Mathematics & Statistics

Academic Year: 2013-2014 Marks: 80
Date: March 2014

Question 1: **[12]**

Question 1: Select and write the correct answer from the given alternatives in each of the following : **[6]**

Question 1.1.1: **[2]**

Which of the following represents direction cosines of the line :

(a)$0, \frac{1}{\sqrt{2}}, \frac{1}{2}$

(b)$0, -\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}$

(c)$0, \frac{\sqrt{3}}{2}, \frac{1}{2}$

(d)$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

Solution: Ans. (C)

$l^2 + m^2 + n^2$

$= (0)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2$

$= \frac{3}{4} + \frac{1}{4} = 1$

Question 1.1.2: **[2]**

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ ans A(Adj A)=KI, then the value of 'K' is

2

- 2

10

-10

Solution: A(Adj A) =|A| . I

$\therefore K = |A|$

$\therefore |A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$

Question 1.1.3: The general solution of the trigonometric equation $\tan^2 \theta = 1$ is ________ **[2]**

$\theta = n\pi \pm \left(\frac{\pi}{3}\right), n \in z$

$\theta = n\pi \pm \frac{\pi}{6}, n \in z$

$\theta = n\pi \pm \frac{\pi}{4}, n \in z$

$0 = n\pi, n \in z$

Solution:

$$\tan^2 \theta = 1 = \tan^2\left(\frac{\pi}{4}\right)$$

$$\tan^2 \theta = \tan^2 \alpha \Rightarrow \theta = n\pi \pm \alpha$$

$$\therefore \theta = n\pi \pm \frac{\pi}{4}$$

Question 1.2 | Attempt any THREE of the following : **[6]**

Question 1.2.1: If $\bar{a}, \bar{b}, \bar{c}$ are the position vectors of the points A, B, C respectively and $2\bar{a} + 3\bar{b} - 5\bar{c} = 0$, then find the ratio in which the point C divides line segment AB. **[2]**

Solution:

$$2\bar{a} + 3\bar{b} - 5\bar{c} = 0$$

$$5\bar{c} = 3\bar{b} + 2\bar{a}$$

$$\bar{c} = \frac{3\bar{b} + 2\bar{a}}{5}$$

$$\bar{c} = \frac{3\bar{b} + 2\bar{a}}{3 + 2}$$

$\therefore$C divides seg AB internally in the ratio 3 : 2

Question 1.2.2: **[2]**

Equation of a plane is $\vec{r}\left(3\hat{i} - 4\hat{j} + 12\hat{k}\right) = 8.$ Find the length of the perpendicular from the origin to the plane.

Solution:

$$\vec{r}\left(3\hat{i}-4\hat{j}+12\hat{k}\right)=8.........(i)$$

$$\vec{n}=3\hat{i}-4\hat{j}+12\hat{k}$$

$$\left|\vec{n}\right|=\sqrt{9+16+144}=13$$

$$\therefore \vec{n}=\frac{3\hat{i}-4\hat{j}+12\hat{k}}{13}$$

$$\therefore \vec{r}.\left(\frac{3\hat{i}-4\hat{j}+12\hat{k}}{13}\right)=\frac{8}{13}...........by(i)$$

$$\vec{r}.\vec{n}=p$$

$\therefore$ Perpendicular distance from the origin is $\frac{8}{13}$

Question 1.2.3: [3]

The Cartestation equation of line is $\frac{x-6}{2}=\frac{y+4}{7}=\frac{z-5}{3}$ find its vector equation.

Solution:

$$\frac{x-6}{2}=\frac{y+4}{7}=\frac{z-5}{3}$$

Line is passing through the point (6,–4,5) with dr^5 2, 7, 3

Equation of line in vector form is

$$\vec{r}=\left(6\vec{i}-4\vec{j}+5\vec{k}\right)+\lambda\left(2\vec{i}+7\vec{j}+3\vec{k}\right)$$

Question 1.2.4: Find the acute angle between the lines whose direction ratios are 5, 12, -13 and 3, - 4, 5. [3]

Solution:

$$\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

$$= \left|\frac{15-48-65}{\sqrt{25+144+169}\sqrt{9+16+25}}\right|$$

$$= \left|-\frac{98}{13\sqrt{2}\times 5\sqrt{2}}\right|$$

$$= \left|-\frac{98}{13\times 5\times 2}\right|$$

$$= \frac{49}{65}$$

$$\theta = \cos^{-1}\left(\frac{49}{65}\right)$$

Question 1.2.5: Write the dual of the following statements: (p ∨ q) ∧ T **[3]**

Solution: Dual of (p ∨ q) ∧ T is (p ∧ q) ∨ F

Question 2: **[14]**

Question 2.1 | Attempt any TWO of the following **[6]**

Question 2.1.1: **[3]**

If the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect each other then find value of k

Solution:

Let $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}=u$ where is any constant.

So for any point on this line has co-ordinates in the form (2u+1,3u-1,4u+1)

$$\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}=v$$

So for any point on this line has co-ordinates in the form (v+3,2v+k,v).

So for any point on this line has co-ordinates in the form (v+3,2v+k,v).

Point of intersection of these two lines will have co-ordinates of the form

(2u +1, 3u −1,4u +1) and (v +3, 2v + k,v) .

