

## SUPPLEMENTARY MATERIAL

### CHAPTER 7

$$7.6.3 \int (px + q)\sqrt{ax^2 + bx + c} \, dx.$$

We choose constants A and B such that

$$\begin{aligned} px + q &= A \left[ \frac{d}{dx}(ax^2 + bx + c) \right] + B \\ &= A(2ax + b) + B \end{aligned}$$

Comparing the coefficients of  $x$  and the constant terms on both sides, we get

$$2aA = p \text{ and } Ab + B = q$$

Solving these equations, the values of A and B are obtained. Thus, the integral reduces to

$$\begin{aligned} A \int (2ax + b)\sqrt{ax^2 + bx + c} \, dx + B \int \sqrt{ax^2 + bx + c} \, dx \\ = AI_1 + BI_2 \end{aligned}$$

$$\text{where } I_1 = \int (2ax + b)\sqrt{ax^2 + bx + c} \, dx$$

Put  $ax^2 + bx + c = t$ , then  $(2ax + b)dx = dt$

$$\text{So } I_1 = \frac{2}{3}(ax^2 + bx + c)^{\frac{3}{2}} + C_1$$

$$\text{Similarly, } I_2 = \int \sqrt{ax^2 + bx + c} \, dx$$

is found, using the integral formulae discussed in [7.6.2, Page 328 of the textbook].

Thus  $\int (px + q)\sqrt{ax^2 + bx + c} dx$  is finally worked out.

**Example 25** Find  $\int x\sqrt{1+x-x^2} dx$

**Solution** Following the procedure as indicated above, we write

$$\begin{aligned} x &= A \left[ \frac{d}{dx}(1+x-x^2) \right] + B \\ &= A(1-2x) + B \end{aligned}$$

Equating the coefficients of  $x$  and constant terms on both sides,

We get  $-2A = 1$  and  $A + B = 0$

Solving these equations, we get  $A = -\frac{1}{2}$  and  $B = \frac{1}{2}$ . Thus the integral reduces to

$$\begin{aligned} \int x\sqrt{1+x-x^2} dx &= -\frac{1}{2} \int (1-2x)\sqrt{1+x-x^2} dx + \frac{1}{2} \int \sqrt{1+x-x^2} dx \\ &= -\frac{1}{2} I_1 + \frac{1}{2} I_2 \end{aligned} \quad (1)$$

Consider  $I_1 = \int (1-2x)\sqrt{1+x-x^2} dx$

Put  $1+x-x^2 = t$ , then  $(1-2x)dx = dt$

Thus  $I_1 = \int (1-2x)\sqrt{1+x-x^2} dx = \int t^{\frac{1}{2}} dt = \frac{2}{3} t^{\frac{3}{2}} + C_1$

$$= \frac{2}{3} (1+x-x^2)^{\frac{3}{2}} + C_1, \text{ where } C_1 \text{ is some constant.}$$

Further, consider 
$$I_2 = \int \sqrt{1+x-x^2} \, dx = \int \sqrt{\frac{5}{4} - \left(x - \frac{1}{2}\right)^2} \, dx$$

Put  $x - \frac{1}{2} = t$ . Then  $dx = dt$

Therefore, 
$$\begin{aligned} I_2 &= \int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2} \, dt \\ &= \frac{1}{2}t\sqrt{\frac{5}{4} - t^2} + \frac{1}{2} \cdot \frac{5}{4} \sin^{-1} \frac{2t}{\sqrt{5}} + C_2 \\ &= \frac{1}{2} \frac{(2x-1)}{2} \sqrt{\frac{5}{4} - \left(x - \frac{1}{2}\right)^2} + \frac{5}{8} \sin^{-1} \left(\frac{2x-1}{\sqrt{5}}\right) + C_2 \\ &= \frac{1}{4}(2x-1)\sqrt{1+x-x^2} + \frac{5}{8} \sin^{-1} \left(\frac{2x-1}{\sqrt{5}}\right) + C_2, \end{aligned}$$

where  $C_2$  is some constant.

Putting values of  $I_1$  and  $I_2$  in (1), we get

$$\begin{aligned} \int x\sqrt{1+x-x^2} \, dx &= -\frac{1}{3}(1+x-x^2)^{\frac{3}{2}} + \frac{1}{8}(2x-1)\sqrt{1+x-x^2} \\ &\quad + \frac{5}{16} \sin^{-1} \left(\frac{2x-1}{\sqrt{5}}\right) + C, \end{aligned}$$

where  $C = -\frac{C_1+C_2}{2}$  is another arbitrary constant.

