CHAPTER 16

Applications of Derivatives

Section-A JEE Advanced/ IIT-JEE

A Fill in the Blanks

1. The larger of cos (ln θ) and ln (cos θ) if $e^{-\pi/2} < \theta < \frac{\pi}{2}$ is .... *(1983 - 1 Mark)*
2. The function $y = 2x^2 - \ln|x|$ is monotonically increasing for values of $x (\neq 0)$ satisfying the inequalities .... and monotonically decreasing for values of x satisfying the inequalities .... *(1983 - 2 Marks)*
3. The set of all x for which $ln(1+x) \leq x$ is equal to .... *(1987 - 2 Marks)*
4. Let P be a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci F_1 and F_2. If A is the area of the triangle PF_1F_2 then the maximum value of A is .... *(1994 - 2 Marks)*
5. Let C be the curve $y^3 - 3xy + 2 = 0$. If H is the set of points on the curve C where the tangent is horizontal and V is the set of the point on the curve C where the tangent is vertical then H=........... and V=............ *(1994 - 2 Marks)*

B True / False

1. If $x - r$ is a factor of the polynomial $f(x) = a_n x^4 + ... + a_0$, repeated m times $(1 < m \leq n)$, then r is a root of f′(x) = 0 repeated m times. *(1983 - 1 Mark)*
2. For $0 < a < x$, the minimum value of the function $\log_a x + \log_x a$ is 2. *(1984 - 1 Mark)*

C MCQs with One Correct Answer

1. If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has *(1983 - 1 Mark)*
   (a) at least one root in [0, 1]
   (b) one root in [2, 3] and the other in [–2, –1]
   (c) imaginary roots
   (d) none of these
2. AB is a diameter of a circle and C is any point on the circumference of the circle. Then *(1983 - 1 Mark)*
   (a) the area of ΔABC is maximum when it is isosceles
   (b) the area of ΔABC is minimum when it is isosceles
   (c) the perimeter of ΔABC is minimum when it is isosceles
   (d) none of these
3. The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point ‘θ’ is such that *(1983 - 1 Mark)*
   (a) it makes a constant angle with the x– axis
   (b) it passes through the origin
   (c) it is at a constant distance from the origin
   (d) none of these
4. If $y = a\ln x + bx^2 + x$ has its extremum values at $x = -1$ and $x = 2$, then *(1983 - 1 Mark)*
   (a) $a = 2, b = -1$ (b) $a = 2, b = -\frac{1}{2}$
   (c) $a = -2, b = \frac{1}{2}$ (d) none of these
5. Which one of the following curves cut the parabola $y^2 = 4ax$ at right angles? *(1994)*
   (a) $x^2 + y^2 = a^2$ (b) $y = e^{-x/2a}$
   (c) $y = ax$ (d) $x^2 = 4ay$
6. The function defined by $f(x) = (x + 2)e^{-x}$ is *(1994)*
   (a) decreasing for all x
   (b) decreasing in $(-\infty, -1)$ and increasing in $(-1, \infty)$
   (c) increasing for all x
   (d) decreasing in $(-1, \infty)$ and increasing in $(-\infty, -1)$
7. The function $f(x) = \frac{\ln(\pi + x)}{\ln(e + x)}$ is *(1995S)*
   (a) increasing on $(0, \infty)$
   (b) decreasing on $(0, \infty)$
   (c) increasing on $(0, \pi/e)$, decreasing on $(\pi/e, \infty)$
   (d) decreasing on $0, \pi/e)$, increasing on $(\pi/e, \infty)$
8. On the interval [0, 1] the function $x^{25}(1-x)^{75}$ takes its maximum value at the point *(1995S)*
   (a) 0 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$
9. The slope of the tangent to a curve $y = f(x)$ at $[x, f(x)]$ is $2x + 1$. If the curve passes through the point (1, 2), then the area bounded by the curve, the x-axis and the line $x = 1$ is *(1995S)*
   (a) $\frac{5}{6}$ (b) $\frac{6}{5}$ (c) $\frac{1}{6}$ (d) 6

10. If $f(x) = \dfrac{x}{\sin x}$ and $g(x) = \dfrac{x}{\tan x}$, where $0 < x \le 1$, then in this interval *(1997 - 2 Marks)*
(a) both $f(x)$ and $g(x)$ are increasing functions
(b) both $f(x)$ and $g(x)$ are decreasing functions
(c) $f(x)$ is an increasing function
(d) $g(x)$ is an increasing function.

11. The function $f(x) = \sin^4 x + \cos^4 x$ increases if *(1999 - 2 Marks)*
(a) $0 < x < \pi/8$ (b) $\pi/4 < x < 3\pi/8$
(c) $3\pi/8 < x < 5\pi/8$ (d) $5\pi/8 < x < 3\pi/4$

12. Consider the following statments in S and R *(2000S)*
S: Both $\sin x$ and $\cos x$ are decreasing functions in the interval $\left(\frac{\pi}{2}, \pi\right)$
R: If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b).
Which of the following is true ?
(a) Both S and R are wrong
(b) Both S and R are correct, but R is not the correct explanation of S
(c) S is correct and R is the correct explanation for S
(d) S is correct and R is wrong

13. Let $f(x) = \int e^x (x-1)(x-2)dx$. Then f decreases in the interval *(2000S)*
(a) $(-\infty, -2)$ (b) $(-2, -1)$
(c) $(1, 2)$ (d) $(2, +\infty)$

14. If the normal to the curve $y = f(x)$ at the point (3,4) makes an angle $\frac{3\pi}{4}$ with the positive x-axis, then $f'(3) =$ *(2000S)*
(a) -1 (b) $-\frac{3}{4}$ (c) $\frac{4}{3}$ (d) 1

15. Let $f(x) = \begin{cases} |x|, & \text{for } 0 < |x| \le 2 \\ 1, & \text{for } x = 0 \end{cases}$ then at $x = 0$, f has *(2000S)*
(a) a local maximum (b) no local maximum
(c) a local minimum (d) no extremum

16. For all $x \in (0,1)$ *(2000S)*
(a) $e^x < 1 + x$ (b) $\log_e(1 + x) < x$
(c) $\sin x > x$ (d) $\log_e x > x$

17. If $f(x) = xe^{x(1-x)}$, then $f(x)$ is *(2001S)*
(a) increasing on $[-1/2, 1]$ (b) decreasing on R
(c) increasing on R (d) decreasing on $[-1/2, 1]$

18. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the coordinate axes, lies in the first quadrant. If its area is 2, then the value of b is *(2001S)*
(a) -1 (b) 3 (c) -3 (d) 1

19. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is *(2001S)*
(a) [0, 1] (b) (0, 1/2] (c) [1/2, 1] (d) (0, 1]

20. The length of a longest interval in which the function $3 \sin x - 4 \sin^3 x$ is increasing, is *(2002S)*
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π

21. The point(s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical, is (are) *(2002S)*
(a) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (b) $\left(\pm\sqrt{\frac{11}{3}}, 1\right)$
(c) $(0, 0)$ (d) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

22. In [0,1] Lagranges Mean Value theorem is NOT applicable to *(2003S)*
(a) $f(x) = \begin{cases} \frac{1}{2} - x & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2 & x \ge \frac{1}{2} \end{cases}$
(b) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$
(c) $f(x) = x|x|$
(d) $f(x) = |x|$

23. Tangent is drawn to ellipse
$\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$ (where $\theta \in (0, \pi/2)$).
Then the value of θ such that sum of intercepts on axes made by this tangent is minimum, is *(2003S)*
(a) $\pi/3$ (b) $\pi/6$ (c) $\pi/8$ (d) $\pi/4$

24. If $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$, then in $(-\infty, \infty)$
(a) $f(x)$ is a strictly increasing function *(2004S)*
(b) $f(x)$ has a local maxima
(c) $f(x)$ is a strictly decreasing function
(d) $f(x)$ is bounded

25. If $f(x) = x^\alpha \log x$ and $f(0) = 0$, then the value of α for which Rolle's theorem can be applied in [0, 1] is *(2004S)*
(a) -2 (b) -1 (c) 0 (d) 1/2

26. If $P(x)$ is a polynomial of degree less than or equal to 2 and S is the set of all such polynomials so that $P(0) = 0$, $P(1) = 1$ and $P'(x) > 0\ \forall\ x \in [0, 1]$, then *(2005S)*
(a) $S = \phi$
(b) $S = ax + (1 - a)x^2\ \forall\ a \in (0, 2)$
(c) $S = ax + (1 - a)x^2\ \forall\ a \in (0, \infty)$
(d) $S = ax + (1 - a)x^2\ \forall\ a \in (0, 1)$

27. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$ *(2007 -3 marks)*
(a) on the left of $x = c$ (b) on the right of $x = c$
(c) at no point (d) at all points

28. Consider the two curves $C_1 : y^2 = 4x$, $C_2 : x^2 + y^2 - 6x + 1 = 0$. Then, ***(2008)***
(a) C_1 and C_2 touch each other only at one point.
(b) C_1 and C_2 touch each other exactly at two points
(c) C_1 and C_2 intersect (but do not touch) at exactly two points
(d) C_1 and C_2 neither intersect nor touch each other

29. The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$ is ***(2008)***
(a) 0 (b) 1 (c) 2 (d) 3

30. Let the function $g : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is ***(2008)***
(a) even and is strictly increasing in $(0, \infty)$
(b) odd and is strictly decreasing in $(-\infty, \infty)$
(c) odd and is strictly increasing in $(-\infty, \infty)$
(d) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

31. The least value of $a \in \mathbb{R}$ for which $4\alpha x^2 + \frac{1}{x} \ge 1$, for all $x > 0$, is ***(JEE Adv. 2016)***
(a) $\frac{1}{64}$ (b) $\frac{1}{32}$ (c) $\frac{1}{27}$ (d) $\frac{1}{25}$

D MCQs with One or More than One Correct

1. Let $P(x) = a_0 + a_1x^2 + a_2x^4 + \ldots\ldots + a_nx^{2n}$ be a polynomial in a real variable x with $0 < a_0 < a_1 < a_2 < \ldots\ldots < a_n$. The function $P(x)$ has
(a) neither a maximum nor a minimum ***(1986 - 2 Marks)***
(b) only one maximum
(c) only one minimum
(d) only one maximum and only one minimum
(e) none of these.

2. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then ***(1986 - 2 Marks)***
(a) $a > 0, b > 0$ (b) $a > 0, b < 0$
(c) $a < 0, b > 0$ (d) $a < 0, b < 0$
(e) none of these.

3. The smallest positive root of the equation, $\tan x - x = 0$ lies in ***(1987 - 2 Marks)***
(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(\frac{\pi}{2}, \pi\right)$
(c) $\left(\pi, \frac{3\pi}{2}\right)$ (d) $\left(\frac{3\pi}{2}, 2\pi\right)$
(e) None of these

4. Let f and g be increasing and decreasing functions, respectively from $[0, \infty)$ to $[0, \infty)$. Let $h(x) = f(g(x))$. If $h(0) = 0$, then $h(x) - h(1)$ is ***(1987 - 2 Marks)***
(a) always zero (b) always negative
(c) always positive (d) strictly increasing
(e) None of these.

5. If $f(x) = \begin{cases} 3x^2+12x-1, & -1 \le x \le 2 \\ 37-x & 2 < x \le 3 \end{cases}$ then: ***(1993 - 2 Marks)***
(a) $f(x)$ is increasing on $[-1, 2]$
(b) $f(x)$ is continues on $[-1, 3]$
(c) $f'(2)$ does not exist
(d) $f(x)$ has the maximum value at $x = 2$

6. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x. Then ***(1998 - 2 Marks)***
(a) h is increasing whenever f is increasing
(b) h is increasing whenever f is decreasing
(c) h is decreasing whenever f is decreasing
(d) nothing can be said in general.

7. If $f(x) = \frac{x^2-1}{x^2+1}$, for every real number x, then the minimum value of f ***(1998 - 2 Marks)***
(a) does not exist because f is unbounded
(b) is not attained even though f is bounded
(c) is equal to 1
(d) is equal to –1

8. The number of values of x where the function $f(x) = \cos x + \cos(\sqrt{2}x)$ attains its maximum is ***(1998 - 2 Marks)***
(a) 0 (b) 1
(c) 2 (d) infinite

9. The function $f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5\, dt$ has a local minimum at $x =$ ***(1999 - 3 Marks)***
(a) 0 (b) 1 (c) 2 (d) 3

10. $f(x)$ is cubic polynomial with $f(2) = 18$ and $f(1) = -1$. Also $f(x)$ has local maxima at $x = -1$ and $f'(x)$ has local minima at $x = 0$, then ***(2006 - 5M, –1)***
(a) the distance between (–1, 2) and $(a, f(a))$, where $x = a$ is the point of local minima is $2\sqrt{5}$
(b) $f(x)$ is increasing for $x \in [1, 2\sqrt{5}]$
(c) $f(x)$ has local minima at $x = 1$
(d) the value of $f(0) = 15$

11. Let $f(x) = \begin{cases} e^x, & 0 \le x \le 1 \\ 2 - e^{x-1}, & 1 < x \le 2 \\ x - e, & 2 < x \le 3 \end{cases}$ and $g(x) = \int_0^x f(t)\,dt, x \in [1, 3]$ then $g(x)$ has ***(2006 - 5M, –1)***
(a) local maxima at $x = 1 + \ln 2$ and local minima at $x = e$
(b) local maxima at $x = 1$ and local minima at $x = 2$
(c) no loeal maxima
(d) no local minima

12. For the function

$$f(x) = x\cos\frac{1}{x},\ x \ge 1, \qquad \textit{(2009)}$$

(a) for at least one x in the interval $[1, \infty)$, $f(x+2) - f(x) < 2$
(b) $\lim_{x\to\infty} f'(x) = 1$
(c) for all x in the interval $[1, \infty)$, $f(x+2) - f(x) > 2$
(d) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

13. If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)\,dt$ for all $x \in (0, \infty)$, then *(2012)*

(a) f has a local maximum at $x = 2$
(b) f is decreasing on (2, 3)
(c) there exists some $c \in (0, \infty)$, such that $f''(c) = 0$
(d) f has a local minimum at $x = 3$

14. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are *(JEE Adv. 2013)*

(a) 24 (b) 32 (c) 45 (d) 60

15. Let $f: (0, \infty) \to R$ be given by $f(x) \int_{\frac{1}{x}}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$. Then *(JEE Adv. 2014)*

(a) $f(x)$ is monotonically increasing on $[1, \infty)$
(b) $f(x)$ is monotonically decreasing on (0, 1)
(c) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$
(d) $f(2^x)$ is an odd function of x on R

16. Let f, g: [–1, 2] → $\mathbb{R}$ be continuous functions which are twice differentiable on the interval (–1, 2). Let the values of f and g at the points –1, 0 and 2 be as given in the following table:

	$x = -1$	$x = 0$	$x = 2$
$f(x)$	3	6	0
$g(x)$	0	1	–1

In each of the intervals (–1, 0) and (0, 2) the function (f – 3g)" never vanishes. Then the correct statement(s) is(are) *(JEE Adv. 2015)*

(a) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$
(b) $f'(x) - 3g'(x) = 0$ has exactly one solution in (–1, 0)
(c) $f'(x) - 3g'(x) = 0$ has exactly one solution in (0, 2)
(d) $f'(x) - 3g'(x) = 0$ has exactly two solutions in (–1, 0) and exactly two solutions in (0, 2)