Equating the x, y and z co-ordinates for both the forms we get three equations

2u+1=v+3

2u-v=2............(1)

3u-1=2v+k

3u-2v=k+1.......(2)

4u+1=v

4u-v=-1...........(3)

Subtracting equation (1)from equation(3) we get,

2u = -3

u=-3/2

Substitute value of u in equation (1) we get,

2(-3/2) - v=2

v=-5

Substitute value of v and in equation (2) we get,

3(-3/2) - 2(-5)=k+1

k=9/2

the value of k is 9/2

Question 2.1.2: Prove that three vectors $\bar{a}, \bar{b}$ and $\bar{c}$ are coplanar, if and only if, there exists a non-zero linear combination $x\bar{a} + y\bar{b} + z\bar{c} = 0$ [3]

Solution: Let $\bar{a}, \bar{b}$ and $\bar{c}$ be coplanar vectors. Then any one of them, say $\bar{a}$, will be the linear combination of $\bar{b}$ and $\bar{c}$.

There exist scalars α and β such that

$\bar{a} = \alpha\bar{b} + \beta\bar{c}$

$\therefore (-1)\bar{a} + \alpha\bar{b} + \beta\bar{c} = \bar{0}$

$i.e \quad x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$

Let x ≠ 0, then divide (1) by x, we get,

$i.e \quad \bar{a} + \left(\frac{y}{x}\right)\bar{b} + \left(\frac{z}{x}\right)\bar{c} = \bar{0}$

$\therefore \bar{a} = \left(-\frac{y}{x}\right)\bar{b} + \left(-\frac{z}{x}\right)\bar{c}$

i.e. $\bar{a} = \alpha\bar{b} + \beta\bar{c}$, where $\alpha = \left(-\frac{y}{x}\right)$ and $\beta = -\frac{z}{x}$ are scalar

therefore $\bar{a}$ is the linear combination of $\bar{b}$ and $\bar{c}$.

Hence, $\bar{a}, \bar{b}, \bar{c}$ are coplanar.

Question 2.1.3: Using truth table prove that : [3]

$\sim p \wedge q \equiv (p \vee q) \wedge \sim p$

Solution:

1	2	3	4	5	6
p	q	~p	$\sim p \wedge q$	$p \vee q$	$(p \vee q) \wedge \sim p$
T	T	F	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	F	F	F

Column (4) and (6) are identical truth value

$\therefore \sim p \wedge q \equiv (p \vee q) \wedge \sim p$

Question 2.2: Attempt any TWO of the following [8]

Question 2.2.1: In any ΔABC, with usual notations, prove that $b^2 = c^2 + a^2 - 2ca\cos B.$

Solution: Consider that for ΔABC, ∠B is in a standard position i.e. vertex B is at the origin and the side BC is along positive x-axis. As ∠B is an angle of a triangle ∠B can be acute or B ∠B can be obtuse.

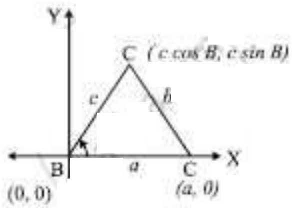


Fig. 1

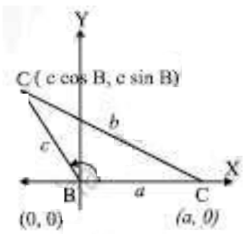


Fig. 2

Using the Cartesian co-ordinate system in both figure (1) and figure (2)
we get B ≡ (0,0)A ≡ (c cos B, c sin B) and C ≡ (a,0)

Now consider l(CA) =b

$\therefore b^2 = (a - c s B)^2 + (0 - c \sin B)^2$, by distance formula

$b^2 = a^2 - 2ac\cos B + c^2\cos^2 B + c^2\sin^2 B$

$b^2 = a^2 - 2ac\cos B + c^2(\sin^2 B + \cos^2 B)$

$b^2 = a^2 + c^2 - 2ac\cos B$

Question 2.2.2: Show that the equation $x^2 - 6xy + 5y^2 + 10x - 14y + 9 = 0$ represents a pair of lines. Find the acute angle between them. Also find the point of intersection of the lines. **[4]**

Solution:

$x^2 - 6xy + 5y^2 + 10x - 14y + 9 = 0$

comaparing with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

we get a=1, h=-3, b=5, g=5, f=-7, c=9

Consider $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

$$\begin{vmatrix} 1 & -3 & 5 \\ -3 & 5 & -7 \\ 5 & -7 & 9 \end{vmatrix}$$

=1(45-49)+3(-27+35)+5(21-25)

=(-4)+3(8)+5(-4)

=-4+24-20=0

Given equation represents a pair of lines

$$\text{Now} \quad \tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right| = \left|\frac{2\sqrt{9-5}}{1+5}\right| = \frac{2}{3}$$

$$\theta = \tan^{-1}\left(\frac{2}{3}\right)$$

The point of intersection= $\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$

$$= \left(\frac{21-25}{5-9}, \frac{-15+7}{5-9}\right)$$

$$= (1, 2)$$

Question 2.2.3: Express the following equations in the matrix form and solve them by method of reduction : **[4]**
2x- y + z = 1, x + 2y + 3z = 8, 3x + y - 4z =1

Solution: The matrix form of given equations is

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ -8 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -15 \\ -8 \end{bmatrix}$$

$$\begin{bmatrix} x + 2y + 3z \\ -5y - 5z \\ -8z \end{bmatrix} = \begin{bmatrix} 8 \\ -15 \\ -8 \end{bmatrix}$$

therefore

x + 2y + 3z = 8 ....(1)

-5y -5z = -15....... (2)

-8z = -8..............(3)

From (3),

z = 1

From (2),

-5y - 5(1) = -15 ... (because z = 1)

