

Pair of Straight Lines

EXERCISE 4.1 [PAGES 119 - 120]

Exercise 4.1 | Q 1.1 | Page 119

Find the combined equation of the following pair of line:

$$2x + y = 0 \text{ and } 3x - y = 0$$

Solution: The combined equation of the lines $2x + y = 0$ and $3x - y = 0$ is

$$(2x + y)(3x - y) = 0$$

$$\therefore 6x^2 - 2xy + 3xy - y^2 = 0$$

$$\therefore 6x^2 + xy - y^2 = 0$$

Exercise 4.1 | Q 1.2 | Page 119

Find the combined equation of the following pair of line:

$$x + 2y - 1 = 0 \text{ and } x - 3y + 2 = 0$$

Solution: The combined equation of the lines $x + 2y - 1 = 0$ and $x - 3y + 2 = 0$ is

$$(x + 2y - 1)(x - 3y + 2) = 0$$

$$\therefore x^2 - 3xy + 2x + 2xy - 6y^2 + 4y - x + 3y - 2 = 0$$

$$\therefore x^2 - xy - 6y^2 + x + 7y - 2 = 0$$

Exercise 4.1 | Q 1.3 | Page 119

Find the combined equation of the following pair of line:

passing through (2, 3) and parallel to the coordinate axes.

Solution: Equations of the coordinate axes are $x = 0$ and $y = 0$

\therefore The equations of the lines passing through (2, 3) and parallel to the coordinate axes are $x = 2$ and $y = 3$.

i.e. $x - 2 = 0$ and $y - 3 = 0$

\therefore their combined equation is

$$(x - 2)(y - 3) = 0$$

$$\therefore xy - 3x - 2y + 6 = 0$$

Exercise 4.1 | Q 1.4 | Page 119

Find the combined equation of the following pair of line:

passing through (2, 3) and perpendicular to the lines $3x + 2y - 1 = 0$ and $x - 3y + 2 = 0$

Solution: Let L_1 and L_2 be the lines passing through the point (2, 3) and perpendicular to the lines $3x + 2y - 1 = 0$ and $x - 3y + 2 = 0$ respectively.

Slopes of the lines $3x + 2y - 1 = 0$ and $x - 3y + 2 = 0$ are $\frac{-3}{2}$ and $\frac{-1}{-3} = \frac{1}{3}$ respectively.

\therefore slopes of the lines L_1 and L_2 pass through the point (2, 3), their equations are

$$y - 3 = \frac{2}{3}(x - 2) \text{ and } y - 3 = -3(x - 2)$$

$$\therefore 3y - 9 = 2x - 4 \text{ and } y - 3 = -3x + 6$$

$$\therefore 2x - 3y + 5 = 0 \text{ and } 3x + y - 9 = 0$$

their combined equation is

$$(2x - 3y + 5)(3x + y - 9) = 0$$

$$\therefore 6x^2 + 2xy - 18x - 9xy - 3y^2 + 27y + 15x + 5y - 45 = 0$$

$$\therefore 6x^2 - 7xy - 3y^2 - 3x + 32y - 45 = 0$$

Exercise 4.1 | Q 1.5 | Page 119

Find the combined equation of the following pair of line:

passing through (-1, 2), one is parallel to $x + 3y - 1 = 0$ and other is perpendicular to $2x - 3y - 1 = 0$

Solution: Let L_1 be the line passing through the point (-1, 2) and parallel to the line $x + 3y - 1 = 0$ whose slope is $-\frac{1}{3}$.

\therefore slope of the line L_1 is $-\frac{1}{3}$

\therefore equation of the line L_1 is

$$y - 2 = -\frac{1}{3}(x + 1)$$

$$\therefore 3y - 6 = -x - 1$$

$$\therefore x + 3y - 5 = 0$$

Let L_2 be the line passing through (-1, 2) and perpendicular to the line $2x - 3y - 1 = 0$ whose slope is

$$\frac{-2}{-3} = \frac{2}{3}$$

\therefore slope of the line L_2 is $-\frac{1}{3}$

\therefore equation of the line L_2 is

$$y - 2 = -\frac{1}{3}(x + 1)$$

$$\therefore 2y - 4 = -3x - 3$$

$$\therefore 3x + 2y - 1 = 0$$

Hence, the equations of the required lines are

$$x + 3y - 5 = 0 \text{ and } 3x + 2y - 1 = 0$$

\therefore their combined equation is

$$(x + 3y - 5)(3x + 2y - 1) = 0$$

$$\therefore 3x^2 + 2xy - x + 9xy + 6y^2 - 3y - 15x - 10y + 5 = 0$$

$$\therefore 3x^2 + 11xy + 6y^2 - 16x - 13y + 5 = 0$$

Exercise 4.1 | Q 2.1 | Page 119

Find the separate equation of the line represented by the following equation:

$$3y^2 + 7xy = 0$$

Solution: $3y^2 + 7xy = 0$

$$\therefore y(3y + 7x) = 0$$

\therefore the separate equations of the lines are $y = 0$ and $7x + 3y = 0$

Exercise 4.1 | Q 2.2 | Page 119

Find the separate equation of the line represented by the following equation:

$$5y^2 + 9y^2 = 0$$

Solution:

$$5y^2 + 9y^2 = 0$$

$$\therefore (\sqrt{5x})^2 - (\sqrt{3y})^2 = 0$$

$$\therefore (\sqrt{5x} + 3y)(\sqrt{5x} - 3y) = 0$$

the separate equations of the lines are $(\sqrt{5x} + 3y) = 0$ and $(\sqrt{5x} - 3y) = 0$

Exercise 4.1 | Q 2.3 | Page 119

Find the separate equation of the line represented by the following equation:

$$x^2 - 4xy = 0$$

Solution: $x^2 - 4xy = 0$

$\therefore x(x - 4y) = 0$

\therefore the separate equations of the lines are $x = 0$ and $x - 4y = 0$

Exercise 4.1 | Q 2.4 | Page 119

Find the separate equations of the lines represented by the equation $3x^2 - 10xy - 8y^2 = 0$

Solution:

Given pairs of lines $3x^2 - 10xy - 8y^2 = 0$

$$3x^2 - 12xy + 2xy - 8y^2 = 0$$

$$3x(x - 4y) + 2y(x - 4y) = 0$$

$$(x - 4y)(3x + 2y) = 0$$

Separated equations

$$3x + 2y = 0 \quad \text{and} \quad x - 4y = 0$$

Exercise 4.1 | Q 2.5 | Page 119

Find the separate equation of the line represented by the following equation:

$$3x^2 - 2\sqrt{3}xy - 3y^2 = 0$$

Solution:

$$3x^2 - 2\sqrt{3}xy - 3y^2 = 0$$

$$\therefore 3x^2 - 3\sqrt{3}xy + \sqrt{3}xy - 3y^2 = 0$$

$$\therefore 3x(x - \sqrt{3}y) + \sqrt{3}y(x - \sqrt{3}y) = 0$$

$$\therefore (x - \sqrt{3}y)(3x + \sqrt{3}y) = 0$$

The separate equations of the lines are

$$x - \sqrt{3}y = 0 \quad \text{and} \quad 3x + \sqrt{3}y = 0$$

Exercise 4.1 | Q 2.6 | Page 119

Find the separate equation of the line represented by the following equation:

$$x^2 + 2(\operatorname{cosec} \alpha)xy + y^2 = 0$$

Solution: $x^2 + 2(\operatorname{cosec} \alpha)xy + y^2 = 0$

i.e. $y^2 + 2(\operatorname{cosec} \alpha)xy + x^2 = 0$

Dividing by x^2 , we get,

$$\left(\frac{y}{x}\right)^2 + 2\operatorname{cosec}\alpha \cdot \left(\frac{y}{x}\right) + 1 = 0$$

$$\therefore \frac{y}{x} = \frac{-2\operatorname{cosec}\alpha \pm \sqrt{4\operatorname{cosec}^2\alpha - 4 \times 1 \times 1}}{2 \times 1}$$

$$= \frac{-2\operatorname{cosec}\alpha \pm 2\sqrt{\operatorname{cosec}^2\alpha - 1}}{2}$$

$$= -\operatorname{cosec} \alpha \pm \cot \alpha$$

$$\therefore \frac{y}{x} = (\cot \alpha - \operatorname{cosec} \alpha) \text{ and}$$

$$\frac{y}{x} = -(\operatorname{cosec} \alpha + \cot \alpha)$$

The separate equations of the lines are

$$(\operatorname{cosec} \alpha - \cot \alpha) x + y = 0 \text{ and } (\operatorname{cosec} \alpha + \cot \alpha) x + y = 0$$

Exercise 4.1 | Q 2.7 | Page 119

Find the separate equation of the line represented by the following equation:

$$x^2 + 2xy \tan \alpha - y^2 = 0$$

Solution:

$$x^2 + 2xy \tan \alpha - y^2 = 0$$

Dividing by y^2

$$\left(\frac{x}{y}\right)^2 + 2\left(\frac{x}{y}\right)\tan\alpha - 1 = 0$$

$$\begin{aligned} \therefore \frac{x}{y} &= \frac{-2\tan\alpha \pm \sqrt{4\tan^2\alpha - 4 \times 1 \times 1}}{2 \times 1} \\ &= \frac{-2\tan\alpha \pm 2\sqrt{\tan^2\alpha - 1}}{2} \end{aligned}$$

$$= -\tan\alpha \pm \sec\alpha$$

$$\left(\frac{x}{y}\right) = (\sec\alpha - \tan\alpha) \text{ and}$$

$$\left(\frac{x}{y}\right) = -(\tan\alpha + \sec\alpha)$$

The separate equations of the lines are

$$(\sec\alpha - \tan\alpha)x + y = 0 \text{ and } (\sec\alpha + \tan\alpha)x - y = 0$$

Exercise 4.1 | Q 3.1 | Page 119

Find the combined equation of the pair of a line passing through the origin and perpendicular to the line represented by following equation:

$$5x^2 - 8xy + 3y^2 = 0$$

Solution: Comparing the equation $5x^2 - 8xy + 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$$a = 5, 2h = -8, b = 3$$

Let m_1 and m_2 be the slopes of the lines represented by $5x^2 - 8xy + 3y^2 = 0$

$$\therefore m_1 + m_2 = \frac{-2h}{b} = \frac{8}{3} \text{ and } m_1m_2 = \frac{a}{b} = \frac{5}{3} \dots(1)$$

Now required lines are perpendicular to these lines

$$\therefore \text{their slopes are } \frac{-1}{m_1} \text{ and } \frac{-1}{m_2}$$

Since these lines are passing through the origin, their separate equations are

$$y = \frac{-1}{m_1}x \text{ and } y = \frac{-1}{m_2}x$$

$$\text{i.e. } m_1y = -x \text{ and } m_2y = -x$$

$$\text{i.e. } x + m_1y = 0 \text{ and } x + m_2y = 0$$

\therefore their combined equation is

$$(x + m_1y)(x + m_2y) = 0$$

$$\therefore x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$$

$$\therefore x^2 + \frac{8}{3}xy + \frac{5}{3}y^2 = 0 \text{ ...[By (1)]}$$

$$\therefore 3x^2 + 8xy + 5y^2 = 0$$

Exercise 4.1 | Q 3.2 | Page 119

Find the combined equation of the pair of a line passing through the origin and perpendicular to the line represented by the following equation:

$$5x^2 + 2xy - 3y^2 = 0$$

Solution: Comparing the equation $5x^2 + 2xy - 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$$a = 5, 2h = 2, b = -3$$

Let m_1 and m_2 be the slopes of the lines represented by $5x^2 + 2xy - 3y^2 = 0$

$$\therefore m_1 + m_2 = \frac{-2h}{b} = \frac{-2}{-3} = \frac{2}{3} \text{ and } m_1m_2 = \frac{a}{b} = \frac{5}{-3} \text{(1)}$$

Now required lines are perpendicular to these lines

$$\therefore \text{their slopes are } \frac{-1}{m_1} \text{ and } \frac{-1}{m_2}$$

Since these lines are passing through the origin, their separate equations are

$$y = \frac{-1}{m_1}x \text{ and } y = \frac{-1}{m_2}x$$

$$\text{i.e. } m_1y = -x \text{ and } m_2y = -x$$

$$\text{i.e. } x + m_1y = 0 \text{ and } x + m_2y = 0$$

\therefore their combined equation is

$$(x + m_1y)(x + m_2y) = 0$$

$$\therefore x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$$

$$\therefore x^2 + \frac{2}{3}xy - \frac{5}{3}y^2 = 0 \text{ ...[By (1)]}$$

$$\therefore 3x^2 + 2xy - 5y^2 = 0$$

Exercise 4.1 | Q 3.3 | Page 119

Find the combined equation of the pair of a line passing through the origin and perpendicular to the line represented by the following equation:

$$xy + y^2 = 0$$

Solution: Comparing the equation $xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$$a = 0, 2h = 1, b = 1$$

Let m_1 and m_2 be the slopes of the lines represented by $xy + y^2 = 0$

$$\therefore m_1 + m_2 = \frac{-2h}{b} = \frac{-1}{1} = -1 \text{ and } m_1m_2 = \frac{a}{b} = \frac{0}{1} = 0 \text{(1)}$$

Now required lines are perpendicular to these lines

$$\therefore \text{their slopes are } \frac{-1}{m_1} \text{ and } \frac{-1}{m_2}$$

Since these lines are passing through the origin, their separate equations are

$$y = \frac{-1}{m_1}x \text{ and } y = \frac{-1}{m_2}x$$

$$\text{i.e. } m_1y = -x \text{ and } m_2y = -x$$

$$\text{i.e. } x + m_1y = 0 \text{ and } x + m_2y = 0$$

∴ their combined equation is

$$(x + m_1y)(x + m_2y) = 0$$

$$\therefore x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$$

$$\therefore x^2 - xy + 0 \cdot y^2 = 0 \dots[\text{By (1)}]$$

$$\therefore x^2 - xy = 0$$

[Note: : Answer in the textbook is incorrect.]

Exercise 4.1 | Q 3.4 | Page 119

Find the combined equation of the pair of a line passing through the origin and perpendicular to the line represented by the following equation:

$$3x^2 - 4xy = 0$$

Solution: Consider $3x^2 - 4xy = 0$

$$\therefore x(3x - 4y) = 0$$

∴ separate equations of the lines are $x = 0$ and $3x - 4y = 0$

Let m_1 and m_2 be the slopes of these lines.

Then m_1 does not exist and $m_2 = \frac{3}{4}$

Now, required lines are perpendicular to these lines.

∴ their slopes are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$

Since m_1 does not exist, $-\frac{1}{m_1} = 0$

Also, $m_2 = \frac{3}{4}$, $-\frac{1}{m_2} = -\frac{4}{3}$

Since these lines are passing through the origin, their

separate equations are $y = 0$ and $y = -\frac{4}{3}x$, i.e. $4x + 3y = 0$

∴ their combined equation is

$$y(4x + 3y) = 0$$

$$\therefore 4xy + 3y^2 = 0$$

Exercise 4.1 | Q 4.1 | Page 119

Find k , if the sum of the slopes of the lines represented by $x^2 + kxy - 3y^2 = 0$ is twice their product.

Solution: Comparing the equation $x^2 + kxy - 3y^2 = 0$ with $ax^2 + 2hxy - by^2 = 0$, we get, $a = 1$, $2h = k$, $b = -3$.

Let m_1 and m_2 be the slopes of the lines represented by $x^2 + kxy - 3y^2 = 0$

$$\therefore m_1 + m_2 = \frac{-2h}{b} = -\frac{k}{-3} = \frac{k}{3}$$

$$\text{and } m_1m_2 = \frac{a}{b} = \frac{1}{-3} = -\frac{1}{3}$$

Now, $m_1 + m_2 = 2(m_1m_2)$...(given)

$$\therefore \frac{k}{3} = 2\left(-\frac{1}{3}\right)$$

$$\therefore k = -2.$$

Exercise 4.1 | Q 4.2 | Page 119

Find k , the slopes of the lines represented by $3x^2 + kxy - y^2 = 0$ differ by 4.

Solution: Comparing the equation $3x^2 + kxy - y^2 = 0$ with $ax^2 + 2hxy - by^2 = 0$, we get, $a = 3$, $2h = k$, $b = -1$.

