-\cos^{-1}x\\ \tan^{-1}(-x)=-\tan^{-1}x\end{bmatrix}$$

$$\Rightarrow \quad \pi-\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)-\tan^{-1}\frac{2x}{1-x^2}=\frac{2\pi}{3}$$

$$\Rightarrow \quad \pi-2\tan^{-1}x-2\tan^{-1}x=\frac{2\pi}{3} \qquad \left[\because 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\tan^{-1}\frac{2x}{1-x^2}\right]$$

$$\Rightarrow \quad 4\tan^{-1}x=\pi-\frac{2\pi}{3}=\frac{\pi}{3}$$

$$\Rightarrow \quad \tan^{-1}x=\frac{\pi}{12} \quad \text{or} \quad x=\tan\frac{\pi}{12}=\tan 15^\circ \qquad \Rightarrow \qquad x=\tan(45^\circ-30^\circ)$$

$$\Rightarrow \quad x=\frac{\tan 45^\circ-\tan 30^\circ}{1+\tan 45^\circ.\tan 30^\circ}$$

$$\Rightarrow \quad x-\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}$$

$$\Rightarrow \quad x=\frac{\sqrt{3}-1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}=\frac{3+1-2\sqrt{3}}{2} \qquad \Rightarrow \quad x=2-\sqrt{3}$$

Q.29. Solve for x: $\tan^{-1}\left(\frac{x-2}{x-1}\right)+\tan^{-1}\left(\frac{x+2}{x+1}\right)=\frac{\pi}{4}$

Ans.

Given, $\tan^{-1}\left(\frac{x-2}{x-1}\right)+\tan^{-1}\left(\frac{x+2}{x+1}\right)=\frac{\pi}{4}$

$$\tan^{-1}\left[\frac{\frac{x-2}{x-1}+\frac{x+2}{x+1}}{1-\frac{x-2}{x-1}\times\frac{x+2}{x+1}}\right]=\tan^{-1}(1) \qquad \left[\begin{array}{l}\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy} \\ \text{and } \tan^{-1}(1)=\frac{\pi}{4}\end{array}\right]$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{(x-2)(x+1)+(x+2)(x-1)}{(x-1)(x+1)}}{\frac{(x-1)(x+1)-(x-2)(x+2)}{(x-1)(x+1)}}\right]=\tan^{-1}(1)$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{x^2+x-2x-2+x^2-x+2x-2}{x^2-1-x^2+4}\right]=\tan^{-1}(1)$$

$$\Rightarrow \quad \frac{2x^2-4}{3}=1 \qquad \Rightarrow \quad 2x^2-4=3$$

$$\Rightarrow \quad x^2=\frac{7}{2} \qquad \Rightarrow \quad x=\pm\sqrt{\frac{7}{2}}$$

Q.30. If $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$, then find x.

Ans.

Here, $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$

$$\Rightarrow \quad (\tan^{-1}x)^2+\left(\frac{\pi}{2}-\tan^{-1}x\right)^2=\frac{5\pi^2}{8}$$

$$\Rightarrow \quad (\tan^{-1}x)^2+(\tan^{-1}x)^2+\frac{\pi^2}{4}-\pi\tan^{-1}x=\frac{5\pi^2}{8}$$

$$\Rightarrow \quad 2(\tan^{-1}x)^2-\pi\tan^{-1}x+\frac{\pi^2}{4}-\frac{5\pi^2}{8}=0$$

$$\Rightarrow \quad 2(\tan^{-1}x)^2-\pi\tan^{-1}x-\frac{3\pi^2}{8}=0 \qquad \ldots(i)$$

Let $\tan^{-1}x=y$, then (i) becomes

$$2y^2-\pi y-\frac{3\pi^2}{8}=0 \qquad \Rightarrow \quad 16y^2-8\pi y-3\pi^2=0$$

$$\Rightarrow \quad 16y^2-12\pi y+4\pi y-3\pi^2=0 \qquad \Rightarrow \quad 4y(4y-3\pi)+\pi(4y-3\pi)=0$$

$$\Rightarrow \quad (4y-3\pi)(4y+\pi)=0 \qquad \Rightarrow \quad y=-\frac{\pi}{4} \text{ or } y=\frac{3\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}x=-\frac{\pi}{4} \qquad \left[\because \frac{3\pi}{4} \text{ does not belongs to domain of } \tan^{-1}x \text{ i.e., } \left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$$

$$\Rightarrow \quad x=\tan\left(-\frac{\pi}{4}\right)=-1$$

Q.31. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}, x, y, z>0$, then find the value of $xy+yz+zx$.