-5y =-10

y = 2

From (1),

x + 2(2)+ 3(1) = 8 ... (because z = 1 and y = 2)

x = 8 -7

x = 1

Thus, x = 1, y = 2, z = 1

Question 3.1 | Attempt any TWO of the following : **[6]**

Question 3.1.1: Show that every homogeneous equation of degree two in x and y, i.e., $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines passing through origin if $h^2 - ab \geq 0$. **[3]**

Solution 1: Consider a homogeneous equation of the second degree in x and y,

$$ax^2 + 2hxy + by^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Case I: If b = 0 (i.e., a ≠ 0, h ≠ 0), then the equation (1) reduce to $ax^2 + 2hxy = 0$
i.e., $x(ax + 2hy) = 0$

Case II: If a = 0 and b = 0 (i.e. h ≠ 0), then the equation (1) reduces to 2hxy = 0, i.e., xy = 0 which represents the coordinate axes and they pass through the origin.

Case III: If b ≠ 0, then the equation (1), on dividing it by b, becomes $\frac{a}{b}x^2 + \frac{2hxy}{b} + y^2 = 0$

$$\therefore y^2 + \frac{2h}{b}xy = -\frac{a}{b}x^2$$

On completing the square and adjusting, we get $y^2 + \frac{2h}{b}xy + \frac{h^2x^2}{b^2} = \frac{h^2x^2}{b^2} - \frac{a}{b}x^2$

$$\left(y + \frac{h}{b}x\right)^2 = \left(\frac{h^2 - ab}{b^2}\right)x^2$$

$$\therefore y + \frac{h}{b}x = \pm\frac{\sqrt{h^2 - ab}}{b}x$$

$$\therefore y = -\frac{h}{b}x \pm \frac{\sqrt{h^2 - ab}}{b}x$$

$$\therefore y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x$$

$\therefore$ equation represents the two lines $y = \left(\frac{-h + \sqrt{h^2 - ab}}{b}\right)x$ and $y = \left(\frac{-h - \sqrt{h^2 - ab}}{b}\right)x$

The above equation are in the form of y = mx
These lines passing through the origin.
Thus the homogeneous equation (1) represents a pair of lines through the origin, if $h^2 - ab \geq 0$.

Solution 2: Consider a homogeneous equation of degree two in x and y

$$ax^2 + 2hxy + by^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

In this equation at least one of the coefficients a, b or h is non zero. We consider two cases.

Case I: If b = 0 then the equation

$ax^2 + 2hxy = 0$

$x(ax + 2hy) = 0$

This is the joint equation of lines x = 0 and (ax+2hy)=0

These lines pass through the origin.

Case II: If b ≠ 0

Multiplying both the sides of equation (i) by b, we get

$abx^2 + 2hbxy + b^2y^2 = 0$

$2hbxy + b^2y^2 = -abx^2$

To make LHS a complete square, we add h^2x^2 on both the sides.

$b^2y^2 + 2hbxy + h^2y^2 = -abx^2 + h^2x^2$

$(by + hx)^2 = (h^2 - ab)x^2$

$(by + hx)^2 = \left[\left(\sqrt{h^2 - ab}\right)x\right]^2$

$(by + hx)^2 - \left[\left(\sqrt{h^2 - ab}\right)x\right]^2 = 0$

$\left[(by + hx) + \left[\left(\sqrt{h^2 - ab}\right)x\right]\right]\left[(by + hx) - \left[\left(\sqrt{h^2 - ab}\right)x\right]\right] = 0$

It is the joint equation of two lines

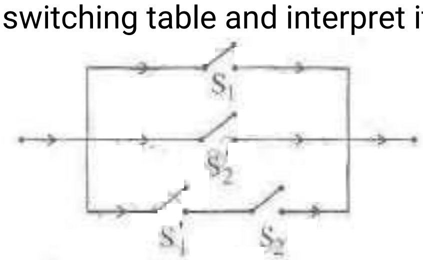
$(by + hx) + \left[\left(\sqrt{h^2 - ab}\right)x = 0 \text{ and } (by + hx) - \left[\left(\sqrt{h^2 - ab}\right)x = 0$

$\left(h + \sqrt{h^2 - ab}\right)x + by = 0 \text{ and } \left(h - \sqrt{h^2 - ab}\right)x + by = 0$

These lines pass through the origin when h^2-ab>0

From the above two cases we conclude that the equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines passing through the origin.

Question 3.1.2: Find the symbolic from of the following switching circuit, construct its switching table and interpret it. **[3]**



Solution: Let

p: The switch S_1 is closed,
q: The switch S_2 is closed.

Switching circuit is (pv~q)v(~p∧q)

The switching table

p	q	~p	~q	pv~q	~p∧ q	(pv~q)v(~p∧q)
1	1	0	0	1	0	1
1	0	0	1	1	0	1
0	1	1	0	0	1	1
0	0	1	1	1	0	1

From the last column of switching table we conclude that the current will always flow through the circuit.