Let m_1 and m_2 be the slopes of the lines represented by $3x^2 + kxy - y^2 = 0$

$$\therefore m_1 + m_2 = \frac{-2h}{b} = -\frac{k}{-1} = k$$

$$\text{and } m_1m_2 = \frac{a}{b} = \frac{3}{-1} = -3$$

$$\therefore (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$$

$$= k^2 - 4(-3)$$

$$= k^2 + 12 \quad \dots(1)$$

But $|m_1 - m_2| = 4$

$$\therefore (m_1 - m_2)^2 = 16 \quad \dots(2)$$

$$\therefore \text{from (1) and (2), } k^2 + 12 = 16$$

$$\therefore k^2 = 4$$

$$\therefore k = \pm 2$$

Exercise 4.1 | Q 4.3 | Page 119

Find k , the slope of one of the lines given by $kx^2 + 4xy - y^2 = 0$ exceeds the slope of the other by 8.

Solution: Comparing the equation $kx^2 + 4xy - y^2 = 0$ with $ax^2 + 2hxy - by^2 = 0$, we get, $a = k$, $2h = 4$, $b = -1$.

Let m_1 and m_2 be the slopes of the lines represented by $kx^2 + 4xy - y^2 = 0$

$$\therefore m_1 + m_2 = \frac{-2h}{b} = -\frac{4}{-1} = 4$$

$$\text{and } m_1 m_2 = \frac{a}{b} = \frac{k}{-1} = -k$$

We are given that $m_2 = m_1 + 8$

$$\therefore m_1 + m_1 + 8 = 4$$

$$\therefore 2m_1 = -4 \quad \therefore m_1 = -2 \quad \dots(1)$$

Also, $m_1(m_1 + 8) = -k$

$$(-2)(-2 + 8) = -k \quad \dots[\text{By (1)}]$$

$$\therefore (-2)(6) = -k$$

$$\therefore -12 = -k$$

$$\therefore k = 12$$

Exercise 4.1 | Q 5.1 | Page 120

Find the condition that the line $4x + 5y = 0$ coincides with one of the lines given by $ax^2 + 2hxy + by^2 = 0$

Solution: The auxiliary equation of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is $bm^2 + 2hm + a = 0$

Given that $4x + 5y = 0$ is one of the lines represented by $ax^2 + 2hxy + by^2 = 0$

The slope of the line $4x + 5y = 0$ is $-\frac{4}{5}$

$\therefore m = -\frac{4}{5}$ is a root of the auxiliary equation $bm^2 + 2hm + a = 0$

$$\therefore b\left(-\frac{4}{5}\right)^2 + 2h\left(-\frac{4}{5}\right) + a = 0$$

$$\therefore \frac{16b}{25} - \frac{8h}{5} + a = 0$$

$$\therefore 16b - 40h + 25a = 0$$

$$\therefore 25a + 16b = 40h$$

This is the required condition.

Exercise 4.1 | Q 5.2 | Page 120

Find the condition that the line $3x + y = 0$ may be perpendicular to one of the lines given by $ax^2 + 2hxy + by^2 = 0$

Solution: The auxiliary equation of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is $bm^2 + 2hm + a = 0$

Since one line is perpendicular to the line $3x + y = 0$ whose slope is $-\frac{3}{1} = -3$

$$\therefore \text{slope of that line} = m = \frac{1}{3}$$

$\therefore m = \frac{1}{3}$ is the root of the auxiliary equation $bm^2 + 2hm + a = 0$.

$$\therefore b\left(\frac{1}{3}\right)^2 + 2h\left(\frac{1}{3}\right) + a = 0$$

$$\therefore \frac{b}{9} + \frac{2h}{3} + a = 0$$

$$\therefore b + 6h + 9a = 0$$

$$\therefore 9a + b + 6h = 0$$

This is the required condition.

Exercise 4.1 | Q 6 | Page 120

If one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is perpendicular to $px + qy = 0$, show that $ap^2 + 2hpq + bq^2 = 0$.

Solution: To prove: $ap^2 + 2hpq + bq^2 = 0$.

Let the slope of the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ be m_1 and m_2

$$\text{Then, } m_1 + m_2 = \frac{-2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

$$\text{Slope of the line } px + qy = 0 \text{ is } \frac{-p}{q}$$

But one of the lines of $ax^2 + 2hxy + by^2 = 0$ is perpendicular to $px + qy = 0$

$$\Rightarrow m_1 = \frac{q}{p}$$

$$\text{Now, } m_1 + m_2 = \frac{-2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

$$\Rightarrow \frac{q}{p} + m_2 = \frac{-2h}{b} \text{ and } \left(\frac{q}{p}\right)m_2 = \frac{a}{b}$$

$$\Rightarrow \frac{q}{p} + m_2 = \frac{-2h}{b} \text{ and } m_2 = \frac{ap}{bq}$$

$$\Rightarrow \frac{q}{p} + \frac{ap}{bq} = \frac{-2h}{b}$$

$$\Rightarrow \frac{bq^2 + ap^2}{pq} = -2h$$

$$\Rightarrow bq^2 + ap^2 = -2h pq$$

$$\Rightarrow ap^2 + 2hpq + bq^2 = 0$$

Exercise 4.1 | Q 7 | Page 120

Find the combined equation of the pair of lines through the origin and making an equilateral triangle with the line $y = 3$.

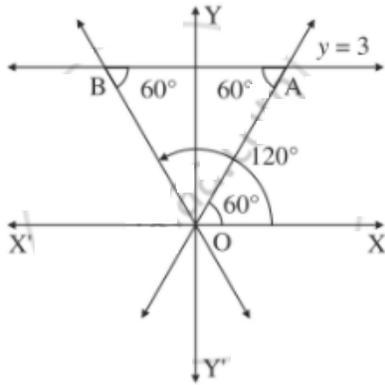
Solution: Let OA and OB be the lines through the origin making an angle of 60° with the line $y = 3$.

\therefore OA and OB make an angle of 60° and 120° with the positive direction of the X-axis.

$$\therefore \text{slope of OA} = \tan 60^\circ = \sqrt{3}$$

\therefore equation of the line OA is

$$y = \sqrt{3}x \text{ i.e. } \sqrt{3}x - y = 0$$



$$\text{Slope of OB} = \tan 120^\circ = \tan (180^\circ - 60^\circ)$$

$$= -\tan 60^\circ = -\sqrt{3}$$

\therefore equation of the line OB is

$$y = -\sqrt{3}x, \text{ i.e. } \sqrt{3}x + y = 0$$

\therefore required joint equation of the lines is

$$(\sqrt{3}x - y)(\sqrt{3}x + y) = 0$$

$$\text{i.e. } 3x^2 - y^2 = 0$$

Exercise 4.1 | Q 8 | Page 120

If the slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is four times the other, show that $16h^2 = 25ab$.

Solution: Let m_1 and m_2 be the slopes of the lines given by $ax^2 + 2hxy + by^2 = 0$

$$\therefore m_1 + m_2 = -\frac{2h}{b}$$

$$\text{and } m_1 m_2 = \frac{a}{b}$$

We are given that $m_2 = 4m_1$

$$\therefore m_1 + 4m_1 = -\frac{2h}{b}$$

$$\therefore 5m_1 = \frac{-2h}{b}$$

$$\therefore m_1 = -\frac{2h}{5b} \quad \dots(1)$$

$$\text{Also, } m_1(4m_1) = \frac{a}{b}$$

$$\therefore 4m_1^2 = \frac{a}{b}$$

$$\therefore m_1^2 = \frac{a}{4b}$$

$$\therefore \left(-\frac{2h}{5b}\right)^2 = \frac{a}{4b} \quad \dots[\text{By(1)}]$$

$$\therefore \frac{4h^2}{25b^2} = \frac{a}{4b}$$

$$\therefore \frac{4h^2}{25b} = \frac{a}{4}, \text{ as } b \neq 0$$

$$\therefore 16h^2 = 25ab$$

This is the required condition.

If one of the lines given by $ax^2 + 2hxy + by^2 = 0$ bisect an angle between the coordinate axes, then show that $(a + b)^2 = 4h^2$.

Solution: The auxiliary equation of the lines given by $ax^2 + 2hxy + by^2 = 0$ is $bm^2 + 2hm + a = 0$.

Since one of the lines bisects an angle between the coordinate axes, that line makes an angle of 45° or 135° with the positive direction of X-axis.

\therefore the slope of that line = $\tan 45^\circ$ or $\tan 135^\circ$

$\therefore m = \tan 45^\circ = 1$

or $m = \tan 135^\circ = \tan (180^\circ - 45^\circ)$

$= -\tan 45^\circ = -1$

$\therefore m = \pm 1$ are the roots of the auxiliary equation $bm^2 + 2hm + a = 0$.

$\therefore b(\pm 1)^2 + 2h(\pm 1) + a = 0$

$\therefore b \pm 2h + a = 0$

$\therefore a + b = \pm 2h$

$\therefore (a + b)^2 = 4h^2$

This is the required condition.

EXERCISE 4.2 [PAGE 124]

Exercise 4.2 | Q 1 | Page 124

. Show that the lines represented by $3x^2 - 4xy - 3y^2 = 0$ are perpendicular to each other.

Solution: Comparing the equation $3x^2 - 4xy - 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get, $a = 3$, $2h = -4$, $b = -3$.

Since $a + b = 3 + (-3) = 0$, the lines represented by $3x^2 - 4xy - 3y^2 = 0$ are perpendicular to each other.

Exercise 4.2 | Q 2 | Page 124

Show that the lines represented by $x^2 + 6xy + 9y^2 = 0$ are coincident.

Solution: Comparing the equation $x^2 + 6xy + 9y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get, $a = 1$, $2h = 6$ i.e. $h = 3$ and $b = 9$.

Since $h^2 - ab = (3)^2 - 1(9)$
 $= 9 - 9 = 0$,

the lines represented by $x^2 + 6xy + 9y^2 = 0$ are coincident.

Exercise 4.2 | Q 3 | Page 124

Find the value of k if lines represented by $kx^2 + 4xy - 4y^2 = 0$ are perpendicular to each other.

Solution: Comparing the equation $kx^2 + 4xy - 4y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get, $a = k$, $2h = 4$ and $b = -4$.

Since lines represented by $kx^2 + 4xy - 4y^2 = 0$ are perpendicular to each other,

$$a + b = 0$$

$$\therefore k - 4 = 0$$

$$\therefore k = 4$$

Exercise 4.2 | Q 4.1 | Page 124

Find the measure of the acute angle between the line represented by:

$$3x^2 - 4\sqrt{3}xy + 3y^2 = 0$$

Solution:

Comparing the equation

$$3x^2 - 4\sqrt{3}xy + 3y^2 = 0 \text{ with}$$

$$ax^2 + 2hxy + by^2 = 0, \text{ we get,}$$

$$a = 3, 2h = -4\sqrt{3} \text{ i.e. } h = -2\sqrt{3} \text{ and } b = 3$$

Let θ be the acute angle between the lines.

$$\begin{aligned} \therefore \tan \theta &= \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| \\ &= \left| \frac{2\sqrt{(-2\sqrt{3})^2 - 3(3)}}{3 + 3} \right| \end{aligned}$$

$$= \left| \frac{2\sqrt{12-9}}{6} \right|$$

$$= \left| \frac{2\sqrt{3}}{6} \right|$$

$$\therefore \tan \theta = \frac{1}{\sqrt{3}} = \tan 30^\circ$$

$$\therefore \theta = 30^\circ$$

Exercise 4.2 | Q 4.2 | Page 124

Find the measure of the acute angle between the line represented by:

$$4x^2 + 5xy + y^2 = 0$$

Solution: Comparing the equation

$$4x^2 + 5xy + y^2 = 0 \text{ with}$$

$$ax^2 + 2hxy + by^2 = 0, \text{ we get,}$$

$$a = 4, 2h = 5 \text{ i.e. } h = 5/2 \text{ and } b = 1$$

Let θ be the acute angle between the lines.

$$\therefore \tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

$$= \left| \frac{2\sqrt{\left(\frac{5}{2}\right)^2 - 4(1)}}{4 + 1} \right|$$

$$= \left| \frac{2\sqrt{\left(\frac{25}{4}\right) - 4}}{5} \right|$$

$$= \left| \frac{2 \times \frac{3}{2}}{5} \right|$$

$$\therefore \tan \theta = \frac{3}{5}$$

$$\therefore \theta = \tan^{-1} \left(\frac{3}{5} \right)$$

Exercise 4.2 | Q 4.3 | Page 124

Find the measure of the acute angle between the line represented by:

$$2x^2 + 7xy + 3y^2 = 0$$

Solution: Comparing the equation

$$2x^2 + 7xy + 3y^2 = 0 \text{ with}$$

$$ax^2 + 2hxy + by^2 = 0, \text{ we get,}$$

$$a = 2, 2h = 7 \text{ i.e. } h = 7/2 \text{ and } b = 3$$

Let θ be the acute angle between the lines.

$$\therefore \tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

$$= \left| \frac{2\sqrt{\left(\frac{7}{2}\right)^2 - 2(3)}}{2 + 3} \right|$$

$$= \left| \frac{2\sqrt{\left(\frac{49}{4}\right) - 6}}{5} \right|$$

$$= \left| \frac{2\sqrt{\left(\frac{49-24}{4}\right)}}{5} \right|$$

$$= \left| \frac{2\sqrt{\left(\frac{25}{4}\right)}}{5} \right|$$

$$= \frac{2 \times \left(\frac{5}{2}\right)}{5}$$

$$= \frac{5}{5}$$

$$\tan \theta = 1$$

$$\therefore \theta = \tan^{-1} 1 = 45^\circ$$

$$\therefore \theta = 45^\circ$$

Exercise 4.2 | Q 4.4 | Page 124

Find the measure of the acute angle between the line represented by:

$$(a^2 + 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$$

Solution: Comparing the equation

$$(a^2 + 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0 \text{ with}$$

$Ax^2 + 2Hxy + By^2 = 0$, we have,

$$A = a^2 + 3b^2, H = 4ab \text{ and } B = b^2 - 3a^2$$

$$\therefore H^2 - AB = 16a^2b^2 - (a^2 - 3b^2)(b^2 - 3a^2)$$

$$= 16a^2b^2 + (a^2 - 3b^2)(3a^2 - b^2)$$

$$= 16a^2b^2 + 3a^4 - 10a^2b^2 + 3b^4$$

$$= 3a^4 + 6a^2b^2 + 3b^4$$

$$= 3(a^4 + 2a^2b^2 + b^4)$$

$$= 3(a^2 + b^2)^2$$

$$\therefore \sqrt{H^2 - AB} = \sqrt{3}(a^2 + b^2)$$

$$\text{Also, } A + B = (a^2 - 3b^2) + (b^2 - 3a^2)$$

$$= -2(a^2 + b^2)$$

Let θ be the acute angle between the lines, then

$$\therefore \tan \theta = \left| \frac{2\sqrt{H^2 - AB}}{A + B} \right|$$

$$= \left| \frac{2\sqrt{3}(a^2 + b^2)}{-2(a^2 + b^2)} \right|$$

$$= \sqrt{3} = \tan 60^\circ$$

$$\therefore \theta = 60^\circ$$

Exercise 4.2 | Q 5 | Page 124

Find the combined equation of lines passing through the origin each of which making an angle of 30° with the line $3x + 2y - 11 = 0$

Solution: The slope of the line $3x + 2y - 11 = 0$ is $m_1 = -3/2$

Let m be the slope of one of the lines making an angle of 30° with the line $3x + 2y - 11 = 0$

The angle between the lines having slopes m and m_1 is 30° .

$$\therefore \tan 30^\circ = \left| \frac{m - m_1}{1 + m \cdot m_1} \right|, \text{ where } \tan 30^\circ = \frac{1}{\sqrt{30}}$$

$$\therefore \frac{1}{\sqrt{30}} = \left| \frac{m - \left(-\frac{3}{2}\right)}{1 + m\left(-\frac{3}{2}\right)} \right|$$

$$\therefore \frac{1}{\sqrt{30}} = \left| \frac{2m + 3}{2 - 3m} \right|$$

On squaring both sides, we get,

$$\frac{1}{3} = \frac{(2m + 3)^2}{(2 - 3m)^2}$$

$$\therefore (2 - 3m)^2 = 3(2m + 3)^2$$

$$\therefore 4 - 12m + 9m^2 = 3(4m^2 + 12m + 9)$$

$$\therefore 4 - 12m + 9m^2 = 12m^2 + 36m + 27$$

$$\therefore 3m^2 + 48m + 23 = 0$$

This is the auxiliary equation of the two lines and their joint equation is obtained by putting $m = y/x$

\therefore the combined equation of the two lines is

$$3\left(\frac{y}{x}\right)^2 + 48\left(\frac{y}{x}\right) + 23 = 0$$

$$\therefore \frac{3y^2}{x^2} + \frac{48y}{x} + 23 = 0$$

$$\therefore 3y^2 + 48xy + 23x^2 = 0$$

$$\therefore 23x^2 + 48xy + 3y^2 = 0$$

Exercise 4.2 | Q 6 | Page 124

If the angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$ is equal to the angle between the lines $2x^2 - 5xy + 3y^2 = 0$, then show that $100(h^2 - ab) = (a + b)^2$.