Insert the following exercises at the end of EXERCISE 7.7 as follows:

$$12. x\sqrt{x+x^2} \quad 13. (x+1)\sqrt{2x^2+3} \quad 14. (x+3)\sqrt{3-4x-x^2}$$

### Answers

$$12. \frac{1}{3}(x^2+x)^{\frac{3}{2}} - \frac{(2x+1)\sqrt{x^2+x}}{8} + \frac{1}{16} \log \left| x + \frac{1}{2} + \sqrt{x^2+x} \right| + C$$

$$13. \frac{1}{6}(2x^2+3)^{\frac{3}{2}} + \frac{x}{2}\sqrt{2x^2+3} + \frac{3\sqrt{2}}{4} \log \left| x + \sqrt{x^2 + \frac{3}{2}} \right| + C$$

$$14. -\frac{1}{3}(3-4x-x^2)^{\frac{3}{2}} + \frac{7}{2} \sin^{-1} \left( \frac{x+2}{\sqrt{7}} \right) + \frac{(x+2)\sqrt{3-4x-x^2}}{2} + C$$

## CHAPTER 10

### 10.7 Scalar Triple Product

Let  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  be any three vectors. The scalar product of  $\vec{a}$  and  $(\vec{b} \times \vec{c})$ , i.e.,  $\vec{a} \cdot (\vec{b} \times \vec{c})$  is called the scalar triple product of  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  in this order and is denoted by  $[\vec{a}, \vec{b}, \vec{c}]$  (or  $[\vec{a}\vec{b}\vec{c}]$ ). We thus have

$$[\vec{a}, \vec{b}, \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$$

#### Observations

1. Since  $(\vec{b} \times \vec{c})$  is a vector,  $\vec{a} \cdot (\vec{b} \times \vec{c})$  is a scalar quantity, i.e.  $[\vec{a}, \vec{b}, \vec{c}]$  is a scalar quantity.
2. Geometrically, the magnitude of the scalar triple product is the volume of a parallelepiped formed by adjacent sides given by the three

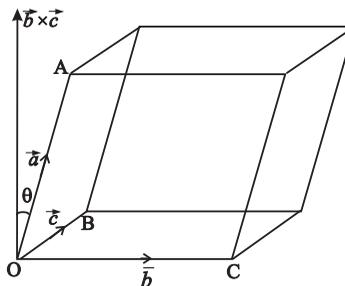


Fig. 10.28

vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  (Fig. 10.28). Indeed, the area of the parallelogram forming the base of the parallelepiped is  $|\vec{b} \times \vec{c}|$ . The height is the projection of  $\vec{a}$  along the normal to the plane containing  $\vec{b}$  and  $\vec{c}$  which is the magnitude of the component of  $\vec{a}$  in the direction of  $\vec{b} \times \vec{c}$  i.e.,  $\frac{|\vec{a} \cdot (\vec{b} \times \vec{c})|}{|\vec{b} \times \vec{c}|}$ . So the required

volume of the parallelepiped is  $\frac{|\vec{a} \cdot (\vec{b} \times \vec{c})|}{|\vec{b} \times \vec{c}|} |\vec{b} \times \vec{c}| = |\vec{a} \cdot (\vec{b} \times \vec{c})|$ ,

3. If  $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ ,  $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$  and  $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ , then

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (b_2c_3 - b_3c_2)\hat{i} + (b_3c_1 - b_1c_3)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}$$

and so

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

4. If  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  be any three vectors, then

$$[\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}]$$

(cyclic permutation of three vectors does not change the value of the scalar triple product).

$$\text{Let } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ and } \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}.$$

Then, just by observation above, we have

$$\begin{aligned}
 [\vec{a}, \vec{b}, \vec{c}] &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
 &= a_1 (b_2 c_3 - b_3 c_2) + a_2 (b_3 c_1 - b_1 c_3) + a_3 (b_1 c_2 - b_2 c_1) \\
 &= b_1 (a_3 c_2 - a_2 c_3) + b_2 (a_1 c_3 - a_3 c_1) + b_3 (a_2 c_1 - a_1 c_2) \\
 &= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\
 &= [\vec{b}, \vec{c}, \vec{a}]
 \end{aligned}$$