Ans.

Given $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2} \quad \Rightarrow \quad \tan^{-1} x + \tan^{-1} y = \frac{\pi}{2} - \tan^{-1} z$

$$\Rightarrow \quad \tan^{-1} x + \tan^{-1} y = \cot^{-1} z \quad \Rightarrow \quad \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}\frac{1}{z}$$

$$\Rightarrow \quad \frac{x+y}{1-xy} = \frac{1}{z} \quad \Rightarrow \quad xz + yz = 1 - xy \quad \Rightarrow \quad xy + yz + zx = 1$$

Q.32. Solve the equation for x: $\sin^{-1}x + \sin^{-1}(1 - x) = \cos^{-1}x$

Ans.

$\sin^{-1}x + \sin^{-1}(1 - x) = \cos^{-1}x$

$$\Rightarrow \quad \sin^{-1}\{x\sqrt{1-(1-x)^2} + (1-x)\sqrt{1-x^2}\} = \sin^{-1}\sqrt{1-x^2}$$

$$[\because \ \sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\} \text{ and } \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}]$$

$$\Rightarrow \quad x\sqrt{1-1+2x-x^2} + \sqrt{1-x^2} - x\sqrt{1-x^2} = \sqrt{1-x^2}$$

$$\Rightarrow \quad x\sqrt{2x-x^2} - x\sqrt{1-x^2} = 0 \quad \Rightarrow \quad x\{\sqrt{2x-x^2} - \sqrt{1-x^2}\} = 0$$

$$\Rightarrow \quad x = 0, \sqrt{2x-x^2} - \sqrt{1-x^2} = 0 \quad \Rightarrow \quad x = 0, \sqrt{2x-x^2} = \sqrt{1-x^2}$$

Now, $\sqrt{2x-x^2} = \sqrt{1-x^2}$

Squaring both sides, we get

$$2x - x^2 = 1 - x^2 \quad \Rightarrow \quad 2x - x^2 - 1 + x^2 = 0$$

$$\Rightarrow \quad 2x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{2}$$

Hence, $x = 0$ and $x = \frac{1}{2}$.

Q.33. Find the value of

$$\cot\ \frac{1}{2}\left[\cos^{-1}\frac{2x}{1+x^2} + \sin^{-1}\frac{1-y^2}{1+y^2}\right],\ |x| < 1,\ y > 0 \text{ and } xy < 1.$$

Long Answer Questions-I (OIQ)

[4 Mark]

Q.1. Prove that : $2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \sin^{-1}\left(\frac{31}{25\sqrt{2}}\right)$

Ans.

$$\text{LHS} = 2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

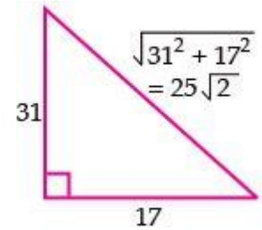
$$= \tan^{-1}\left(\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right) + \tan^{-1}\left(\frac{1}{7}\right) \qquad \left[\because 2\tan^{-1}x = \tan\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right) \qquad \left[\because \tan^{-1}x + \tan^{-1}y = \tan\frac{x+y}{1-\text{xy}} \text{ and } \text{xy} < 1\right]$$

$$= \tan^{-1}\left(\frac{28+3}{21} \times \frac{21}{21-4}\right)$$

$$= \tan^{-1}\left(\frac{31}{17}\right)$$



Let : $\tan^{-1}\left(\frac{31}{17}\right) = \theta \quad \Rightarrow \quad \tan\theta = \frac{31}{17}$

$\Rightarrow \quad \sin\theta = \frac{31}{25\sqrt{2}} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{31}{25\sqrt{2}}\right)$

$\Rightarrow \quad \tan^{-1}\left(\frac{31}{17}\right) = \sin^{-1}\left(\frac{31}{25\sqrt{2}}\right)$

$\Rightarrow$ LHS = RHS

Q.2. Does the following trigonometric equation have any solutions? If yes, obtain the solution(s):

$$\tan^{-1}\left(\frac{x+1}{x-1}\right) + \tan^{-1}\left(\frac{x-1}{x}\right) = -\tan^{-1}7$$

Ans.