Solution:

The acute angle θ between the lines $ax^2 + 2hxy + by^2 = 0$ is given by

$$\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| \dots(1)$$

Comparing the equation $2x^2 - 5xy + 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get.

$$a = 2, 2h = -5, \text{ i.e. } h = -\frac{5}{2} \text{ and } b = 3$$

Let α be the acute angle between the lines $2x^2 - 5xy + 3y^2 = 0$

$$\therefore \tan \alpha = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

$$= \left| \frac{2\sqrt{\left(\frac{5}{2}\right)^2 - 2(3)}}{2 + 3} \right|$$

$$= \left| \frac{\left(2\frac{\sqrt{25}}{4} - 6\right)}{5} \right|$$

$$= \left| \frac{2 \times \frac{1}{2}}{5} \right|$$

$$\therefore \tan \alpha = \frac{1}{5} \quad \dots(2)$$

If $\theta = \alpha$, then $\tan \theta = \tan \alpha$

$$\therefore \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| = \frac{1}{5} \quad \dots[\text{By (1) and (2)}]$$

$$\therefore \frac{4(h^2 - ab)}{(a + b)^2} = \frac{1}{25}$$

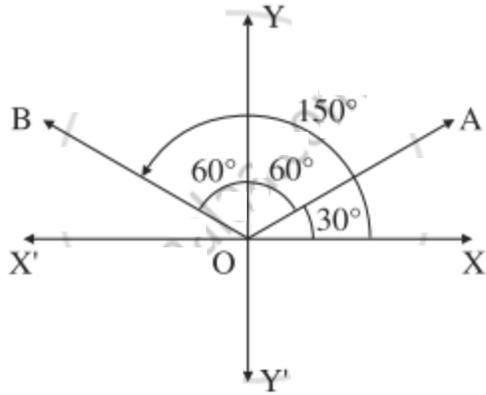
$$\therefore 100(h^2 - ab) = (a + b)^2$$

This is the required condition.

Exercise 4.2 | Q 7 | Page 124

Find the combined equation of lines passing through the origin and each of which making an angle of 60° with the Y-axis.

Solution:



Let OA and OB be the lines through the origin making an angle of 60° with the Y-axis.
Then OA and OB make an angle of 30° and 150° with the positive direction of X-axis.

$$\therefore \text{slope of OA} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

\therefore equation of the line OA is

$$y = \frac{1}{\sqrt{3}}x \text{ i.e. } x - \sqrt{3}y = 0$$

Slope of OB = $\tan 150^\circ = \tan (180^\circ - 30^\circ)$

$$= -\tan 30^\circ = -\frac{1}{\sqrt{3}}$$

\therefore equation of the line OB is

$$y = -\frac{1}{\sqrt{3}}x \text{ i.e. } x + \sqrt{3}y = 0$$

\therefore required combined equation is

$$(x - \sqrt{3}y)(x + \sqrt{3}y) = 0$$

$$\text{i.e. } x^2 - 3y^2 = 0$$

EXERCISE 4.3 [PAGES 127 - 128]

Exercise 4.3 | Q 1.1 | Page 127

Find the joint equation of the pair of the line through the point (2, -1) and parallel to the lines represented by $2x^2 + 3xy - 9y^2 = 0$.

Solution: The combined equation of the given lines is

$$2x^2 + 3xy - 9y^2 = 0$$

$$\text{i.e. } 2x^2 + 6xy - 3xy - 9y^2 = 0$$

$$\text{i.e. } 2x(x + 3y) - 3y(x + 3y) = 0$$

$$\text{i.e. } (x + 3y)(2x - 3y) = 0$$

\therefore their separate equations are

$$x + 3y = 0 \text{ and } 2x - 3y = 0$$

$$\therefore \text{ their slopes are } m_1 = \frac{-1}{3} \text{ and } m_2 = \frac{-2}{-3} = \frac{2}{3}$$

The slopes of the lines parallel to these lines are m_1 and m_2 i.e. $-\frac{1}{3}$ and $\frac{2}{3}$.

\therefore the equations of the lines with these slopes and through the point (2, -1) are

$$y + 1 = -\frac{1}{3}(x - 2) \text{ and } y + 1 = \frac{2}{3}(x - 2)$$

$$\text{i.e. } 3y + 3 = -x + 2 \text{ and } 3y + 3 = 2x - 4$$

$$\text{i.e. } x + 3y + 1 = 0 \text{ and } 2x - 3y - 7 = 0$$

\therefore the joint equation of these lines is

$$(x + 3y + 1)(2x - 3y - 7) = 0$$

$$\therefore 2x^2 - 3xy - 7x + 6xy - 9y^2 - 21y + 2x - 3y - 7 = 0$$

$$\therefore 2x^2 + 3xy - 9y^2 - 5x - 24y - 7 = 0$$

Exercise 4.3 | Q 1.2 | Page 127

Find the joint equation of the pair of the line through the point (2, -3) and parallel to the lines represented by $x^2 + xy - y^2 = 0$.

Solution: The combined equation of the given lines is

$$x^2 + xy - y^2 = 0 \quad \dots(1)$$

with $ax^2 + 2hxy + by^2 = 0$, we get,

$$a = 1, 2h = 1, b = -1$$

Let m_1 and m_2 be the slopes of the lines represented by (1).

$$\text{Then } m_1 + m_2 = -\frac{2h}{b} = \frac{-1}{-1} = 1 \text{ and } m_1 m_2 = \frac{a}{b} = \frac{1}{-1} = -1 \quad \dots(2)$$

The slopes of the lines parallel to these lines are m_1 and m_2 .

\therefore the equations of the lines with these slopes and through the point (2, -3) are

$$y + 3 = m_1(x - 2) \text{ and } y + 3 = m_2(x - 2)$$

$$\text{i.e. } m_1(x - 2) - (y + 3) = 0 \text{ and } m_2(x - 2) - (y + 3) = 0$$

\therefore the joint equation of these lines is

$$[m_1(x - 2) - (y + 3)][m_2(x - 2) - (y + 3)] = 0$$

$$\therefore m_1 m_2 (x - 2)^2 - m_1(x - 2)(y + 3) - m_2(x - 2)(y + 3) + (y + 3)^2 = 0$$

$$\therefore m_1 m_2 (x - 2)^2 - (m_1 + m_2)(x - 2)(y + 3) + (y + 3)^2 = 0$$

$$\therefore -(x - 2)^2 - (x - 2)(y + 3) + (y + 3)^2 = 0 \quad \dots[\text{By (2)}]$$

$$\therefore (x - 2)^2 + (x - 2)(y + 3) - (y + 3)^2 = 0$$

$$\therefore (x^2 - 4x + 4) + (xy + 3x - 2y - 6) - (y^2 + 6y + 9) = 0$$

$$\therefore x^2 - 4x + 4 + xy + 3x - 2y - 6 - y^2 - 6y - 9 = 0$$

$$\therefore x^2 + xy - y^2 - x - 8y - 11 = 0$$

Exercise 4.3 | Q 2 | Page 127

Show that the equation $x^2 + 2xy + 2y^2 + 2x + 2y + 1 = 0$ does not represent a pair of lines.

Solution: Comparing the equation

$$x^2 + 2xy + 2y^2 + 2x + 2y + 1 = 0 \text{ with}$$

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \text{ we get,}$$

$$a = 1, h = 1, b = 2, g = 1, f = 1, c = 1.$$

The given equation represents a pair of lines, if

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \text{ and } h^2 - ab \geq 0$$

$$\text{Now, } D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(2 - 1) - 1(1 - 1) + 1(1 - 2)$$

$$= 1 - 0 - 1 = 0$$

$$\text{and } h^2 - ab = (1)^2 - 1(2) = -1 < 0$$

\therefore given equation does not represent a pair of lines.

Exercise 4.3 | Q 3 | Page 127

Show that the equation $2x^2 - xy - 3y^2 - 6x + 19y - 20 = 0$ represents a pair of lines.

Solution: Comparing the equation

$$2x^2 - xy - 3y^2 - 6x + 19y - 20 = 0$$

with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,

$$a = 2, h = -\frac{1}{2}, b = -3, f = \frac{19}{2} \text{ and } c = -20$$

$$\therefore D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -\frac{1}{2} & -3 \\ -\frac{1}{2} & -3 & \frac{19}{2} \\ -3 & \frac{19}{2} & -20 \end{vmatrix}$$

Taking $\frac{1}{2}$ common from each row, we get,

$$\begin{aligned}
D &= \frac{1}{8} \begin{vmatrix} 4 & -1 & -6 \\ -1 & -6 & 19 \\ -6 & 19 & -40 \end{vmatrix} \\
&= \frac{1}{8} [4(240 - 361) + 1(40 + 114) - 6(-19 - 36)] \\
&= \frac{1}{8} [4(-121) + 154 - 6(-55)] \\
&= \frac{11}{8} [4(-11) + 14 - 6(-5)] \\
&= \frac{11}{8} (-44 + 14 + 30) = 0
\end{aligned}$$

$$\text{Also, } h^2 - ab = \left(-\frac{1}{2}\right)^2 - 2(-3) = \frac{1}{4} + 6 = \frac{25}{4} > 0$$

\therefore the given equation represents a pair of lines.

Exercise 4.3 | Q 4 | Page 127

Show that the equation $2x^2 + xy - y^2 + x + 4y - 3 = 0$ represents a pair of lines. Also, find the acute angle between them.

Solution: Comparing the equation

$$2x^2 + xy - y^2 + x + 4y - 3 = 0 \text{ with}$$

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get,

$$a = 2, h = \frac{1}{2}, b = -1, g = \frac{1}{2}, f = 2, c = -3.$$

$$\therefore D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 2 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & 2 \\ \frac{1}{2} & 2 & -3 \end{vmatrix}$$

Taking $\frac{1}{2}$ common from each row, we get,

$$\begin{aligned}
 D &= \frac{1}{8} \begin{vmatrix} 4 & 1 & 1 \\ 1 & -2 & 4 \\ 1 & 4 & -6 \end{vmatrix} \\
 &= \frac{1}{8} [4(12 - 16) - 1(-6 - 4) + 1(4 + 2)] \\
 &= \frac{1}{8} [4(-4) - 1(-10) + 1(6)] \\
 &= \frac{1}{8} (-16 + 10 + 6) = 0
 \end{aligned}$$

$$\text{Also, } h^2 - ab = \left(\frac{1}{2}\right)^2 - 2(-1) = \frac{1}{4} + 2 = \frac{9}{4} > 0$$

\therefore the given equation represents a pair of lines.

Let θ be the acute angle between the lines

$$\begin{aligned}
 \therefore \tan \theta &= \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| \\
 &= \left| \frac{2\sqrt{\left(\frac{1}{2}\right)^2 - 2(-1)}}{2 - 1} \right| \\
 &= \left| \frac{2\sqrt{\frac{1}{4} + 2}}{1} \right| \\
 &= 2 \times \frac{3}{2} = 3
 \end{aligned}$$

$$\therefore \theta = \tan^{-1}(3)$$

Find the separate equation of the line represented by the following equation:

$$(x - 2)^2 - 3(x - 2)(y + 1) + 2(y + 1)^2 = 0$$

Solution: $(x - 2)^2 - 3(x - 2)(y + 1) + 2(y + 1)^2 = 0$

$$\therefore (x - 2)^2 - 2(x - 2)(y + 1) - (x - 2)(y + 1) + 2(y + 1)^2 = 0$$

$$\therefore (x - 2)[(x - 2) - 2(y + 1)] - (y + 1)[(x - 2) - 2(y + 1)] = 0$$

$$\therefore (x - 2)(x - 2 - 2y - 2) - (y + 1)(x - 2 - 2y - 2) = 0$$

$$\therefore (x - 2)(x - 2y - 4) - (y + 1)(x - 2y - 4) = 0$$

$$\therefore (x - 2y - 4)(x - 2 - y - 1) = 0$$

$$\therefore (x - 2y - 4)(x - y - 3) = 0$$

\therefore the separate equations of the lines are

$$x - 2y - 4 = 0 \text{ and } x - y - 3 = 0$$

Alternative Method:

$$(x - 2)^2 - 3(x - 2)(y + 1) + 2(y + 1)^2 = 0 \quad \dots(1)$$

Put $x - 2 = X$ and $y + 1 = Y$

\therefore (1) becomes,

$$X^2 - 3XY + 2Y^2 = 0$$

$$\therefore X^2 - 2XY - XY + 2Y^2 = 0$$

$$\therefore X(X - 2Y) - Y(X - 2Y) = 0$$

$$\therefore (X - 2Y)(X - Y) = 0$$

\therefore the separate equations of the lines are

$$X - 2Y = 0 \text{ and } X - Y = 0$$

$$\therefore (x - 2) - 2(y + 1) = 0 \text{ and } (x - 2) - (y + 1) = 0$$

$$\therefore x - 2y - 4 = 0 \text{ and } x - y - 3 = 0$$

Exercise 4.3 | Q 5.2 | Page 127

Find the separate equation of the line represented by the following equation:

$$10(x + 1)^2 + (x + 1)(y - 2) - 3(y - 2)^2 = 0$$

Solution: $10(x + 1)^2 + (x + 1)(y - 2) - 3(y - 2)^2 = 0 \quad \dots(1)$

Put $x + 1 = X$ and $y - 2 = Y$

\therefore (1) becomes

$$10X^2 + XY - 3Y^2 = 0$$

$$10X^2 + 6XY - 5XY - 3Y^2 = 0$$

$$2X(5X + 3Y) - Y(5X + 3Y) = 0$$

$$(2X - Y)(5X + 3Y) = 0$$

$$5X + 3Y = 0 \quad \text{and} \quad 2X - Y = 0$$

$$5X + 3Y = 0$$

$$5(x + 1) + 3(y - 2) = 0$$

$$5x + 5 + 3y - 6 = 0$$

$$\therefore \mathbf{5x + 3y - 1 = 0}$$

$$2X - Y = 0$$

$$2(x + 1) - (y - 2) = 0$$

$$2x + 2 - y + 2 = 0$$

$$\therefore \mathbf{2x - y + 4 = 0}$$

Exercise 4.3 | Q 6.1 | Page 127

Find the value of k, if the following equations represent a pair of line:

$$3x^2 + 10xy + 3y^2 + 16y + k = 0$$

Solution: Comparing the given equation with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

we get, $a = 3$, $h = 5$, $b = 3$, $g = 0$, $f = 8$, $c = k$.

Now, given equation represents a pair of lines.

$$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\therefore (3)(3)(k) + 2(8)(0)(5) - 3(8)^2 - 3(0)^2 - k(5)^2 = 0$$

$$\therefore 9k + 0 - 192 - 0 - 25k = 0$$

$$\therefore -16k - 192 = 0$$

$$\therefore -16k = 192$$

$$\therefore k = -12$$

Exercise 4.3 | Q 6.2 | Page 127

Find the value of k, if the following equations represent a pair of line:

$$kxy + 10x + 6y + 4 = 0$$

Solution: Comparing the given equation with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

we get, $a = 0$, $h = k/2$, $b = 0$, $g = 5$, $f = 3$, $c = 4$

Now, given equation represents a pair of lines.

$$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\therefore (0)(0)(4) + 2(3)(5)\left(\frac{k}{2}\right) - 0(3)^2 - 0(5)^2 - 4\left(\frac{k}{2}\right)^2 = 0$$

$$\therefore 0 + 15k - 0 - 0 - k^2 = 0$$

$$\therefore 15k - k^2 = 0$$

$$\therefore -k(k - 15) = 0$$

$$\therefore k = 0 \text{ or } k = 15$$

If $k = 0$, then the given equation becomes

$10x + 6y + 4 = 0$ which does not represent a pair of lines.

$$\therefore k \neq 0$$

Hence, $k = 15$.

Exercise 4.3 | Q 6.3 | Page 127

Find the value of k, if the following equations represent a pair of line:

$$x^2 + 3xy + 2y^2 + x - y + k = 0$$

Solution: Comparing the given equation with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

we get, $a = 1$, $h = \frac{3}{2}$, $b = 2$, $g = \frac{1}{2}$, $f = -\frac{1}{2}$, $c = k$.