Question 3.1.3: If A, B, C, D are (1, i, l), (2, l ,3), (3; 2, 2) and (3, 3, 4) respectively., then find the volume of the parallelized with AB, AC and AD as concurrent edges **[3]**

Solution:

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ be the position vectors of points A(1,1,1),B(2,1,3),C(3, 2, 2) and D(3,3, 4)

$$a = \hat{i} + \hat{j} + \hat{k}$$

$$\bar{b} = 2\hat{i} + \hat{j} + 3\hat{k}$$

$$\bar{c} = 3\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\bar{d} = 3\hat{i} + 3\hat{j} + 4\hat{k}$$

Given that vectors $\overline{AB}, \overline{AC}$ and $\overline{AD}$ represent the concurrent edges of a palallelopiped ABCD.

$$\overline{AB} = \bar{b} - \bar{a} = 2\hat{i} + \hat{j} + 3\hat{k} - \hat{i} - \hat{j} - \hat{k} = \hat{i} + 2\hat{k}$$

$$\overline{AC} = \bar{c} - \bar{a} = 3\hat{i} + 2\hat{j} + 2\hat{k} - \hat{i} - \hat{j} - \hat{k} = 2\hat{i} + \hat{j} + \hat{k}$$

$$\overline{AD} = \bar{d} - \bar{a} = 3\hat{i} + 3\hat{j} + 4\hat{k} - \hat{i} - \hat{j} - \hat{k} = 2\hat{i} + 2\hat{j} + 3\hat{k}$$

Consider, $\overline{AB}.\left(\overline{AC} \times \overline{AD}\right) = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 3 \end{vmatrix}$

=1(3-2)+2(4-2)=1+4=5

therefore Volume of parallelopiped with AB,AC and AD as concurrent edges is

$$V = \left[\overline{AB}.\left(\overline{AC} \times \overline{AD}\right)\right] = 5\text{cubic unit}$$

Question 3.2 | Attempt any TWO of the following **[8]**

Question 3.2.1: Find the equation of the plane passing through the line of intersection of planes 2x - y + z = 3 and 4x- 3y + 5z + 9 = 0 and parallel to the line **[4]**

$$\frac{x+1}{2}=\frac{y+3}{4}=\frac{z-3}{5}$$

Solution: Given planes are 2x-y+z =3, 4x-3y+5z+9 =0
Equation of required plane passing through their intersection is
(2x - y + z -3) +λ(4x-3y+5z + 9)= 0 .....(1)
(2 + 4λ) x +(-1-3λ) y+(1+5λ) z+(-3+ 9λ)= 0

Directio ratios of the normal to the above plane are $2+4\lambda,\ -1-3\lambda$ and $1+5\lambda$

Plane is parallel to the line $\frac{x+1}{2}=\frac{y+3}{4}=\frac{z-3}{5}$

Direction ratios of line are 2, 4,5
Given that required plane is parallel to given line.
∴normal of the plane is perpendicular to the given line

$$(2+4\lambda)2+(-1-3\lambda)4+(1+5\lambda)5=0$$

$$4+8\lambda-4-12\lambda+5+25\lambda=0$$

$$21\lambda+5=0$$

$$\therefore \lambda=-\frac{5}{21}$$

Substituting λ in (1)
∴ Equation of plane is

(2x-y+z-3)-5/21(4x-3y+5z+9)=0

42x-21y+21z-63-20x+15y-25z-45=0

22x-6y-4z-108=0

11x-3y-2z-54=0

Question 3.2.2: Minimize :Z=6x+4y
Subject to : 3x+2y ≥12
x+y ≥5
0 ≤x ≤4
0 ≤ y ≤ 4

Solution: 3x+2y ≥12

Points : (4, 0) and (0, 6), Non-origin side

x+y ≥5

Points : (5, 9) and (0, 5), Non-origin side

0 ≤x ≤4

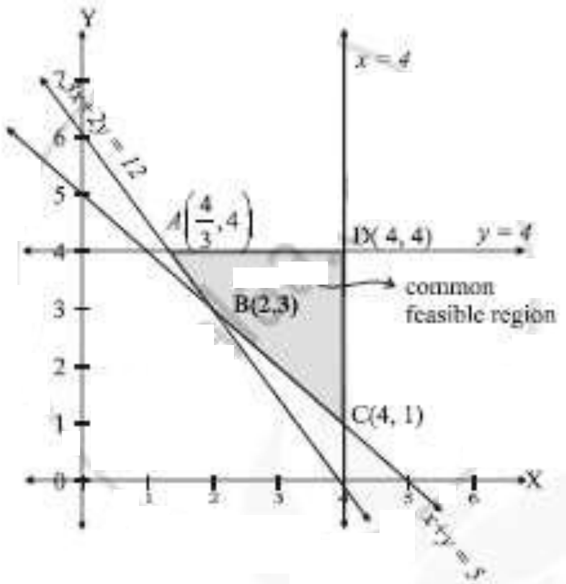
Parallel to y-axis, point (4, 0), origin side

$0 \le y \le 4$

Parallel to x-axis, point (0, 4), origin side

$x \ge 0, y \ge 0$

x-axis and y-axis, first quadrant only.



A is the intersection of 3x+2y =12 and y= 4

x=4/3 and y=4

A(4/3, 4)

B is intersection of 3x + 2 y = 12 and x + y= 5

x=2, y=3

B(2,3)

C is the intersection of x = 4 and x + y = 5

x=4, y=1

C(4,1)

D is the intersection of x = 4 and y = 4

D (4, 4)

End Points	value of z=6x+4y
A(4/3, 4)	8+16=24
B(2, 3)	12+12=24
C(4, 1)	24+4=28
D(4, 4)	24+16=40

Z is minimum 24 on the segment AB joining A(4/3 ,4) and (2,3)

Question 3.2.3: **[4]**

Show that:

$$\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(\frac{33}{65}\right)$$

Solution:

Let a = $\cos^{-1}\left(\frac{4}{5}\right)$ and b = $\cos^{-1}\left(\frac{12}{13}\right)$

