Mathematics
Class XII
Sample Paper – 10 Solution

SECTION – A

1. By observation we find that

2 + x = 10

x = 8.

2. $\frac{d}{dx}\left(\cos\sqrt{x}\right)$

$=-\left(\sin\sqrt{x}\right)\frac{d}{dx}\left(\sqrt{x}\right)$

$\frac{dy}{dx}=\frac{-\sin\sqrt{x}}{2\sqrt{x}}$

3. DE:

$\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+\frac{dy}{dx}+y\sin y=0$

It is linear, since $y\times\sin y$ is product of two different functions, and their individual power is one.

4. Let θ be the angles between, the given two lines

So, the angle between them given their direction cosines is given by

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1{}^2+b_1{}^2+c_1{}^2}\sqrt{a_2{}^2+b_2{}^2+c_2{}^2}}$$

substituting we get

$a_1=2$

$a_2=3$

$b_1=1$

$b_2=2$

$c_1=-3$

$c_2=-1$

$$\theta=\cos^{1}\left(\frac{11}{14}\right)$$

OR

Let θ be the angles between, the given two lines

So, the angle between them given their direction cosines is given by

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1{}^2+b_1{}^2+c_1{}^2}\sqrt{a_2{}^2+b_2{}^2+c_2{}^2}}$$

substituting we get

$$a_1a_2+b_1b_2+c_1c_2=(2)(-1)+(7)(2)+(-3)(4)=0$$

$$\theta=\cos^{1}(0)=\frac{\pi}{2}$$

SECTION – B

5. Let X be the non-empty set for which P(X) is the power set.

$ARB \Leftrightarrow A \subset B$

i. $ARA \Leftrightarrow A \subset A$, every set is a subset of itself. R is reflexive

ii. If $A, B, C \in P(X)$

$ARB \Leftrightarrow A \subset B$, $BRC \Leftrightarrow B \subset C$

$A \subset B$ and $B \subset C \Rightarrow A \subset C$

So ARC; Hence R is transitive.

iii. $ARB \Leftrightarrow A \subset B$ does not imply $B \subset A$

So $B \not R A$

R is not symmetric

R is reflexive, transitive but not symmetric $\Rightarrow$R is not an equivalence relation

6. We have,

$2A - 3B + 5C = O$

$2A = 3B - 5C$

$$2A = 3\begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$$

$$2A = \begin{bmatrix} -6 & 6 & 0 \\ 9 & 3 & 12 \end{bmatrix} - \begin{bmatrix} 10 & 0 & -10 \\ 35 & 5 & 30 \end{bmatrix}$$

$$2A = \begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix}$$

$$A = \frac{1}{2}\begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix}$$

$$A = \begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}$$

7.

$$\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$$

$$\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$$

$$= 2\int \frac{-(\cos^2 x - \sin^2 x)}{2\sin x \cos x} dx$$

$$= -2\int \frac{\cos 2x}{\sin 2x} dx$$

$$= -\int \frac{2\cos 2x}{\sin 2x} dx$$

$$= -\log|\sin 2x| + C \qquad ....(\because \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c)$$

8.

$$\int \frac{dx}{5 - 8x - x^2}$$

$$= \int \frac{dx}{-(x^2 + 8x - 5)}$$

$$= \int \frac{dx}{-(x^2 + 8x + 16 - 16 - 5)}$$

$$= \int \frac{dx}{-[(x+4)^2 - 21]}$$

$$= \int \frac{dx}{(\sqrt{21})^2 - (x+4)^2}$$

$$= \frac{1}{2\sqrt{21}} \log\left|\frac{\sqrt{21} + (x+4)}{(x+4) - \sqrt{21}}\right| + C$$

$$= \frac{1}{2\sqrt{21}} \log\left|\frac{x + 4 + \sqrt{21}}{x + 4 - \sqrt{21}}\right| + C$$

OR

Let I = $\int \frac{x^2+1}{x^4+1}dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$

(Dividing numerator and denominator by x^2)

$$I = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}\,dx$$

Substituting $x - \frac{1}{x} = t$, $\Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$ we get,

$$I = \int \frac{dt}{t^2+2} = \int \frac{dt}{t^2+\left(\sqrt{2}\right)^2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{x-\frac{1}{x}}{\sqrt{2}}\right]$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{x^2-1}{\sqrt{2}x}\right]+c$$