Now, given equation represents a pair of lines.

$$\therefore \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\text{i.e. } \begin{vmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & k \end{vmatrix} = 0$$

Taking out $\frac{1}{2}$ common from each row, we get,

$$\frac{1}{8} \begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & -1 \\ 1 & -1 & 2k \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & -1 \\ 1 & -1 & 2k \end{vmatrix} = 0$$

$$\therefore 2(8k - 1) - 3(6k + 1) + 1(-3 - 4) = 0$$

$$\therefore 16k - 2 - 18k - 3 - 7 = 0$$

$$\therefore -2k - 12 = 0$$

$$\therefore -2k = 12 \quad \therefore k = -6$$

Exercise 4.3 | Q 7 | Page 128

Find p and q , if the equation $px^2 - 8xy + 3y^2 + 14x + 2y + q = 0$ represents a pair of perpendicular lines.

Solution: The given equation represents a pair of lines perpendicular to each other

$$\therefore (\text{coefficient of } x^2) + (\text{coefficient of } y^2) = 0$$

$$\therefore p + 3 = 0 \quad \therefore p = -3$$

With this value of p , the given equation is

$$-3x^2 - 8xy + 3y^2 + 14x + 2y + q = 0$$

Comparing this equation with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$a = -3, h = -4, b = 3, g = 7, f = 1 \text{ and } c = q$$

$$\therefore D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} -3 & -4 & 7 \\ -4 & 3 & 1 \\ 7 & 1 & q \end{vmatrix}$$

$$= -3(3q - 1) + 4(-4q - 7) + 7(-4 - 21)$$

$$= -9q + 3 - 16q - 28 - 175$$

$$= -25q - 200$$

$$= -25(q + 8)$$

Since the given equation represents a pair of lines, $D = 0$

$$\therefore -25(q + 8) = 0$$

$$\therefore q = -8$$

Hence, $p = -3$ and $q = -8$.

Exercise 4.3 | Q 8 | Page 128

Find p and q , if the equation $2x^2 + 8xy + py^2 + qx + 2y - 15 = 0$ represents a pair of parallel lines.

Solution: The given equation is $2x^2 + 8xy + py^2 + qx + 2y - 15 = 0$

Comparing it with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,

$$a = 2, h = 4, b = p, g = q/2, f = 1, c = -15$$

Since the lines are parallel, $h^2 = ab$

$$\therefore (4)^2 = 2p$$

$$\therefore p = 8$$

Since the given equation represents a pair of lines

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \text{ where } b = p = 8$$

$$\text{i.e. } \begin{vmatrix} 2 & 4 & \frac{q}{2} \\ 4 & 8 & 1 \\ \frac{q}{2} & 1 & -15 \end{vmatrix} = 0$$

$$\text{i.e. } 2(-120 - 1) - 4\left(-60 - \frac{q}{2}\right) + \frac{q}{2}(4 - 4q) = 0$$

$$\text{i.e. } -242 + 240 + 2q + 2q - 2q^2 = 0$$

$$\text{i.e. } -2q^2 + 4q - 2 = 0$$

$$\text{i.e. } q^2 - 2q + 1 = 0$$

$$\text{i.e. } (q - 1)^2 = 0$$

$$\therefore q - 1 = 0$$

$$\therefore q = 1$$

Hence, $p = 8$ and $q = 1$

Exercise 4.3 | Q 9 | Page 128

Equations of pairs of opposite sides of a parallelogram are $x^2 - 7x + 6 = 0$ and $y^2 - 14y + 40 = 0$. Find the joint equation of its diagonals.

Solution: Let ABCD be the parallelogram such that the combined equation of sides AB and CD is $x^2 - 7x + 6 = 0$ and the combined equation of sides BC and AD $y^2 - 14y + 40 = 0$

The separate equations of the lines represented by $x^2 - 7x + 6 = 0$, i.e. $(x - 1)(x - 6) = 0$ are $x - 1 = 0$ and $x - 6 = 0$

Let equation of the side AB be $x - 1 = 0$ and equation of side CD be $x - 6 = 0$

The separate equations of the lines represented by $y^2 - 14y + 40 = 0$, i.e. $(y - 4)(y - 10) = 0$ are $y - 4 = 0$ and $y - 10 = 0$

Let equation of the side BC be $y - 4 = 0$ and equation of side AD be $y - 10 = 0$

Coordinates of the vertices of the parallelogram are A(1, 10), B(1, 4), C(6, 4) and D(6, 10)

∴ equation of the diagonal AC is

$$\frac{y - 10}{x - 1} = \frac{10 - 4}{1 - 6} = \frac{6}{-5}$$

$$\therefore 5y + 50 = 6x - 6$$

$$\therefore 6x + 5y - 56 = 0$$

and equation of the diagonal BD is

$$\frac{y - 4}{x - 1} = \frac{4 - 10}{1 - 6} = \frac{-6}{-5} = \frac{6}{5}$$

$$\therefore 5y - 20 = 6x - 6$$

$$\therefore 6x - 5y + 14 = 0$$

Hence, the equations of the diagonals are $6x + 5y - 56 = 0$ and $6x - 5y + 14 = 0$

∴ the joint equation of the diagonals is

$$(6x + 5y - 56)(6x - 5y + 14) = 0$$

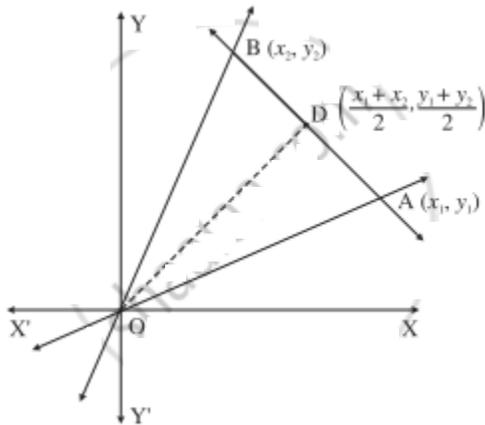
$$\therefore 36x^2 - 30xy + 84x + 30xy - 25y^2 + 70y - 336x + 280y - 784 = 0$$

$$\therefore 36x^2 - 25y^2 - 252x + 350y - 784 = 0$$

[Note: Answer in the textbook is incorrect]

ΔOAB is formed by the lines $x^2 - 4xy + y^2 = 0$ and the line AB . The equation of line AB is $2x + 3y - 1 = 0$. Find the equation of the median of the triangle drawn from O .

Solution:



Let D be the midpoint of seg AB where A is (x_1, y_1) and B is (x_2, y_2) .

Then D has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$.

The joint (combined) equation of the lines OA and OB is $x^2 - 4xy + y^2 = 0$ and the equation of the line AB is $2x + 3y - 1 = 0$

\therefore points A and B satisfy the equations $2x + 3y - 1 = 0$ and $x^2 - 4xy + y^2 = 0$ simultaneously.

We eliminate x from the above equations, i.e., put $x = \frac{1 - 3y}{2}$ in the equation $x^2 - 4xy + y^2 = 0$, we get,

$$\therefore \left(\frac{1-3y}{2}\right)^2 - 4\left(\frac{1-3y}{2}\right)y + y^2 = 0$$

$$\therefore (1-3y)^2 - 8(1-3y)y + 4y^2 = 0$$

$$\therefore 1 - 6y + 9y^2 - 8y + 24y^2 + 4y^2 = 0$$

$$\therefore 37y^2 - 14y + 1 = 0$$

The roots y_1 and y_2 of the above quadratic equation are the y-coordinates of the points A and B.

$$\therefore y_1 + y_2 = -\frac{b}{a} = \frac{14}{37}$$

$$\therefore \text{y-coordinate of D} = \frac{y_1 + y_2}{2} = \frac{7}{37}$$

Since D lies on the line AB, we can find the x-coordinate of D as

$$2x + 3\left(\frac{7}{37}\right) - 1 = 0$$

$$\therefore 2x = 1 - \frac{21}{37} = \frac{16}{37}$$

$$\therefore x = \frac{8}{37}$$

$$\therefore \text{D is } \left(\frac{8}{37}, \frac{7}{37}\right)$$

$$\therefore \text{equation of the median OD is } \frac{x}{8/37} = \frac{y}{7/37}$$

$$\text{i.e. } 7x - 8y = 0$$

Find the coordinates of the points of intersection of the lines represented by $x^2 - y^2 - 2x + 1 = 0$

Solution: Consider, $x^2 - y^2 - 2x + 1 = 0$

$$\therefore (x^2 - 2x + 1) - y^2 = 0$$

$$\therefore (x - 1)^2 - y^2 = 0$$

$$\therefore (x - 1 + y)(x - 1 - y) = 0$$

$$\therefore (x + y - 1)(x - y - 1) = 0$$

\therefore separate equations of the lines are

$$x + y - 1 = 0 \text{ and } x - y + 1 = 0$$

To find the point of intersection of the lines, we have to solve

$$x + y - 1 = 0 \quad \dots(1)$$

$$\text{and } x - y + 1 = 0 \quad \dots(2)$$

Adding (1) and (2), we get,

$$2x = 0$$

$$\therefore x = 0$$

Substituting $x = 0$ in (1), we get,

$$0 + y - 1 = 0$$

$$\therefore y = 1$$

\therefore coordinates of the point of intersection of the lines are (0, 1).

[**Note:** Answer in the textbook is incorrect.]

MISCELLANEOUS EXERCISE 4 [PAGES 129 - 130]

Miscellaneous Exercise 4 | Q 1.01 | Page 129

Choose correct alternatives:

If the equation $4x^2 + hxy + y^2 = 0$ represents two coincident lines, then $h = \underline{\hspace{2cm}}$

1. ± 2

2. ± 3

3. ± 4

4. ± 5

Solution: If the equation $4x^2 + hxy + y^2 = 0$ represents two coincident lines, then $h = \underline{\pm}$

4.

Miscellaneous Exercise 4 | Q 1.02 | Page 129

Choose correct alternatives:

If the lines represented by $kx^2 - 3xy + 6y^2 = 0$ are perpendicular to each other, then

1. $k = 6$
2. $k = -6$
3. $k = 3$
4. $k = -3$

Solution: $k = -6$

Miscellaneous Exercise 4 | Q 1.03 | Page 129

Choose correct alternatives:

Auxiliary equation of $2x^2 + 3xy - 9y^2 = 0$ is

1. $2m^2 + 3m - 9 = 0$
2. $9m^2 - 3m - 2 = 0$
3. $2m^2 - 3m + 9 = 0$
4. $-9m^2 - 3m + 2 = 0$

Solution: Auxiliary equation of $2x^2 + 3xy - 9y^2 = 0$ is $9m^2 - 3m - 2 = 0$.

Miscellaneous Exercise 4 | Q 1.04 | Page 129

Choose correct alternatives:

The difference between the slopes of the lines represented by $3x^2 - 4xy + y^2 = 0$ is 2

1. 2
2. 1
3. 3
4. 4

Solution: The difference between the slopes of the lines represented by $3x^2 - 4xy + y^2 = 0$ is 2.

Miscellaneous Exercise 4 | Q 1.05 | Page 129

Choose correct alternatives:

If two lines $ax^2 + 2hxy + by^2 = 0$ make angles α and β with X-axis, then $\tan(\alpha + \beta) =$ _____.

1. $h/a+b$
2. $h/a-b$
3. $2h/a+b$

4. $2h/a-b$

Solution:

If two lines $ax^2 + 2hxy + by^2 = 0$ make angles α and β with X-axis, then $\tan(\alpha + \beta) = \frac{2h}{a-b}$

Explanation:

$$m_1 = \tan \alpha, m_2 = \tan \beta$$

$$\begin{aligned} \therefore \tan(\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta} \\ &= \frac{m_1 + m_2}{1 - m_1 m_2} = \frac{-2h/b}{1 - (a/b)} = \frac{2h}{a-b} \end{aligned}$$

Miscellaneous Exercise 4 | Q 1.06 | Page 129

Choose correct alternatives:

If the slope of one of the two lines given by $\frac{x^2}{a} + \frac{2xy}{h} + \frac{y^2}{b} = 0$ is twice that of the other, then $ab : h^2 = \underline{\hspace{2cm}}$.

1. 1 : 2
2. 2 : 1
3. 8 : 9
4. 9 : 8

Solution:

If the slope of one of the two lines given by $\frac{x^2}{a} + \frac{2xy}{h} + \frac{y^2}{b} = 0$ is twice that of the other, then $ab : h^2 = \underline{\mathbf{9 : 8}}$.

Explanation:

$$m_1 + m_2 = \frac{-2b}{h} \text{ and } m_1 m_2 = \frac{b}{a}$$

where $m_1 = 2m_2$

$$\therefore 2m_2 + m_2 = -\frac{2b}{h} \text{ and } 2m_2 \times m_2 = \frac{b}{a}$$

$$\therefore m_2 = \frac{-2b}{3h} \text{ and } m_2^2 = \frac{b}{2a}$$

$$\therefore \left(\frac{-2b}{3h}\right)^2 = \frac{b}{2a}$$

$$\therefore \frac{4b^2}{9h^2} = \frac{b}{2a}$$

$$\therefore \frac{ab}{h^2} = \frac{9}{8}$$

Miscellaneous Exercise 4 | Q 1.07 | Page 130

Choose correct alternatives:

The joint equation of the lines through the origin and perpendicular to the pair of lines $3x^2 + 4xy - 5y^2 = 0$ is _____.

1. $5x^2 + 4xy - 3y^2 = 0$
2. $3x^2 + 4xy - 5y^2 = 0$
3. $3x^2 - 4xy + 5y^2 = 0$
4. $5x^2 + 4xy + 3y^2 = 0$

Solution: The joint equation of the lines through the origin and perpendicular to the pair of lines $3x^2 + 4xy - 5y^2 = 0$ is $5x^2 + 4xy - 3y^2 = 0$.

Miscellaneous Exercise 4 | Q 1.08 | Page 130

Choose correct alternatives:

If acute angle between lines $ax^2 + 2hxy + by^2 = 0$ is, $\pi/4$, then $4h^2 =$ _____.

1. $a^2 + 4ab + b^2$
2. $a^2 + 6ab + b^2$
3. $(a + 2b)(a + 3b)$
4. $(a - 2b)(2a + b)$

Solution:

If acute angle between lines $ax^2 + 2hxy + by^2 = 0$ is, $\frac{\pi}{4}$, then $4h^2 = \underline{a^2 + 6ab + b^2}$.

Miscellaneous Exercise 4 | Q 1.09 | Page 130

Choose correct alternatives:

If the equation $3x^2 - 8xy + qy^2 + 2x + 14y + p = 1$ represents a pair of perpendicular lines, then the values of p and q are respectively.

1. - 3 and - 7
2. - 7 and - 3
3. 3 and 7
4. - 7 and 3

Solution: - 7 and - 3

Miscellaneous Exercise 4 | Q 1.1 | Page 130

Choose correct alternatives:

The area of triangle formed by the lines $x^2 + 4xy + y^2 = 0$ and $x - y - 4 = 0$ is

1. $4/\sqrt{3}$ sq units
2. $8/\sqrt{3}$ sq units
3. $16/\sqrt{3}$ sq units
4. $15/\sqrt{3}$ sq units

Solution: The area of triangle formed by the lines $x^2 + 4xy + y^2 = 0$ and $x - y - 4 = 0$ is $8/\sqrt{3}$ sq units

Miscellaneous Exercise 4 | Q 1.11 | Page 130

Choose correct alternatives:

The combined equation of the coordinate axes is

1. $x + y = 0$
2. $xy = k$
3. $xy = 0$
4. $x - y = k$

Solution: The combined equation of the coordinate axes is $\underline{xy = 0}$.

Miscellaneous Exercise 4 | Q 1.12 | Page 130

Choose correct alternatives:

If $h^2 = ab$, then slopes of lines $ax^2 + 2hxy + by^2 = 0$ are in the ratio

1. 1:2
2. 2:1
3. 2:3
4. **1:1**

Solution: If $h^2 = ab$, then slopes of lines $ax^2 + 2hxy + by^2 = 0$ are in the ratio **1:1**.

Hint: If $h^2 = ab$, then lines are coincident. Therefore slopes of the lines are equal.

Miscellaneous Exercise 4 | Q 1.13 | Page 130

Choose correct alternatives:

If slope of one of the lines $ax^2 + 2hxy + by^2 = 0$ is 5 times the slope of the other, then $5h^2 =$ _____

1. ab
2. $2ab$
3. $7ab$
4. **$9ab$**

Solution: If slope of one of the lines $ax^2 + 2hxy + by^2 = 0$ is 5 times the slope of the other, then $5h^2 =$ **$9ab$** .

Miscellaneous Exercise 4 | Q 1.14 | Page 130

Choose correct alternatives:

If distance between lines $(x - 2y)^2 + k(x - 2y) = 0$ is 3 units, then $k =$ _____.