Let a = $\cos^{-1}\left(\frac{4}{5}\right)$

cos a = $\frac{4}{5}$

We know that

$\sin^2 a = 1 - \cos^2 a$

sin a = $\sqrt{1-\cos^2 a}$

$$=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}$$

$$=\sqrt{\frac{25-16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

Let b = $\cos^{-1}\left(\frac{12}{13}\right)$

cos b = $\frac{12}{13}$

W know that

$\sin^2 b = 1 - \cos^2 b$

sin b = $\sqrt{1-\cos^2 b}$

$$= \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{1-\frac{144}{169}}$$

$$= \sqrt{\frac{169-144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

We know that

cos (a+b) = cos a cos b - sin a sin b

Putting values

cos a = $\frac{4}{5}$, sin a = $\frac{3}{5}$

& cos b = $\frac{12}{13}$, sin b = $\frac{5}{13}$

cos (a+b) = $\frac{4}{5} \times \frac{12}{13} \times \frac{3}{5} \times \frac{5}{13}$

$$= \frac{48}{65} - \frac{3}{13}$$

$$= \frac{48-15}{65}$$

$$= \frac{33}{65}$$

∴ cos (a+b) = $\frac{33}{65}$

a + b = $\cos^{-1}\left(\frac{33}{65}\right)$

$$\cos^{-1}\frac{4}{5}+\cos^{-1}\left(\frac{12}{15}\right)=\cos^{-1}\left(\frac{33}{65}\right)$$

Hence LH.S = R.H.S

Hence proved.

Question 4.1 | Select an write the correct answer from the given alternatives in each of the following: [6]

Question 4.1.1: [3]

If y =1 - cos θ , x = 1 - sin θ , then $\frac{dy}{dx}$ $at \quad \theta=\frac{\pi}{4}$ is _______

Solution:

$$\frac{dy}{dx}=\sin\theta$$

$$\frac{dx}{d\theta}=-\cos\theta$$

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=-\frac{\sin\theta}{\cos\theta}=-\tan\theta$$

$$\frac{dy}{dx}=-\tan\left(\frac{\pi}{4}\right)$$

$$\therefore\frac{dy}{dx}=-1$$

Question 4.1.2: [3]

The integrating factor of linear differential equation $\frac{dy}{dx}+y\sec x=\tan x$ is

(a)secx- tan x

(b) sec x · tan x

(c)sex+tanx

(d) secx.cotx

Solution:

$$\frac{dy}{dx}+P.y=Q$$

therefore P=secx

$$I.f=e^{\int\sec x dx}=e^{\log|sex+\tan x|}$$

=secx+tanx

Question 4.1.3: The equation of tangent to the curve $y = 3x^2 - x + 1$ at the point (1, 3) is **[2]**
(a) y=5x+2
(b)y=5x-2
(c)y=1/5x+2
(d)y=1/5x-2

Solution: y = 5x – 2

$$\frac{dy}{dx} = 6x - 1 \quad \text{at} \quad (1, 3)$$

Slope of the tangent at (1, 3) = (6 - 1) = 5

Equation of tangent is $y - y_1 = m(x - x_1)$

y - 3 = 5(x - 1)

5x - y - 2 = 0
y = 5x - 2

Question 4.2 | Attempt any THREE of the following: **[6]**

Question 4.2.1: Examine the continuity of the function **[2]**
f(x) =sin x- cos x, for x ≠ 0

=- 1, for x =0
At the point x = 0

Solution:

$$f(0) = -1$$

$$\text{and} \ \lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} (\sin x - \cos x) = -1$$

$$\therefore f(0) = \lim_{x \to 0} f(x)$$

Hence f (x) is continuous at x = 0

Question 4.2.2: Verify Rolle's Theorem for the function **[2]**
$f(x)=x^2-5x+9$ on [1,4]

Solution: The function f given as $f(x)=x^2-5x+9$ is a polynomial function. Hence

(i) it is continuous on [1,4]
(ii) differentiable on (1,4).

$$Now, f(1) = 1^2 - 5(1) + 9 = 1 - 5 + 9 = 5$$

$$\text{and } f(4) = 4^2 - 5(4) + 9 = 16 - 20 + 9 = 5$$

f (1)=f(4)

Thus, the function f satisfies all the conditions of the Rolle's theorem.

therefore there exists c ∈ (1, 4) such that f '(c)= 0

$$Now, f(x) = x^2 - 5x + 9$$

$$\therefore f\prime(x) = \frac{d}{dx}\left(x^2 - 5x + 9\right) = 2x - 5 \times 1 + 0$$

=2x-5

f'(c)=2c-5

f'(c)=0 gives, 2c-5=0

$$c = \frac{5}{2} \in (1, 4)$$

Hence, the Rolle's theorem is verified

Question 4.2.3: **[2]**

Evaluate : $\int \sec^n x \tan x dx$

Solution:

$$I = \int \sec^{n-1} x \sec x \tan x dx$$

Let secx=t

$$\therefore \sec x \tan x dx = dt$$

$$I = \int t^{n-1} dt$$

$$= \frac{t^n}{n} + c$$

$$= \frac{\sec^n x}{n} + C$$

Question 4.2.4: The probability mass function (p.m.f.) of X is given below: **[2]**

X=x	1	2	3
P (X= x)	1/5	2/5	2/5

Find E(X^2)

Solution:

x	P(x)	xP(x)	x^2P(x)
1	1/5	1/5	1/5
2	2/5	4/5	8/5
3	2/5	6/5	18/5
			$\sum x^2P(x) = \frac{27}{5}$