9. Equation of parabola having vertex at the origin and axis along positive y axis is given by

$x^2 = 4ay$, where a is a parameter

Differentiating w.r.t. x

$$\frac{d}{dx}x^2 = 4a\frac{d}{dx}y$$

$$2x = 4a\frac{dy}{dx}$$

$$a = \frac{x}{2\frac{dy}{dx}}$$

substituting value of a in equation $x^2 = 4ay$

$$x^2 = 4 \times \frac{x}{2 \times \frac{dy}{dx}} \times y$$

$$x\frac{dy}{dx} = 2y$$

which is the required differential equation

10. $\left(\vec{x} - \vec{a}\right).\left(\vec{x} + \vec{a}\right) = 15$

$$\Rightarrow \vec{x}.\vec{x} - \vec{a}.\vec{a} = 15$$

$$\Rightarrow \left|\vec{x}\right|^2 - \left|\vec{a}\right|^2 = 15$$

$$\Rightarrow \left|\vec{x}\right|^2 - 1 = 15$$

$$\Rightarrow \left|\vec{x}\right|^2 = 16$$

$$\Rightarrow \left|\vec{x}\right| = \pm 4$$

OR

Distance between the parallel planes

is given by $\dfrac{|d-k|}{|\vec{n}|}$

$\vec{r}.\ 6\hat{i}-3\hat{j}+9\hat{k}\ +13=0$

$\Rightarrow \vec{r}.\ 2\hat{i}-\hat{j}+3\hat{k}\ =-\dfrac{13}{3}$

$\vec{r}.\ 2i-1\hat{j}+3\hat{k}\ =4$ and $\vec{r}.\ 2\hat{i}-\hat{j}+3\hat{k}\ =-\dfrac{13}{3}$

$\therefore$ the distance between the given parallel planes

is $\dfrac{\left|4-\left(-\dfrac{13}{3}\right)\right|}{\sqrt{2\times2+(-1\times-1)+3\times3}}$

$=\dfrac{\left|4+\left(\dfrac{13}{3}\right)\right|}{\sqrt{4+1+9}}=\dfrac{\dfrac{25}{3}}{\sqrt{14}}=\dfrac{25}{3\sqrt{14}}$

11. $P(I)=\dfrac{70}{100};\ \ P(II)=\dfrac{30}{100};$

$E:$ standard quality;

$P(E/I)=\dfrac{30}{100};\ P(E/II)=\dfrac{90}{100}$

$P(II/E)=\dfrac{P(II).P(E/II)}{P(I).P(E/I)+P(II).P(E/II)}$

$=\dfrac{\dfrac{30}{100}\times\dfrac{90}{100}}{\dfrac{70}{100}\times\dfrac{30}{100}+\dfrac{30}{100}\times\dfrac{90}{100}}=\dfrac{9}{16}$

12. Probability of getting a six = $\frac{1}{6}$

Probability of getting any other number = $\frac{5}{6}$

Following are the proceedings of game

1. got a six, hence game over
2. got some other number, then six appears
3. got six in 3rd chance
4. did not get six at all

Expected earnings

$$=\frac{1}{6}\times 1+\frac{5}{6}\times(-1)\times\frac{1}{6}(1)+\frac{5}{6}\times(-1)\times\frac{5}{6}\times(-1)\times\frac{5}{6}(-1)+\frac{5}{6}\times(-1)\times\frac{5}{6}\times(-1)\times\frac{1}{6}(1)$$
$$=\frac{1}{6}+\frac{25}{216}-\frac{5}{36}-\frac{125}{216}=\frac{36}{216}+\frac{25}{216}-\frac{30}{216}-\frac{125}{216}=\frac{-94}{216}$$

OR

Let the coin be tossed n times,

Probability of getting a head = $\frac{1}{2}$

Probability of getting no head in n trials is = $\left(\frac{1}{2}\right)^n$

$\therefore$ Probability of getting atleast one head = 1- $\left(\frac{1}{2}\right)^n$ ...(1)

Probability of getting at least one head must be > 90% = 0.9 (given) ...(2)

From (1) and (2)

$$1-\left(\frac{1}{2}\right)^n > 0.9$$

$$\Rightarrow \left(\frac{1}{2}\right)^n < 1 - 0.9$$

$$\Rightarrow \left(\frac{1}{2}\right)^n < 0.1 \qquad \Rightarrow n \geq 4$$

SECTION – C

13. $z_1Rz_2 = \dfrac{z_1 - z_2}{z_1 + z_2}$ is real

(i) R is reflexive : $z_1Rz_1 = \dfrac{z_1 - z_1}{z_1 + z_1} = 0, 0 \in$ real numbers

(ii) R is symmetric: $z_1Rz_2 = \dfrac{z_1 - z_2}{z_1 + z_2} \in$ real numbers, $\Rightarrow \dfrac{z_2 - z_1}{z_2 + z_1} \in$ real numbers

$\Rightarrow z_2R\, z_1$

(iii) R is transitive: Let $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2, z_3 = a_3 + ib_3 \in C$,
such that

$z_1Rz_2 = \dfrac{z_1 - z_2}{z_1 + z_2} \in$ real numbers and $z_2Rz_3 = \dfrac{z_2 - z_3}{z_2 + z_3} \in$ real numbers

$z_1Rz_2 = \dfrac{z_1 - z_2}{z_1 + z_2} \in$ real numbers

$$\Rightarrow \frac{(a_1 + ib_1) - (a_2 + ib_2)}{(a_1 + ib_1) + (a_2 + ib_2)} \in \text{real numbers} \Rightarrow \frac{(a_1 - a_2) + i(b_1 - b_2)}{(a_1 + a_2) + i(b_1 + b_2)} \in \text{real numbers}$$

$$\Rightarrow \left[\frac{(a_1 - a_2) + i(b_1 - b_2)}{(a_1 + a_2) + i(b_1 + b_2)} \times \frac{(a_1 + a_2) + i(b_1 + b_2)}{(a_1 + a_2) + i(b_1 + b_2)}\right] \in \text{real numbers}$$