1. ± 3
2. $\pm 5\sqrt{5}$
3. 0
4. **$\pm 3\sqrt{5}$**

Solution: If distance between lines $(x - 2y)^2 + k(x - 2y) = 0$ is 3 units, then $k = \pm 3\sqrt{5}$

Explanation:

$$(x - 2y)^2 + k(x - 2y) = 0$$

$$\therefore (x - 2y)(x - 2y + k) = 0$$

\therefore equations of the lines are $x - 2y = 0$ and $x - 2y + k = 0$ which are parallel to each other.

$$\therefore \left| \frac{k - 0}{\sqrt{1 + 4}} \right| = 3$$

$$\therefore k = \pm 3\sqrt{5}$$

MISCELLANEOUS EXERCISE 4 [PAGES 130 - 132]

Miscellaneous Exercise 4 | Q 1.01 | Page 130

Find the joint equation of the line:

$$x - y = 0 \text{ and } x + y = 0$$

Solution: Find the joint equation of the line $x - y = 0$ and $x + y = 0$ is

$$(x - y)(x + y) = 0$$

$$\therefore x^2 - y^2 = 0$$

Miscellaneous Exercise 4 | Q 1.02 | Page 130

Find the joint equation of the line:

$$x + y - 3 = 0 \text{ and } 2x + y - 1 = 0$$

Solution: Find the joint equation of the line $x + y - 3 = 0$ and $2x + y - 1 = 0$

$$(x + y - 3)(2x + y - 1) = 0$$

$$\therefore 2x^2 + xy - x + 2xy + y^2 - y - 6x - 3y + 3 = 0$$

$$\therefore 2x^2 + 3xy + y^2 - 7x - 4y + 3 = 0$$

Miscellaneous Exercise 4 | Q 1.03 | Page 130

Find the joint equation of the line passing through the origin having slopes 2 and 3.

Solution: We know that the equation of the line passing through the origin and having slope m is $y = mx$. Equations of the lines passing through the origin and having slopes 2 and 3 are $y = 2x$ and $y = 3x$ respectively. i.e. their equations are

$$2x - y = 0 \text{ and } 3x - y = 0 \text{ respectively.}$$

\therefore their joint equation is

$$(2x - y)(3x - y) = 0$$

$$\therefore 6x^2 - 2xy - 3xy + y^2 = 0$$

$$\therefore 6x^2 - 5xy + y^2 = 0$$

Miscellaneous Exercise 4 | Q 1.04 | Page 130

Find the joint equation of the line passing through the origin and having inclinations 60° and 120° .

Solution: Slope of the line having inclination θ is $\tan \theta$.

Inclinations of the given lines are 60° and 120°

\therefore their slopes are $m_1 = \tan 60^\circ = \sqrt{3}$ and

$$m_2 = \tan 120^\circ = \tan (180^\circ - 60^\circ)$$

$$= -\tan 60^\circ = -\sqrt{3}.$$

Since the lines pass through the origin, their equations are

$$y = \sqrt{3}x \text{ and } y = -\sqrt{3}x$$

$$\text{i.e. } \sqrt{3}x - y = 0 \text{ and } \sqrt{3}x + y = 0$$

\therefore the joint equation of these lines is

$$\left(\sqrt{3}x - y\right)\left(\sqrt{3}x + y\right) = 0$$

$$\therefore 3x^2 - y^2 = 0$$

Miscellaneous Exercise 4 | Q 1.05 | Page 130

Find the joint equation of the line passing through $(1, 2)$ and parallel to the coordinate axes

Solution: Equations of the coordinate axes are $x = 0$ and $y = 0$

\therefore the equations of the lines passing through $(1, 2)$ and parallel to the coordinate axes are $x = 1$ and $y = 2$.

$$\text{i.e. } x - 1 = 0 \text{ and } y - 2 = 0$$

\therefore their combined equation is

$$(x - 1)(y - 2) = 0$$

$$\therefore x(y - 2) - 1(y - 2) = 0$$

$$\therefore xy - 2x - y + 2 = 0$$

Miscellaneous Exercise 4 | Q 1.06 | Page 130

Find the joint equation of the line passing through (3, 2) and parallel to the lines $x = 2$ and $y = 3$.

Solution: Equations of the lines passing through (3, 2) and parallel to the lines $x = 2$ and $y = 3$ are $x = 3$ and $y = 2$.

i.e. $x - 3 = 0$ and $y - 2 = 0$

∴ their joint equation is

$$(x - 3)(y - 2) = 0$$

$$\therefore xy - 2x - 3y + 6 = 0$$

Miscellaneous Exercise 4 | Q 1.07 | Page 131

Find the joint equation of the line passing through (-1, 2) and perpendicular to $x + 2y + 3 = 0$ and $3x - 4y - 5 = 0$

Solution: Let L_1 and L_2 be the lines passing through the origin and perpendicular to the lines $x + 2y + 3 = 0$ and $3x - 4y - 5 = 0$ respectively.

Slopes of the lines $x + 2y + 3 = 0$ and $3x - 4y - 5 = 0$ are $-\frac{1}{2}$ and $-\frac{3}{-4} = \frac{3}{4}$ respectively.

∴ slopes of the lines L_1 and L_2 are 2 and $\frac{-4}{3}$ respectively.

Since the lines L_1 and L_2 pass through the point (-1, 2), their equations are

$$\therefore (y - y_1) = m(x - x_1)$$

$$\therefore (y - 2) = 2(x + 1)$$

$$\Rightarrow y - 1 = 2x + 2$$

$$\Rightarrow 2x - y + 4 = 0 \text{ and}$$

$$\therefore (y - 2) = \left(\frac{-4}{3}\right)(x + 1)$$

$$\Rightarrow 3y - 6 = (-4)(x + 1)$$

$$\Rightarrow 3y - 6 = -4x - 4$$

$$\Rightarrow 4x + 3y - 6 + 4 = 0$$

$$\Rightarrow 4x + 3y - 2 = 0$$

their combined equation is

$$\therefore (2x - y + 4)(4x + 3y - 2) = 0$$

$$\therefore 8x^2 + 6xy - 4x - 4xy - 3y^2 + 2y + 16x + 12y - 8 = 0$$

$$\therefore 8x^2 + 2xy + 12x - 3y^2 + 14y - 8 = 0$$

Miscellaneous Exercise 4 | Q 1.08 | Page 131

Find the joint equation of the line passing through the origin and having slopes $1 + \sqrt{3}$ and $1 - \sqrt{3}$

Solution:

Let l_1 and l_2 be the two lines. Slopes of l_1 is $1 + \sqrt{3}$ and that of l_2 is $1 - \sqrt{3}$

Therefore the equation of a line (l_1) passing through the origin and having slope is

$$y = (1 + \sqrt{3})x$$

$$\therefore (1 + \sqrt{3})x - y = 0 \quad \dots(1)$$

Similarly, the equation of the line (l_2) passing through the origin and having slope is

$$y = (1 - \sqrt{3})x$$

$$\therefore (1 - \sqrt{3})x - y = 0 \dots(2)$$

From (1) and (2) the required combined equation is

$$\left[(1 + \sqrt{3})x - y \right] \left[(1 - \sqrt{3})x - y \right] = 0$$

$$\therefore (1 + \sqrt{3})x \left[(1 - \sqrt{3})x - y \right] - y \left[(1 - \sqrt{3})x - y \right] = 0$$

$$\therefore (1 - \sqrt{3})(1 + \sqrt{3})x^2 - (1 + \sqrt{3})xy - (1 - \sqrt{3})xy + y^2 = 0$$

$$\therefore \left((1)^2 - (\sqrt{3})^2 \right) x^2 - \left[(1 + \sqrt{3}) + (1 - \sqrt{3}) \right] xy + y^2 = 0$$

$$\therefore (1 - 3)x^2 - 2xy + y^2 = 0$$

$$\therefore -2x^2 - 2xy + y^2 = 0$$

$$\therefore 2x^2 + 2xy - y^2 = 0$$

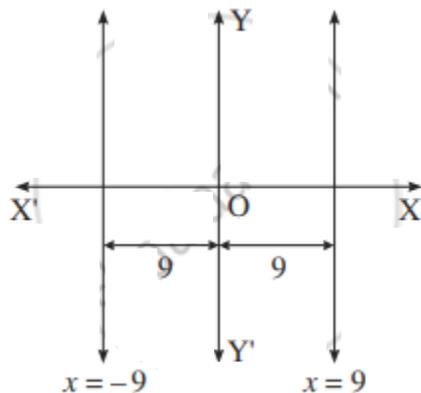
This is the required combined equation.

Miscellaneous Exercise 4 | Q 1.09 | Page 131

Find the joint equation of the line which are at a distance of 9 units from the Y-axis.

Solution: Equations of the lines, which are parallel to the Y-axis and at a distance of 9 units from it, are $x = 9$ and $x = -9$

i.e. $x - 9 = 0$ and $x + 9 = 0$



\therefore their combined equation is

$$(x - 9)(x + 9) = 0$$

$$\therefore x^2 - 81 = 0$$

Miscellaneous Exercise 4 | Q 1.1 | Page 131

Find the joint equation of the line passing through the point (3, 2), one of which is parallel to the line $x - 2y = 2$, and other is perpendicular to the line $y = 3$.

Solution:

Let L_1 be the line passes through (3, 2) and parallel to the line $x -$

$2y = 2$ whose slope is $\frac{-1}{-2} = \frac{1}{2}$

\therefore slope of the line L_1 is $\frac{1}{2}$

\therefore equation of the line L_2 is

$$y - 2 = \frac{1}{2}(x - 3)$$

$$\therefore 2y - 4 = x - 3$$

$$\therefore x - 2y + 1 = 0$$

Let L_2 be the line passes through (3, 2) and perpendicular to the line $y = 3$.

\therefore equation of the line L_2 is of the form $x = a$. Since L_2 passes through (3, 2), $3 = a$.

\therefore equation of the line L_2 is $x = 3$, i.e. $x - 3 = 0$

Hence, the equations of the required lines are

$$x - 2y + 1 = 0 \text{ and } x - 3 = 0$$

\therefore their joint equation is

$$(x - 2y + 1)(x - 3) = 0$$

$$\therefore x^2 - 2xy + x - 3x + 6y - 3 = 0$$

$$\therefore x^2 - 2xy - 2x + 6y - 3 = 0$$

Miscellaneous Exercise 4 | Q 1.11 | Page 131

Find the joint equation of the line passing through the origin and perpendicular to the lines $x + 2y = 19$ and $3x + y = 18$

Solution: Let L_1 and L_2 be the lines passing through the origin and perpendicular to the lines $x + 2y = 19$ and $3x + y = 18$ respectively.

Slopes of the lines $x + 2y = 19$ and $3x + y = 18$ are $-1/2$ and $-3/1 = -3$ respectively.

\therefore slopes of the lines L_1 and L_2 are 2 and $1/3$ respectively.

Since the lines L_1 and L_2 pass through the origin, their equations are

$$y = 2x \text{ and } y = 1/3 x$$

$$\text{i.e. } 2x - y = 0 \text{ and } x - 3y = 0$$

\therefore their combined equation is

$$(2x - y)(x - 3y) = 0$$

$$\therefore 2x^2 - 6xy - xy + 3y^2 = 0$$

$$\therefore 2x^2 - 7xy + 3y^2 = 0$$

Miscellaneous Exercise 4 | Q 2.1 | Page 131

Show that the following equations represents a pair of line:

$$x^2 + 2xy - y^2 = 0$$

Solution: Comparing the equation $x^2 + 2xy - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$$a = 1, 2h = 2 \text{ i.e, } h = 1, \text{ and } b = -1$$

$$\therefore h^2 - ab = (1)^2 - 1(-1) = 1 + 1 = 2 > 0$$

Since the equation $x^2 + 2xy - y^2 = 0$ is a homogeneous equation of second degree and $h^2 - ab > 0$, the given equation represents a pair of lines which are real and distinct.

Miscellaneous Exercise 4 | Q 2.2 | Page 131

Show that the following equations represents a pair of line:

$$4x^2 + 4xy + y^2 = 0$$

Solution: Comparing the equation $4x^2 + 4xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$$a = 4, 2h = 4 \text{ i.e, } h = 2, \text{ and } b = 1$$

$$\therefore h^2 - ab = (2)^2 - 4(1) = 4 - 4 = 0$$

Since the equation $4x^2 + 4xy + y^2 = 0$ is a homogeneous equation of second degree and $h^2 - ab = 0$, the given equation represents a pair of lines which are real and coincident.

Miscellaneous Exercise 4 | Q 2.3 | Page 131

Show that the following equations represent a pair of line:

$$x^2 - y^2 = 0$$

Solution: Comparing the equation $x^2 - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$$a = 1, 2h = 0 \text{ i.e, } h = 0, \text{ and } b = -1$$

$$\therefore h^2 - ab = (0)^2 - 1(-1) = 0 + 1 = 1 > 0$$

Since the equation $x^2 - y^2 = 0$ is a homogeneous equation of second degree and $h^2 - ab > 0$, the given equation represents a pair of lines which are real and distinct.

Miscellaneous Exercise 4 | Q 2.4 | Page 131

Show that the following equations represent a pair of line:

$$x^2 + 7xy - 2y^2 = 0$$

Solution: Comparing the equation $x^2 + 7xy - 2y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$$a = 1, 2h = 7 \text{ i.e, } h = \frac{7}{2}, \text{ and } b = -2$$

$$\therefore h^2 - ab = \left(\frac{7}{2}\right)^2 - 1(-2)$$

$$= \frac{49}{4} + 2$$

$$= \frac{57}{4} \text{ i.e. } 14.25 = 14 > 0$$

Since the equation $x^2 + 7xy - 2y^2 = 0$ is a homogeneous equation of second degree and $h^2 - ab > 0$, the given equation represents a pair of lines which are real and distinct.

Miscellaneous Exercise 4 | Q 2.5 | Page 131

Show that the following equations represent a pair of line:

$$x^2 - 2\sqrt{3}xy - y^2 = 0$$

Solution:

Comparing the equation $x^2 - 2\sqrt{3}xy - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$$a = 1, 2h = -2\sqrt{3} \text{ i.e, } h = -\sqrt{3}, \text{ and } b = 1$$

$$\therefore h^2 - ab = \left(-\sqrt{3}\right)^2 - 1 \quad (1)$$

$$= 3 - 1 = 2 > 0$$

Since the equation $x^2 - 2\sqrt{3}xy - y^2 = 0$ is a homogeneous equation of second degree and $h^2 - ab > 0$, the given equation represents a pair of lines which are real and distinct.

Miscellaneous Exercise 4 | Q 3.1 | Page 131

Find the separate equation of the line represented by the following equation:

$$6x^2 - 5xy - 6y^2 = 0$$

Solution: $6x^2 - 5xy - 6y^2 = 0$

$$\therefore 6x^2 - 9xy + 4xy - 6y^2 = 0$$

$$\therefore 3x(2x - 3y) + 2y(2x - 3y) = 0$$

$$\therefore (2x - 3y)(3x + 2y) = 0$$

the separate equations of the lines are

$$2x - 3y = 0 \text{ and } 3x + 2y = 0.$$

Miscellaneous Exercise 4 | Q 3.2 | Page 131

Find the separate equation of the line represented by the following equation:

$$x^2 - 4y^2 = 0$$

Solution: $x^2 - 4y^2 = 0$

$$\therefore x^2 - (2y)^2 = 0$$

$$\therefore (x - 2y)(x + 2y) = 0$$

the separate equations of the lines are

$$x - 2y = 0 \text{ and } x + 2y = 0$$

Miscellaneous Exercise 4 | Q 3.3 | Page 131

Find the separate equation of the line represented by the following equation:

$$3x^2 - y^2 = 0$$

Solution:

$$3x^2 - y^2 = 0$$

$$\therefore (\sqrt{3x})^2 - y^2 = 0$$

$$\therefore (\sqrt{3x} - y)(\sqrt{3x} + y) = 0$$

the separate equations of the lines are

$$\sqrt{3x} - y = 0 \text{ and } \sqrt{3x} + y = 0$$

Miscellaneous Exercise 4 | Q 3.4 | Page 131

Find the separate equation of the line represented by the following equation:

$$2x^2 + 2xy - y^2 = 0$$

Solution: $2x^2 + 2xy - y^2 = 0$

The auxiliary equation is $-m^2 + 2m + 2 = 0$

$$\therefore m^2 - 2m - 2 = 0$$

$$\therefore m = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2 \times 1}$$

$$= \frac{2 \pm \sqrt{4 + 8}}{2}$$

$$= \frac{2 \pm 2\sqrt{3}}{2}$$

$$= 1 \pm \sqrt{3}$$

$\therefore m_1 = 1 + \sqrt{3}$ and $m_2 = 1 - \sqrt{3}$ are the slopes of the lines.