17. Let f : $\mathbb{R} \to (0, \infty)$ and g : $\mathbb{R} \to \mathbb{R}$ be twice differentiable functions such that f" and g" are continuous functions on $\mathbb{R}$. Suppose f '(2) = g(2) = 0, f"(2) ≠ 0 and g'(2) ≠ 0. If $\lim_{x\to 2}\frac{f(x)g(x)}{f'(x)g'(x)} = 1$, then *(JEE Adv. 2016)*

(a) f has a local minimum at x=2
(b) f has a local maximum at x=2
(c) $f''(2) > f(2)$
(d) $f(x) - f''(x) = 0$ for at least one x ∈ $\mathbb{R}$

E Subjective Problems

1. Prove that the minimum value of $\frac{(a+x)(b+x)}{(c+x)}$, $a, b > c, x > -c$ is $(\sqrt{a-c} + \sqrt{b-c})^2$. *(1979)*

2. Let x and y be two real variables such that $x > 0$ and $xy = 1$. Find the minimum value of $x+y$. *(1981 - 2 Marks)*

3. For all x in [0, 1], let the second derivative $f''(x)$ of a function $f(x)$ exist and satisfy $|f''(x)| < 1$. If $f(0) = f(1)$, then show that $|f'(x)| < 1$ for all x in [0, 1]. *(1981 - 4 Marks)*

4. Use the function $f(x) = x^{1/x}, x > 0$. to determine the bigger of the two numbers e^π and π^e *(1981 - 4 Marks)*

5. If $f(x)$ and $g(x)$ are differentiable function for $0 \le x \le 1$ such that $f(0) = 2, g(0) = 0, f(1) = 6; g(1) = 2$, then show that there exist c satisfying $0 < c < 1$ and $f'(c) = 2g'(c)$. *(1982 - 2 Marks)*

6. Find the shortest distance of the point (0, c) from the parabola $y = x^2$ where $0 \le c \le 5$. *(1982 - 2 Marks)*

7. If $ax^2 + \frac{b}{x} \ge c$ for all positive x where $a > 0$ and $b > 0$ show that $27ab^2 \ge 4c^3$. *(1982 - 2 Marks)*

8. Show that $1 + x\ln(x + \sqrt{x^2+1}) \ge \sqrt{1+x^2}$ for all $x \ge 0$ *(1983 - 2 Marks)*

9. Find the coordinates of the point on the curve $y = \frac{x}{1+x^2}$ where the tangent to the curve has the greatest slope. *(1984 - 4 Marks)*

10. Find all the tangents to the curve $y = \cos(x+y),\ -2\pi \le x \le 2\pi$, that are parallel to the line $x + 2y = 0$. *(1985 - 5 Marks)*

11. Let $f(x) = \sin^3 x + \lambda \sin^2 x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the intervals in which λ should lie in order that $f(x)$ has exactly one minimum and exactly one maximum. *(1985 - 5 Marks)*

12. Find the point on the curve $4x^2 + a^2y^2 = 4a^2,\ 4 < a^2 < 8$ that is farthest from the point (0, – 2). *(1987 - 4 Marks)*

13. Investigate for maxima and minima the function ***(1988 - 5 Marks)***

$$f(x)=\int_1^x [2(t-1)(t-2)^3+3(t-1)^2(t-2)^2]dt$$

14. Find all maxima and minima of the function

$y=x(x-1)^2, 0 \le x \le 2$ ***(1989 - 5 Marks)***

Also determine the area bounded by the curve $y=x\,(x-1)^2$, the y-axis and the line $y=2$.

15. Show that $2\sin x+\tan x \ge 3x$ where $0 \le x < \frac{\pi}{2}$. ***(1990 - 4 Marks)***

16. A point P is given on the circumference of a circle of radius r. Chord QR is parallel to the tangent at P. Determine the maximum possible area of the triangle PQR. ***(1990 - 4 Marks)***

17. A window of perimeter P (including the base of the arch) is in the form of a rectangle surmounded by a semi circle. The semi- circular portion is fitted with coloured glass while the rectangular part is fitted with clear glass transmits three times as much light per square meter as the coloured glass does.

What is the ratio for the sides of the rectangle so that the window transmits the maximum light? ***(1991 - 4 Marks)***

18. A cubic $f(x)$ vanishes at $x=2$ and has relative minimum / maximum at $x=-1$ and $x=\frac{1}{3}$ if $\int_{-1}^{1} f\,dx=\frac{14}{3}$, find the cubic $f(x)$. ***(1992 - 4 Marks)***

19. What normal to the curve $y=x^2$ forms the shortest chord? ***(1992 - 6 Marks)***

20. Find the equation of the normal to the curve

$y=(1+x)^y+\sin^{-1}(\sin^2 x)$ at $x=0$ ***(1993 - 3 Marks)***

21. Let $f(x)=\begin{cases} -x^3+\dfrac{(b^3-b^2+b-1)}{(b^2+3b+2)}, & 0\le x<1 \\ 2x-3, & 1\le x\le 3 \end{cases}$ ***(1993 - 5 Marks)***

Find all possible real values of b such that $f(x)$ has the smallest value at $x=1$.

22. The curve $y=ax^3+bx^2+cx+5$, touches the x-axis at $P(-2, 0)$ and cuts the y axis at a point Q, where its gradient is 3. Find a, b, c. ***(1994 - 5 Marks)***

23. The circle $x^2+y^2=1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis and the line segment PQ at S. Find the maximum area of the triangle QSR. ***(1994 - 5 Marks)***

24. Let (h, k) be a fixed point, where $h>0, k>0$. A straight line passing through this point cuts the positive direction of the coordinate axes at the points P and Q. Find the minimum area of the triangle OPQ, O being the origin. ***(1995 - 5 Marks)***

25. A curve $y=f(x)$ passes through the point $P(1,1)$. The normal to the curve at P is $a(y-1)+(x-1)=0$. If the slope of the tangent at any point on the curve is proportional to the ordinate of the point, determine the equation of the curve. Also obtain the area bounded by the y-axis, the curve and the normal to the curve at P. ***(1996 - 5 Marks)***

26. Determine the points of maxima and minima of the function

$f(x)=\frac{1}{8}\ln x - bx+x^2, x>0$, where $b\ge 0$ is a constant. ***(1996 - 5 Marks)***

27. Let $f(x)=\begin{cases} xe^{ax}, & x\le 0 \\ x+ax^2-x^3, & x>0 \end{cases}$ ***(1996 - 3 Marks)***

Where a is a positive constant. Find the interval in which $f'(x)$ is increasing.

28. Let $a+b=4$, where $a<2$, and let $g(x)$ be a differentiable function.

If $\frac{dg}{dx}>0$ for all x, prove that $\int_0^a g(x)\,dx+\int_0^b g(x)dx$ increases as $(b-a)$ increases. ***(1997 - 5 Marks)***

29. Suppose $f(x)$ is a function satisfying the following conditions

(a) $f(0)=2, f(1)=1$, ***(1998 - 8 Marks)***

(b) f has a minimum value at $x=5/2$, and

(c) for all x,

$$f'(x)=\begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

where a, b are some constants. Determine the constants a, b and the function $f(x)$.

30. A curve C has the property that if the tangent drawn at any point P on C meets the co-ordinate axes at A and B, then P is the mid-point of AB. The curve passes through the point (1, 1). Determine the equation of the curve. ***(1998 - 8 Marks)***

31. Suppose $p(x)=a_0+a_1x+a_2x^2+\ldots\ldots+a_nx^n$. If $|p(x)|\le|e^{x-1}-1|$ for all $x\ge 0$, prove that

$|a_1+2a_2+\ldots..+na_n|\le 1$. ***(2000 - 5 Marks)***

32. Let $-1\le p\le 1$. Show that the equation $4x^3-3x-p=0$ has a unique root in the interval[1/2, 1] and identify it. ***(2001 - 5 Marks)***

33. Find a point on the curve $x^2+2y^2=6$ whose distance from the line $x+y=7$, is minimum. ***(2003 - 2 Marks)***

34. Using the relation $2(1-\cos x)<x^2, x\ne 0$ or otherwise, prove that $\sin(\tan x)\ge x, \forall\ x\in\left[0, \frac{\pi}{4}\right]$ ***(2003 - 4 Marks)***

35. If the function $f : [0,4] \to R$ is differentiable then show that

(i) For $a, b \in (0,4)$, $(f(4))^2 - (f(0))^2 = 8f'(a)f(b)$

(ii) $\int_0^4 f(t)dt = 2[\alpha f(\alpha^2) + \beta f(\beta^2)] \forall\ 0 < \alpha, \beta < 2$

(2003 - 4 Marks)

36. If $P(1) = 0$ and $\frac{dP(x)}{dx} > P(x)$ for all $x \geq 1$ then prove that $P(x) > 0$ for all $x > 1$. *(2003 - 4 Marks)*

37. Using Rolle's theorem, prove that there is at least one root in $(45^{1/100}, 46)$ of the polynomial $P(x) = 51x^{101} - 2323(x)^{100} - 45x + 1035$. *(2004 - 2 Marks)*

38. Prove that for $x \in \left[0, \frac{\pi}{2}\right]$, $\sin x + 2x \geq \frac{3x(x+1)}{\pi}$. Explain the identity if any used in the proof. *(2004 - 4 Marks)*

39. If $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$, for all $x_1, x_2 \in R$. Find the equation of tangent to the cuve $y = f(x)$ at the point (1, 2). *(2005 - 2 Marks)*

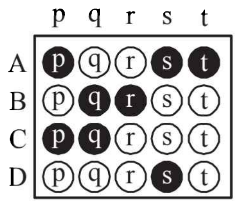
40. If $p(x)$ be a polynomial of degree 3 satisfying $p(-1) = 10$, $p(1) = -6$ and $p(x)$ has maxima at $x = -1$ and $p'(x)$ has minima at $x = 1$. Find the distance between the local maxima and local minima of the curve. *(2005 - 4 Marks)*

41. For a twice differentiable function $f(x)$, $g(x)$ is defined as $g(x) = (f'(x)^2 + f''(x))\, f(x)$ on $[a, e]$. If for $a < b < c < d < e$, $f(a) = 0$, $f(b) = 2$, $f(c) = -1$, $f(d) = 2$, $f(e) = 0$ then find the minimum number of zeros of $g(x)$. *(2006 - 6M)*

F Match the Following

Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :

If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.



1. In this questions there are entries in columns I and II. Each entry in column I is related to exactly one entry in column II. Write the correct letter from column II against the entry number in column I in your answer book.

Let the functions defined in column I have domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *(1992 - 2 Marks)*

Column I	Column II
(A) $x + \sin x$	(p) increasing
(B) $\sec x$	(q) decreasing
	(r) neither increasing nor decreasing

G Comprehension Based Questions

PASSAGE - 1

If a continuous function f defined on the real line R, assumes positive and negative values in R then the equation $f(x) = 0$ has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum value is negative then the equation $f(x) = 0$ has a root in R.

Consider $f(x) = ke^x - x$ for all real x where k is a real constant.

1. The line $y = x$ meets $y = ke^x$ for $k \leq 0$ at *(2007 - 4 marks)*

(a) no point (b) one point

(c) two points (d) more than two points

2. The positive value of k for which $ke^x - x = 0$ has only one root is *(2007 - 4 marks)*

(a) $\frac{1}{e}$ (b) 1

(c) e (d) $\log_e 2$

3. For $k > 0$, the set of all values of k for which $ke^x - x = 0$ has two distinct roots is *(2007 - 4 marks)*

(a) $\left(0, \frac{1}{e}\right)$ (b) $\left(\frac{1}{e}, 1\right)$

(c) $\left(\frac{1}{e}, \infty\right)$ (d) $(0, 1)$

PASSAGE - 2

Let $f(x) = (1 - x)^2 \sin^2 x + x^2$ for all $x \in IR$ and let

$g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t\right) f(t)dt$ for all $x \in (1, \infty)$. *(2012)*

4. Consider the statements:
P : There exists some $x \in$ R such that $f(x) + 2x = 2(1+x^2)$
Q : There exists some $x \in$ R such that $2f(x) + 1 = 2x(1+x)$
Then
(a) both P and Q are true
(b) P is true and Q is false
(c) P is false and Q is true
(d) both P and Q are false

5. Which of the following is true?
(a) g is increasing on $(1, \infty)$
(b) g is decreasing on $(1, \infty)$
(c) g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$
(d) g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

PASSAGE - 3

Let $f : [0, 1] \to \mathbb{R}$ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x, x \in [0, 1]$.

6. Which of the following is true for $0 < x < 1$? ***(JEE Adv. 2013)***
(a) $0 < f(x) < \infty$ (b) $-\frac{1}{2} < f(x) < \frac{1}{2}$
(c) $-\frac{1}{4} < f(x) < 1$ (d) $-\infty < f(x) < 0$

7. If the function $e^{-x} f(x)$ assumes its minimum in the interval $[0, 1]$ at $x = \frac{1}{4}$, which of the following is true?
(a) $f'(x) < f(x), \frac{1}{4} < x < \frac{3}{4}$ ***(JEE Adv. 2013)***
(b) $f'(x) > f(x), 0 < x < \frac{1}{4}$
(c) $f'(x) < f(x), 0 < x < \frac{1}{4}$
(d) $f'(x) < f(x), \frac{3}{4} < x < 1$

I Integer Value Correct Type

1. The maximum value of the function
$f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set
A= $\{x \mid x^2 + 20 \leq 9x\}$ is ***(2009)***

2. Let $p(x)$ be a polynomial of degree 4 having extremum at
$x = 1, 2$ and $\lim_{x\to 0}\left(1 + \frac{p(x)}{x^2}\right) = 2.$
Then the value of p (2) is ***(2009)***

3. Let f be a real-valued differentiable function on **R** (the set of all real numbers) such that $f(1) = 1$. If the y-intercept of the tangent at any point $P(x, y)$ on the curve $y = f(x)$ is equal to the cube of the abscissa of P, then find the value of f(–3) ***(2010)***

4. Let f be a function defined on **R** (the set of all real numbers) such that $f'(x) = 2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4$ for all $x \in$ **R**.
If g is a function defined on **R** with values in the interval $(0, \infty)$ such that
$f(x) = ln(g(x))$, for all $x \in R$
then the number of points in R at which g has a local maximum is ***(2010)***

5. let $f : IR \to IR$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is ***(2012)***

6. Let $p(x)$ be a real polynomial of least degree which has a local maximum at $x = 1$ and a local minimum at $x = 3$. If $p(1) = 6$ and $p(3) = 2$, then $p'(0)$ is ***(2012)***

7. A vertical line passing through the point (h, 0) intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \max_{1/2 \leq h \leq 1} \Delta(h)$ and $\Delta_2 = \min_{1/2 \leq h \leq 1} \Delta(h)$,
then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$ ***(JEE Adv. 2013)***

8. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1, 3) is ***(JEE Adv. 2014)***

9. A cylindrical container is to be made from certain solid material with the following constraints: It has a fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container.
If the volume of the material used to make the container is minimum when the inner radius of the container is 10 mm, then the value of $\frac{V}{250\pi}$ is ***(JEE Adv. 2015)***

GP_3480

Section-B JEE Main / AIEEE

1. The maximum distance from origin of a point on the curve $x = a \sin t - b \sin\left(\frac{at}{b}\right)$

$y = a \cos t - b \cos\left(\frac{at}{b}\right)$, both $a, b > 0$ is [2002]