$\Rightarrow$ The imaginary part $= \left[(a_1 - a_2) \times (b_1 + b_2) + (b_1 - b_2) \times (a_1 + a_2)\right] = 0$

$\Rightarrow 2[a_1b_2 - a_2b_1] = 0 \Rightarrow [a_1b_2 - a_2b_1] \Rightarrow a_1b_2 = a_2b_1$...(i)

Similary, $z_2Rz_3 = \dfrac{z_2 - z_3}{z_2 + z_3} \in R \Rightarrow a_2b_3 = a_3b_2$....(ii)

Multiplying, (i) and (ii),

$a_1b_2a_2b_3 = a_2b_1a_3b_2$

Case I : When $b_2a_2 \neq 0$

$a_1b_2a_2b_3 = a_2b_1a_3b_2 \Rightarrow a_1b_3 = b_1a_3 \Rightarrow z_2 \in R$

Case II : When $b_2a_2 = 0 \Rightarrow z_2 = 0 \in R$

$\Rightarrow$ R is transitive

The given relation R is (i) Reflexive (ii) Symmetric (iii) Transitive

$\Rightarrow$ The given relation R is an Equivalence relation

OR

$f: N \rightarrow N$

$$f(n)=\begin{cases}\dfrac{n+1}{2} & \text{if } n \text{ is odd}\\ \dfrac{n}{2} & \text{if } n \text{ is even}\end{cases}$$

Let $f(n_1)=f(n_2)$

Case 1: n_1, n_2 are odd

Let $f(n_1)=f(n_2)$

$\Rightarrow \dfrac{n_1+1}{2}=\dfrac{n_2+1}{2}$

$\Rightarrow n_1=n_2$

Case 2: n_1, n_2 are even

$f(n_1)=f(n_2) \Rightarrow \dfrac{n_1}{2}=\dfrac{n_2}{2} \Rightarrow n_1=n_2$

Case3: n_1 is odd and n_2 is even

$f(n_1)=f(n_2) \Rightarrow \dfrac{n_1+1}{2}=\dfrac{n_2}{2}$

$\Rightarrow n_1+1=n_2$

$\Rightarrow n_1 \neq n_2$

Hence,

$f(n_1)=f(n_2)$ does not imply $n_1=n_2 \ \forall \ n_1, n_2 \in N$

$\therefore f$ is not one $-$ one

Function f is onto and hence, f is surjective.

So f is not bijective.

14. L.H.S. $= 2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{5\sqrt{2}}{7} + 2\tan^{-1}\frac{1}{8}$

$$= 2\left(\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\right) + \sec^{-1}\frac{5\sqrt{2}}{7}$$

$$= 2\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= 2\tan^{-1}\left(\frac{13}{39}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)}\right) + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right) = \tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right) = \tan^{-1}1$$

$$= \frac{\pi}{4}$$

15. Consider $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

$R_1 \to R_1 + (R_2 + R_3)$

$$\Rightarrow \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$\Rightarrow (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$C_1 \to C_1 - C_3; C_2 \to C_2 - C_3$

$$\Rightarrow (5x+4)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 4-x & 2x \\ x-4 & x-4 & x+4 \end{vmatrix}$$

$$= (5x+4)(4-x)^2 \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 2x \\ -1 & -1 & x+4 \end{vmatrix}$$

$$= (5x+4)(4-x)^2$$

16. $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ w.r.t. $\cos^{-1}x^2$

Let

$$u=\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$$

and $v=\cos^{-1}x^2$

$$u=\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$$

putting $x^2=\cos\theta$

$$=\tan^{-1}\left\{\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right\}$$

$$u=\tan^{-1}\left\{\frac{\sqrt{2\cos^2\frac{\theta}{2}}-\sqrt{2\sin^2\frac{\theta}{2}}}{\sqrt{2\cos^2\frac{\theta}{2}}+\sqrt{2\sin^2\frac{\theta}{2}}}\right\}$$

$$u=\tan^{-1}\left\{\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}\right\}$$

$$u=\tan^{-1}\left\{\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right\}$$

$$u=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}$$

$$u=\frac{\pi}{4}-\frac{\theta}{2}=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x^2$$

$$\frac{du}{dx}=\frac{x}{\sqrt{1-x^4}}$$

Now,

$$v=\cos^{-1}x^2$$

$$\frac{dv}{dx}=\frac{-2x}{\sqrt{1-x^4}},\ \frac{du}{dv}=\frac{\frac{x}{\sqrt{1-x^4}}}{\frac{-2x}{\sqrt{1-x^4}}}=-\frac{1}{2}$$