$E(x^2) = \sum x^2P(x) = \frac{27}{5}$

Question 4.2.5: Given that X~ B(n = 10, p), if E(X) = 8. find the value of p. **[2]**

Solution:

$X \sim B(n = 10, p)$

$\therefore E(X) = np$

$8 = 10p$

$p = 0.8 = \frac{4}{5}$

Question 5.1 | Attempt any TWO of' the following : **[6]**

Question 5.1.1: If y=f(u) is a differentiable function of u and u = g(x) is a differentiable function of x then prove that y = f (g(x)) is a differentiable function of x and **[3]**

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Solution: Let δx be a small increment in x.
Let δy and δu be the corresponding increments in y and u respectively

As δx → 0, δy → 0, δu → 0.
As u is differentiable function, it is continuous.

Consider the incrementary ratio $\frac{\delta y}{\delta x}$

We have , $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$

Taking limit as δx → 0, on both sides,

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left(\frac{\partial t y}{\delta u} \times \frac{\delta u}{\delta x} \right)$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x} ...(1)$$

Since y is a differentiable function of u , $\lim_{\delta u \to 0} \frac{\delta y}{\delta u}$ exists

and $\lim_{\delta u \to 0} \frac{\delta y}{\delta x}$ exists as u is a differentiable function of x.

Hence, R.H.S. of (1) exists

$$\text{now} \quad \lim_{\delta u \to 0} \frac{\delta y}{\delta u} = \frac{dy}{du} \quad \text{and} \quad \lim_{\delta u \to 0} \frac{\delta u}{\delta x} = \frac{du}{dx}$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{du} \times \frac{du}{dx}$$

Since R.H.S. exists, L.H.S. of (1) also exists and

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Question 5.1.2: Obtain the differential equation by eliminating arbitrary constants A, B from the equation - y = A cos (log x) + B sin (log x) **[3]**

Solution:

$$y = A\cos(\log x) + B\sin(\log x)$$

Diff. w.r.t x

$$\frac{dy}{dx} = -A\frac{\sin(\log x)}{x} + B\frac{\cos(\log x)}{x}$$

$$\frac{dy}{dx} = \frac{-A\sin(\log x) + B\cos(\log x)}{x}$$

$$x.\frac{dy}{dx} = -A\sin(\log x) + B\cos(\log x)$$

Again diff. w.r.t. x

$$x.\frac{d^2y}{dx^2} + \frac{dy}{dx} = -A\frac{\cos(\log x)}{x} - B\frac{\sin(\log x)}{x}$$

$$x.\frac{d^2y}{dx^2}+\frac{dy}{dx}=-\frac{A\cos(\log x)+B\sin(\log x)}{x}$$

$$x.\frac{d^2y}{dx^2}+\frac{dy}{dx}=-\frac{y}{x}$$

$$x^2.\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

Question 5.1.3: [3]

Evaluate : $\int \frac{x^2}{(x^2+2)(2x^2+1)}dx$

Solution:

Let$I=\int \frac{x^2}{(x^2+2)(2x^2+1)}dx$

consider$\frac{x^2}{(x^2+2)(2x^2+1)}$

Put x^2= t (For finding partial fractions only)

$$\frac{t}{(t+2)(2t+1)}=\frac{A}{t+2}+\frac{B}{2t+1}$$

t=A(2t+1)+B(t+2)

On Solving we get A=2/3, B=-1/3

$$\frac{t}{(t+2)(2t+1)}=\frac{\frac{2}{3}}{t+2}+\frac{-\frac{1}{3}}{2t+1}$$

$$\frac{x^2}{(x^2+2)(2x^2+1)}=\frac{\frac{2}{3}}{t+2}+\frac{-\frac{1}{3}}{2t+1}$$

$$I=\int\left[\frac{\frac{2}{3}}{t+2}+\frac{-\frac{1}{3}}{2t+1}\right]dx$$

$$=\frac{2}{3}\int\frac{1}{x^2+2}dx-\frac{1}{3}\int\frac{1}{2x^2+1}dx$$

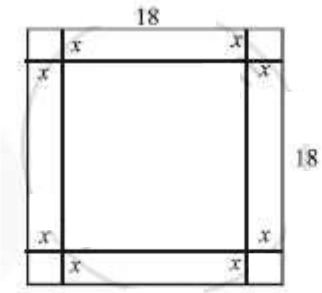
$$=\frac{2}{3}\int\frac{1}{x^2+\left(\sqrt{2}\right)^2}dx-\frac{1}{6}\int\frac{1}{x^2+\left(\frac{1}{\sqrt{2}}\right)^2}dx$$

$$=\frac{\sqrt{2}}{3}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)-\frac{1}{3\sqrt{2}}\tan^{-1}\left(\sqrt{2}x\right)+c$$

Question 5.2: Attempt any TWO of the following : **[8]**

Question 5.2.1: An open box is to be made out of a piece of a square card board of sides 18 cms. by cutting off equal squares from the comers and turning up the sides. Find the maximum volume of the box. **[4]**

Solution:



Let each side of the square cut off from each corner be x cm.
Then the base of the box will be of side 18 - 2x cm and the height of the box will be x cm
Then volume of box V = (18 - 2x)(18 - 2x)
V = $(18 - 2x)^2x$
V = 4x3 + 324x - 72x2 ..(i)
Differentiating w.r t to x , we get

$$\frac{dV}{dx} = 12x^2 + 324 - 144x$$

$$\frac{dV}{dx} = 12(x^2 - 12x + 27) \quad ...(ii)$$

For maximum volume $\frac{dV}{dx} = 0$

$\Rightarrow 12(x^2 - 12x + 27) = 0$

$\Rightarrow x^2 - 9x -3x + 27 = 0$

$\Rightarrow (x - 9)(x - 3) = 0$

$\Rightarrow x = 9, 3$

Again differentiating, we get

$$\frac{d^2V}{dx^2} = 2x - 12 \quad ....(iii)$$

At x = 9.

$$\frac{d^2V}{dx^2} = +ve$$

∴ V is minimum at x = 9 at x = 3.

$$\frac{d^2V}{dx^2} = -ve$$

∴ V is maximum at x = 3.