\therefore their separate equations are

$$y = m_1x \text{ and } y = m_2x$$

$$\text{i.e. } y = (1 + \sqrt{3})x \text{ and } y = (1 - \sqrt{3})x$$

$$\text{i.e. } (\sqrt{3} + 1)x - y = 0 \text{ and } (\sqrt{3} - 1)x + y = 0$$

Miscellaneous Exercise 4 | Q 4.1 | Page 131

Find the joint equation of the pair of a line through the origin and perpendicular to the lines given by

$$x^2 + 4xy - 5y^2 = 0$$

Solution: Comparing the equation $x^2 + 4xy - 5y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
 $a = 1, 2h = 4, b = -5$

Let m_1 and m_2 be the slopes of the lines represented by $x^2 + 4xy - 5y^2 = 0$

$$\therefore m_1 + m_2 = \frac{-2h}{b} = \frac{4}{5} \text{ and } m_1m_2 = \frac{a}{b} = \frac{-1}{5} \quad \dots(1)$$

Now, required lines are perpendicular to these lines

$$\therefore \text{their slopes are } \frac{-1}{m_1} \text{ and } \frac{1}{m_2}$$

Since these lines are passing through the origin, their separate equations are

$$y = \frac{-1}{m_1}x \text{ and } y = \frac{1}{m_2}x$$

$$\text{i.e. } m_1y = -x \text{ and } m_2y = -x$$

$$\text{i.e. } x + m_1y = 0 \text{ and } x + m_2y = 0$$

\therefore their combined equation is

$$(x + m_1y)(x + m_2y) = 0$$

$$\therefore x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$$

$$\therefore x^2 + \frac{4}{5}xy - \frac{1}{5}y^2 = 0 \quad \dots[\text{By(1)}]$$

$$\therefore 5x^2 + 4xy - y^2 = 0$$

Miscellaneous Exercise 4 | Q 4.2 | Page 131

Find the joint equation of the pair of a line through the origin and perpendicular to the lines given by

$$2x^2 - 3xy - 9y^2 = 0$$

Solution: Comparing the equation $2x^2 - 3xy - 9y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
 $a = 2, 2h = -3, b = -9$

Let m_1 and m_2 be the slopes of the lines represented by $2x^2 - 3xy - 9y^2 = 0$

$$\therefore m_1 + m_2 = \frac{-2h}{b} = -\frac{3}{9} \text{ and } m_1m_2 = \frac{a}{b} = -\frac{2}{9} \quad \dots(1)$$

Now, required lines are perpendicular to these lines

$$\therefore \text{their slopes are } \frac{-1}{m_1} \text{ and } -\frac{1}{m_2}$$

Since these lines are passing through the origin, their separate equations are

$$y = \frac{-1}{m_1}x \text{ and } y = \frac{-1}{m_2}x$$

$$\text{i.e. } m_1y = -x \text{ and } m_2y = -x$$

$$\text{i.e. } x + m_1y = 0 \text{ and } x + m_2y = 0$$

\therefore their combined equation is

$$(x + m_1y)(x + m_2y) = 0$$

$$\therefore x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$$

$$\therefore x^2 + \left(-\frac{3}{9}\right)xy + \left(-\frac{2}{9}\right)y^2 = 0 \quad \dots[\text{By}(1)]$$

$$\therefore 9x^2 - 3xy - 2y^2 = 0$$

Miscellaneous Exercise 4 | Q 4.3 | Page 131

Find the joint equation of the pair of a line through the origin and perpendicular to the lines given by

$$x^2 + xy - y^2 = 0$$

Solution: Comparing the equation $x^2 + xy - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
 $a = 1, 2h = 1, b = -1$

Let m_1 and m_2 be the slopes of the lines represented by $x^2 + xy - y^2 = 0$

$$\therefore m_1 + m_2 = \frac{-2h}{b} = \frac{-1}{-1} = 1 \text{ and } m_1m_2 = \frac{a}{b} = \frac{1}{-1} = -1 \quad \dots(1)$$

Now, required lines are perpendicular to these lines

$$\therefore \text{their slopes are } \frac{-1}{m_1} \text{ and } -\frac{1}{m_2}$$

Since these lines are passing through the origin, their separate equations are

$$y = \frac{-1}{m_1}x \text{ and } y = \frac{-1}{m_2}x$$

$$\text{i.e. } m_1y = -x \text{ and } m_2y = -x$$

$$\text{i.e. } x + m_1y = 0 \text{ and } x + m_2y = 0$$

\therefore their combined equation is

$$(x + m_1y)(x + m_2y) = 0$$

$$\therefore x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$$

$$\therefore x^2 + 1xy + (-1)y^2 = 0 \quad \dots[\text{By(1)}]$$

$$\therefore x^2 + xy - y^2 = 0$$

Miscellaneous Exercise 4 | Q 5.1 | Page 131

Find k , if the sum of the slopes of the lines given by $3x^2 + kxy - y^2 = 0$ is zero.

Solution: Comparing the equation $3x^2 + kxy - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$$a = 3, 2h = k, b = -1$$

Let m_1 and m_2 be the slopes of the lines represented by $3x^2 + kxy - y^2 = 0$.

$$\therefore m_1 + m_2 = \frac{-2h}{b} = \frac{-k}{-1} = k$$

$$\text{Now, } m_1 + m_2 = 0 \quad \dots(\text{Given})$$

$$\therefore k = 0$$

Miscellaneous Exercise 4 | Q 5.2 | Page 131

Find k, if the sum of the slopes of the lines given by $x^2 + kxy - 3y^2 = 0$ is equal to their product.

Solution: Comparing the equation $x^2 + kxy - 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
 $a = 1, 2h = k, b = -3$

Let m_1 and m_2 be the slopes of the lines represented by $x^2 + kxy - 3y^2 = 0$.

$$\therefore m_1 + m_2 = \frac{-2h}{b} = \frac{-k}{-3} = \frac{k}{3}$$

$$m_1 m_2 = \frac{a}{b} = \frac{1}{-3} = \frac{-1}{3}$$

Now, $m_1 + m_2 = m_1 m_2$...(Given)

$$\therefore \frac{k}{3} = \frac{-1}{3}$$

$$\therefore k = -1.$$

Miscellaneous Exercise 4 | Q 5.3 | Page 131

Find k, if the slope of one of the lines given by $3x^2 - 4xy + ky^2 = 0$ is 1.

Solution: The auxiliary equation of the lines given by $3x^2 - 4xy + ky^2 = 0$ is $km^2 - 4m + 3 = 0$

Given, slope of one of the lines is 1.

$\therefore m = 1$ is the root of the auxiliary equation $km^2 - 4m + 3 = 0$

$$\therefore k(1)^2 - 4(1) + 3 = 0$$

$$\therefore k - 4 + 3 = 0$$

$$\therefore k = 1$$

Miscellaneous Exercise 4 | Q 5.4 | Page 131

Find k, if one of the lines given by $3x^2 - kxy + 5y^2 = 0$ is perpendicular to the line $5x + 3y = 0$.

Solution:

The auxiliary equation of the lines given by $3x^2 - kxy + 5y^2 = 0$ is $5m^2 - km + 3 = 0$

Now, one line is perpendicular to the line $5x + 3y = 0$, whose slope is $-\frac{5}{3}$

$$\therefore \text{slope of that line} = m = \frac{3}{5}$$

$\therefore m = \frac{3}{5}$ is the root of the auxiliary equation $5m^2 - km + 3 = 0$

$$\therefore 5\left(\frac{3}{5}\right)^2 - k\left(\frac{3}{5}\right) + 3 = 0$$

$$\therefore \frac{9}{5} - \frac{3k}{5} + 3 = 0$$

$$\therefore 9 - 3k + 15 = 0$$

$$\therefore 3k = 24$$

$$\therefore k = 8$$

Miscellaneous Exercise 4 | Q 5.5 | Page 131

Find k if the slope of one of the lines given by $3x^2 + 4xy + ky^2 = 0$ is three times the other.

Solution: $3x^2 + 4xy + ky^2 = 0$

\therefore divide by x^2

$$\frac{3x^2}{x^2} + \frac{4xy}{x^2} + \frac{ky^2}{x^2} = 0$$

$$3 + \frac{4y}{x} + \frac{ky^2}{x^2} = 0 \quad \dots(1)$$

$$\therefore y = mx$$

$$\therefore \frac{y}{x} = m$$

Put $\frac{y}{x} = m$ in equation (1)

Comparing the equation $km^2 + 4m + 3 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$$a = k, 2h = 4, b = 3$$

$$m_1 = 3m_2 \quad \dots(\text{given condition})$$

$$m_1 + m_2 = \frac{-2h}{k} = -\frac{4}{k}$$

$$m_1 m_2 = \frac{a}{b} = \frac{3}{k}$$

$$m_1 + m_2 = -\frac{4}{k}$$

$$4m_2 = -\frac{4}{k} \quad \dots(m_1 = 3m_2)$$

$$m_2 = -\frac{1}{k}$$

$$m_1 m_2 = \frac{3}{k}$$

$$3m_2^2 = \frac{3}{k} \quad \dots(m_1 = 3m_2)$$

$$3\left(-\frac{1}{k}\right)^2 = \frac{3}{k} \quad \dots(m_2 = -\frac{1}{k})$$

$$\frac{1}{k^2} = \frac{1}{k}$$

$$k^2 = k$$

$$k = 1 \text{ or } k = 0$$

Miscellaneous Exercise 4 | Q 5.6 | Page 131

Find k , if the slopes of lines given by $kx^2 + 5xy + y^2 = 0$ differ by 1.

Solution: Comparing the equation $kx^2 + 5xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$$a = k, 2h = 5 \text{ i.e. } h = \frac{5}{2}$$

$$m_1 + m_2 = \frac{-2h}{b} = -\frac{5}{1} = -5$$

$$\text{and } m_1 m_2 = \frac{a}{b} = \frac{k}{1} = k$$

the slope of the line differ by $(m_1 - m_2) = 1 \dots(1)$

$$\therefore (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$$

$$(m_1 - m_2)^2 = (-5)^2 - 4(k)$$

$$(m_1 - m_2)^2 = 25 - 4k$$

$$1 = 25 - 4k \quad \dots[\text{By}(1)]$$

$$4k = 24$$

$$k = 6$$

Miscellaneous Exercise 4 | Q 5.7 | Page 131

Find k , if one of the lines given by $6x^2 + kxy + y^2 = 0$ is $2x + y = 0$.

Solution: The auxiliary equation of the lines represented by $6x^2 + kxy + y^2 = 0$ is $m^2 + km + 6 = 0$

Since one of the line is $2x + y = 0$ whose slope is $m = -2$.

$\therefore m = -2$ is the root of the auxiliary equation $m^2 + km + 6 = 0$.

$$\therefore (-2)^2 + k(-2) + 6 = 0$$

$$\therefore 4 - 2k + 6 = 0$$

$$\therefore 2k = 10$$

$$\therefore k = 5$$

Miscellaneous Exercise 4 | Q 6 | Page 131

Find the joint equation of the pair of lines which bisect angles between the lines given

by $x^2 + 3xy + 2y^2 = 0$

Solution: $x^2 + 3xy + 2y^2 = 0$

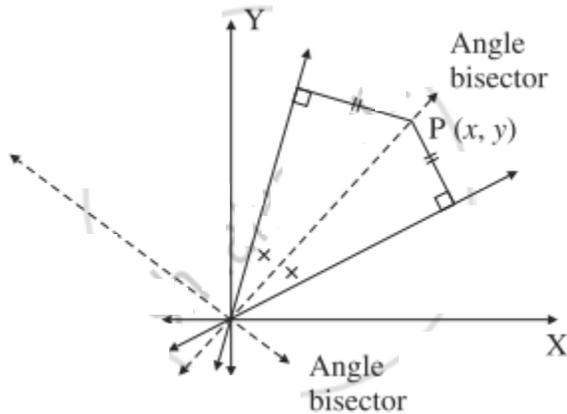
$$\therefore x^2 + 2xy + xy + 2y^2 = 0$$

$$\therefore x(x + 2y) + y(x + 2y) = 0$$

$$\therefore (x + 2y)(x + y) = 0$$

\therefore separate equations of the lines represented by $x^2 + 3xy + 2y^2 = 0$ are $x + 2y = 0$ and $x + y = 0$

Let $P(x, y)$ be any point on one of the angle bisector. Since the points on the angle bisectors are equidistant from both the lines,



the distance of $P(x, y)$ from the line $x + 2y = 0$

= the distance of $P(x, y)$ from the line $x + y = 0$

$$\therefore \left| \frac{x + 2y}{\sqrt{1 + 4}} \right| = \left| \frac{x + y}{\sqrt{1 + 1}} \right|$$

$$\therefore \frac{(x + 2y)^2}{5} = \frac{(x + y)^2}{2}$$

$$\therefore 2(x + 2y)^2 = 5(x + y)^2$$

$$\therefore 2(x^2 + 4xy + 4y^2) = 5(x^2 + 2xy + y^2)$$

$$\therefore 2x^2 + 8xy + 8y^2 = 5x^2 + 10xy + 5y^2$$

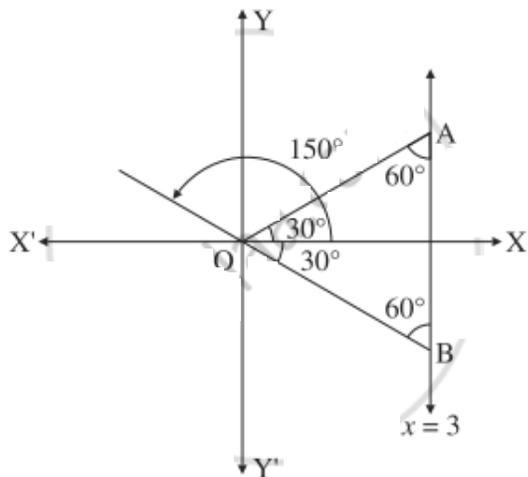
$$\therefore 3x^2 + 2xy - 3y^2 = 0$$

This is the required joint equation of the lines which bisect the angles between the lines represented by $x^2 + 3xy + 2y^2 = 0$

Miscellaneous Exercise 4 | Q 7 | Page 131

Find the joint equation of the pair of lines through the origin and making an equilateral triangle with the line $x = 3$.

Solution:



Let OA and OB be the lines through the origin making an angle of 60° with the line $x = 3$.

\therefore OA and OB make an angle of 30° and 150° with the positive direction of X-axis.

$$\therefore \text{slope of OA} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\therefore \text{equation of the line OA is } y = \frac{1}{\sqrt{3}}x$$

$$\therefore \sqrt{3}y = x$$

$$\therefore x - \sqrt{3}y = 0$$

Slope of OB = $\tan 150^\circ = \tan (180^\circ - 30^\circ)$

$$= -\tan 30^\circ = -\frac{1}{\sqrt{3}}$$

$$\therefore \text{equation of the line OB is } y = \frac{-1}{\sqrt{3}}x$$

$$\therefore \sqrt{3}y = -x$$

$$\therefore x + \sqrt{3}y = 0$$

\therefore required combined equation of the lines is

$$(x - \sqrt{3}y)(x + \sqrt{3}y) = 0$$

$$\text{i.e. } x^2 - 3y^2 = 0$$

Miscellaneous Exercise 4 | Q 8 | Page 131

Show that the lines $x^2 - 4xy + y^2 = 0$ and $x + y = 10$ contain the sides of an equilateral triangle. Find the area of the triangle.

Solution: We find the joint equation of the pair of lines OA and OB through origin, each making an angle of 60° with $x + y = 10$ whose slope is -1 .

Let OA(or OB) has slope m .

\therefore its equation is $y - mx \dots(1)$

$$\text{Also, } \tan 60^\circ = \left| \frac{m - (-1)}{1 + m(-1)} \right|$$

$$\therefore \sqrt{3} = \left| \frac{m + 1}{1 - m} \right|$$

Squaring both sides, we get,

$$3 = \frac{(m + 1)^2}{(1 - m)^2}$$

$$\therefore 3(1 - 2m + m^2) = m^2 + 2m + 1$$

$$\therefore 3 - 6m + 3m^2 = m^2 + 2m + 1$$

$$\therefore 2m^2 - 8m + 2 = 0$$

$$\therefore m^2 - 4m + 1 = 0$$

$$\therefore \left(\frac{y}{x}\right) - 4\left(\frac{y}{x}\right) + 1 = 0 \dots[\text{By}(1)]$$

$$\therefore y^2 - 4xy + x^2 = 0$$

$\therefore x^2 - 4xy + y^2 = 0$ is the joint equation of the two lines through the origin each making an angle of 60° with $x + y = 10$

$\therefore x^2 - 4xy + y^2 = 0$ and $x + y = 10$ form a triangle OAB which is equilateral.