OR

we have,

$$x=\frac{\sin^3 t}{\sqrt{\cos 2t}} \text{ and } y=\frac{\cos^3 t}{\sqrt{\cos 2t}}$$

so,

$$x=\sin^3 t(\cos 2t)^{\frac{-1}{2}} \text{ and } y=\cos^3 t(\cos 2t)^{\frac{-1}{2}}$$

differentiating w.r.t. t

$$\frac{dx}{dt}=\frac{3\sin^2 t\cos t}{\sqrt{\cos 2t}}+\sin^3 t\times\frac{-1}{2}\times(\cos 2t)^{\frac{-3}{2}}\frac{d}{dx}(\cos 2t)$$

$$=\frac{3\sin^2 t.\cos t.\cos 2t+\sin^3 t.\sin 2t}{(\cos 2t)^{\frac{3}{2}}}$$

$$=\frac{3\sin^2 t.\cos t.(1-2\sin^2 t)+\sin^3 t.(2\sin t.\cos t)}{(\cos 2t)^{\frac{3}{2}}}$$

$$=\frac{3\sin^2 t.\cos t-4\sin^4 t.\cos t}{(\cos 2t)^{\frac{3}{2}}}$$

$$=\frac{\sin 2t.\sin 3t}{2(\cos 2t)^{\frac{3}{2}}}$$

$$\frac{dy}{dt}=\frac{-3\cos^2 t\sin t}{\sqrt{\cos 2t}}+\cos 3t\times\frac{-1}{2}\times(\cos 2t)^{\frac{-3}{2}}\frac{d}{dx}(\cos 2t)$$

$$=\frac{-3\cos^2 t.\sin t.\cos 2t+\cos^3 t.\sin 2t}{(\cos 2t)^{\frac{3}{2}}}$$

$$=\frac{3\cos^2 t.\sin t-4\cos^4 t.\sin t}{(\cos 2t)^{\frac{3}{2}}}$$

$$=\frac{-\sin 2t.\cos 3t}{(\cos 2t)^{\frac{3}{2}}}$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{-\sin 2t.\cos 3t}{(\cos 2t)^{\frac{3}{2}}}}{\frac{\sin 2t.\sin 3t}{2(\cos 2t)^{\frac{3}{2}}}}=-\cos 3t$$

17. Let

$$u=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) \text{ and } v=\tan^{-1}x$$

putting $x=\tan\theta$

$$u=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}\right)$$

$$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$u=\frac{\theta}{2}=\frac{1}{2}\tan^{-1}x$$

differentiating w.r.t.x

$$\frac{du}{dx}=\frac{1}{2\left(1+x^2\right)}$$

$$\frac{dv}{dx}=\frac{1}{1+x^2}$$

$$\frac{du}{dv}=\frac{\frac{1}{2\left(1+x^2\right)}}{\frac{1}{1+x^2}}=\frac{1}{2}$$

18. Here, $x = a\sin^3 t$, $y = b\cos^3 t$ ....(1)

Differentiating (1) w.r.t. t

$$\frac{dx}{dt} = 3a\sin^2 t \times \cos t \text{ and}$$

$$\frac{dy}{dt} = -3b\cos^2 t \times \sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-3b\cos^2 t \times \sin t}{3a\sin^2 t \times \cos t} = -\frac{b}{a}\cot t$$

$\therefore$ Slope of the tangent at $t = \frac{\pi}{2}$

$$\left.\frac{dy}{dx}\right]_{\frac{\pi}{2}} = -\frac{b}{a}\cot\frac{\pi}{2} = 0$$

Hence, equation of tangent is given by

$$y - b\cos^3\frac{\pi}{2} = 0 \text{ or } y = 0$$

19.

Given that

$$I = \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)} d\theta$$

Put, $\sin\theta = t$

$\Rightarrow \cos\theta d\theta = dt$

$$I = \int \frac{\cos\theta}{(4+\sin^2\theta)\left[5-4(1-\sin^2\theta)\right]} d\theta$$

$$I = \int \frac{dt}{(4+t^2)\left[5-4(1-t^2)\right]} I = \int \frac{dt}{(4+t^2)(5-4+4t^2)}$$

$$I = \int \frac{dt}{(4+t^2)(1+4t^2)}$$

Using partial fraction,

$$\frac{1}{(4+t^2)(1+4t^2)} = \frac{A}{4+t^2} + \frac{B}{1+4t^2} \quad \text{......(i)}$$

$$\frac{1}{(4+t^2)(1+4t^2)} = \frac{A(1+4t^2)+B(4+t^2)}{(4+t^2)(1+4t^2)}$$

$1 = A(1+4t^2)+B(4+t^2)$

Consider, $t^2 = m$

$1 = A(1+4m)+B(4+m)$

Put, $m = -4$

$1 = A(1-16)$

$1 = A(-15)$

$A = \frac{-1}{15}$

Put, $m = \frac{-1}{4}$

$1 = B\left(4-\frac{1}{4}\right)$

$1 = B\times\frac{15}{4}$

$B = \frac{4}{15}$

Put A and B in (i),

$$\int\left(\frac{-1}{15(4+t^2)} + \frac{4}{15\times 4(\frac{1}{4}+t^2)}\right)dt$$

$$\frac{-1}{15}\times\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + \frac{1}{15}\times\frac{1}{\frac{1}{2}}\tan^{-1}(2t)+C$$

$$\frac{-1}{30}\tan^{-1}\left(\frac{\sin\theta}{2}\right) + \frac{2}{15}\times\tan^{-1}(2\sin\theta)+C$$

20.