Similarly, the reader may verify that

$$[\vec{a}, \vec{b}, \vec{c}] = [\vec{a}, \vec{b}, \vec{c}]$$

$$\text{Hence } [\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}]$$

5. In scalar triple product  $\vec{a} \cdot (\vec{b} \times \vec{c})$ , the dot and cross can be interchanged. Indeed,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = [\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}] = \vec{c} \cdot (\vec{a} \times \vec{b}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

6.  $[\vec{a}, \vec{b}, \vec{c}] = -[\vec{a}, \vec{c}, \vec{b}]$ . Indeed

$$[\vec{a}, \vec{b}, \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$= \vec{a} \cdot (-\vec{c} \times \vec{b})$$

$$= -(\vec{a} \cdot (\vec{c} \times \vec{b}))$$

$$= -[\vec{a}, \vec{c}, \vec{b}]$$

7.  $[\vec{a}, \vec{a}, \vec{b}] = 0$  Indeed

$$\begin{aligned} [\vec{a}, \vec{a}, \vec{b}] &= [\vec{a}, \vec{b}, \vec{a}] \\ &= [\vec{b}, \vec{a}, \vec{a}] \\ &= \vec{b} \cdot (\vec{a} \times \vec{a}) \\ &= \vec{b} \cdot \vec{0} = 0. \end{aligned} \quad (\text{as } \vec{a} \times \vec{a} = \vec{0})$$

**Note:** The result in 7 above is true irrespective of the position of two equal vectors.

### 10.7.1 Coplanarity of Three Vectors

**Theorem 1** Three vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are coplanar if and only if  $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$ .

**Proof** Suppose first that the vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are coplanar.

If  $\vec{b}$  and  $\vec{c}$  are parallel vectors, then,  $\vec{b} \times \vec{c} = \vec{0}$  and so  $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$ .

If  $\vec{b}$  and  $\vec{c}$  are not parallel then, since  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are coplanar,  $\vec{b} \times \vec{c}$  is perpendicular to  $\vec{a}$ .

$$\text{So } \vec{a} \cdot (\vec{b} \times \vec{c}) = 0.$$

Conversely, suppose that  $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$ . If  $\vec{a}$  and  $\vec{b} \times \vec{c}$  are both non-zero, then we conclude that  $\vec{a}$  and  $\vec{b} \times \vec{c}$  are perpendicular vectors. But  $\vec{b} \times \vec{c}$  is perpendicular to both  $\vec{b}$  and  $\vec{c}$ . Therefore,  $\vec{a}$  and  $\vec{b}$  and  $\vec{c}$  must lie in the plane, i.e. they are coplanar.

If  $\vec{a} = 0$ , then  $\vec{a}$  is coplanar with any two vectors, in particular with  $\vec{b}$  and  $\vec{c}$ . If  $(\vec{b} \times \vec{c}) = 0$ , then  $\vec{b}$  and  $\vec{c}$  are parallel vectors and so,  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are coplanar since any two vectors always lie in a plane determined by them and a vector which is parallel to any one of it also lies in that plane.

**Note:** Coplanarity of four points can be discussed using coplanarity of three vectors.

Indeed, the four points A, B, C and D are coplanar if the vectors  $\overline{AB}$ ,  $\overline{AC}$  and  $\overline{AD}$  are coplanar.

**Example 26** Find  $\vec{a} \cdot (\vec{b} \times \vec{c})$ , if  $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ ,  $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$  and  $\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$ .

**Solution** We have  $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = -10$ .

**Example 27** Show that the vectors

$\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ ,  $\vec{b} = 2\hat{i} + 3\hat{j} - 4\hat{k}$  and  $\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$  are coplanar.

**Solution** We have  $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 0$ .

Hence, in view of Theorem 1,  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are coplanar vectors.

**Example 28** Find  $\lambda$  if the vectors

$\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$ ,  $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$  and  $\vec{c} = \lambda\hat{i} + 7\hat{j} + 3\hat{k}$  are coplanar.

**Solution** Since  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  are coplanar vectors, we have  $[\vec{a}, \vec{b}, \vec{c}] = 0$ , i.e.,

$$\begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & -1 \\ \lambda & 7 & 3 \end{vmatrix} = 0.$$

$$\Rightarrow 1(-3 + 7) - 3(6 + \lambda) + 1(14 + \lambda) = 0$$

$$\Rightarrow \lambda = 0.$$

**Example 29** Show that the four points A, B, C and D with position vectors

$4\hat{i} + 5\hat{j} + \hat{k}$ ,  $-(\hat{j} + \hat{k})$ ,  $3\hat{i} + 9\hat{j} + 4\hat{k}$  and  $4(-\hat{i} + \hat{j} + \hat{k})$ , respectively are coplanar.