$$\tan^{-1}\left(\frac{x+1}{x-1}\right)+\tan^{-1}\left(\frac{x-1}{x}\right)=-\tan^{-1}7$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\left(\frac{x+1}{x-1}\right)+\left(\frac{x-1}{x}\right)}{1-\left(\frac{x+1}{x-1}\right)\left(\frac{x-1}{x}\right)}\right]=-\tan^{-1}7,\ \text{if}\ \left(\frac{x+1}{x-1}\right)\left(\frac{x-1}{x}\right)<1....(*)$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{x(x+1)+(x-1)^2}{(x-1)x-(x+1)(x-1)}\right]=-\tan^{-1}7$$

$$\Rightarrow \quad \frac{(x^2+x)+(x^2+1-2x)}{(x^2-x)-(x^2-1)}=\tan\left[-\tan^{-1}7\right]$$

$$\Rightarrow \quad \frac{2x^2-x+1}{-x+1}=-7 \quad \Rightarrow \quad 2x^2-8x+8=0 \quad \Rightarrow \quad 2(x^2-4x+4)=0$$

$$\Rightarrow \quad (x-2)^2=0 \quad \Rightarrow \quad x=2$$

Let us now verify whether $x = 2$ satisfies the condition (*)

For $x = 2$,

$$\left(\frac{x+1}{x-1}\right)\left(\frac{x-1}{x}\right)=3\times\frac{1}{2}=\frac{3}{2}\ \text{which is not less than 1}$$

Hence, this value does not satisfy the condition (*)

i.e., there is no solution of the given trigonometric equation.

Q.3. Simplify: $\tan^{-1}\left(\frac{3\sin 2\alpha}{5+3\cos 2\alpha}\right)+\tan^{-1}\left(\frac{1}{4}\tan\alpha\right),\ \text{where}\ -\frac{\pi}{2}<\alpha<\frac{\pi}{2}$

Ans.

We have, $\tan^{-1}\left[\frac{3\times\frac{2\tan\alpha}{1+\tan^2\alpha}}{5+3\left(\frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right)}\right]+\tan^{-1}\left(\frac{1}{4}\tan\alpha\right)$

$$=\tan^{-1}\left[\frac{6\tan\alpha}{8+2\tan^2\alpha}\right]+\tan^{-1}\left(\frac{1}{4}\tan\alpha\right)$$

$$=\tan^{-1}\left[\frac{3\tan\alpha}{4+\tan^2\alpha}\right]+\tan^{-1}\left(\frac{1}{4}\tan\alpha\right)$$

$$=\tan^{-1}\left[\frac{\frac{3\tan\alpha}{4+\tan^2\alpha}+\frac{1}{4}\tan\alpha}{1-\frac{3\tan\alpha}{4+\tan^2\alpha}\times\frac{1}{4}\tan\alpha}\right]=\tan^{-1}\left[\frac{16\tan\alpha+\tan^3\alpha}{16+\tan^2\alpha}\right]$$

$$=\tan^{-1}\left[\frac{\tan\alpha(16+\tan^2\alpha)}{(16+\tan^2\alpha)}\right]$$

$$=\tan^{-1}(\tan\alpha)=\alpha. \qquad \left[\because\ \alpha\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$$

Q.4. Prove that: $\tan^{-1}\left(\frac{63}{16}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$

Ans.

Let $\sin^{-1}\frac{5}{13} = \theta$ and $\cos^{-1}\frac{3}{5} = \varphi$

$\Rightarrow \quad \sin\theta = \frac{5}{13}$ and $\cos\varphi = \frac{3}{5} \qquad \left[\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ and } \varphi \in [0, \pi]\right]$

$\Rightarrow \quad \cos\theta = +\sqrt{1 - \left(\frac{5}{13}\right)^2}$

and $\quad \sin\varphi = +\sqrt{1 - \left(\frac{3}{5}\right)^2} \qquad \left[\begin{array}{l} \because\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and } \varphi \in [0, \pi] \\ \Rightarrow\ \cos\theta \text{ and } \sin\varphi are + \text{ve} \end{array}\right]$

$\Rightarrow \quad \cos\theta = \frac{12}{13}$ and $\sin\varphi = \frac{4}{5}$

$\therefore \quad \tan\theta = \frac{5}{13} \times \frac{13}{12} = \frac{5}{12}, \quad \tan\varphi = \frac{4}{5} \times \frac{5}{3} = \frac{4}{3}$

Now $\quad \tan(\theta + \varphi) = \frac{\tan\theta + \tan\varphi}{1 - \tan\theta.\ \tan\varphi}$

$$= \frac{\frac{5}{12} + \frac{4}{3}}{1 - \frac{5}{12} \times \frac{4}{3}} = \frac{15 + 48}{36} \times \frac{36}{36 - 20} = \frac{63}{16}$$

$\Rightarrow \quad \tan(\theta + \varphi) = \frac{63}{16} \quad \Rightarrow \quad \theta + \varphi = \tan^{-1}\frac{63}{16}$

$\Rightarrow \quad \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5} = \tan^{-1}\frac{63}{16}$

Q.5. Which is greater, tan 1 or $\tan^{-1} 1$?