Let $I = \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$ ....(i)

$$I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)\sin x \cos x}{\sin^4 x + \cos^4 x}\, dx \quad \left(\text{Applying } \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right) \text{..(ii)}$$

Adding, we get

$$2I = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}$$

Divide numerator & denominator by $\cos^4 x$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\tan x.\sec^2 x}{\tan^4 x + 1}\, dx$$

Put $\tan^2 x = t$

$2\tan x.\sec^2 x\, dx = dt$

$x = 0, t = 0$

$x = \frac{\pi}{2}, t = \infty$

$$2I = \frac{\pi}{2} \times \frac{1}{2}$$

$$I = \frac{\pi}{8} \tan^{-1} x \Big]_0^{\infty}$$

$$= \frac{\pi}{8}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{16}$$

21. $xdy - ydx = \sqrt{x^2 + y^2}dx$

put y = vx

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

we get

$$\Rightarrow v + x\frac{dv}{dx} = \frac{\sqrt{x^2 + v^2x^2} + vx}{x}$$

$$\Rightarrow x\frac{dv}{dx} = \sqrt{1 + v^2}$$

$$\Rightarrow \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

int ergrating

$$\int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \log\left|v + \sqrt{1 + v^2}\right| = \log|x| + \log c$$

$$\Rightarrow \left|v + \sqrt{1 + v^2}\right| = |cx|$$

$$\Rightarrow \left|\frac{y}{x} + \sqrt{1 + \frac{y}{x^2}^{2}}\right| = |cx|$$

$$\Rightarrow \left\{y + \sqrt{x^2 + y^2}\right\}^2 = c^2x^4$$

as required

OR

We have

$$2ye^{\frac{x}{y}}dx+\left(y-2xe^{\frac{x}{y}}\right)dy=0$$

$$\frac{dx}{dy}=\frac{2xe^{\frac{x}{y}}-y}{2ye^{\frac{x}{y}}}$$

sub $x=vy$

$$\frac{dx}{dy}=v+y\frac{dv}{dy}$$

so,

$$\Rightarrow v+y\frac{dv}{dy}=\frac{2ve^{v}-1}{2e^{v}}$$

$$\Rightarrow y\frac{dv}{dy}=-\frac{1}{2e^{v}}$$

$$\Rightarrow 2ye^{v}dv=-dy$$

$$\Rightarrow 2\int e^{v}dv=-\int\frac{1}{y}dv$$

$$\Rightarrow 2e^{v}=-\log|y|+\log c$$

$$\Rightarrow 2e^{v}=-\log\left|\frac{y}{c}\right|$$

$$\Rightarrow 2e^{\frac{x}{y}}=\log\left|\frac{c}{y}\right|$$

as required

22. If given lines are $\vec{r}_1 = \hat{i} + \hat{j} + \lambda\left(2\hat{i} - \hat{j} + \hat{k}\right)$ and $\vec{r}_2 = 2\hat{i} + \hat{j} - \hat{k} + \mu\left(3\hat{i} - 5\hat{j} + 2\hat{k}\right)$

Comparing with $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$, $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$

$\vec{a}_1 = \hat{i} + \hat{j}, \quad \vec{b}_1 = 2\hat{i} - \hat{j} + \hat{k}$

$\vec{a}_2 = 2\hat{i} + \hat{j} - \hat{k}, \quad \vec{b}_2 = 3\hat{i} - 5\hat{j} + 2\hat{k}$

$\Rightarrow \vec{a}_2 - \vec{a}_1 = \hat{i} - \hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = 3\hat{i} - \hat{j} - 7\hat{k}$$

$|\vec{b}_1 \times \vec{b}_2| = \sqrt{9+1+49} = \sqrt{59}$

Shortest distance

$$= \left| \frac{\left(\vec{b}_1 \times \vec{b}_2\right).\left(\vec{a}_2 - \vec{a}_1\right)}{\vec{b}_1 \times \vec{b}_2} \right| = \left| \frac{(3\hat{i} - \hat{j} - 7k).\left(\hat{i} - k\right)}{\vec{b}_1 \times \vec{b}_2} \right|$$

$$= \frac{10}{\sqrt{59}}$$

23. We know that equation of plane passing through 3 points.

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-3 & y+1 & z-2 \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 0$$

$\Rightarrow \; x-3 \;\; 12-0 \; - \; y+1 \;\; 8+8 \; + \; z-2 \;\; 0+12 \; = 0$

$\Rightarrow 12x - 36 - 16y - 16 + 12z - 24 = 0$

$\Rightarrow 12x - 16y + 12z - 76 = 0$

$\Rightarrow 3x - 4y + 3z - 19 = 0$

Also, perpendicular distance of $P(6, 5, 9)$ to the plane $3x - 4y + 3z - 19 = 0$

$$= \frac{|3\times6 - 4\times5 + 3\times9 - 19|}{\sqrt{9+16+9}}$$

$$= \frac{6}{\sqrt{34}} \text{ units}$$

SECTION – D

24. Let $A = IA$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

so,

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

$R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

$R_3 \to \frac{1}{2} R_3$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$$

$R_1 \rightarrow R_1 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$$

$R_2 \rightarrow R_2 - 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A$$

Hence,

$$A^{1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

OR

We have $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ and $f(x) = x^3 - 23x - 40$

$\therefore f(A) = A^3 - 23A - 40I$

Now,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$A.A = A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$$

$$A^2.A = A^3 = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$$

$$A.A = A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$$

$$A^2.A = A^3 = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$$

$$f(A) = A^3 - 23A - 40I$$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - 23\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Hence, A is the root of the polynomials $f(x) = x^3 - 23x - 40$.