**Solution** We know that the four points A, B, C and D are coplanar if the three vectors  $\overline{AB}$ ,  $\overline{AC}$  and  $\overline{AD}$  are coplanar, i.e., if

$$[\overline{AB}, \overline{AC} \text{ and } \overline{AD}] = 0$$

Now  $\overline{AB} = -(\hat{j} + \hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = -4\hat{i} - 6\hat{j} - 2\hat{k}$

$$\overline{AC} = (3\hat{i} + 9\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = -\hat{i} + 4\hat{j} + 3\hat{k}$$

and  $\overline{AD} = 4(-\hat{i} + \hat{j} + \hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = -8\hat{i} - \hat{j} + 3\hat{k}$

Thus  $[\overline{AB}, \overline{AC} \text{ and } \overline{AD}] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = 0.$

Hence A, B, C and D are coplanar.

**Example 30** Prove that  $[\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}] = 2 [\vec{a}, \vec{b}, \vec{c}]$ .

**Solution** We have

$$\begin{aligned} [\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}] &= (\vec{a} + \vec{b}) \cdot ((\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})) \\ &= (\vec{a} + \vec{b}) \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}) \\ &= (\vec{a} + \vec{b}) \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}) \quad (\text{as } \vec{c} \times \vec{c} = \vec{0}) \\ &= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{b} \times \vec{a}) + \vec{a} \cdot (\vec{c} \times \vec{a}) + \vec{b} \cdot (\vec{b} \times \vec{c}) + \vec{b} \cdot (\vec{b} \times \vec{a}) + \vec{b} \cdot (\vec{c} \times \vec{a}) \\ &= [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{a}] + [\vec{a}, \vec{c}, \vec{a}] + [\vec{b}, \vec{b}, \vec{c}] + [\vec{b}, \vec{b}, \vec{a}] + [\vec{b}, \vec{c}, \vec{a}] \\ &= 2[\vec{a}, \vec{b}, \vec{c}] \quad (\text{Why ?}) \end{aligned}$$

**Example 31** Prove that  $[\vec{a}, \vec{b}, \vec{c} + \vec{d}] = [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{d}]$

**Solution** We have

$$\begin{aligned} [\vec{a}, \vec{b}, \vec{c} + \vec{d}] &= \vec{a} \cdot (\vec{b} \times (\vec{c} + \vec{d})) \\ &= \vec{a} \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{d}) \\ &= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{b} \times \vec{d}) \\ &= [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{d}]. \end{aligned}$$

**Exercise 10.5**

- Find  $[\vec{a} \vec{b} \vec{c}]$  if  $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ ,  $\vec{b} = 2\hat{i} - 3\hat{j} + \hat{k}$  and  $\vec{c} = 3\hat{i} + \hat{j} - 2\hat{k}$   
(Ans. 24)
- Show that the vectors  $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ ,  $\vec{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}$  and  $\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$  are coplanar.
- Find  $\lambda$  if the vectors  $\hat{i} - \hat{j} + \hat{k}$ ,  $3\hat{i} + \hat{j} + 2\hat{k}$  and  $\hat{i} + \lambda\hat{j} - 3\hat{k}$  are coplanar.  
(Ans.  $\lambda = 15$ )
- Let  $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ ,  $\vec{b} = \hat{i}$  and  $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ . Then
  - If  $c_1 = 1$  and  $c_2 = 2$ , find  $c_3$  which makes  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  coplanar (Ans.  $c_3 = 2$ )
  - If  $c_2 = -1$  and  $c_3 = 1$ , show that no value of  $c_1$  can make  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  coplanar.
- Show that the four points with position vectors  $4\hat{i} + 8\hat{j} + 12\hat{k}$ ,  $2\hat{i} + 4\hat{j} + 6\hat{k}$ ,  $3\hat{i} + 5\hat{j} + 4\hat{k}$  and  $5\hat{i} + 8\hat{j} + 5\hat{k}$  are coplanar.
- Find  $x$  such that the four points A (3, 2, 1) B (4,  $x$ , 5), C (4, 2, -2) and D (6, 5, -1) are coplanar. (Ans.  $x = 5$ )
- Show that the vectors  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  coplanar if  $\vec{a} + \vec{b}$ ,  $\vec{b} + \vec{c}$  and  $\vec{c} + \vec{a}$  are coplanar.