Ans.

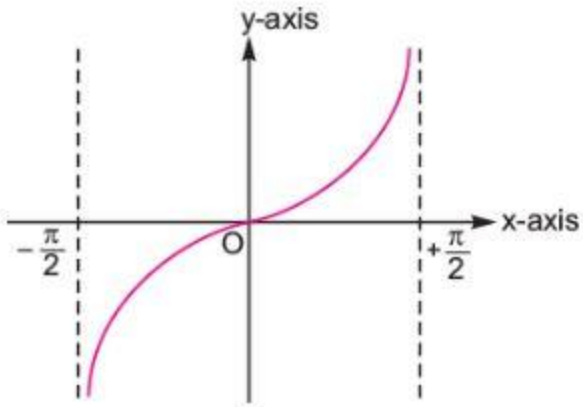
From figure, we can see that tan x is increasing function in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Now, $1 > \frac{\pi}{4}$

$\Rightarrow \quad \tan 1 > \tan\frac{\pi}{4} \quad [\because \tan x \text{ is increa sin } g \text{ function}]$

$\Rightarrow \quad \tan 1 > 1$

$\Rightarrow \quad \tan 1 > 1 > \frac{\pi}{4}$



$\Rightarrow \quad$ tan 1 > 1 > tan−1 1

$\Rightarrow \quad$ tan 1 > tan−1 1

Q.6. If $ax + b\,(\sec(\tan^{-1} x)) = c$ and $ay + b\,(\sec(\tan^{-1} y)) = c$, then find the value of $\frac{x+y}{1-xy}$.

Ans.

Let $\tan^{-1} x = \alpha$ and $\tan^{-1} y = \beta$

$\Rightarrow \quad \tan \alpha = x \quad$ and $\quad \tan \beta = y$

Now, given equation becomes

$$a \tan \alpha + b (\sec \alpha) = c \quad \text{and} \quad a \tan \beta + b (\sec \beta) = c$$

$\Rightarrow \quad a \tan \alpha + b \sec \alpha = c \quad$ and $\quad a \tan \beta + b \sec \beta = c$

$\Rightarrow \quad \alpha$ and β are the roots of $a \tan \theta + b \sec \theta = c$

Again, $\because \quad a \tan \theta + b \sec \theta = c \quad \Rightarrow \quad b \sec \theta = c - a \tan \theta$

$\Rightarrow \quad b^2 \sec^2 \theta = (c - a \tan \theta)^2$ [Squaring both side]

$\Rightarrow \quad b^2 \sec^2 \theta = c^2 - 2ac \tan \theta + a^2 \tan^2 \theta$

$\Rightarrow \quad b^2 (1 + \tan^2 \theta) = c^2 - 2ac \tan \theta + a^2 \tan^2 \theta$

$\Rightarrow \quad - b^2 - b^2 \tan^2 \theta + c^2 - 2ac \tan \theta + a^2 \tan^2 \theta = 0$

$\Rightarrow \quad (a^2 - b^2) \tan^2 \theta - 2ac \tan \theta + (c^2 - b^2) = 0$

Since $\tan \alpha$, $\tan \beta$ are roots of quadratic equation with variable $\tan \theta$.

Q.7. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = p$, then prove that:

$$x\sqrt{1 - x^2} + y\sqrt{1 - y^2} + z\sqrt{1 - z^2} = 2\,xyz$$

Ans.

Let $\sin^{-1} x = A \quad \Rightarrow \quad \sin A = x$

$\sin^{-1} y = B \quad \Rightarrow \quad \sin B = y$

$\sin^{-1} z = C \quad \Rightarrow \quad \sin C = z$

Given, $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = p$

$\Rightarrow \quad A + B + C = p \Rightarrow 2A + 2B + 2C = 2\pi$

$\therefore \quad \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$ [Using trigonometric property]

$\Rightarrow \quad 2 \sin A \cos A + 2 \sin B \cos B + 2 \sin C \cos C = 4 \sin A \sin B \sin C$

$\Rightarrow \quad 2 \sin A.\sqrt{1 - \sin^2 A} + 2 \sin B.\sqrt{1 - \sin^2 B} + 2 \sin C.\sqrt{1 - \sin^2 C} = 4 \sin A \sin B \sin C$

$\Rightarrow \quad 2x\sqrt{1 - x^2} + 2y\sqrt{1 - y^2} + 2z\sqrt{1 - z^2} = 4\,xyz$

$\Rightarrow \quad x\sqrt{1 - x^2} + y\sqrt{1 - y^2} + z\sqrt{1 - z^2} = 2\,xyz$

Hence proved.