Let seg OM perpendicular line AB whose equation is $x + y = 10$

$$\therefore OM = \left| \frac{-10}{\sqrt{1+1}} \right| = 5\sqrt{2}$$

$$\begin{aligned} \therefore \text{area of equilateral } \Delta OAB &= \frac{(OM)^2}{\sqrt{3}} = \frac{(5\sqrt{2})^2}{\sqrt{3}} \\ &= \frac{50}{\sqrt{3}} \text{ sq units.} \end{aligned}$$

Miscellaneous Exercise 4 | Q 9 | Page 131

If the slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is three times the other, prove that $3h^2 = 4ab$.

Solution: Let m_1 and m_2 be the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$

We are given that $m_2 = 3m_1$

$$\therefore m_1 + 3m_1 = -\frac{2h}{b}$$

$$\therefore 4m_1 = -\frac{2h}{b}$$

$$\therefore m_1 = -\frac{h}{2b}$$

$$\text{Also, } m_1(3m_1) = \frac{a}{b}$$

$$\therefore 3m_1^2 = \frac{a}{b}$$

$$\therefore 3\left(-\frac{h}{2b}\right)^2 = \frac{a}{b} \quad \dots[\text{By (1)}]$$

$$\therefore \frac{3h^2}{4b^2} = \frac{a}{b}$$

$$\therefore 3h^2 = 4ab, \text{ as } b \neq 0$$

Miscellaneous Exercise 4 | Q 10 | Page 132

Find the combined equation of bisectors of angles between the lines represented by $5x^2 + 6xy - y^2 = 0$.

Solution: Comparing the equation $5x^2 + 6xy - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$$a = 5, 2h = 6, b = -1$$

Let m_1 and m_2 be the slopes of the lines represented by $5x^2 + 6xy - y^2 = 0$.

$$\therefore m_1 + m_2 = \frac{-2h}{b} = \frac{-6}{-1} = 6 \text{ and } m_1 m_2 = \frac{a}{b} = \frac{5}{-1} = -5 \quad \dots(1)$$

The separate equations of the lines are

$$y = m_1 x \text{ and } y = m_2 x, \text{ where } m_1 \neq m_2$$

$$\text{i.e. } m_1 x - y = 0 \text{ and } m_2 x - y = 0.$$

Let $P(x, y)$ be any point on one of the bisector of the angles between the lines.

\therefore the distance of P from the line $m_1 x - y = 0$ is equal to the distance of P from the line $m_2 x - y = 0$.

$$\therefore \left| \frac{m_1 x - y}{\sqrt{m_1^2 + 1}} \right| = \left| \frac{m_2 x - y}{\sqrt{m_2^2 + 1}} \right|$$

Squaring both sides, we get,

$$\frac{(m_1 x - y)^2}{m_1^2 + 1} = \frac{(m_2 x - y)^2}{m_2^2 + 1}$$

$$\therefore (m_2^2 + 1)(m_1 x - y)^2 = (m_1^2 + 1)(m_2 x - y)^2$$

$$\therefore (m_2^2 + 1)(m_1^2 x^2 - 2m_1 xy + y^2) = (m_1^2 + 1)(m_2^2 x^2 - 2m_2 xy + y^2)$$

\therefore

$$m_1^2 m_2^2 x^2 - 2m_1 m_2^2 xy + m_2^2 y^2 + m_1^2 x^2 - 2m_1 xy + y^2 = m_1^2 m_2^2 x^2 - 2m_1^2 m_2 xy + m_1^2 y^2 + m_2^2 x^2 - 2m_2 xy + y^2$$

$$\therefore (m_1^2 - m_2^2)x^2 + 2m_1 m_2(m_1 - m_2)xy - 2(m_1 - m_2)xy - (m_1^2 - m_2^2)y^2 = 0$$

Dividing throughout by $m_1 - m_2 (\neq 0)$, we get,

$$(m_1 + m_2)x^2 + 2m_1 m_2 xy - 2xy - (m_1 + m_2)y^2 = 0$$

$$\therefore 6x^2 - 10xy - 2xy - 6y^2 = 0 \quad \dots[\text{By (1)}]$$

$$\therefore 6x^2 - 12xy - 6y^2 = 0$$

$$\therefore x^2 - 2xy - y^2 = 0$$

This is the joint equation of the bisectors of the angles between the lines represented by $5x^2 + 6xy - y^2 = 0$.

Miscellaneous Exercise 4 | Q 11 | Page 132

Find an if the sum of the slope of lines represented by $ax^2 + 8xy + 5y^2 = 0$ is twice their product.

Solution: Comparing the equation $ax^2 + 8xy + 5y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get, $a = a$, $2h = 8$, $b = 5$

Let m_1 and m_2 be the slopes of the lines represented by $ax^2 + 8xy + 5y^2 = 0$.

$$\therefore m_1 + m_2 = \frac{-2h}{b} = -\frac{8}{5}$$

$$\text{and } m_1m_2 = \frac{a}{b} = \frac{a}{5}$$

$$\text{Now, } (m_1 + m_2) = 2(m_1m_2)$$

$$-\frac{8}{5} = 2\left(\frac{a}{5}\right)$$

$$a = 4$$

Miscellaneous Exercise 4 | Q 12 | Page 132

If the line $4x - 5y = 0$ coincides with one of the lines given by $ax^2 + 2hxy + by^2 = 0$, then show that $25a + 40h + 16b = 0$

Solution: The auxiliary equation of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is $bm^2 + 2hm + a = 0$.

Given that $4x - 5y = 0$ is one of the lines represented by $ax^2 + 2hxy + by^2 = 0$.

The slope of the line $4x - 5y = 0$ is $\frac{-4}{-5} = \frac{4}{5}$

$\therefore m = \frac{4}{5}$ is a root of the auxiliary equation $bm^2 + 2hm + a = 0$.

$$\therefore b\left(\frac{4}{5}\right)^2 + 2h\left(\frac{4}{5}\right) + a = 0$$

$$\therefore \frac{16b}{25} + \frac{8h}{5} + a = 0$$

$$\therefore 16b + 40h + 25a = 0 \text{ i.e.}$$

$$\therefore 25a + 40h + 16b = 0$$

Miscellaneous Exercise 4 | Q 13.1 | Page 132

Show that the following equation represents a pair of line. Find the acute angle between them:

$$9x^2 - 6xy + y^2 + 18x - 6y + 8 = 0$$

Solution: Comparing this equation with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \text{ we get,}$$

$$a = 9, h = -3, b = 1, g = 9, f = -3 \text{ and } c = 8$$

$$\therefore D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} 9 & -3 & 9 \\ -3 & 1 & -3 \\ 9 & -3 & 8 \end{vmatrix}$$

$$= 9(8 - 9) + 3(-24 + 27) + 9(9 - 9)$$

$$= 9(-1) + 3(3) + 9(0)$$

$$= -9 + 9 + 0 = 0$$

$$\text{and } h^2 - ab = (-3)^2 - 9(1) = 9 - 9 = 0$$

∴ the given equation represents a pair of lines.

Let θ be the acute angle between the lines.

$$\therefore \tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

$$= \left| \frac{2\sqrt{(-3)^2 - 9(1)}}{10} \right|$$

$$= \left| \frac{2\sqrt{9 - 9}}{10} \right| = 0$$

$$\therefore \tan \theta = \tan 0^\circ$$

$$\therefore \theta = 0^\circ.$$

Miscellaneous Exercise 4 | Q 13.2 | Page 132

Show that the following equation represents a pair of line. Find the acute angle between them:

$$2x^2 + xy - y^2 + x + 4y - 3 = 0$$

Solution: Comparing this equation with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \text{ we get,}$$

$$a = 2, h = \frac{1}{2}, b = -1, g = \frac{1}{2}, f = 2 \text{ and } c = -3$$

$$\therefore D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} 2 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & 2 \\ \frac{1}{2} & 2 & -3 \end{vmatrix}$$

$$= 2(3 - 4) - \frac{1}{2} \left(-\frac{3}{2} - 1 \right) + \frac{1}{2} \left(1 + \frac{1}{2} \right)$$

$$= -2 + \frac{3}{4} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4}$$

$$= -2 + 1 + 1$$

$$= -2 + 2 = 0$$

∴ the given equation represents a pair of lines.

Let θ be the acute angle between the lines.

$$\therefore \tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

$$= \left| \frac{2\sqrt{\left(\frac{1}{2}\right)^2 - (2)(-1)}}{2 - 1} \right|$$

$$= \left| \frac{2\sqrt{\frac{1}{4} + 2}}{1} \right|$$

$$= 2\sqrt{\frac{9}{4}} = 3$$

$$\therefore \tan \theta = \tan 3$$

$$\therefore \theta = \tan^{-1}(3)$$

Miscellaneous Exercise 4 | Q 13.3 | Page 132

Show that the following equation represents a pair of line. Find the acute angle between them:

$$(x - 3)^2 + (x - 3)(y - 4) - 2(y - 4)^2 = 0$$

Solution: Put $x - 3 = X$ and $y - 4 = Y$ in the given equation, we get,

$$X^2 + XY - 2Y^2 = 0$$

Comparing this equation with $ax^2 + 2hxy + by^2 = 0$, we get,

$$a = 1, h = 1/2, b = -2$$

This is the homogeneous equation of second degree

$$\begin{aligned} \text{and } h^2 - ab &= \left(\frac{1}{2}\right)^2 - 1(-2) \\ &= \frac{1}{4} + 2 = \frac{9}{4} > 0 \end{aligned}$$

Hence, it represents a pair of lines passing through the new origin (3, 4).

Let θ be the acute angle between the lines.

$$\therefore \tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

here $a = 1, 2h = 1$, i.e. $h = \frac{1}{2}$ and $b = -2$

$$\begin{aligned} \therefore \tan \theta &= \left| \frac{2\sqrt{\left(\frac{1}{2}\right)^2 - 1(-2)}}{1 - 2} \right| \\ &= \left| \frac{2\left(\sqrt{\frac{1}{4} + 2}\right)}{-1} \right| \\ &= \left| \frac{2 \times \frac{3}{2}}{-1} \right| \end{aligned}$$

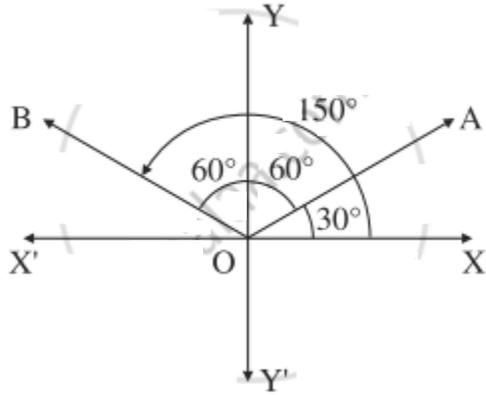
$$\therefore \tan \theta = 3$$

$$\therefore \theta = \tan^{-1}(3)$$

Miscellaneous Exercise 4 | Q 14 | Page 132

Find the combined equation of lines passing through the origin and each of which making an angle of 60° with the Y-axis.

Solution:



Let OA and OB be the lines through the origin making an angle of 60° with the Y-axis. Then OA and OB make an angle of 30° and 150° with the positive direction of X-axis.

$$\therefore \text{slope of OA} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

\therefore equation of the line OA is

$$y = \frac{1}{\sqrt{3}}x \text{ i.e. } x - \sqrt{3}y = 0$$

Slope of OB = $\tan 150^\circ = \tan (180^\circ - 30^\circ)$

$$= -\tan 30^\circ = -\frac{1}{\sqrt{3}}$$

\therefore equation of the line OB is

$$y = -\frac{1}{\sqrt{3}}x \text{ i.e. } x + \sqrt{3}y = 0$$

\therefore required combined equation is

$$(x - \sqrt{3}y)(x + \sqrt{3}y) = 0$$

$$\text{i.e. } x^2 - 3y^2 = 0$$

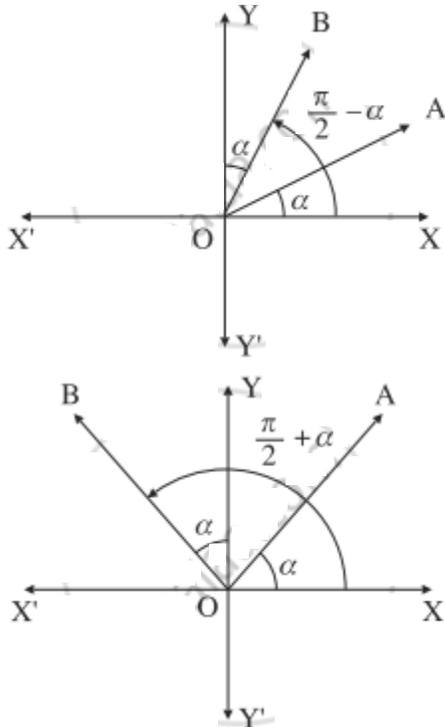
Miscellaneous Exercise 4 | Q 15 | Page 132

If the lines represented by $ax^2 + 2hxy + by^2 = 0$ make angles of equal measure with the coordinate axes, then show that $a \pm b$.

OR

Show that, one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ will make an angle of the same measure with the X-axis as the other makes with the Y-axis, if $a = \pm b$.

Solution:



Let OA and OB be the two lines through the origin represented by $ax^2 + 2hxy + by^2 = 0$.

Since these lines make angles of equal measure with the coordinate axes, they make angles α and $\pi/2 - \alpha$ with the positive direction of X-axis or α and $\pi/2 + \alpha$ with the positive direction of X-axis.

\therefore slope of the line OA = $m_1 = \tan \alpha$

and slope of the line OB = m_2

$$= \tan\left(\frac{\pi}{2} - \alpha\right) \text{ or } \tan\left(\frac{\pi}{2} + \alpha\right)$$

i.e. $m_2 = \cot \alpha$ or $m_2 = -\cot \alpha$

$\therefore m_1 m_2 = \tan \alpha \times \cot \alpha = 1$

$$\text{OR } m_1 m_2 = \tan \alpha (-\cot \alpha) = -1$$

$$\text{i.e. } m_1 m_2 = \pm 1$$

$$\text{But } m_1 m_2 = ab$$

$$\therefore a/b = \pm 1$$

$$\therefore a = \pm b$$

This is the required condition.

Miscellaneous Exercise 4 | Q 16 | Page 132

Show that the combined equation of the pair of lines passing through the origin and each making an angle α with the line $x + y = 0$ is $x^2 + 2(\sec 2\alpha)xy + y^2 = 0$

Solution: Let OA and OB be the required lines.

Let OA (or OB) has slope m .

$$\therefore \text{its equation is } y = mx \quad \dots(1)$$

It makes an angle α with $x + y = 0$ whose slope is -1 .

$$\therefore \tan \alpha = \left| \frac{m + 1}{1 + m(-1)} \right|$$

Squaring both sides, we get,

$$\tan^2 \alpha = \frac{(m + 1)^2}{(1 - m)^2}$$

$$\therefore \tan^2 \alpha (1 - 2m + m^2) = m^2 + 2m + 1$$

$$\therefore \tan^2 \alpha - 2m \tan^2 \alpha + m^2 \tan^2 \alpha = m^2 + 2m + 1$$

$$\therefore (\tan^2 \alpha - 1)m^2 - 2(1 + \tan^2 \alpha)m + (\tan^2 \alpha - 1) = 0$$

$$\therefore m^2 - 2\left(\frac{1 + \tan^2\alpha}{\tan^2\alpha - 1}\right)m + 1 = 0$$

$$\therefore m^2 + 2\left(\frac{1 + \tan^2\alpha}{1 - \tan^2\alpha}\right)m + 1 = 0$$

$$\therefore m^2 + 2(\sec 2\alpha)m + 1 = 0 \dots \left[\because \cos 2\alpha = \frac{1 - \tan^2\alpha}{1 + \tan^2\alpha} \right]$$

$$\therefore \frac{y^2}{x^2} + 2(\sec 2\alpha)\frac{y}{x} + 1 = 0$$

$$\therefore y^2 + 2xy \sec 2\alpha + x^2 = 0 \dots [\text{By (1)}]$$

$$\therefore y^2 + 2xy \sec 2\alpha + x^2 = 0$$

$$\therefore x^2 + 2(\sec 2\alpha)xy + y^2 = 0 \text{ is the required equation.}$$

Miscellaneous Exercise 4 | Q 17 | Page 132

Show that the line $3x + 4y + 5 = 0$ and the lines $(3x + 4y)^2 - 3(4x - 3y)^2 = 0$ form the sides of an equilateral triangle.