25. $f(x) = (x-2)^4(x+1)^3$

$$f'(x) = 3(x-2)^4(x+1)^2 + 4(x+1)^3(x-2)^3$$
$$= (x-2)^3(x+1)^2[3(x-2)+4(x+1)]$$
$$= (x-2)^3(x+1)^2[3x-6+4x+4]$$
$$= (x-2)^3(x+1)^2[7x-2]$$

$f'(x) = 0 \Rightarrow (x-2)^3(x+1)^2[7x-2] \Rightarrow x = -1, \frac{2}{7}, 2$

Let us examine the behaviour of $f'(x)$, slightly to the left and right of each of these three values of x

(i)

$x = -1$:

When $x < -1$; $f'(x) > 0$

When $x > -1$; $f'(x) > 0$

$\Rightarrow x = -1$ is neither a point of local maxima nor minima

$\Rightarrow$ It may be a point of inflexion

(ii)

$x = \frac{2}{7}$

When $x < \frac{2}{7}$; $f'(x) > 0$

When $x > \frac{2}{7}$; $f'(x) < 0$

$\Rightarrow x = \frac{2}{7}$ is a point of local maxima

$$f\left(\frac{2}{7}\right) = \left(\frac{2}{7} - 2\right)^4\left(\frac{2}{7}+1\right)^3 = \left(\frac{-12}{7}\right)^4\left(\frac{9}{7}\right)^3 = \frac{2^8 \times 3^{10}}{7^7}$$

$\Rightarrow$ The local maximum value is $\frac{2^8 \times 3^{10}}{7^7}$

(iii)

$x = 2$

When $x < 2$; $f'(x) < 0$

When $x > 2$; $f'(x) > 0$

$\Rightarrow x = 2$ is a point of local minima

$f(2) = (2-2)^4(2+1)^3 = 0$

$\Rightarrow$ The local minimum value is 0

26. Curve1 is circle, $x^2 + y^2 = 4$, vertex = (0,0), Radius = 2

Curve2 is parabola, $y^2 = 3(2x-1)$, vertex $= \left(\frac{1}{2}, 0\right)$

On solving the two equations, we get

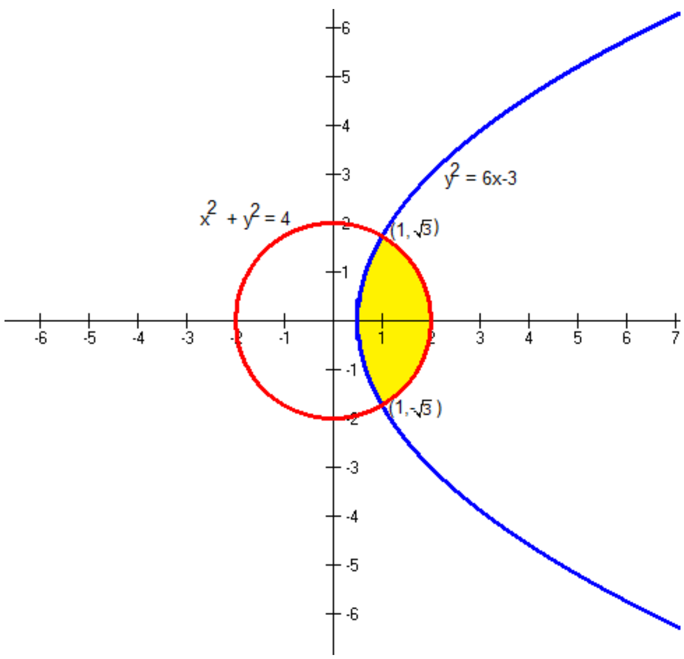
$x^2 + 3(2x-1) = 4$

$\Rightarrow (x+7)(x-1) = 0$

$\Rightarrow x = 1, -7$

$x = -7$ is not possible since y^2 must be positive

Hence, x = 1



The region is symmetric about the x-axis. The region above the x-axis, bounded by the parallel lines x = 1/2, x = 1 and x = 1, x = 2.