(a) $a - b$ (b) $a + b$ (c) $\sqrt{a^2 + b^2}$ (d) $\sqrt{a^2 - b^2}$

2. If $2a + 3b + 6c = 0, (a, b, c \in R)$ then the quadratic equation $ax^2 + bx + c = 0$ has [2002]

(a) at least one root in [0, 1] (b) at least one root in [2, 3]
(c) at least one root in [4, 5] (d) none of these

3. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals [2003]

(a) $\frac{1}{2}$ (b) 3 (c) 1 (d) 2

4. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is [2004]

(a) $\left(\frac{9}{8}, \frac{9}{2}\right)$ (b) $(2, -4)$ (c) $\left(\frac{-9}{8}, \frac{9}{2}\right)$ (d) $(2, 4)$

5. A function $y = f(x)$ has a second order derivative $f''(x) = 6(x - 1)$. If its graph passes through the point (2,1) and at that point the tangent to the graph is $y = 3x - 5$, then the function is [2004]

(a) $(x+1)^2$ (b) $(x-1)^3$ (c) $(x+1)^3$ (d) $(x-1)^2$

6. The normal to the curve $x = a(1 + \cos\theta), y = a \sin\theta$ at 'θ' always passes through the fixed point [2004]

(a) (a, a) (b) $(0, a)$ (c) $(0, 0)$ (d) $(a, 0)$

7. If $2a + 3b + 6c = 0$, then at least one root of the equation $ax^2 + bx + c = 0$ lies in the interval [2004]

(a) (1,3) (b) (1,2) (c) (2,3) (d) (0, 1)

8. Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is [2005]

(a) $2ab$ (b) ab (c) $\sqrt{ab}$ (d) $\frac{a}{b}$

9. The normal to the curve [2005]
$x = a(\cos\theta + \theta \sin\theta), y = a(\sin\theta - \theta\cos\theta)$ at any point θ' is such that

(a) it passes through the origin
(b) it makes an angle $\frac{\pi}{2} + \theta$ with the x- axis
(c) it passes through $\left(a\frac{\pi}{2}, -a\right)$
(d) it is at a constant distance from the origin

10. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of ice is 5 cm, then the rate at which the thickness of ice decreases is [2005]

(a) $\frac{1}{36\pi}$ cm/min. (b) $\frac{1}{18\pi}$ cm/min.
(c) $\frac{1}{54\pi}$ cm/min. (d) $\frac{5}{6\pi}$ cm/min

11. If the equation $a_nx^n + a_{n-1}x^{n-1} + \ldots\ldots\ldots\ldots + a_1x = 0$ $a_1 \neq 0, n \geq 2$, has a positive root $x = \alpha$, then the equation $na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots\ldots\ldots + a_1 = 0$ has a positive root, which is [2005]

(a) greater than α
(b) smaller than α
(c) greater than or equal to α
(d) equal to α

12. The function $f(x) = \frac{x}{2} + \frac{2}{x}$ has a local minimum at [2006]

(a) $x = 2$ (b) $x = -2$
(c) $x = 0$ (d) $x = 1$

13. A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides having fence are of same length x. The maximum area enclosed by the park is [2006]

(a) $\frac{3}{2}x^2$ (b) $\sqrt{\frac{x^3}{8}}$ (c) $\frac{1}{2}x^2$ (d) πx^2

14. A value of c for which conclusion of Mean Value Theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3] is [2007]

(a) $\log_3 e$ (b) $\log_e 3$ (c) $2\log_3 e$ (d) $\frac{1}{2}\log_3 e$

15. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in **[2007]**

(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$

16. If p and q are positive real numbers such that $p^2 + q^2 = 1$, then the maximum value of $(p+q)$ is **[2007]**

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\sqrt{2}$ (d) 2.

17. Suppose the cubic $x^3 - px + q$ has three distinct real roots where $p > 0$ and $q > 0$. Then which one of the following holds? **[2008]**

(a) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$

(b) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$

(c) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

(d) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

18. How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have? **[2008]**

(a) 7 (b) 1 (c) 3 (d) 5

19. Let $f(x) = x|x|$ and $g(x) = \sin x$.

Statement-1 : gof is differentiable at $x = 0$ and its derivative is continuous at that point.

Statement-2 : gof is twice differentiable at $x = 0$. **[2009]**

(a) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is false.
(c) Statement-1 is false, Statement-2 is true.
(d) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

20. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$: **[2009]**

(a) P(–1) is not minimum but P(1) is the maximum of P
(b) P(–1) is the minimum but P(1) is not the maximum of P
(c) Neither P(–1) is the minimum nor P(1) is the maximum of P
(d) P(–1) is the minimum and P(1) is the maximum of P

21. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is **[2010[**

(a) $y = 1$ (b) $y = 2$ (c) $y = 3$ (d) $y = 0$

22. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} k - 2x, & \text{if } x \le -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$$

If f has a local minimum at $x = -1$, then a possible value of k is **[2010]**

(a) 0 (b) $-\frac{1}{2}$ (c) –1 (d) 1

23. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function defined by

$$f(x) = \frac{1}{e^x + 2e^{-x}}$$ **[2010]**

Statement -1 : $f(c) = \frac{1}{3}$, for some $c \in \mathbf{R}$.

Statement -2 : $0 < f(x) \le \frac{1}{2\sqrt{2}}$, for all $x \in \mathbf{R}$

(a) Statement -1 is true, Statement -2 is true ; Statement -2 is **not** a correct explanation for Statement -1.
(b) Statement -1 is true, Statement -2 is false.
(c) Statement -1 is false, Statement -2 is true .
(d) Statement - 1 is true, Statement 2 is true ; Statement -2 is a correct explanation for Statement -1.

24. The shortest distance between line $y - x = 1$ and curve $x = y^2$ is **[2011]**

(a) $\frac{3\sqrt{2}}{8}$ (b) $\frac{8}{3\sqrt{2}}$ (c) $\frac{4}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{4}$

25. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t} \sin t \, dt$. Then f has **[2011]**

(a) local minimum at π and 2π
(b) local minimum at π and local maximum at 2π
(c) local maximum at π and local minimum at 2π
(d) local maximum at π and 2π

26. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is : **[2012]**

(a) $\frac{9}{7}$ (b) $\frac{7}{9}$ (c) $\frac{2}{9}$ (d) $\frac{9}{2}$

27. Let $a, b \in R$ be such that the function f given by $f(x) = \ln|x| + bx^2 + ax, x \ne 0$ has extreme values at $x = -1$ and $x = 2$

Statement-1 : f has local maximum at $x = -1$ and *at* $x = 2$.

Statement-2 : $a = \frac{1}{2}$ and $b = \frac{-1}{4}$ **[2012]**

(a) Statement-1 is false, Statement-2 is true.
(b) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.
(c) Statement-1 is true, statement-2 is true; statement-2 is **not** a correct explanation for Statement-1.
(d) Statement-1 is true, statement-2 is false.

28. A line is drawn through the point (1,2) to meet the coordinate axes at *P* and *Q* such that it forms a triangle *OPQ*, where O is the origin. If the area of the triangle *OPQ* is least, then the slope of the line *PQ* is : **[2012]**

(a) $-\frac{1}{4}$ (b) -4 (c) -2 (d) $-\frac{1}{2}$

29. The intercepts on *x*-axis made by tangents to the curve, $y=\int_0^x |t|\,dt,\ x\in R$, which are parallel to the line $y = 2x$, are equal to : **[JEE M 2013]**

(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 4

30. If *f* and *g* are differentiable functions in [0, 1] satisfying $f(0)=2=g(1), g(0)=0$ and $f(1)=6$, then for some $c\in]0,1[$ **[JEE M 2014]**

(a) $f'(c)=g'(c)$ (b) $f'(c)=2g'(c)$
(c) $2f'(c)=g'(c)$ (d) $2f'(c)=3g'(c)$

31. Let f(x) be a polynomial of degree four having extreme values at x = 1 and x = 2. If $\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right]=3$, then f(2) is equal to : **[JEE M 2015]**

(a) 0 (b) 4 (c) -8 (d) -4

32. Consider :

$$f(x)=\tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x\in\left(0,\frac{\pi}{2}\right).$$

A normal to y = f(x) at $x=\frac{\pi}{6}$ also passes through the point : **[JEE M 2016]**

(a) $\left(\frac{\pi}{6},0\right)$ (b) $\left(\frac{\pi}{4},0\right)$ (c) $(0,0)$ (d) $\left(0,\frac{2\pi}{3}\right)$

33. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then: **[JEE M 2016]**

(a) $x=2r$ (b) $2x=r$
(c) $2x=(\pi+4)r$ (d) $(4-\pi)x=\pi r$

16 Applications of Derivatives

Section-A : JEE Advanced/ IIT-JEE

A **1.** $\cos(\ln\theta)$ **2.** $x \in \left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$; $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$ **3.** $x \geq 0$ **4.** abe

5. ϕ, $\{(1, 1)\}$

B **1.** F **2.** F

C	**1.** (a)	**2.** (a)	**3.** (c)	**4.** (b)	**5.** (d)	**6.** (d)
	7. (b)	**8.** (b)	**9.** (a)	**10.** (c)	**11.** (b)	**12.** (d)
	13. (c)	**14.** (d)	**15.** (d)	**16.** (b)	**17.** (a)	**18.** (c)
	19. (d)	**20.** (a)	**21.** (d)	**22.** (a)	**23.** (c)	**24.** (a)
	25. (d)	**26.** (b)	**27.** (a)	**28.** (b)	**29.** (c)	**30.** (c)
	31. (c)					
D	**1.** (c)	**2.** (b, c)	**3.** (c)	**4.** (a)	**5.** (a, b, c)	**6.** (a, c)
	7. (d)	**8.** (b)	**9.** (b, d)	**10.** (b, c)	**11.** (a, b)	**12.** (b, c, d)
	13. (a, b, c, d)	**14.** (a, c)	**15.** (a, c, d)	**16.** (b, c)	**17.** (a, d)	

E **2.** 2 **4.** e^{π} **6.** $\sqrt{c-\frac{1}{4}}$ **9.** $(0, 0)$ **10.** $2x+4y-\pi=0$

$2x+4y+3\pi=0$

11. $\lambda \in \left(-\frac{3}{2}, 0\right) \cup \left(0, \frac{3}{2}\right)$ **12.** $(0, 2)$

13. f is min at $x=\frac{7}{5}$ and max. at $x=1$ **14.** $\frac{10}{3}$ sq. units **16.** $\frac{3\sqrt{3}}{4}r^2$

17. $6+\pi:6$ **18.** x^3+x^2-x+2 **19.** $x+\sqrt{2}y=\sqrt{2}$ or $x-\sqrt{2}y=-\sqrt{2}$

20. $x+y=1$ **21.** $b \in (-2, -1) \cup (1, \infty)$ **22.** $a=-\frac{1}{2}, b=\frac{-3}{4}, c=3$

23. $\frac{4\sqrt{3}}{9}$ sq. units **24.** 2 kh **25.** $y=e^{a(x-1)}$; 1 sq. unit

26. min at $x=\frac{1}{4}(b+\sqrt{b^2-1})$, max at $x=\frac{1}{4}(b-\sqrt{b^2-1})$ **27.** $\left(\frac{-2}{a}, \frac{a}{3}\right)$

29. $a=\frac{1}{4}, b=\frac{-5}{4}, c=2$, $f(x)=\frac{1}{4}x^2-\frac{5}{4}x+2$ **30.** $xy=1$

33. $(2, 1)$ **39.** $y=2$ **40.** $4\sqrt{65}$ **41.** 6

F **1.** (A) $\rightarrow$ p (B) $\rightarrow$ r

G **1.** (c) **2.** (a) **3.** (a) **4.** (c) **5.** (b) **6.** (d) **7.** (c)

I **1.** 7 **2.** 0 **3.** 9 **4.** 1 **5.** 5 **6.** 9 **7.** 9 **8.** 8
9. 4

Section-B : JEE Main/ AIEEE

1. (b) **2.** (a) **3.** (d) **4.** (a) **5.** (b) **6.** (d) **7.** (d) **8.** (a) **9.** (d) **10.** (b) **11.** (b) **12.** (a)
13. (c) **14.** (c) **15.** (d) **16.** (c) **17.** (a) **18.** (b) **19.** (b) **20.** (a) **21.** (c) **22.** (c) **23.** (d)
24. (a) **25.** (c) **26.** (c) **27.** (b) **28.** (c) **29.** (a) **30.** (b) **31.** (a) **32.** (d) **33.** (a)

Section-A JEE Advanced/ IIT-JEE

A. Fill in the Blanks

1. We have $e^{-\pi/2} < \theta < \pi/2 \Rightarrow -\frac{\pi}{2} < \ln\theta < \ln \pi/2$

$\Rightarrow \cos(-\pi/2) < \cos(\ln\theta) < \cos(\ln \pi/2)$

$\Rightarrow \cos(\ln\theta) > 0$ .... (1)

Also $-1 \le \cos\theta \le 1 \forall\ \theta$

$\therefore\ -1 \le \ln(\cos\theta) \le 0\ \forall\ 0 < \cos\theta \le 1$

$\Rightarrow \ln(\cos\theta) \le 0$ .... (2)

From (1) and (2) we get, $\cos(\ln\theta) > \ln(\cos\theta)$

$\therefore$ $\cos(\ln\theta)$ is larger.

2. $y = 2x^2 - \ln|x| \Rightarrow \frac{dy}{dx} = 4x - \frac{1}{x} = \frac{(2x+1)(2x-1)}{x}$

Critical points are 0, 1/2, –1/2

Clearly $f(x)$ is increasing on $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$ and

$f(x)$ is decreasing on $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$.

3. Let $f(x) = \log(1+x) - x$ for $x > -1$

$$f'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x}$$

We observe that,

$f'(x) > 0$ if $-1 < x < 0$ and $f'(x) < 0$ if $x > 0$

Therefore f increases in $(-1, 0)$ and decreases in $(0, \infty)$.

Also $f(0) = \log 1 - 0 = 0$

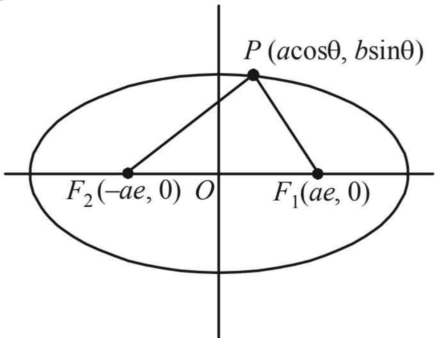
$\therefore\ x \ge 0 \Rightarrow f(x) \le f(0)$

$\Rightarrow \log(1+x) - x \le 0 \Rightarrow \log(1+x) \le x$

Thus we get, $\log(1+x) \le x, \forall\ x \ge 0$

4. Let $P(a\cos\theta, b\sin\theta)$ be any point on the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci $F_1(ae, 0)$ and $F_2(-ae, 0)$



Then area of ΔPF_1F_2 is given by

$$A = \frac{1}{2}\begin{vmatrix} a\cos\theta & b\sin\theta & 1 \\ ae & 0 & 1 \\ -ae & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2}|-b\sin\theta(ae+ae)| = abe|\sin\theta|$$

$\because\ |\sin\theta| \le 1$

$\therefore\ A_{max} = abe$

5. The given curve is $C: y^3 - 3xy + 2 = 0$

Differentiating it with respect to x, we get

$$3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{-x+y^2}$$

$\therefore$ Slope of tangent to C at point (x_1, y_1) is

$$\frac{dy}{dx} = \frac{y_1}{-x_1 + y_1^2}$$

For horizontal tangent, $\frac{dy}{dx} = 0 \Rightarrow y_1 = 0$

For $y_1 = 0$ in C, we get no value of x_1

$\therefore$ There is no point on C at which tangent is horizontal

$\therefore\ H = \phi$

For vertical tangent $\frac{dy}{dx} = \frac{1}{0} \Rightarrow -x_1 + y_1^2 = 0 \Rightarrow x_1 = y_1^2$

From C, $y_1^3 - 3y_1^3 + 2 = 0$

$\Rightarrow\ y_1^3 = 1 \Rightarrow y_1 = 1 \Rightarrow x_1 = 1$

$\therefore$ There is only one point (1, 1) at which vertical tangent can be drawn

$\therefore\ V = \{(1, 1)\}$

B. True / False

1. If $(x - r)$ is a factor of $f(x)$ repeated m times then $f'(x)$ is a polynomial with $(x - r)$ as factor repeated $(m - 1)$ times.