Solution: The slope of the line $3x + 4y + 5 = 0$ is $m_1 = -3/4$

Let m be the slope of one of the line making an angle of 60° with the line $3x + 4y + 5 = 0$. The angle between the lines having slope m and m_1 is 60° .

$$\therefore \tan 60^\circ = \left| \frac{m - m_1}{1 + m \cdot m_1} \right|, \text{ where } \tan 60^\circ = \sqrt{3}$$

$$\therefore \sqrt{3} = \left| \frac{m - \left(-\frac{3}{4}\right)}{1 + m\left(-\frac{3}{4}\right)} \right|$$

$$\therefore \sqrt{3} = \left| \frac{4m + 3}{4 - 3m} \right|$$

On squaring both sides, we get,

$$3 = \frac{(4m + 3)^2}{(4 - 3m)^2}$$

$$\therefore 3(4 - 3m)^2 = (4m + 3)^2$$

$$\therefore 3(16 - 24m + 9m^2) = 16m^2 + 24m + 9$$

$$\therefore 48 - 72m + 27m^2 = 16m^2 + 24m + 9$$

$$\therefore 11m^2 - 96m + 39 = 0$$

This is the auxiliary equation of the two lines and their joint equation is obtained by putting $m = y/x$.

\therefore the combined equation of the two lines is

$$11\left(\frac{y}{x}\right)^2 - 96(y/x) + 39 = 0$$

$$\therefore \frac{11y^2}{x^2} - \frac{96y}{x} + 39 = 0$$

$$\therefore 11y^2 - 96xy + 39x^2 = 0$$

$$\therefore 39x^2 - 96xy + 11y^2 = 0$$

$\therefore 39x^2 - 96xy + 11y^2 = 0$ is the joint equation of the two lines through the origin each making an angle of 60° with the line $3x + 4y + 5 = 0$

The equation $39x^2 - 96xy + 11y^2 = 0$ can be written as: $-39x^2 + 96xy - 11y^2 = 0$

$$\text{i.e. } (9x^2 - 48x^2) + (24xy + 72xy) + (16y^2 - 27y^2) = 0$$

$$\text{i.e. } (9x^2 + 24xy + 16y^2) - 3(16x^2 - 24xy + 9y^2) = 0$$

$$\text{i.e. } (3x + 4y)^2 - 3(4x - 3y)^2 = 0$$

Hence, the line $3x + 4y + 5 = 0$ and the lines $(3x + 4y)^2 - 3(4x - 3y)^2$ form the sides of an equilateral triangle.

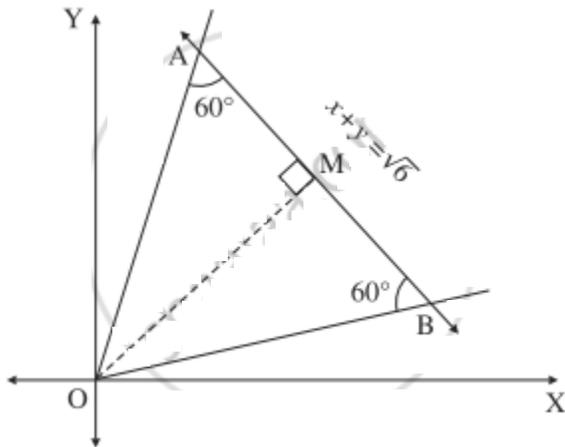
Miscellaneous Exercise 4 | Q 18 | Page 132

Show that the lines $x^2 - 4xy + y^2 = 0$ and the line $x + y = \sqrt{6}$ form an equilateral triangle.

Find its area and perimeter.

Solution: $x^2 - 4xy + y^2 = 0$ and $x + y = \sqrt{6}$ form a triangle OAB which is equilateral.

Let OM be the perpendicular from the origin O to AB whose equation is $x + y = \sqrt{6}$



$$\therefore OM = \left| \frac{-\sqrt{6}}{\sqrt{1+1}} \right| = \sqrt{3}$$

$$\begin{aligned} \therefore \text{area of } \Delta OAB &= \frac{OM^2}{\sqrt{3}} \\ &= \frac{(\sqrt{3})^2}{\sqrt{3}} = \sqrt{3} \text{ sq.units.} \end{aligned}$$

In right-angled triangle OAM,

$$\sin 60^\circ = \frac{OM}{OA}$$

$$\therefore \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{OA}$$

\therefore length of the each side of the equilateral triangle OAB = 2 units.

\therefore perimeter of $\Delta OAB = 3 \times$ length of each side

$$= 3 \times 2 = 6 \text{ units}$$

Miscellaneous Exercise 4 | Q 19 | Page 132

If the slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is square of the slope of the other line, show that $a^2b + ab^2 + 8h^3 = 6abh$.

Solution: Let m be the slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$.

Then the other line has slope m^2

$$\therefore m + m^2 = \frac{-2h}{b} \quad \dots(1) \text{ and}$$

$$(m)(m^2) = \frac{a}{b}$$

$$\text{i.e. } m^3 = \frac{a}{b} \quad \dots(2)$$

$$\therefore (m + m^2)^3 = m^3 + (m^2)^3 + 3(m)(m^2)(m + m^2) \dots[\because (p + q)^3 = p^3 + q^3 + 3pq(p + q)]$$

$$\therefore \left(\frac{-2h}{b}\right)^3 = \frac{a}{b} + \frac{a^2}{b^2} + 3\frac{a}{b}\left(\frac{-2h}{b}\right)$$

$$\therefore \frac{-8h^3}{b^3} = \frac{a}{b} + \frac{a^2}{b^2} - \frac{6ah}{b^2}$$

Multiplying by b^3 , we get,

$$-8h^3 = ab^2 + a^2b - 6abh$$

$$\therefore a^2b + ab^2 + 8h^3 = 6abh$$

This is the required condition.

Miscellaneous Exercise 4 | Q 20 | Page 132

Prove that the product of length of perpendiculars drawn from

$P(x_1, y_1)$ to the lines represented by $ax^2 + 2hxy + by^2 = 0$ is

$$\left| \frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{a-b^2 + 4h^2}} \right|$$

Solution: Let m_1 and m_2 be the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$

$$\therefore m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b} \quad \dots(1)$$

The separate equations of the lines represented by $ax^2 + 2hxy + by^2 = 0$ are

$$y = m_1x \text{ and } y = m_2x$$

$$\text{i.e. } m_1x - y = 0 \text{ and } m_2x - y = 0$$

Length of perpendicular from $P(x_1, y_1)$ on

$$m_1x - y = 0 \text{ is } \left| \frac{m_1x_1 - y_1}{\sqrt{m_1^2 + 1}} \right|$$

Length of perpendicular from $P(x_1, y_1)$ on

$$m_2x - y = 0 \text{ is } \left| \frac{m_2x_1 - y_1}{\sqrt{m_2^2 + 1}} \right|$$

\therefore product of lengths of perpendiculars

$$= \left| \frac{m_1x_1 - y_1}{\sqrt{m_1^2 + 1}} \right| \times \left| \frac{m_2x_1 - y_1}{\sqrt{m_2^2 + 1}} \right|$$

$$= \left| \frac{m_1m_2x_1^2 - (m_1 + m_2)x_1y_1 + y_1^2}{\sqrt{m_1^2m_2^2 + m_1^2 + m_2^2 + 1}} \right|$$

$$= \frac{m_1 m_2 x_1^2 - (m_1 + m_2)x_1 y_1 + y_1^2}{\sqrt{m_1^2 m_2^2 + (m_1 + m_2)^2 - 2m_1 m_2 + 1}}$$

$$= \left| \frac{\frac{a}{b} \cdot x_1^2 - \frac{-2h}{b} x_1 y_1 + y_1^2}{\sqrt{\frac{a^2}{b^2} + \frac{-2h}{b} - \frac{2a}{b} + 1}} \right| \quad \dots(\text{By (1)})$$

$$= \left| \frac{ax_1^2 + 2hx_1 y_1 + by_1^2}{\sqrt{a^2 + 4h^2 - 2ab + b^2}} \right|$$

$$= \left| \frac{ax_1^2 + 2hx_1 y_1 + by_1^2}{\sqrt{(a^2 - 2ab + b^2) + 4h^2}} \right|$$

$$= \left| \frac{ax_1^2 + 2hx_1 y_1 + by_1^2}{\sqrt{(a - b)^2 + 4h^2}} \right|$$

Miscellaneous Exercise 4 | Q 21 | Page 132

Show that the difference between the slopes of the lines given by $(\tan^2\theta + \cos^2\theta)x^2 - 2xy \tan \theta + (\sin^2\theta)y^2 = 0$ is two.

Solution: Comparing the equation

$$(\tan^2\theta + \cos^2\theta)x^2 - 2xy \tan \theta + (\sin^2\theta)y^2 = 0$$

with $ax^2 + 2hxy + by^2 = 0$, we get,

$$a = \tan^2\theta + \cos^2\theta,$$

$$2h = -2\tan \theta$$

$$b = \sin^2\theta$$

Let m_1 and m_2 be the slopes of the lines represented by the given equation.

$$\therefore m_1 + m_2 = \frac{-2h}{b} = - \left[\frac{-2 \tan \theta}{\sin^2 \theta} \right] = \frac{2 \tan \theta}{\sin^2 \theta} \quad \dots(1)$$

$$\text{and } m_1 m_2 = \frac{a}{b} = \frac{\tan^2 \theta + \cos^2 \theta}{\sin^2 \theta} \theta \quad \dots(2)$$

$$\therefore (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$$

$$= \left(\frac{2 \tan \theta}{\sin^2 \theta} \right)^2 - 4 \left(\frac{\tan^2 \theta + \cos^2 \theta}{\sin^2 \theta} \right)$$

$$= \frac{4 \tan^2 \theta}{\sin^4 \theta} - 4 \left(\frac{\tan^2 \theta + \cos^2 \theta}{\sin^2 \theta} \right)$$

$$= \frac{4 \left(\frac{\sin^2 \theta}{\cos^2 \theta} \right)}{\sin^4 \theta} - 4 \left[\frac{\left(\frac{\sin^2 \theta}{\cos^2 \theta} + \cos^2 \theta \right)}{\sin^2 \theta} \right]$$

$$= \frac{4}{\sin^2 \theta \cdot \cos^2 \theta} - \frac{4(\sin^2 \theta + \cos^4 \theta)}{\sin^2 \theta \cdot \cos^2 \theta}$$

$$= 4 \left[\frac{1 - \sin^2 \theta - \cos^4 \theta}{\sin^2 \theta \cdot \cos^2 \theta} \right]$$

$$= 4 \left[\frac{\cos^2 \theta - \cos^4 \theta}{\sin^2 \theta \cdot \cos^2 \theta} \right]$$

$$= 4 \left[\frac{\cos^2 \theta (1 - \cos^2 \theta)}{\sin^2 \theta \cdot \cos^2 \theta} \right] = 4$$

$$\therefore |m_1 - m_2| = 2$$

\therefore the slopes differ by 2.

Miscellaneous Exercise 4 | Q 22 | Page 132

Find the condition that the equation $ay^2 + bxy + ex + dy = 0$ may represent a pair of lines.

Solution: Comparing the equation $ay^2 + bxy + ex + dy = 0$ with $Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$, we get,

$$A = 0, H = \frac{b}{2}, B = a, G = \frac{e}{2}, F = \frac{d}{2}, C = 0$$

The given equation represents a pair of lines,

$$\text{if } \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0$$

$$\text{i.e. if } \begin{vmatrix} 0 & \frac{b}{2} & \frac{e}{2} \\ \frac{b}{2} & a & \frac{d}{2} \\ \frac{e}{2} & \frac{d}{2} & 0 \end{vmatrix} = 0$$

$$\text{i.e. if } 0 - \frac{b}{2} \left(0 - \frac{ed}{4} \right) + \frac{e}{2} \left(\frac{bd}{4} - \frac{ae}{2} \right) = 0$$

$$\text{i.e. if } \frac{bed}{8} + \frac{bed}{8} - \frac{ae^2}{4} = 0$$

$$\text{i.e. if } bed - ae^2 = 0$$

$$\text{i.e. if } e(bd - ae) = 0$$

$$\text{i.e. if } e = 0 \quad \text{or} \quad bd - ae = 0$$

$$\text{i.e. if } e = 0 \quad \text{or} \quad bd = ae$$

This is the required condition.

Miscellaneous Exercise 4 | Q 23 | Page 132

If the lines given by $ax^2 + 2hxy + by^2 = 0$ form an equilateral triangle with the line $lx + my = 1$, show that $(3a + b)(a + 3b) = 4h^2$.

Solution: Since the lines $ax^2 + 2hxy + by^2 = 0$ form an equilateral triangle with the line $lx + my = 1$, the angle between the lines $ax^2 + 2hxy + by^2 = 0$ is 60° .

$$\therefore \tan 60^\circ = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

$$\therefore \sqrt{3} = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

$$\therefore 3(a + b)^2 = 4(h^2 - ab)$$

$$\therefore 3(a^2 + 2ab + b^2) = 4h^2 - 4ab$$

$$\therefore 3a^2 + 6ab + 3b^2 + 4ab = 4h^2$$

$$\therefore 3a^2 + 10ab + 3b^2 = 4h^2$$

$$\therefore 3a^2 + 9ab + ab + 3b^2 = 4h^2$$

$$\therefore 3a(a + 3b) + b(a + 3b) = 4h^2$$

$$\therefore (3a + b)(a + 3b) = 4h^2$$

This is the required condition.

Miscellaneous Exercise 4 | Q 24 | Page 132

If the line $x + 2 = 0$ coincides with one of the lines represented by the equation $x^2 + 2xy + 4y + k = 0$, then prove that $k = -4$.

Solution: One of the lines represented by $x^2 + 2xy + 4y + k = 0$ is $x + 2 = 0$(1)

Let the other line represented by (1) be $ax + by + c = 0$

$$\therefore \text{their combined equation is } (x + 2)(ax + by + c) = 0$$

$$\therefore ax^2 + bxy + cx + 2ax + 2by + 2c = 0$$

$$\therefore ax^2 + bxy + (2a + c)x + 2by + 2c = 0 \quad \dots(2)$$

As the equations (1) and (2) are the combined equations of the same two lines, they are identical.

∴ by comparing their corresponding coefficients, we get,

$$\frac{a}{1} = \frac{b}{2} = \frac{2b}{4} = \frac{2c}{k} \text{ and } 2a + c = 0$$

$$\therefore a = \frac{2c}{k} \text{ and } c = -2a$$

$$\therefore a = \frac{2(-2a)}{k}$$

$$\therefore 1 = \frac{-4}{k}$$

$$\therefore k = -4$$

Miscellaneous Exercise 4 | Q 25 | Page 132

Prove that the combined of the pair of lines passing through the origin and perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$.

Solution: Let m_1 and m_2 be the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$.

$$\therefore m_1 + m_2 = \frac{-2h}{b} \text{ and } m_1 m_2 = \frac{a}{b} \dots(1)$$

Now, the required lines are perpendicular to these lines.

$$\therefore \text{their slopes are } -\frac{1}{m_1} \text{ and } -\frac{1}{m_2}$$

Since these lines are passing through the origin, their separate equations are

$$y = -\frac{1}{m_1}x \text{ and } y = -\frac{1}{m_2}x$$

$$\text{i.e. } m_1 y = -x \text{ and } m_2 y = -x$$

$$\text{i.e. } x + m_1 y = 0 \text{ and } x + m_2 y = 0$$

∴ their combined equation is

$$(x + m_1 y)(x + m_2 y) = 0$$

$$\therefore x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$$

$$\therefore x^2 \frac{-2h}{b} x + \frac{a}{b} y^2 = 0 \quad \dots[\text{By}(1)]$$

$$\therefore bx^2 - 2hxy + ay^2 = 0$$

Miscellaneous Exercise 4 | Q 26 | Page 132

If equation $ax^2 - y^2 + 2y + c = 1$ represents a pair of perpendicular lines, then find a and c .

Solution: The given equation represents a pair of lines perpendicular to each other.

$$\therefore \text{coefficient of } x^2 + \text{coefficient of } y^2 = 0$$

$$\therefore a - 1 = 0$$

$$\therefore a = 1$$

With this value of a , the given equation is

$$x^2 - y^2 + 2y + c - 1 = 0$$

Comparing this equation with

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0, \text{ we get,}$$

$$A = 1, H = 0, B = -1, G = 0, F = 1, C = c - 1$$

Since the given equation represents a pair of lines,

$$D = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & c-1 \end{vmatrix} = 0$$

$$\therefore 1(-c + 1 - 1) - 0 + 0 = 0$$

$$\therefore -c = 0$$

$$\therefore c = 0$$

Hence, $a = 1, c = 0$.