$$\text{Required Area} = 2\left\{\int_{1/2}^{1} y_{C_2}\,dx + \int_{1}^{2} y_{C_1}\,dx\right\}$$

$$= 2\left\{\sqrt{3}\int_{1/2}^{1}\sqrt{2x-1}\,dx + \int_{1}^{2}\sqrt{4-x^2}\,dx\right\}$$

$$= 2\left\{\left[\sqrt{3}.\frac{2}{3}.\frac{(2x-1)^{3/2}}{2}\right]_{1/2}^{1} + \left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right]_{1}^{2}\right\}$$

$$= \left\{\left[\frac{1}{\sqrt{3}}(2x-1)^{3/2}\right]_{1/2}^{1} + \left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right]_{1}^{2}\right\}$$

$$= 2\left[\frac{1}{\sqrt{3}} + 2\sin^{-1}(1) - \frac{\sqrt{3}}{2} - 2\sin^{-1}\frac{1}{2}\right]$$

$$= 2\left[\pi - \frac{\pi}{3} + \frac{2-3}{2\sqrt{3}}\right] = 2\left[-\frac{1}{2\sqrt{3}} + \frac{2\pi}{3}\right]$$

$$= \left[-\frac{1}{\sqrt{3}} + \frac{4\pi}{3}\right] \text{sq. units.}$$

OR

Given : $y = \frac{5}{2}x - 5; x + y - 9 = 0; y = \frac{3}{4}x - \frac{3}{2}$

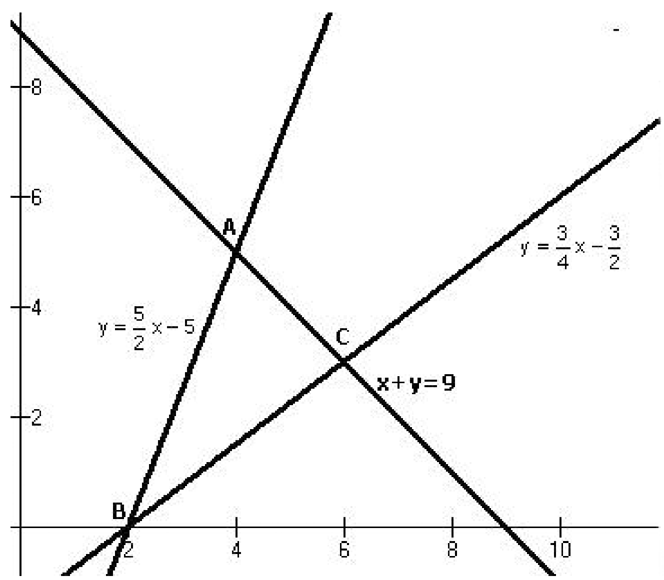
Point of intersection of the pair of lines $y = \frac{5}{2}x - 5; x + y - 9 = 0$ is (4,5)

Point of intersection of the pair of lines $y = \frac{5}{2}x - 5; y = \frac{3}{4}x - \frac{3}{2}$ is (2,0)

Point of intersection of the pair of lines $y = \frac{3}{4}x - \frac{3}{2}; x + y - 9 = 0$ is (6,3)

The area bounded by the 3 lines is the area of the triangle formed by the 3 lines.

Let ΔABC have vertices A(2,0) B(4,5) and C (6,3)



Area (△ABC) = area under segment AB + area under segment BC

- area under segment AC

$$\int_2^4\left(\frac{5}{2}x-5\right)dx+\int_4^6\left(-x+9\right)dx-\int_2^6\left(\frac{3}{4}x-\frac{3}{2}\right)dx$$

$$=\left[\frac{5x^2}{4}-5x\right]_2^4+\left[\frac{-x^2}{2}+9x\right]_4^6-\left[\frac{3x^2}{8}-\frac{3x}{2}\right]_2^6$$

$$=\left(\frac{5\times4^2}{4}-5\times4\right)-\left(\frac{5\times2^2}{4}-5\times2\right)+\left(\frac{-6^2}{2}+9\times6\right)-\left(\frac{-4^2}{2}+9\times4\right)-\left(\frac{3\times6^2}{8}-\frac{3\times6}{2}\right)+\left(\frac{3\times2^2}{8}-\frac{3\times2}{2}\right)$$

$$=20-20-\left(5-10\right)+\left(-18+54\right)-\left(-8+36\right)-\left(\frac{27}{2}-9\right)+\left(\frac{3}{2}-3\right)$$

$$=5+36-28-\frac{9}{2}-\frac{3}{2}=7 \text{ square units}$$

27. Equation of the plane passing through the intersection

of the planes x + y + z = 1 and 2x + 3y + 4z = 5 is :

$$\left(x+y+z-1\right)+\lambda\left(2x+3y+4z-5\right)=0$$

$$\left(1+2\lambda\right)x+\left(1+3\lambda\right)y+\left(1+4\lambda\right)z-\left(1+5\lambda\right)=0$$

This plane has to be perpendicular to the plane x - y + z = 0.