$\therefore$ Statement is false.

2. Given that $0 < a < x$.

Let $f(x) = \log_a x + \log_x a = \log_a x + \frac{1}{\log_a x} \ge 2$

But equality holds for $\log_a x = 1$

$\Rightarrow x = a$ which is not possible.

$\therefore\ f(x) > 2$

$\therefore\ f_{min}$ cannot be 2.

$\therefore$ Statement is false.

C. MCQs with ONE Correct Answer

1. **(a)** Consider the function $f(x) = ax^3 + bx^2 + cx$ on [0, 1] then being a polynomial. It is continuous on [0, 1], differentiable on (0, 1) and

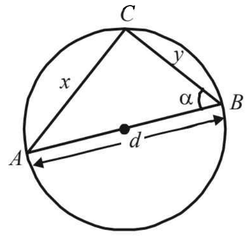
$f(0) = f(1) = 0$ [as given $a + b + c = 0$]

$\therefore$ By Rolle's theorem $\exists\ x \in (0,1)$ such that

$f'(x) = 0 \Rightarrow 3ax^2 + 2bx + c = 0$

Thus equation $3ax^2 + 2bx + c = 0$ has at least one root in [0, 1].

2. **(a)** Area of ΔABC, $A = \frac{1}{2} \times d\cos\alpha \times d\sin\alpha = \frac{d^2}{4}\sin 2\alpha$



which is max. when $\sin 2\alpha = 1$
i.e. $\alpha = 45°$
$\therefore$ ΔABC is an isosceles triangle.

3. **(c)** $\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta \quad \ldots(1)$

$\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta \quad \ldots(2)$

Dividing (2) by (1), we get

$\frac{dy}{dx} = \tan\theta$ (slope of tangent)

$\therefore$ Slope of normal $= -\cot\theta$
$\therefore$ Equation of normal is

$y - a(\sin\theta - \theta\cos\theta) = -\frac{\cos\theta}{\sin\theta}(x - a(\cos\theta + \theta\sin\theta))$

$\Rightarrow y\sin\theta - a\sin^2\theta + a\sin\theta\cos\theta$
$= -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$

$\Rightarrow x\cos\theta + y\sin\theta = a$

As θ varies inclination is not constant.
$\therefore$ (a) is not correct.
Clearly does not pass through (0, 0).

It's distance from origin $= \left|\frac{a}{\sqrt{\cos^2\theta + \sin^2\theta}}\right| = a$

which is constant

4. **(b)** $y = a\ln x + bx^2 + x$
has its extremum values at $x = -1$ and 2

$\therefore \frac{dy}{dx} = 0$ at $x = -1$ and 2

$\Rightarrow \frac{a}{x} + 2bx + 1 = 0$ or $2bx^2 + x + a = 0$

has -1 and 2 as its roots.

$\therefore 2b - 1 + a = 0 \quad \ldots(1)$
$8b + 2 + a = 0 \quad \ldots(2)$

Solving (1) and (2) we get $a = 2, b = -1/2$.

5. **(d)** For $y^2 = 4ax$, y-axis is tangent at (0, 0), while for $x^2 = 4ay$, x-axis is tangent at (0, 0). Thus the two curves cut each other at right angles.

6. **(d)** $f'(x) = -(x+2)e^{-x} + e^{-x} = -(x+1)e^{-x} = 0 \Rightarrow x = -1$

For $x \in (-\infty, -1), f'(x) > 0$ and for $x \in (-1, \infty), f'(x) < 0$

$\therefore f(x)$ is increasing on $(-\infty, -1)$ and decreasing on $(-1, -\infty)$.

7. **(b)** We have $f(x) = \frac{\ln(\pi + x)}{\ln(e + x)}$

$\therefore f'(x) = \frac{\left(\frac{1}{\pi + x}\right)\ln(e+x) - \frac{1}{(e+x)}\ln(\pi + x)}{[\ln(e+x)]^2}$

$= \frac{(e+x)\ln(e+x) - (\pi+x)\ln(\pi+x)}{(e+x)(\pi+x)(\ln(e+x))^2}$

< 0 on $(0, \infty)$ since $1 < e < \pi$

$\therefore f(x)$ decreases on $(0, \infty)$.

8. **(b)** Let $y = x^{25}(1-x)^{75}$

$\Rightarrow \frac{dy}{dx} = 25x^{24}(1-x)^{75} - 75x^{25}(1-x)^{74}$
$= 25x^{24}(1-x)^{74}(1 - x - 3x) = 25x^{24}(1-x)^{74}(1-4x)$

For maximum value of y, $\frac{dy}{dx} = 0$

$\Rightarrow x = 0,\ 1,\ 1/4,\ x = 1/4 \in (0,1)$

Also at $x = 0, y = 0$, at $x = 1, y = 0$, and at $x = 1/4, y > 0$

$\therefore$ Max. value of y occurs at $x = 1/4$

9. **(a)** Slope of tangent at $(x, f(x))$ is $2x + 1$

$\Rightarrow f'(x) = 2x + 1 \Rightarrow f(x) = x^2 + x + c$

Also the curve passes through (1, 2)

$\therefore f(1) = 2$

$\Rightarrow 2 = 1 + 1 + c \Rightarrow c = 0, \therefore f(x) = x^2 + x$

$\therefore$ Required area $= \int_0^1 (x^2 + x)\,dx$

$= \left(\frac{x^3}{3} + \frac{x^2}{2}\right)_0^1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

10. **(c)** We have $f(x) = \frac{x}{\sin x}, 0 < x \le 1$

$\Rightarrow f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$

where $\sin^2 x$ is always +ve, when $0 < x \le 1$. But to check Nr., we again let

$h(x) = \sin x - x\cos x$

$\Rightarrow h'(x) = x\sin x > 0$ for $0 < x \le 1 \Rightarrow h(x)$ is increasing

$\Rightarrow h(0) < h(x)$, when $0 < x \le 1$

$\Rightarrow 0 < \sin x - x\cos x$, when $0 < x \le 1$

$\Rightarrow \sin x - x\cos x > 0$, when $0 < x \le 1$

$\Rightarrow f'(x) > 0,\ x \in (0, 1]$

$\Rightarrow f(x)$ is increasing on (0, 1]

Again $g(x) = \frac{x}{\tan x}$

$\Rightarrow g'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x}$, when $0 < x \le 1$

Here $\tan^2 x > 0$ But to check Nr. we consider

$p(x) = \tan x - x\sec^2 x$

$p'(x) = \sec^2 x - \sec^2 x - x.2\sec x.\sec x\tan x$

$\Rightarrow p'(x) = -2x\sec^2 x\tan x < 0$ for $0 < x \le 1$

$\Rightarrow p(x)$ is decreasing, when $0 < x \le 1$

$\Rightarrow p(0) > p(x) \Rightarrow 0 > \tan x - x\sec^2 x$

$\therefore g'(x) < 0$

Hence $g(x)$ is decreasing when $0 < x \le 1$.

11. **(b)** We are given $f(x) = \sin^4 x + \cos^4 x$

$\Rightarrow f'(x) = 4\sin^3 x\cos x - 4\cos^3 x\sin x$

$= -4 \sin x \cos x (\cos^2 x - \sin^2 x)$
$= -2. \sin 2x \cos 2x = -\sin 4x$
Now for $f(x)$ to be increasing function
$f'(x) > 0 \Rightarrow -\sin 4x > 0 \Rightarrow \sin 4x < 0$

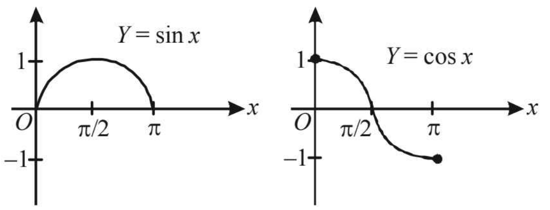
$\Rightarrow \pi < 4x < 2\pi \Rightarrow \frac{\pi}{4} < x < \frac{\pi}{2}$

Since, If $f(x)$ increasing on $(\pi/4, \pi/2)$

$\frac{\pi}{4} = \frac{4\pi}{8} > \frac{3\pi}{8}$

It will be increasing on $(\pi/4, 3\pi/8)$.

12. **(d)** From graph it is clear that both $\sin x$ and $\cos x$ in the interval $(\pi/2, \pi)$ are decreasing function.



$\therefore$ S is correct.
To disprove R let us consider the counter example :
$f(x) = \sin x$ on $(0, \pi/2)$ so that $f'(x) = \cos x$
Again from graph it is clear that $f(x)$ is increasing on $(0, \pi/2)$ but $f'(x)$ is decreasing on $(0, \pi/2)$
$\therefore$ R is wrong.

13. **(c)** $f(x) = \int e^x (x-1)(x-2)\,dx$
For decreasing function, $f'(x) < 0$
$\Rightarrow e^x (x-1)(x-2) < 0 \Rightarrow (x-1)(x-2) < 0$
$\Rightarrow 1 < x < 2, \qquad \because e^x > 0 \, \forall x \in R$

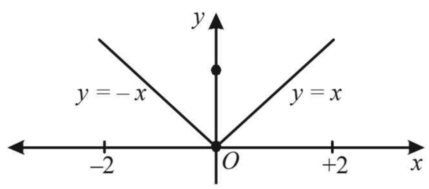
14. **(d)** Slope of tangent $y = f(x)$ is $\frac{dy}{dx} = f'(x)_{(3,4)}$

Therefore, slope of normal $= -\frac{1}{f'(x)_{(3,4)}} = -\frac{1}{f'(3)}$

but $-\frac{1}{f'(3)} = \tan\left(\frac{3\pi}{4}\right)$ (given)

or $-\frac{1}{f'(3)} = \tan\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = -1 \Rightarrow f'(3) = 1$

15. **(d)** It is clear from figure that at $x = 0$, $f(x)$ is not differentiable.

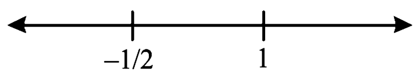


$\Rightarrow f(x)$ has neither maximum nor minimum at $x = 0$.

16. **(b)** Let $f(x) = e^x - 1 - x$ then $f'(x) = e^x - 1 > 0$ for $x \in (0,1)$
$\therefore$ $f(x)$ is an increasing function.
$\therefore$ $f(x) > f(0), \forall x \in (0,1)$
$\Rightarrow e^x - 1 - x > 0 \Rightarrow e^x > 1 + x$
$\therefore$ (a) does not hold.

(b) Let $g(x) = \log(1+x) - x$
then $g'(x) = \frac{1}{1+x} - 1 = -\frac{x}{1+x} < 0, \forall x \in (0,1)$
$\therefore$ $g(x)$ is decreasing on (0, 1) $\therefore x > 0$
$\Rightarrow g(x) < g(0)$
$\Rightarrow \log(1+x) - x < 0 \Rightarrow \log(1+x) < x$
$\therefore$ (b) holds. Similarly it can be shown that (c) and (d) do not hold.

17. **(a)** $f(x) = xe^{x(1-x)}$
$\Rightarrow f'(x) = e^{x(1-x)} + (1-2x)\,x\,e^{x(1-x)}$
$= -e^{x(1-x)}(2x^2 - x - 1) = -e^{x(1-x)}(2x+1)(x-1)$



$\therefore$ $f(x)$ is increasing on [–1/2, 1]

18. **(c)** Tangent to $y = x^2 + bx - b$ at (1, 1) is
$y - 1 = (2+b)(x-1) \Rightarrow (b+2)x - y = b+1$

x-intercept $= \frac{b+1}{b+2}$ and y-intercept $= -(b+1)$

Given $Ar(\Delta) = 2 \Rightarrow \frac{1}{2}\left(\frac{b+1}{b+2}\right)[-(b+1)] = 2$

$\Rightarrow b^2 + 2b + 1 = -4(b+2) \Rightarrow b^2 + 6b + 9 = 0$
$\Rightarrow (b+3)^2 = 0 \Rightarrow b = -3$

19. **(d)** $f(x) = (1+b^2)x^2 + 2bx + 1$
It is a quadratic expression with coeff. of $x^2 = 1 + b^2 > 0$.
$\therefore$ $f(x)$ represents an upward parabola whose min value is $\frac{-D}{4a}$, D being the discriminant.

$\therefore m(b) = -\frac{4b^2 - 4(1+b^2)}{4(1+b^2)} \Rightarrow m(b) = \frac{1}{1+b^2}$

For range of m (b) :

$\frac{1}{1+b^2} > 0$ also $b^2 \geq 0 \Rightarrow 1 + b^2 \geq 1$

$\Rightarrow \frac{1}{1+b^2} \leq 1$. Thus $m(b) = (0, 1]$

20. **(a)** $3 \sin x - 4 \sin^3 x = \sin 3x$ which increases for

$3x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow x \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$ whose length is $\frac{\pi}{3}$.

21. **(d)** The given curve is $y^3 + 3x^2 = 12y$

$\Rightarrow 3y^2 \frac{dy}{dx} + 6x = 12\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2x}{4 - y^2}$

For vertical tangents $\frac{dy}{dx} = \frac{1}{0} \Rightarrow 4 - y^2 = 0 \Rightarrow y = \pm 2$

For y = 2, $x^2 = \frac{24-8}{3} = \frac{16}{3} \Rightarrow x = \pm\frac{4}{\sqrt{3}}$

For $y = -2$, $x^2 = \frac{-24+8}{3} = -ve$ (not possible)

$\therefore$ Req. points are $(\pm 4/\sqrt{3}, 2)$.

22. **(a)** There is only one function in option (a) whose critical point $\frac{1}{2} \in (0,1)$ for the rest of the parts critical point

$0 \notin (0, 1)$. It can be easily seen that functions in options (b), (c) and (d) are continuous on [0, 1] and differentiable in (0, 1).