Q.8. If $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \pi$, then prove that $a + b + c = abc$.

Ans.

Firstly, let us assume

$\tan^{-1} a = \alpha \Rightarrow \tan \alpha = a$

$\tan^{-1} b = \beta \Rightarrow \tan \beta = b$

$\tan^{-1} c = \gamma \Rightarrow \tan \gamma = c$

Now, given that

$\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \pi \Rightarrow \alpha + \beta + \gamma = \pi$

$\therefore \quad \alpha + \beta = \pi - \gamma$

Taking tangent on both sides, we have

$\tan(\alpha + \beta) = \tan(\pi - \gamma)$

$\Rightarrow \quad \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha . \tan\beta} = -\tan\gamma$

$\Rightarrow \quad \tan\alpha + \tan\beta = -\tan\gamma(1 - \tan\alpha . \tan\beta)$

$\Rightarrow \quad \tan\alpha + \tan\beta = -\tan\gamma + \tan\alpha . \tan\beta . \tan\gamma$

$\Rightarrow \quad \tan\alpha + \tan\beta + \tan\gamma = \tan\alpha . \tan\beta . \tan\gamma$

Thus, $a + b + c = 2abc$ Hence proved.

Q.9. Show that: $2\tan^{-1}\left\{\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4} - \frac{\beta}{2}\right)\right\} = \tan^{-1}\frac{\sin\alpha . \cos\beta}{\cos\alpha . \sin\beta}$

Ans.

$$\text{LHS} \quad = 2\tan^{-1}\left\{\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\}$$

$$= \tan^{-1}\frac{2\tan\frac{\alpha}{2}\,.\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}{1-\tan^2\frac{\alpha}{2}\,.\tan^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)} \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\frac{2\tan\frac{\alpha}{2}\,\frac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}}{1-\tan^2\frac{\alpha}{2}\,.\left(\frac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}\right)^2} \qquad \left[\because\ \tan(a-b) = \frac{\tan a-\tan b}{1+\tan a\tan b}\right]$$

$$= \tan^{-1}\left(\frac{2\tan\frac{\alpha}{2}\,.\frac{\left(1-\tan\frac{\beta}{2}\right)\left(1+\tan\frac{\beta}{2}\right)}{\left(1+\tan\frac{\beta}{2}\right)^2}}{\frac{\left(1+\tan\frac{\beta}{2}\right)^2-\tan^2\frac{\alpha}{2}\,.\left(1-\tan\frac{\beta}{2}\right)^2}{\left(1+\tan\frac{\beta}{2}\right)^2}}\right)$$

$$= \tan^{-1}\left[\frac{2\tan\frac{\alpha}{2}\,.\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1+\tan\frac{\beta}{2}\right)^2-\tan^2\frac{\alpha}{2}\,.\left(1-\tan\frac{\beta}{2}\right)^2}\right]$$

$$= \tan^{-1}\left[\frac{2\tan\frac{\alpha}{2}\,.\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1+\tan^2\frac{\beta}{2}\right)\left(1-\tan^2\frac{\alpha}{2}\right)+2\tan\frac{\beta}{2}\,.\left(1+\tan^2\frac{\alpha}{2}\right)}\right]$$

$$= \tan^{-1}\left(\frac{\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}\cdot\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}{\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}+\frac{2\tan\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}\right)$$

$$= \tan^{-1}\left(\frac{\sin\alpha.\cos\beta}{\cos\alpha+\sin\beta}\right) \qquad \left[\text{Dividing } N^r \text{ and } D^r \text{ by } \left(1+\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)\right]$$

= RHS

Q10. Solve the equation $\tan^{-1}\sqrt{x^2+x}+\sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$.

Ans.

Given equation exists, if

$$x^2 + x \geq 0 \quad \text{and} \quad 0 < \sqrt{x^2+x+1} \leq 1 \qquad [\because x^2+x+1 \text{ is always greater than zero}]$$

Now,

$$x^2 + x \geq 0 \quad \text{and} \quad x^2 + x + 1 \leq 1$$

$$\Rightarrow \quad x^2 + x \geq 0 \quad \text{and} \quad x^2 + x \leq 0$$

$$\Rightarrow \quad x^2 + x = 0 \quad \textit{i.e.,} \quad x(x+1) = 0$$

Hence, $x = 0$ and -1 are the solutions of the given equation.