Thus,

$$\left(1+2\lambda\right)1+\left(1+3\lambda\right)\left(-1\right)+\left(1+4\lambda\right)1=0$$

$$1+2\lambda-1-3\lambda+1+4\lambda=0$$

$$1+3\lambda=0$$

$$\lambda=-\frac{1}{3}$$

Thus, the equation of the plane is :

$$\left(1+2\left(-\frac{1}{3}\right)\right)x+\left(1+3\left(-\frac{1}{3}\right)\right)y+\left(1+4\left(-\frac{1}{3}\right)\right)z-\left(1+5\left(-\frac{1}{3}\right)\right)=0$$

$$\left(1-\frac{2}{3}\right)x+\left(1-1\right)y+\left(1-\frac{4}{3}\right)z-\left(1-\frac{5}{3}\right)=0$$

$$\frac{x}{3}-\frac{z}{3}+\frac{2}{3}=0$$

$$x-z=-2$$

Thus, the distance of this plane form the origin is :

$$\left|\frac{-\left(-2\right)}{\sqrt{1^2+0^2+1^2}}\right|=\left|\frac{2}{\sqrt{2}}\right|=\sqrt{2}$$

OR

Any point in the line is

$2+3\lambda, -4+4\lambda, 2+2\lambda$

The vector equation of the plane is given as

$\vec{r}\cdot\left(\hat{i}-2\hat{j}+k\right)=0$

Thus the cartesian equation of the plane is $x-2y+z=0$

Since the point lies in the plane

$(2+3\lambda)1+(-4+4\lambda)(-2)+(2+2\lambda)1=0$

$\Rightarrow 2+8+2+3\lambda-8\lambda+2\lambda=0$

$\Rightarrow 12-3\lambda=0$

$\Rightarrow 12=3\lambda$

$\Rightarrow \lambda=4$

Thus, the point of intersection of the line and the plane is: $2+3\times4, -4+4\times4, 2+2\times4$

$\Rightarrow 14,12,10$

Distance between $(2, 12, 5)$ and $(14, 12, 10)$ is:

$d=\sqrt{(14-2)^2+(12-12)^2+(10-5)^2}$

$\Rightarrow d=\sqrt{144+25}$

$\Rightarrow d=\sqrt{169}$

$\Rightarrow d=13$ units

28. Let x hectares of land be allocated to crop X and y hectares to crop Y.

Total profit = Rs. $(10500x + 9000y)$

Linear programming problem is,

Max. $Z = 10500x + 9000y$

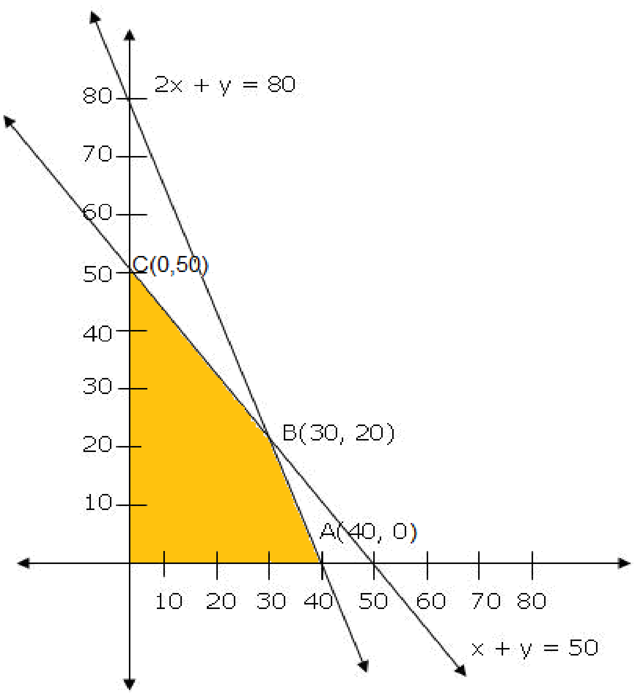
s.t $\quad x + y \le 50 \quad$ (constraint related to land)

$20x + 10y \le 800 \quad$ (constraint related to use of hectare)

$2x + y \le 80$

$x \ge 0, y \ge 0$

Graphically the problem can be represented as



Graph

Corner point	z = 10500x + 9000y
0(0, 0)	0
A(40, 0)	420000
B(30, 20)	495000 → Maximize
C(0, 50)	450000

Hence, society will get the maximum profit of Rs. 4,95,000 by allocating 30 hectares for crop X and 20 hectares for crop Y.

29. N = 2

Success = ‘throwing a doublet with a pair of dice’.

p = P (throwing a doublet with a pair of dice)

$$p = \frac{6}{36} = \frac{1}{6}$$

$$q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

$$P(X = 0) = {}^2C_0 p^0 q^2 = \left(\frac{5}{6}\right)^2 = \frac{25}{36}$$

$$P(X = 1) = {}^2C_1 p^1 q^1 = 2\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = \frac{10}{36}$$

$$P(X = 2) = {}^2C_2 p^2 q^0 = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$$

Hence the distribution of X is:

X	0	1	2
P(X)	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$
XP(X)	0	$\frac{10}{36}$	$\frac{2}{36}$
$X^2P(X)$	0	$\frac{10}{36}$	$\frac{4}{36}$

$$\mu = \sum_{i=1}^{n} p_i x_i$$

$$\therefore \mu = \frac{12}{36} = \frac{1}{3} = 0.33$$

$$\sigma^2 = \sum p_i (x_i - \mu)^2$$

$$= \sum p_i x_i^2 - \mu^2$$

$$= \frac{14}{36} - \frac{1}{9} = \frac{7}{18} - \frac{1}{9} = \frac{5}{18} = 0.28$$