Now for $f(x)=\begin{cases}\left(\frac{1}{2}-x\right), & x<1/2\\ \left(\frac{1}{2}-x\right)^2, & x\ge 1/2\end{cases}$

Here $f'\left(\frac{1^-}{2}\right)=-1$ and $f'(1/2^+)=-2\left(\frac{1}{2}-\frac{1}{2}\right)=0$

$\therefore$ $f'\left(\frac{1^-}{2}\right)\ne f'(1/2^+)$

$\therefore$ f is not differentiable at $1/2 \in (0,1)$

$\therefore$ LMV is not applicable for this function in [0, 1]

23. (c) Equation of tangent to the ellipse $\frac{x^2}{27}+y^2=1$ at $(3\sqrt{3}\cos\theta, \sin\theta), \theta\in(0,\pi/2)$ is $\frac{\sqrt{3}\,x\cos\theta}{9}+y.\sin\theta=1$

$\therefore$ Intercept on x-axis $=\frac{9}{\sqrt{3}\cos\theta}$;

Intercept on y-axis $=\frac{1}{\sin\theta}$

$\therefore$ Sum of intercepts $=S=3\sqrt{3}\sec\theta+\text{cosec}\,\theta$

For min. value of S, $\frac{dS}{d\theta}=0$

$\Rightarrow$ $3\sqrt{3}\sec\theta\tan\theta-\text{cosec}\,\theta\cot\theta=0$

$\Rightarrow$ $\frac{3\sqrt{3}\sin\theta}{\cos^2\theta}-\frac{\cos\theta}{\sin^2\theta}=0 \Rightarrow 3\sqrt{3}\sin^3\theta-\cos^3\theta=0$

$\Rightarrow$ $\tan^3\theta=\frac{1}{3\sqrt{3}}=\left(\frac{1}{\sqrt{3}}\right)^3$

$\Rightarrow$ $\tan\theta=\frac{1}{\sqrt{3}}=\tan\pi/6\Rightarrow\theta=\pi/6$

24. (a) $f(x)=x^3+bx^2+cx+d, 0<b^2<c$

$f'(x)=3x^2+2bx+c$

Discriminant $=4b^2-12c=4(b^2-3c)<0$

$\therefore$ $f'(x)>0\ \forall\ x\in R$

$\Rightarrow$ $f(x)$ is strictly increasing $\forall\ x\in R$

25. (d) For Rolle's theorem in $[a, b]$

$f(a)=f(b)$, In [0, 1] $\Rightarrow f(0)=f(1)=0$

$\because$ The function has to be continuous in [0, 1]

$\Rightarrow$ $f(0)=\lim\limits_{x\to 0^+} f(x)=0\Rightarrow \lim\limits_{x\to 0} x^\alpha\log x=0$

$\Rightarrow$ $\lim\limits_{x\to 0}\frac{\log x}{x^{-\alpha}}=0$

Applying L' Hospital's Rule

$\lim\limits_{x\to 0}\frac{1/x}{-\alpha x^{-\alpha-1}}=0\Rightarrow\lim\limits_{x\to 0}\frac{-x^\alpha}{\alpha}=0\Rightarrow\alpha>0$

26. (b) Let the polynominal be $P(x)=ax^2+bx+c$

Given $P(0)=0$ and $P(1)=1 \Rightarrow c=0$ and $a+b=1$

$\Rightarrow a=1-b$

$\therefore$ $P(x)=(1-b)x^2+bx$

$\Rightarrow$ $P'(x)=2(1-b)x+b$

Given $P'(x)>0, \forall\ x\in[0,1]$

$\Rightarrow$ $2(1-b)x+b>0$

$\Rightarrow$ When $x=0, b>0$ and when $x=1, b<2$

$\Rightarrow$ $0<b<2$

$\therefore$ $S=\{(1-a)x^2+ax, a\in(0,2)\}$

27. (a) The equation of tangent to the curve $y=e^x$ at (c, e^c) is

$y-e^c=e^c(x-c)$ ...(1)

and equation of line joining $(c-1, e^{c-1})$ and $(c+1, e^{c+1})$ is

$y-e^{c-1}=\frac{e^{c+1}-e^{c-1}}{(c+1)-(c-1)}[x-(c-1)]$

$\Rightarrow$ $y-e^{c-1}=\frac{e^c(e-e^{-1})}{2}[x-c+1]$ ...(2)

Subtracting equation (1) from (2), we get

$e^c-e^{c-1}=e^c(x-c)\left[\frac{e-e^{-1}-2}{2}\right]+e^c\left(\frac{e-e^{-1}}{2}\right)$

$\Rightarrow$ $x-c=\frac{\left[1-e^{-1}-\left(\frac{e-e^{-1}}{2}\right)\right]}{\frac{e-e^{-1}-2}{2}}=\frac{2-e-e^{-1}}{e-e^{-1}-2}$

$=\frac{e+e^{-1}-2}{2-(e-e^{-1})}=\frac{\frac{e+e^{-1}}{2}-1}{1-\frac{e-e^{-1}}{2}}=\frac{+ve}{-ve}=-ve$

$\Rightarrow$ $x-c<0\Rightarrow x<c$

$\therefore$ The two lines meet on the left of line $x=c$.

28. (b) The given curves are

$C_1: y^2=4x$ ...(1) and $C_2: x^2+y^2-6x+1=0$...(2)

Solving (1) and (2) we get

$x^2+4x-6x+1=0 \Rightarrow x=1$ and $\Rightarrow y=2$ or -2

$\therefore$ Points of intersection of the two curves are (1, 2) and (1, –2).

For C_1, $\frac{dy}{dx}=\frac{2}{y}$

$\therefore$ $\left(\frac{dy}{dx}\right)_{(1,2)}=1=m_1$ and $\left(\frac{dy}{dx}\right)_{(1,-2)}=-1=m_1'$

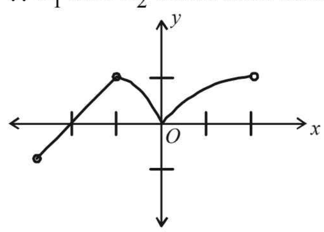
For C_2, $\frac{dy}{dx}=\frac{3-x}{y}$ $\therefore$ $\left(\frac{dy}{dx}\right)_{(1,2)}=1=m_2$

and $\left(\frac{dy}{dx}\right)_{(1,-2)}=-1=m_2'$

$\because m_1=m_2$ and $m_1'=m_2'$

$\therefore$ C_1 and C_2 touch each other at two points.

29. (c)



The given function is

$$f(x)=\begin{cases}(2+x)^3, & -3<x\le -1\\ x^{2/3}, & -1<x-2\end{cases}$$

The graph of $y=f(x)$ is as shown in the figure. From graph, clearly, there is one local maximum (at $x=-1$) and one local minima (at $x=0$)

$\therefore$ total number of local maxima or minima = 2.

30. **(c)** Given that $g(u)=2\tan^{-1}(e^u)-\frac{\pi}{2}$

$$\therefore\ g(-u)=2\tan^{-1}\left(e^{-u}\right)-\frac{\pi}{2}=2\tan^{-1}\left(\frac{1}{e^u}\right)-\frac{\pi}{2}$$

$$=2\cot^{-1}\left(e^u\right)-\frac{\pi}{2}=2\left[\frac{\pi}{2}-\tan^{-1}\left(e^u\right)\right]-\frac{\pi}{2}$$

$$=\pi-2\tan^{-1}(e^u)-\frac{\pi}{2}=\frac{\pi}{2}-2\tan^{-1}(e^u)$$

$=-g(u)$ $\quad\therefore$ g is an odd function.

Also $g'(u)=\frac{2e^u}{1+e^{2u}}>0$, $\forall u\in(-\infty,\infty)$

$\therefore$ g is strictly increasing on $(-\infty,\infty)$.

31. **(c)** Let $f(x)=4\alpha x^2+\frac{1}{x}$

For $x>0$, $f_{min}=1$

$$f'(x)=8\alpha x-\frac{1}{x^2}=0\Rightarrow x=\frac{1}{2\alpha^{1/3}}$$

$$f_{min}=1\Rightarrow 4\alpha\left(\frac{1}{2\alpha^{1/3}}\right)^2+2\alpha^{1/3}=1$$

$$\Rightarrow 3\alpha^{\frac{1}{3}}=1 \text{ or } \alpha=\frac{1}{27}$$

D. MCQs with ONE or MORE THAN ONE Correct

1. **(c)** We have $P'(x)=2a_1x+4a_2x^3+\ldots+2na_nx^{2n-1}$

$P'(x)=0\Rightarrow x=0$

$$P''(x)=2a_1+12a_2x^2+\ldots+2n(2n-1)a_nx^{2n-2}$$

$P''(x)|_{x=0}=+ve$ as $a_1>0$

$\therefore$ $P(x)$ has only one minimum at $x=0$.

2. **(b, c)** Let the line $ax+by+c=0$ be normal to the curve $xy=1$ at the point (x',y'), then

$x'y'=1\ldots(1)$ [$pt(x',y')$ lies on the curve]

Also differentiating the curve $xy=1$ with respect to x we get $y+x\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=-\frac{y}{x}$

$\Rightarrow \left.\frac{dy}{dx}\right)_{(x',y')}=\frac{-y'}{x'}$, $\therefore$ Slope of normal $=\frac{x'}{y'}$

Also equation of normal suggests, slope of normal $=\frac{-a}{b}$

$\therefore$ We must have,

$$\frac{x'}{y'}=-\frac{a}{b}\qquad\ldots(2)$$

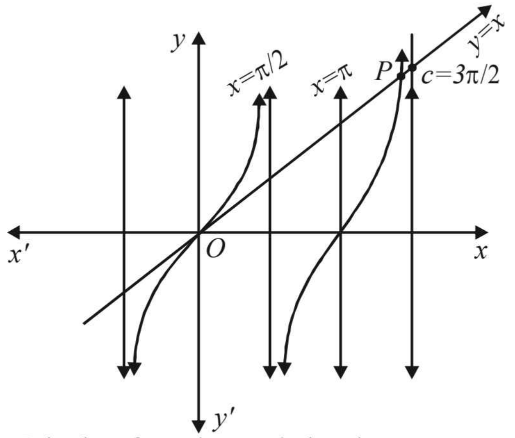
Now from eq. (1), $x'y'>0\Rightarrow x',y'$ are of same sign

$\Rightarrow$ $\frac{x'}{y'}=+ve\Rightarrow-\frac{a}{b}=+ve\Rightarrow\frac{a}{b}=-ve$

$\Rightarrow$ a and b are of opposite sign.

$\Rightarrow$ either $a<0$ and $b>0$ or $a>0$ and $b<0$.

3. **(c)**



It is clear from the graph that the curves $y=\tan x$ and $y=x$ intersect at P in $(\pi,3\pi/2)$.

Thus the smallest +ve root of $\tan x-x=0$ lies in $(\pi,3\pi/2)$.

4. **(a)** Since g is decreasing in $[0,\infty)$

$\therefore$ For $x\ge y$, $\quad g(x)\le g(y)$ ....(1)

Also $g(x), g(y)\in[0,\infty)$ and f is increasing from $[0,\infty)$ to $[0,\infty)$.

$\therefore$ For $g(x), g(y)\in[0,\infty)$ such that $g(x)\le g(y)$

$\Rightarrow$ $f(g(x))\le f(g(y))$, where $x\ge y$ $\Rightarrow$ $h(x)\le h(y)$

$\Rightarrow$ h is decreasing function from $[0,\infty)$ to $[0,\infty)$

$\therefore$ $h(x)\le h(0)$, $\forall\ x\ge 0$

But $h(0)=0$ (given)

$\therefore$ $h(x)\le 0\ \forall\ x\ge 0$ ...(2)

Also $h(x)\ge 0\ \forall\ x\ge 0$ ...(3)

[as $h(x)\in[0,\infty)$]

From (2) and (3), we get $h(x)=0$, $\forall\ x\ge 0$

Hence, $h(x)-h(1)=0-0=0\ \forall\ x\ge 0$

5. **(a, b, c)** We are given that

$$f(x)=\begin{cases}3x^2+2x-1, & -1\le x\le 2\\ 37-x, & 2<x\le 3\end{cases}$$

Then on $[-1,2]$, $f'(x)=6x+12$

For $-1\le x\le 2, -6\le 6x\le 12$

$\Rightarrow$ $6\le 6x+12\le 24$

$\Rightarrow$ $f'(x)>0, \forall x\in[-1,2]$

$\therefore$ f is increasing on $[-1,2]$

Also $f(x)$ being polynomial for $x\in[-1,2)\cup(2,3]$ $f(x)$ is cont. on $[-1,3]$ except possibly at

At $x=2$,

$$\text{LHL} = \lim_{h\to0} f(2-h) = \lim_{h\to0} 3(2-h)^2+12(2-h)-1 = 35$$

$$\text{RHL} = \lim_{h\to0} f(2+h) = \lim_{h\to0} 37-(2+h) = 35$$

and $f(2) = 3.2^2 + 12.2 - 1 = 35$

LHL = RHL = $f(2)$

$\Rightarrow$ $f(x)$ is continuous at $x = 2$

Hence $f(x)$ is continuous on $[-1, 3]$

Again at $x = 2$

$$RD = \lim_{h\to0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to0}\frac{37-(2+h)-35}{h} = 1$$

$$LD = \lim_{h\to0}\frac{f(2)-f(2-h)}{h}$$

$$= \lim_{h\to0}\frac{35-3(2-h)^2-12(2-h)+1}{h}$$

$$= \lim_{h\to0}\frac{-3h^2+24h}{h} = 24$$

As $LD \neq RD$

$\therefore$ $f'(2)$ does not exist. Hence $f(x)$ can not have max. value at $x = 2$.

6. **(a, c)** We have

$$h'(x) = f'(x)[1-2f(x)+3f(x))^2]$$

$$= 3f'(x)\left[(f(x))^2 - \frac{2}{3}f(x)+\frac{1}{3}\right]$$

$$= 3f'(x)[\{f(x)-1/3\}^2+2/9]$$

Note that $h'(x) < 0$ whenever $f'(x) < 0$ and $h'(x) > 0$ whenever $f'(x) > 0$, thus, $h(x)$ increases (decreases) whenever $f(x)$ increases (decreases).

7. **(d)** $f(x) = \dfrac{x^2-1}{x^2+1} = \dfrac{(x^2+1)-2}{x^2+1} = 1 - \dfrac{2}{x^2+1}$

For $f(x)$ to be min $\dfrac{2}{x^2+1}$ should be max, which is so if $x^2 + 1$ is min. And $x^2 + 1$ is min at $x = 0$.

$\therefore\ f_{\min} = \dfrac{0-1}{0+1} = -1$

8. **(b)** The maximum value of $f(x) = \cos x + \cos(\sqrt{2x})$ is 2 which occurs at $x = 0$. Also, there is no value of x for which this value will be attained again.

9. **(b,d)** $\dfrac{dy}{dx} = f'(x) \Rightarrow x(e^x-1)(x-1)(x-2)^3(x-3)^5 = 0$

Critical points are 0, 1, 2, 3. Consider change of sign of $\dfrac{dy}{dx}$ at $x = 3$.

$x < 3$, $\dfrac{dy}{dx} = -$ve and $x > 3$, $\dfrac{dy}{dx} = +$ve

Change is from –ve to +ve, hence minimum at $x = 3$. Again minimum and maximum occur alternately.