∴ Maximum volume V = (18−6)(18−6) × 3

= 12 x 12 x 3 = 432 cm^3

Question 5.2.2: **[4]**

Prove that :

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

Solution: Since 'a' lies between 0 and 2a,
We have

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx, \;\ldots\ldots\; \left(by \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right)$$

$$= I_1 + I_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(say)}$$

$$I_2 = \int_a^{2a} f(x)dx$$

Put x=2a-t

therefore dx=-dt

When x=a,2a-t=a

t=a

When x =2a, 2a- t = 2a

t= 0

$$I_2 = \int_0^{2a} f(x)dx = \int_a^0 f(2a-t)(-dt)$$

$$= -\int_a^0 f(2a-t)dt = \int_0^a f(2a-t)dt \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; \left(By \int_a^b f(x)dx = -\int_b^a f(x)dx\right)$$

$$= \int_0^a f(2a-x)dx \;\ldots\ldots\ldots\ldots.\; \left(By \int_a^b f(X)dx = \int_a^b f(t)dt\right)$$

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

$$= \int_0^a [f(x) + f(2a-x)]dx$$

Question 5.2.3: If the function f (x) is continuous in the interval [-2, 2], find the values of a and b where **[4]**

$$f(x) = \frac{\sin ax}{x} - 2, f \text{ or } -2 \le x \le 0$$
$$= 2x+1, f \text{ or } 0 \le x \le 1$$
$$= 2b\sqrt{x^2+3} - 1, f \text{ or } 1 < x \le 2$$

Solution: Since the function f (x) is continuous in the interval [-2,2]

f is continuous at in x = 0 and x = 1
(i) continuity at x = 0

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\frac{\sin ax}{x} - 2 \right)$$

$$= \lim_{x \to 0} \left(\frac{\sin ax}{ax} a - 2 \right)$$

=a(1)-2

=a-2

f (x)= 2x +1, for 0<= x <=1 ...(i)
f(0)=2(0)+1=1

f is continuous at x=0

$$\lim_{x \to 0^-} f(x) = f(0)$$

a-2=1

a=3

(ii) Continuity at x = 1

From (i), f(1)=3

$$\lim_{x \to 1} f(x) = \lim_{x \to 1^+} \left(2b\sqrt{x^2+3} - 1 \right)$$

$$= 2b \lim_{x \to 1} \sqrt{x^2+3} - 1$$

$$= 2b\sqrt{1+3} - 1 = 4b - 1$$

f is continuous at x = 1

$$\lim_{x \to 1} f(x) = f(1)$$

4b-1=3

4b=4

b=1

Question 6.1: Attempt any TWO of the following [6]

Question 6.1.1: [3]

Solve the differential equation $\frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x}$

Solution:

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$$

Put y = vx

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore (1) becomes, v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x}$$

$$\therefore v + x\frac{dv}{dx} = v + \sqrt{1 + v^2}$$

$$\therefore \frac{1}{\sqrt{1 + v^2}}dv = \frac{1}{x}dx$$

Integrating, we get,

$$\int \frac{1}{\sqrt{1 + v^2}}dv = \int \frac{1}{x}dx + c_1$$

$$\therefore \log\left|v + \sqrt{1 + v^2}\right| = \log|x| + \log c, where c_1 = \log c$$

$$\therefore \frac{y + \sqrt{x^2 + y^2}}{x} = cx$$

$\left(y + \sqrt{x^2 + y^2}\right) = cx^2$ is the general solution

Question 6.1.2: A fair coin is tossed 8 times. Find the probability that it shows heads at least once **[3]**

Solution: Let X = Number of heads
p = probability of getting head in one toss

p=1/2

$$q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$$

Given n=8

$$x \sim B\left(8, \frac{1}{2}\right)$$

The p.m.f. of X is given as

$$P(X = x) = {}^nC_x p^x q^{n-x}$$

$$i.e P(x) = {}^8C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^)(8 - x), x = 0, 1, 2, 3, \ldots., 8$$

P (getting heads at least once)

P (getting heads at least once)

$$P[X \geq 1] = 1 - P[X = 0]$$

$$= 1 - P(0) = 1 - {}^{8}C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{8-0}$$

$$= 1 - \left(\frac{1}{2}\right)^8 = 1 - \frac{1}{256} = \frac{255}{256}$$

$$P[X \geq 1] = 0.996$$

Question 6.1.3: [3]

If $x^p y^q = (x+y)^{p+q}$ then Prove that $\frac{dy}{dx} = \frac{y}{x}$

Solution:

$x^p y^q = (x+y)^{p+q}$

Taking log both side

$$p \log x + q \log y = (p+q) \log(x+y)$$

Diff. w.r.t. x

$$\frac{p}{x} + \frac{q}{y}\frac{dy}{dx} = \frac{p+q}{x+y} + \left(\frac{p+q}{x+y}\right)\frac{dy}{dx}$$

$$\frac{q}{y}\frac{dy}{dx} - \left(\frac{p+q}{x+y}\right)\frac{dy}{dx} = \frac{p+q}{x+y} - \frac{p}{x}$$

$$\left(\frac{q}{y} - \frac{p+q}{x+y}\right)\frac{dy}{dx} = \left(\frac{p+q}{x+y} - \frac{p}{x}\right)$$

$$\left(\frac{qx-py}{y}\right)\frac{dy}{dx} = \left(\frac{qx-py}{x}\right)$$

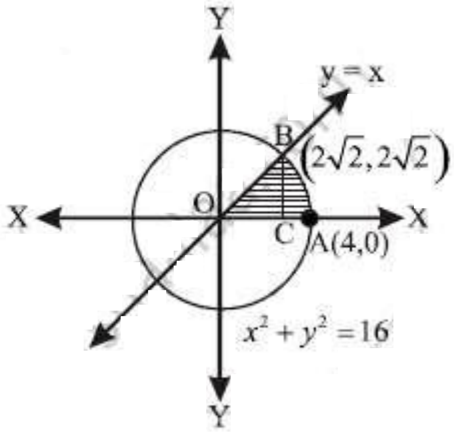
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

Question 6.2 | Attempt any TWO of the following : [8]

Question 6.2.1: Find the area of the sector of a circle bounded by the circle $x^2 + y^2 = 16$ and the line y = x in the first quadrant. [4]

Solution:



Given that $x^2 + y^2 = 16$ ...(i)

y=x ....(ii)

By equation (i) & (ii)

$x = \pm 2\sqrt{2}$

$y = \pm 2\sqrt{2}$

But required area in first quadrant

$x = y = 2\sqrt{2}$

$From dig. area = Area of \Delta OBC + Area of region CABC$

$= \int_o^{2\sqrt{2}} x dx + \int_{2\sqrt{2}}^{4} \sqrt{16 - x^2} dx$

$= \frac{1}{2}[x^2]_0^{2\sqrt{2}} + \left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_{\sqrt{2}}^{4}$

$= 4 + 8 \times \frac{\pi}{2} - 4 - 8 \times \frac{\pi}{4} = 2\pi sq. units$

Question 6.2.2: **[4]**

Prove that $\int \sqrt{x^2 - a^2} dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$

Solution:

Let $I = \int \sqrt{x^2 - a^2} dx$

$I = \int \sqrt{x^2 - a^2}.1. dx$

$$I=\sqrt{x^2-a^2}.\int dx-\int\left[\frac{d}{dx}\left(\sqrt{x^2-a^2}\right)\int dx\right]dx$$

$$I=x\sqrt{x^2-a^2}-\int\left[\frac{2x}{2\sqrt{x^2-a^2}}x\right]dx$$

$$I=x\sqrt{x^2-a^2}-\int\left[\frac{x^2}{\sqrt{x^2-a^2}}\right]dx$$

$$I=x\sqrt{x^2-a^2}-\int\left[\frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}\right]dx$$

$$I=x\sqrt{x^2-a^2}-\int\frac{x^2-a^2}{\sqrt{x^2-a^2}}dx+a^2\int\frac{dx}{\sqrt{x^2-a^2}}$$

$$I=x\sqrt{x^2-a^2}-\int\sqrt{x^2-a^2}dx+a^2\int\frac{dx}{\sqrt{x^2-a^2}}$$

$$I=x\sqrt{x^2-a^2}-I+a^2\int\frac{dx}{\sqrt{x^2-a^2}}$$

$$2I=x\sqrt{x^2-a^2}+a^2\log\left|x+\sqrt{x^2-a^2}\right|+C\prime$$

$$I=\frac{x\sqrt{x^2-a^2}}{2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+\frac{C'}{2}$$

$$I=\frac{x\sqrt{x^2-a^2}}{2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$$

Question 6.2.3: A random variable X has the following probability distribution: **[4]**

X=x	0	1	2	3	4	5	6
P[X=x]	k	3k	5k	7k	9k	11k	13k

(a) Find k
(b) find P(O <X< 4)
(c) Obtain cumulative distribution function (c. d. f.) of X.

Solution: sum P(x)=1

P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)=1

K+3x +5k +7k +9k +11k +13k=1

49k=1

k=1/49

P(0<x<4)=P(1)+P(2)+p(3)

=3k+5k+7k

=15k

=15/49

$$F(0) = P(0) = \frac{1}{49}$$

$$F(1) = P(0) + P(1) = \frac{1}{49} + \frac{3}{49} = \frac{4}{49}$$

$$F(2) = P(0) + P(1) + P(2) = \frac{1}{49} + \frac{3}{49} + \frac{5}{49} = \frac{9}{49}$$

$$F(3) = P(0) + P(1) + P(2) + P(3) = \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49} = \frac{16}{49}$$

$$F(4) = P(0) + P(1) + P(2) + P(3) + P(4) = \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49} + \frac{9}{49} = \frac{25}{49}$$

$$F(5) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5) = \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49} + \frac{9}{49} + \frac{11}{49} = \frac{36}{49}$$

$$F(6) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49} + \frac{9}{49} + \frac{11}{49} + \frac{13}{49} = \frac{49}{49}$$

∴ Cumulative distribution function (c.d.f.) of x

X	0	1	2	3	4	5	6
F(x)	1/49	4/49	9/49	16/46	25/49	36/49	1