$\therefore$ 2nd minimum is at $x = 1$

10. **(b, c)** Let $f(x) = ax^3 + bx^2 + cx + d$

Then, $f(2) = 18 \Rightarrow 8a + 4b + 2c + d = 18$ ...(1)

$f(1) = -1 \Rightarrow a + b + c + d = -1$ ...(2)

$f(x)$ has local max. at $x = -1$

$\Rightarrow 3a - 2b + c = 0$ ...(3)

$f'(x)$ has local min. at $x = 0 \Rightarrow b = 0$ ...(4)

Solving (1), (2), (3) and (4), we get

$$f(x) = \frac{1}{4}(19x^3 - 57x + 34) \Rightarrow f(0) = \frac{17}{2}$$

Also $f'(x) = \dfrac{57}{4}(x^2-1) > 0, \forall x > 1$

Also $f'(x) = 0 \Rightarrow x = 1, -1$

$f''(-1) < 0, f''(1) > 0 \Rightarrow x = -1$ is a point of local max. and $x = 1$ is a point of local min. Distance between $(-1, 2)$ and $(1, f(1))$, i.e. $(1, -1)$ is $\sqrt{13} \neq 2\sqrt{5}$

11. **(a, b)** $\because\ g(x) = \int_0^x f(t)\,dt$

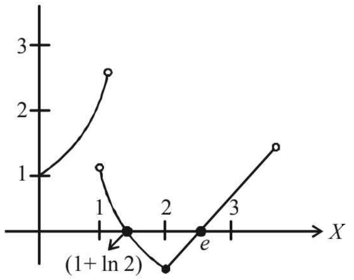
$$\Rightarrow g'(x) = f(x) = \begin{cases} e^x, & 0 \le x \le 1 \\ 2-e^{x-1}, & 1 < x \le 2 \\ x-e, & 2 < x \le 3 \end{cases}$$

$\therefore\ g'(x) = 0 \Rightarrow e^{x-1} = 2$ or $x - e = 0$

$\Rightarrow x - 1 = \log 2$ or $x = e \Rightarrow x = 1 + \ln 2$ or e

$$g''(x) = \begin{cases} e^x, & 0 \le x \le 1 \\ -e^{x-1}, & 1 < x \le 2 \\ 1, & 2 < x \le 3 \end{cases}$$

$\therefore\ g''(1+\ln 2) = -2$ and $g''(e) = 1 \Rightarrow g(x)$ has local max. at $x = 1 + \ln 2$ and local min. at $x = e$.



Graph of $g'(x)$

Also graph of $g'(x)$ suggests, $g(x)$ has local max. at $x = 1$ and local min. at $x = 2$

12. **(b, c, d)** We have, $f(x) = x\cos\dfrac{1}{x}$, $x \ge 1$

$\therefore\ f'(x) = \cos\dfrac{1}{x} + \dfrac{1}{x}\sin\dfrac{1}{x}$

$\lim_{x\to\infty} f'(x) = \cos 0 + (0) \times$ (some finite value)

$\Rightarrow \lim_{x\to\infty} f'(x) = 1$

Also $f''(x) = \dfrac{1}{x^2}\sin\dfrac{1}{x} - \dfrac{1}{x^2}\sin\dfrac{1}{x} - \dfrac{1}{x^3}\cos\dfrac{1}{x}$

$\Rightarrow f''(x) = \dfrac{-1}{x^3}\cos\dfrac{1}{x} < 0,\ \forall x \in [1, \infty)$

$\Rightarrow$ $f'(x)$ is strictly decreasing in $[1, \infty)$

$\therefore$ $f'(x) > \lim_{x\to\infty} f'(x)$

$\Rightarrow$ $\dfrac{f(x+2)-f(x)}{(x+2)-x} > 1 \Rightarrow f(x+2)-f(x) > 2$

13. (a, b, c, d)

We have $f(x) = \int_0^x e^{t^2}(t-2)(t-3)dt$

$\Rightarrow f'(x) = e^{x^2}.(x-2)(x-3)$ $f'(x) = 0 \Rightarrow x = 2, 3$

$f''(x) = e^{x^2}.2x(x^2-5x+6) + e^{x^2}(2x-5)$

$f''(2) = -$ ve and $f''(3) = +$ ve

$\therefore x = 2$ is a point of local maxima

and $x = 3$ is a point of local minima

Also or $x \in (2, 3) f'(x) < 0$

$\Rightarrow f$ is decreasing on $(2, 3)$

Also we observe $f''(0) < 0$ and $f''(1) > 0$

$\therefore$ There exists some $C \in (0, 1)$ such that $f''(C) = 0$

$\therefore$ All the options are correct.

14. (a, c) Let $L = 8x$, $B = 15x$ and y be the length of square cut off from each corner. Then volume of box

$= (8x-2y)(15x-2y)y$

$V = 120x^2y - 46xy^2 + 4y^3$

$\dfrac{dV}{dy} = 120x^2 - 92xy + 12y^2$

Now $\dfrac{dV}{dy} = 0$ at $y = 5$ for maximum value of V.

$\Rightarrow$ $[30x^2 - 23xy + 3y^2]_{y=5} = 0$

$\Rightarrow$ $6x^2 - 23x + 15 = 0 \Rightarrow x = 3, \dfrac{5}{6}$

For $x = 3$, sides are 45 and 24.

15. (a, c, d) $f(x) = \int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \dfrac{dt}{t}$

$\therefore$ $f'(x) = \dfrac{e^{-\left(x+\frac{1}{x}\right)}}{x} + \dfrac{x}{x^2} e^{-\left(\frac{1}{x}+x\right)} = \dfrac{2}{x} e^{-\left(x+\frac{1}{x}\right)}$

For $x \in [1, \infty)$, $f'(x) > 0$

$\therefore f$ is monotonically increasing on $[1, \infty)$

(a) is correct.

For $x \in (0, 1)$, $f'(x) > 0$

$\therefore$ (b) is not correct

$f(x) + f\left(\dfrac{1}{x}\right) = \int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)}\dfrac{dt}{t} + \int_{x}^{1/x} e^{-\left(t+\frac{1}{t}\right)}\dfrac{dt}{t} = 0$

$\therefore$ (c) is correct.

Replacing x by 2^x in $f(x) + f\left(\dfrac{1}{x}\right) = 0$

We get $f(2^x) + f\left(2^{-x}\right) = 0$ or $f(2^x) = -f(2^{-x})$

$\therefore$ $f(2^x)$ is an odd function.

$\therefore$ (d) is correct.

16. (b, c) Let $h(x) = f(x) - 3g(x)$

$h(-1) = h(0) = h(2) = 3$

$\therefore$ By Rolle's theorem $h'(x) = 0$ has atleast one solution in $(-1, 0)$ and atleast one solution in $(0, 2)$ But $h''(x)$ never vanishes in $(-1, 0)$ and $(0, 2)$ therefore $h'(x) = 0$ should have exactly one solution in each interval.

17. (a, d) $\lim_{x\to 2} \dfrac{f(x)g(x)}{f'(x)g'(x)} = 1$ $\left[\dfrac{0}{0}\text{form}\right]$

$\Rightarrow$ $\lim_{x\to 2} \dfrac{f'(x)g(x)+f(x)g'(x)}{f''(x)g'(x)+f'(x)g''(x)} = 1$

$\Rightarrow$ $\dfrac{g'(2)f(2)}{f''(2)g'(2)} = 1 \Rightarrow f(2) = f''(2)$

$\therefore$ $f(x) - f''(x) = 0$ for atleast one $x \in R$.

$\because$ Range of $f(x)$ is $(0, \infty)$

$\therefore$ $f(x) > 0, \forall x \in R$

$\Rightarrow$ $f(2) > 0 \Rightarrow f''(2) > 0$

$\Rightarrow$ f has a local minimum at $x = 2$

E. Subjective Problems

1. $f(x) = \dfrac{(a+x)(b+x)}{(c+x)}, a, b > c, x > -c$

$= \dfrac{(a-c+x+c)(b-c+x+c)}{x+c}$

$= \dfrac{(a-c)(b-c)}{x+c} + (x+c) + a + b - 2c$

$\Rightarrow$ $f'(x) = \dfrac{-(a-c)(b-c)}{(x+c)^2} + 1$

$\therefore$ $f'(x) = 0 \Rightarrow x = -c \pm \sqrt{(a-c)(b-c)}$

$\Rightarrow$ $x = -c + \sqrt{(a-c)(b-c)}$ [+ve sign is taken $\because x > -c$]

Also $f''(x) = \dfrac{2(a-c)(b-c)}{(x+c)^3} > 0$ for $a, b > c$ and $x > -c$

$\therefore$ $f(x)$ is least at $x = -c + \sqrt{(a-c)(b-c)}$

$\therefore$ $f_{min} = \dfrac{(a-c)(b-c)}{\sqrt{(a-c)(b-c)}} + \sqrt{(a-c)(b-c)}$

$+ (a-c) + (b-c)$

$= (a-c) + (b-c) + 2\sqrt{(a-c)(b-c)}$

$= (\sqrt{a-c} + \sqrt{b-c})^2$

2. Given that x and y are two real variables such that $x > 0$ and $xy = 1$.

To find the minimum value of $x + y$.

Let $S = x + y$

$\Rightarrow$ $S = x + \dfrac{1}{x}$ (using $xy = 1$)

$\therefore$ $\dfrac{dS}{dx} = 1 - \dfrac{1}{x^2}$

For minimum value of S, $\dfrac{dS}{dx} = 0$

$\Rightarrow \quad 1-\frac{1}{x^2}=0 \Rightarrow x=\pm 1$

But $x>0$, $\therefore x=1$

Now $\frac{d^2S}{dx^2}=\frac{2}{x^3}$

$\Rightarrow \left.\frac{d^2S}{dx^2}\right|_{x=1}=2=+ve$

$\therefore$ S is minimum when $x=1$ $\therefore S_{\min}=1+\frac{1}{1}=2$

3. We are given that

$x\in[0,1], |f''(x)|<1$ and $f(0)=f(1)$

To prove that $|f'(x)|<1, \forall x\in[0,1]$

Here $f(x)$ is continuous on [0, 1], differentiable on (0, 1) and $f(0)=f(1)$

$\therefore$ By Rolle's thm.,

$\exists c\in(0,1)$ such that $f'(c)=0$ ...(1)

Now there may be three cases for $x\in[0,1]$

(i) $x=c$ (ii) $x>c$ (iii) $x<c$

Case I : For $x=c$.

If $x=c$ then $f'(x)=0<1$ [from (1)]

Hence the result $|f'(x)|<1$ is obtained in this case.

Case II : For $x>c$

Consider the interval $[c, x]$.

As $f'(x)$ is continuous on $[c, x]$ and differentiable on (c, x)

$\therefore$ By LMV $f''(\alpha)=\frac{f'(x)-f'(c)}{x-c}$ where $\alpha\in(c,x)$

$\Rightarrow f'(x)=(x-c)f''(\alpha)$ $[\because f'(c)=0]$

Now, $x,c\in[0,1]$ and $x>c$

$\therefore \quad x-c<1$ .... (i)

also $|f''(x)|<1, \forall x$ (given)

$\therefore \quad |f''(\alpha)|<1$ .... (ii)

Combining (i) and (ii), $(x-c)|f''(\propto)|<1$

$\therefore \quad |f'(x)|<1$. Hence the result in this case.

Case III : For $x<c$

Consder the interval $[x, c]$.

As $f'(x)$ is continuous on $[x, c]$ and differentiable on (x, c)

$\therefore$ By LMV for $\beta\in(x,c)$

$f''(\beta)=\frac{f'(c)-f'(x)}{c-x} \Rightarrow f'(x)=-(c-x)f''(\beta)$

[Using $f'(c)=0$]

$\therefore \quad |f'(x)|=|(c-x)f''(\beta)|$

as $x,c\in[0,1]$ and $x<c$

$\therefore \quad 0<c-x<1$ also $|f''(\beta)|<1$ as $|f''(x)|<1, \forall x$

$\therefore \quad |(c-x)f''(\beta)|<1$

$\therefore \quad |f''(x)|<1$ hence the result in this case.

Combining all the three cases we get

$|f'(x)|<1, \forall x\in[0,1]$

4. $f(x)=x^{1/x},\ x>0$

Let $y=x^{1/x} \Rightarrow \log y=\frac{1}{x}\log x$

Differentiating w.r.t. x we get

$\frac{1}{y}\frac{dy}{dx}=\frac{\frac{1}{x}.x-1.\log x}{x^2} \Rightarrow \frac{dy}{dx}=\frac{y(1-\log x)}{x^2}$

For max/min value put $\frac{dy}{dx}=0$

$\Rightarrow \frac{y(1-\log x)}{x^2}=0 \Rightarrow \log x=1 \Rightarrow x=e$

Also, $\frac{d^2y}{dx^2}=\frac{\left(\frac{dy}{dx}(1-\log x)-\frac{1}{x}y\right)x^2-2xy(1-\log x)}{x^4}$

$\left.\frac{d^2y}{dx^2}\right|_{x=e}=\left(\frac{-xy}{x^4}\right)_{x=e}$

$\left[\text{Using } \frac{dy}{dx}=0, 1-\log x=0 \text{ at } x=e\right]$

$=\frac{-e^{1/e}}{e^3}=-ve$

$\therefore$ y is max at $x=e$

$\therefore$ $e^{1/e}$ is the max. value of $f(x)$.

$\therefore \quad x^{1/x}<e^{1/e}, \forall x$

$\therefore$ Put $x=\pi$, we get, $\pi^{1/\pi}<e^{1/e}$

$\Rightarrow$ Raising to the power πe on both sides we get

$\pi^e<e^\pi$ or $e^\pi>\pi^e$

5. Given that $f(x)$ and $g(x)$ are differentiable for $x\in[0,1]$ such that $f(0)=2$; $f(1)=6$, $g(0)=0$; $g(1)=2$

To show that $\exists c\in(0,1)$ such that $f'(c)=2g'(c)$

Let us consider $h(x)=f(x)-2g(x)$

Then $h(x)$ is continuous on [0, 1] and differentiable on (0, 1)

Also $h(0)=f(0)-2g(0)=2-2\times 0=2$

$h(1)=f(1)-2g(1)=6-2\times 2=2$

$\therefore \quad h(0)=h(1)$

$\therefore$ All the conditions of Rolle's theorem are satisfied for $h(x)$ on [0, 1]

$\therefore \quad \exists c\in(0,1)$ such that $h'(c)=0$

$\Rightarrow f'(c)-2g'(c)=0 \Rightarrow f'(c)=2g'(c)$

6. $(0, c), y=x^2, 0\le c\le 5$.

Any point on parabola is (x, x^2)

Distance between (x, x^2) and (0, 1) is

$D=\sqrt{x^2+(x^2-c)^2}$

To minimum D we consider

$D^2=x^4-(2c-1)x^2+c^2=\left(x^2-\frac{2c-1}{2}\right)^2+c-\frac{1}{4}$

which is minimum when $x^2-\frac{2c-1}{2}=0 \Rightarrow x^2=\frac{2c-1}{2}$

$\Rightarrow D_{\min}=\sqrt{c-\frac{1}{4}}$

7. Given $ax^2 + \frac{b}{x} \ge c$ ...(1)

$\forall x > 0, a > 0, b > 0$

To show that $27ab^2 \ge 4c^3$.
Let us consider the function $f(x) = ax^2 + b/x$

then $f'(x) = 2ax - \frac{b}{x^2} = 0$

$\Rightarrow x^3 = b/2a \Rightarrow x = (b/2a)^{1/3}$

$\therefore \quad f''(x) = 2a + \frac{2b}{x^3}$

$\Rightarrow \quad f''\left(\left(\frac{b}{2a}\right)^{1/3}\right) = 2a + \frac{2b}{b} \times 2a = 6a > 0$

$\therefore \quad f$ is minimum at $x = \left(\frac{b}{2a}\right)^{1/3}$

As (1) is true $\forall\ x$

$\therefore$ so is for $x = \left(\frac{b}{2a}\right)^{1/3}$

$\Rightarrow \quad a\left(\frac{b}{2a}\right)^{2/3} + \frac{b}{(b/2a)^{1/3}} \ge c$

$\Rightarrow \quad \frac{a\left(\frac{b}{2a}\right) + b}{(b/2a)^{1/3}} \ge c \Rightarrow \frac{3b}{2}\left(\frac{2a}{b}\right)^{1/3} \ge c$

As a, b are +ve, cubing both sides we get

$\frac{27b^3}{8}.\frac{2a}{b} \ge c^3 \Rightarrow 27ab^2 \ge 4c^3$ Hence proved.

8. To show

$1 + x\ln(x + \sqrt{x^2+1}) \ge \sqrt{1+x^2}$ for $x \ge 0$

Consider $f(x) = 1 + x\ln(x + \sqrt{x^2+1}) - \sqrt{1+x^2})$

Here, $f'(x) = \ln(x + \sqrt{x^2+1}) + \frac{x}{x + \sqrt{x^2+1}}\left[1 + \frac{x}{\sqrt{x^2+1}}\right] - \frac{x}{\sqrt{1+x^2}}$

$= \ln(x + \sqrt{x^2+1})$

As $x + \sqrt{x^2+1} \ge 1$ for $x \ge 1$

$\therefore \quad \ln(x + \sqrt{x^2+1}) \ge 0$

$\therefore \quad f'(x) \ge 0, \forall\, x \ge 0$

Hence $f(x)$ is increasing function.

Now for $x \ge 0 \Rightarrow f(x) \ge f(0)$

$\Rightarrow \quad 1 + x\ln(x + \sqrt{x^2+1}) - \sqrt{1+x^2} \ge 0$

$\Rightarrow \quad 1 + x\ln(x + \sqrt{x^2+1}) \ge \sqrt{1+x^2}$

9. Equation of the curve is given by

$y = \frac{x}{1+x^2}$ ...(1)

Differentiating with respect to x, we get

$\frac{dy}{dx} = \frac{1+x^2 - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$

Again let $f(x) = \frac{1-x^2}{(1+x^2)^2} = \frac{dy}{dx}$

Now, $f'(x) = \frac{(1+x^2)^2(-2x) - (1-x^2)2(1+x^2)(2x)}{(1+x^2)^4}$

$= \frac{(1+x^2)(-2x) - (1-x^2)2.2x}{(1+x^2)^3} = \frac{x(2x^2-6)}{(1+x^2)^3}$

For the greatest value of slope, we have

$f'(x) = \frac{x(2x^2-6)}{(1+x^2)^3} = 0 \Rightarrow x = 0, \pm\sqrt{3}$

Again we find,

$f''(x) = \frac{12x^2(3-x^2)}{(1+x^2)^4} - \frac{6(1-x^2)}{(1+x^2)^3}$

$\therefore \quad f''(0) = -6$ and $f''(\pm\sqrt{3}) = \frac{3}{16}$

Thus, second order derivative at $x = 0$ is negative and second order derivative at $x = \pm\sqrt{3}$ is positive.
Therefore, the tangent to the curve has maximum slope at (0, 0).

10. Equation of given curve $y = \cos(x+y)$, $-2\pi \le x \le 2\pi$
Differentiating with respect to x,

$\frac{dy}{dx} = -\sin(x+y).\left[1 + \frac{dy}{dx}\right]$

$\Rightarrow \quad [1 + \sin(x+y)]\frac{dy}{dx} = -\sin(x+y)$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{\sin(x+y)}{1+\sin(x+y)}$ ...(1)

Since the tangent to given curve is parallel to $x + 2y = 0$

$\therefore \quad \frac{-\sin(x+y)}{1+\sin(x+y)} = -\frac{1}{2}$ [For parallel line $m_1 = m_2$]

$\Rightarrow \quad 2\sin(x+y) = 1 + \sin(x+y)$

$\Rightarrow \quad \sin(x+y) = 1$

Thus, $\cos(x+y) = 0$

Using equation of curve and above result, we get, $y = 0$

$\Rightarrow \quad \sin x = 1 \Rightarrow x = n\pi + (-1)^n \pi/2, n \in Z \Rightarrow x = \pi/2, -3\pi/2$

which belong to the interval $[-2\pi, 2\pi]$

Thus the points on curve at which tangets are parallel to given line are $(\pi/2, 0)$ and $(-3\pi/2, 0)$

The equation of tangent at $(\pi/2, 0)$ is

$y - 0 = -\frac{1}{2}(x - \pi/2)$

$\Rightarrow 2y = -x + \pi/2 \Rightarrow 2x + 4y - \pi = 0$

The equation of tangent at $(-3\pi/2, 0)$ is

$y - 0 = -\frac{1}{2}(x + 3\pi/2)$

$\Rightarrow 2y = -x - 3\pi/2 \Rightarrow 2x + 4y + 3\pi = 0$

Thus the required equations of tangents are
$2x + 4y - \pi = 0$ and $2x + 4y + 3\pi = 0$.

11. The given function is,
$f(x) = \sin^3 x + \lambda\sin^2 x$ for $-\pi/2 < x < \pi/2$

$\therefore \quad f'(x) = 3\sin^2 x\cos x + 2\lambda\sin x\cos x$

$$= \frac{1}{2}\sin 2x(3\sin x + 2\lambda)$$

So, from $f'(x) = 0$, we get $x = 0$

or $\quad 3\sin x + 2\lambda = 0$

Also, $f''(x) = \cos 2x\,(3\sin x + 2\lambda) + \frac{3}{2}\sin 2x\cos x$

Therefore, for $\lambda = \frac{-3}{2}\sin x$, we have

$$f''(x) = 3\sin x\cos^2 x = -2\lambda\cos^2 x$$

Now, if $0 < x < \pi/2$, then $-3/2 < \lambda < 0$ and therefore $f''(x) > 0$.

$\Rightarrow$ $f(x)$ has one minimum for this value of λ.

Also for $x = 0$, we have $f''(0) = 2\lambda < 0$, That is $f(x)$ has a maximum at $x = 0$

Again if $-\pi/2 < x < 0$, then $0 < \lambda < 3/2$ and therefore $f''(x) = -2\lambda\cos^2 x < 0$.

So that $f(x)$ has a maximum.

Also for $x = 0, f''(a) = 2\lambda > 0$ so that $f(x)$ has a minimum.

Thus, for exactly one maximum and minimum value of $f(x)$, λ must lie in the interval

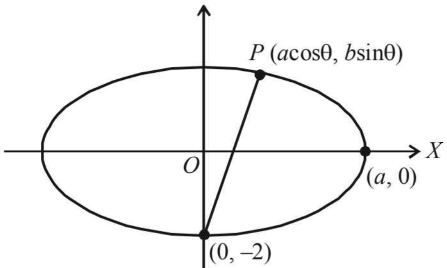
$$-3/2 < \lambda < 0 \text{ or } 0 < \lambda < 3/2$$

i.e., $\lambda \in (-3/2, 0) \cup (0, 3/2)$.

12. The equation of given curve can be expressed as

$$\frac{x^2}{a^2} + \frac{y^2}{4} = 1 \text{ where } 4 < a^2 < 8$$

Clearly it is the question of an ellipse



Let us consider a point $P\,(a\cos\theta, 2\sin\theta)$ on the ellipse.

Let the distance of $P\,(a\cos\theta, 2\sin\theta)$ from $(0, -2)$ is L.

Then, $L^2 = (a\cos\theta - 0)^2 + (2\sin\theta + 2)^2$

$\Rightarrow$ Differentiating with respect to θ, we have

$$\frac{d(L^2)}{d\theta} = \cos\theta[-2a^2\sin\theta + 8\sin\theta + 8]$$

For max. or min. value of L we should have

$$\frac{d(L^2)}{d\theta} = 0$$

$\Rightarrow$ $\cos\theta\,[-2a^2\sin\theta + 8\sin\theta + 8] = 0$

$\Rightarrow$ Either $\cos\theta = 0$

or $\quad (8 - 2a^2)\sin\theta + 8 = 0 \Rightarrow \theta = \frac{\pi}{2}$ or $\sin\theta = \frac{4}{a^2 - 4}$

Since $a^2 < 8 \Rightarrow a^2 - 4 < 4$

$\Rightarrow \quad \frac{4}{a^2 - 4} > 1 \Rightarrow \sin\theta > 1$ which is not possible

Also $\frac{d^2(L^2)}{d\theta^2} = \cos\theta[-2a^2\cos\theta + 8\cos\theta]$

$$+ (-\sin\theta)\,[-2a^2\sin\theta + 8\sin\theta + 8]$$

At $\theta = \frac{\pi}{2}, \frac{d^2(L^2)}{d\theta^2} = 0 - [16 - 2a^2] = 2(a^2 - 8) < 0$

as $a^2 < 8$

$\therefore$ L is max. at $\theta = \pi/2$ and the farthest point is $(0, 2)$.

13. We have,

$$f(x) = \int_1^x [2(t-1)(t-2)^3 + (t-1)^2 3(t-2)^2]dt$$

Then using the theorem,

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} g(t)dt\right] = g[\psi(x)]\psi'(x) - g[\phi(x)]\phi'(x)$$

We get,

$$f'(x) = 2\,(x-1)\,(x-2)^3 + 3\,(x-1)^2\,(x-2)^2$$
$$= (x-1)\,(x-2)^2\,(2x - 4 + 3x - 3)$$
$$= (x-1)\,(x-2)^2\,(5x-7)$$

For extreme values $f'(x) = 0 \Rightarrow x = 1, 2, 7/5$

Now, $f''(x) = (x-2)^2\,(5x-7) + 2\,(x-1)\,(x-2)\,(5x-7) + 5\,(x-1)\,(x-2)^2$

At $x = 1$, $f''(x) = 1\,(-2) = -2 < 0$

$\therefore$ f is max. at $x = 1$

At $x = 2$ $f''(x) = 0$

$\therefore$ f is neither maximum nor minimum at $x = 2$.

At $\quad x = 7/5$

$$f''(x) = 5\left(\frac{7}{5} - 1\right)\left(\frac{7}{5} - 2\right)^2 = 5 \times \frac{2}{5} \times \frac{9}{25} = \frac{18}{25} > 0$$

$\therefore$ $f(x)$ is minimum at $x = 7/5$.

14. We have $y = x\,(x-1)^2,\ 0 \le x \le 2$

$$\frac{dy}{dx} = (x-1)^2 + 2x(x-1) = (x-1)(3x-1)$$

For max. or min. $\frac{dy}{dx} = 0$

$\Rightarrow$ $(x-1)\,(3x-1) = 0 \Rightarrow x = 1, 1/3$

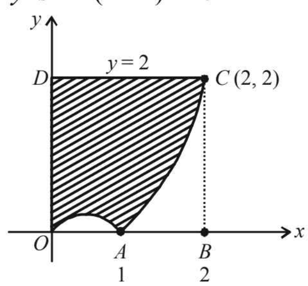
$$\frac{d^2y}{dx^2} = 3x - 1 + 3(x-1) = 6x - 4$$

At $x = 1$, $\frac{d^2y}{dx^2} = 2(+ve)$ $\therefore$ y is min. at $x = 1$

At $x = 1/3, \frac{d^2y}{dx^2} = -2(-ve)$ $\therefore$ y is max. at $x = 1/3$

$\therefore$ Max value of y is $= \frac{1}{3}\left(\frac{1}{3} - 1\right)^2 = \frac{4}{27}$

Min value of y is $= 1\,(1-1)^2 = 0$



Now the curve cuts the axis x at $(0, 0)$ and $(1, 0)$. When x increases from 1 to 2, y also increases and is +ve.

When $y = 2$, $x\,(x-1)^2 = 2$

$\Rightarrow$ $x = 2$

Using max./min. values of y and points of intersection with x-axis, we get the curve as in figure and shaded area is the required area.

$\therefore$ The required area = Area of square $OBCD - \int_0^2 y\,dx$

$= 2\times 2 - \int_0^2 x(x-1)^2\,dx$

$= 4 - \left[\left(x\frac{(x-1)^3}{3}\right)_0^2 - \frac{1}{3}\int_0^2 (x-1)^3 .1\,dx\right]$

$= 4 - \left[\frac{x}{3}(x-1)^3 - \frac{(x-1)^4}{12}\right]_0^2$

$= 4 - \left[\frac{2}{3} - \frac{1}{12} + \frac{1}{12}\right] = 4 - \frac{2}{3} = \frac{10}{3}$ sq.units.

15. Let $f(x) = 2\sin x + \tan x - 3x$ on $0 \le x < \pi/2$

then $f'(x) = 2\cos x + \sec^2 x - 3$

and $f''(x) = -2\sin x + 2\sec^2 x \tan x$

$= 2\sin x\,[\sec^3 x - 1]$

for $0 \le x < \pi/2$ $\quad f''(x) \ge 0$

$\Rightarrow$ $f'(x)$ is an increasing function on $0 \le x < \pi/2$.

$\therefore$ For $x \ge 0, \Rightarrow f'(x) \ge f'(0)$

$\Rightarrow$ $f'(x) \ge 0$ for $0 \le x < \pi/2$

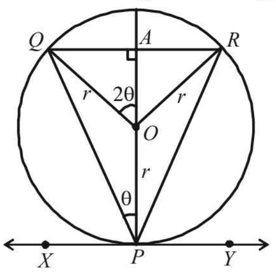
$\Rightarrow$ $f(x)$ is an increasing function on $0 \le x < \pi/2$

$\therefore$ For $x \ge 0,\ f(x) \ge f(0)$

$\Rightarrow$ $2\sin x + \tan x - 3x \ge 0,\quad 0 \le x < \pi/2$

$\Rightarrow$ $2\sin x + \tan x \ge 3x,\quad 0 \le x < \pi/2$ Hence proved

16. As $QR \parallel XY$ diameter through P is $\perp$ QR.



Now area of ΔPQR is given by $A = \frac{1}{2}QR.AP$

But $QR = 2.QA = 2r\sin 2\theta$

and $PA = OA + OP = r\cos 2\theta + r$

$\therefore$ $A = \frac{1}{2}.2r\sin 2\theta.(r + r\cos 2\theta)$

$= r^2.\ 2\sin\theta\cos\theta\ .\ 2\cos^2\theta = 4r^2\sin\theta\cos^3\theta$

For max. value of area, $\frac{dA}{d\theta} = 0$

$\Rightarrow$ $4r^2[\cos^4\theta - 3\sin^2\theta\cos^2\theta] = 0$

$\Rightarrow$ $\cos^2\theta(\cos^2\theta - 3\sin^2\theta) = 0 \Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30°$

Also $\frac{d^2A}{d\theta^2} = 4r^2[-4\cos^3\theta\sin\theta - 6\sin\theta\cos^3\theta$

$+ 6\sin^3\theta\cos\theta]$

$= 4r^2\,[-10\sin\theta\cos^3\theta + 6\sin^3\theta\cos\theta]$

$\left.\frac{d^2A}{d\theta^2}\right|_{\theta=30°} = 4r^2\left[-10.\frac{1}{2}.\frac{3\sqrt{3}}{8} + 6.\frac{1}{8}.\frac{\sqrt{3}}{2}\right]$

$= 4r^2\left[\frac{-15\sqrt{3}}{8} + \frac{3\sqrt{3}}{8}\right] = 4r^2\left(\frac{-12\sqrt{3}}{8}\right) = -ve$

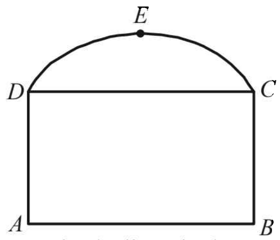
$\therefore$ A is maximum at $\theta = 30°$

And $A_{max} = 4r^2\sin 30°\cos^3 30° = 4r^2 \times \frac{1}{2} \times \frac{3\sqrt{3}}{8} = \frac{3\sqrt{3}}{4}r^2$

17. Let $ABCEDA$ be the window as shown in the figure and let

$AB = x\,m$

$BC = y\,m$



Then its perimeter including the base DC of arch

$= \left(2x + 2y + \frac{\pi x}{2}\right)m$

$\therefore$ $P = \left(2 + \frac{\pi}{2}\right)x + 2y$ ...(1)

Now, area of rectangle $ABCD = xy$

and area of arch $DCED = \frac{\pi}{2}\left(\frac{x}{2}\right)^2$

Let λ be the light transmitted by coloured glass per sq. m. Then 3λ will be the light transmitted by clear glass per sq. m.

Hence the area of light transmitted $= 3\lambda(xy) + \lambda\left[\frac{\pi}{2}\left(\frac{x}{2}\right)^2\right]$

$\Rightarrow$ $A = \lambda\left[3xy + \frac{\pi x^2}{8}\right]$ ......(2)

Substituting the value of y from (1) in (2), we get

$A = \lambda\left[3x\frac{1}{2}\left[P - \left(\frac{4+\pi}{2}\right)x\right] + \frac{\pi x^2}{8}\right]$

$= \lambda\left[\frac{3Px}{2} - \frac{3(4+\pi)}{4}x^2 + \frac{\pi x^2}{8}\right]$

$\therefore$ $\frac{dA}{dx} = \lambda\left[\frac{3P}{2} - \frac{3(4+\pi)}{2}x + \frac{\pi x}{4}\right]$

For A to be maximum $\frac{dA}{dx} = 01$

$\Rightarrow$ $x = \frac{\frac{3P}{2}}{\frac{-\pi}{4} + \left(\frac{12+3\pi)}{2}\right)}$

$\Rightarrow$ $x = \frac{3P}{2} \times \frac{4}{5\pi + 24} \Rightarrow x = \frac{6P}{5\pi + 24}$

Also $\frac{d^2A}{dx^2} = \lambda\left[\frac{-3(4+\pi)}{2} + \frac{\pi}{4}\right] < 0$

$\therefore$ A is max when $x = \frac{6P}{5\pi + 24}$

[Using value of P from (1)]

$\Rightarrow\ (5\pi+24-12-3\pi)x=12y \Rightarrow (2\pi+12)x=12y$

$\Rightarrow\ \frac{y}{x}=\frac{\pi+6}{6}$

$\therefore$ The required ratio of breadth to length of the rectangle $=6+\pi:6$

18. Let $f(x)=ax^3+bx^2+cx+d$

ATQ, $f(x)$ vanishes at $x=-2$

$\Rightarrow\ -8a+4b-2c+d=0$ ...(1)

$f'(x)=3ax^2+2bx+c$

Againg ATQ, $f(x)$ has relative max./min at

$x=-1$ and $x=\frac{1}{3}$

$\Rightarrow\ f'(-1)=0=f'(1/3)$

$\Rightarrow\ 3a-2b+c=0$ ...(2)

and $a+2b+3c=0$ ...(3)

Also, $\int_{-1}^{1} f(x)=\frac{14}{3}$

$\Rightarrow\ \left(\frac{ax^4}{4}+\frac{bx^3}{3}+\frac{cx^2}{2}+dx\right)_{-1}^{1}=\frac{14}{3}$

$\Rightarrow\ \left[\frac{a}{4}+\frac{b}{3}+\frac{c}{2}+d\right]-\left[\frac{a}{4}-\frac{b}{3}+\frac{c}{2}-d\right]=\frac{14}{3} \Rightarrow \frac{b}{3}+d=\frac{7}{3}$

$\Rightarrow b+3d=7$ ...(4)

From (1), (2), (3), (4) on solving, we get

$a=1, b=1, c=-1, d=2$

$\therefore$ The required cubic is x^3+x^2-x+2.

19. The given curve is $y=x^2$ ...(1)

Consider any point $A\ (t, t^2)$ on (1) at which normal chord drawn is shortest.

Then eq. of normal to (1) at $A\ (t, t^2)$ is

$y-t^2=-\frac{1}{\left(\frac{dy}{dx}\right)_{(t,t^2)}}(x-t)$ [where $\frac{dy}{dx}=2x$ from (1)]

$y-t^2=-\frac{1}{2t}(x-t)$

$\Rightarrow\ x+2ty=t+2t^3$ ...(2)

This normal meets the curve again at point B which can be obtained by solving (1) and (2) as follows :

Putting $y=x^2$ in (2), we get

$2tx^2+x-(t+2t^3)=0,$

$D=1+8t(t+2t^3)=1+8t^2+16t^4=(1+4t^2)^2$

$\therefore\ x=\frac{-1+1+4t^2}{4t}, \frac{-1-1-4t^2}{4t}=t, -\frac{1}{2t}-t$

$\therefore\ y=t^2, t^2+\frac{1}{4t^2}+1$

Thus, $B\left(-t-\frac{1}{2t}, t^2+\frac{1}{4t^2}+1\right)$

$\therefore$ Length of normal chord

$AB=\sqrt{\left(2t+\frac{1}{2t}\right)^2+\left(\frac{1}{4t^2}+1\right)^2}$

Consider $Z=AB^2=\left(2t+\frac{1}{2t}\right)^2+\left(\frac{1}{4t^2}+1\right)^2$

$\Rightarrow\ Z=\frac{1}{16t^4}+\frac{3}{4t^2}+3+4t^2$

For shortest chord, we have to minimize Z, and for that $\frac{dZ}{dt}=0$

$\Rightarrow\ -\frac{1}{4t^5}-\frac{3}{2t^3}+8t=0 \Rightarrow -1-6t^2+32t^6=0$

$\Rightarrow\ 32(t^2)^3-6t^2-1=0 \Rightarrow (2t^2-1)(16t^4+8t^2+1)=0$

$\Rightarrow\ t^2=\frac{1}{2}$ (leaving –ve values of t^2)

$\Rightarrow\ t=\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}},$

$\frac{d^2Z}{dt^2}=\frac{5}{4t^6}+\frac{9}{2t^4}+8$

$\left.\frac{d^2Z}{dt^2}\right|_{t=\frac{1}{\sqrt{2}}}=+ve$ also $\left.\left|\frac{d^2Z}{dt^2}\right|\right|_{t=-\frac{1}{\sqrt{2}}}=+ve$

$\therefore$ Z is minimum at $t=\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$

For $t=\frac{1}{\sqrt{2}}$ normal chord is (from (2)) $x+\sqrt{2}y=\sqrt{2}$

For $t=-\frac{1}{\sqrt{2}}$ normal chord is $x-\sqrt{2}y=-\sqrt{2}$

20. The given curve is $y=(1+x)^y+\sin^{-1}(\sin^2 x)$

Here at $x=0, y=(1+0)^y+\sin^{-1}(0) \Rightarrow y=1$

$\therefore$ Point at which normal has been drawn is (0, 1).

For slope of normal we need to find dy/dx, and for that we consider the curve as $y=u+v \Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$

where $u=(1+x)^y$ ...(i)

and $v=\sin^{-1}(\sin^2 x)$ ...(ii)

Taking log on both sides of equation (i) we get

$\log u=y\log(1+x)$

$\Rightarrow\ \frac{1}{u}\frac{du}{dx}=\frac{y}{1+x}+\log(1+x).\frac{dy}{dx}$

$\Rightarrow\ \frac{du}{dx}=(1+x)^y\left[\frac{y}{1+x}+\log(1+x)\frac{dy}{dx}\right]$

Also $v=\sin^{-1}(\sin^2 x)$

$\Rightarrow\ \frac{dv}{dx}=\frac{1}{\sqrt{1-\sin^4 x}}.2\sin x\cos x$

$\Rightarrow\ \frac{dv}{dx}=\frac{2\sin x}{\sqrt{1+\sin^2 x}}.$

Thus, we get,

$\frac{dy}{dx}=(1+x)^y\left[\frac{y}{1+x}+\log(1+x)\frac{dy}{dx}\right]+\frac{2\sin x}{\sqrt{1+\sin^2 x}}$

$\Rightarrow\ [1-(1+x)^y\log(1+x)]\frac{dy}{dx}$

$=y(1+x)^{y-1}+\frac{2\sin x}{\sqrt{1+\sin^2 x}}$

$\Rightarrow\ \frac{dy}{dx}=\frac{y(1+x)^{y-1}+\frac{2\sin x}{\sqrt{1+\sin^2 x}}}{1-(1+x)^y\log(1+x)}$

$\left.\frac{dy}{dx}\right|_{(0,1)} = 1$, $\therefore$ Slope of normal = – 1

$\therefore$ Equation of normal to given curve at (0, 1) is
$y - 1 = -1\,(x - 0) \Rightarrow x + y = 1.$

21. We have, $f(x) = \begin{cases} -x^3 + \dfrac{b^3 - b^2 + b - 1}{b^3 + 3b + 2}, & 0 \le x < 1 \\ 2x - 3, & 1 \le x \le 3 \end{cases}$

We can see from definition of the function, that
$f(1) = 2(1) - 3 = -1$
Also $f(x)$ is increasing on [1, 3], $f'(x)$ being $2 > 0$.
$\therefore$ $f(1) = -1$ is the smallest value of $f(x)$
Again $f'(x) = -3x^2$ for $x \in [0, 1]$ such that $f'(x) < 0$
$\Rightarrow$ $f(x)$ is decreasing on [0, 1]
$\therefore$ For fixed value of b, its smallest occur when $x \to 1$

i.e., $\lim\limits_{h\to 0} f(1-h) = \lim\limits_{h\to 0} -(1-h)^3 + \dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}$

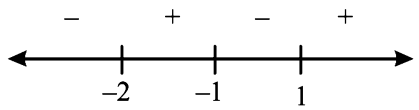
$= -1 + \dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}$

As given that the smallest value of $f(x)$ occur at $x = 1$
$\therefore$ Any other smallest value $\ge f(1)$

$\Rightarrow -1 + \dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} \ge -1$

$\Rightarrow \dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} \ge 0 \Rightarrow \dfrac{(b^2+1)(b-1)}{(b+2)(b+1)} \ge 0$

$\Rightarrow (b-1)(b+1)(b+2) \ge 0$



$\Rightarrow b \in (-2, -1) \cup (1, \infty).$

22. Given that $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-2, 0)$

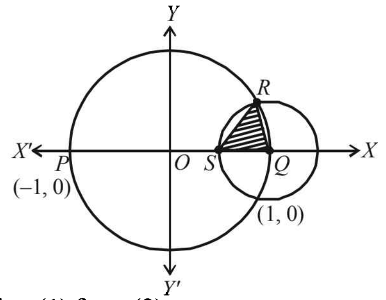
$\Rightarrow \left(\dfrac{dy}{dx}\right)_{x=-2} = 0$ and $P(-2, 0)$ lies on curve

$\Rightarrow 3ax^2 + 2bx + c]_{x=-2} = 0$
$\Rightarrow 12a - 4b + c = 0$ ...(1)
and $-8a + 4b - 2c + 5 = 0$ ...(2)
[$\because$ (– 2, 0) lies on curve]
Also the curve cuts the y-axis at Q
$\therefore$ For $x = 0, y = 5$ $\therefore Q(0, 5)$
At Q gradient of the curve is 3

$\Rightarrow \left.\dfrac{dy}{dx}\right|_{x=0} = 3 \Rightarrow 3x^2 + 2bx + c]_{x=0} = 3$

$\Rightarrow c = 3$ ...(3)
Solving (1), (2) and (3), we get
$a = -1/2, b = -3/4$ and $c = 3$.

23. The given circle is $x^2 + y^2 = 1$ ...(1)
which intersect x-axis at $P(-1, 0)$ and $Q(1, 0)$.
Let radius of circle with centre at $Q(1, 0)$ be r, where r is variable.
Then equation of this circle is,
$(x - 1)^2 + y^2 = r^2$ ...(2)



Subtracting (1) from (2) we get
$(x-1)^2 - x^2 = (r^2 - 1)$

$\Rightarrow -2x + 1 = r^2 - 1 \Rightarrow x = 1 - \dfrac{r^2}{2}$

Substituting this value of x in (2), we get

$\dfrac{r^4}{4} + y^2 = r^2 \Rightarrow y = \pm r\sqrt{1 = \dfrac{r^2}{4}}$

$\therefore R\left(1 - \dfrac{r^2}{2}, r\sqrt{1 - \dfrac{r^2}{4}}\right)$ point being above x-axis.

$\therefore$ Area of $\Delta QRS = \dfrac{1}{2} SQ \times$ ordinate of point R

$\Rightarrow A = \dfrac{1}{2} \times r \times r\sqrt{1 - \dfrac{r^2}{4}}$

A will be max. if A^2 is max.

$A^2 = \dfrac{r^4}{4}\left(1 - \dfrac{r^2}{4}\right) = \dfrac{r^4}{4} - \dfrac{r^6}{16}$

Differentiating A^2 w.r. to r, we get $\dfrac{dA^2}{dr} = r^3 - \dfrac{3}{8} r^5$

For A^2 to be max. $\dfrac{dA^2}{dr} = 0$

$\Rightarrow r^3\left(1 - \dfrac{3}{8} r^2\right) = 0 \Rightarrow r = \dfrac{2\sqrt{2}}{\sqrt{3}}$

$\dfrac{d^2(A^2)}{dr^2} = 3r^2 - \dfrac{15}{8} r^4$

$\Rightarrow \left.\dfrac{d^2(A^2)}{dr^2}\right|_{r^2 = \frac{8}{3}} = 3 \times \dfrac{8}{3} - \dfrac{15}{8} \times \dfrac{64}{9} = -ve$

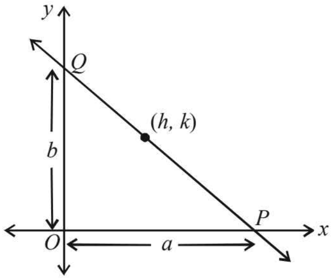
$\therefore$ A^2 and hence A is max. when, $r = \dfrac{2\sqrt{2}}{\sqrt{3}}$

$\therefore$ Max. area $= \sqrt{\dfrac{1}{4}\left(\dfrac{2\sqrt{2}}{\sqrt{3}}\right)^4 - \dfrac{1}{16}\left(\dfrac{2\sqrt{2}}{\sqrt{3}}\right)^6}$

$= \sqrt{\dfrac{1}{4} \times \dfrac{64}{9} - \dfrac{1}{16} \times \dfrac{512}{27}} = \sqrt{\dfrac{16}{9} - \dfrac{32}{27}}$

$= \dfrac{4}{3\sqrt{3}} = \dfrac{4\sqrt{3}}{9}$ sq. units.

24. Let the given line be $\dfrac{x}{a} + \dfrac{y}{b} = 1$, so that it makes an intercept of a units on x-axis and b units on y-axis. As it passes through the fixed point (h, k), therefore we must have



$\Rightarrow \frac{k}{b} = 1 - \frac{h}{a} \quad \Rightarrow b = \frac{ak}{a-h}$ ...(1)

Now Area of $\Delta OPQ = A = \frac{1}{2}ab$

$\therefore \quad A = \frac{1}{2}a\left(\frac{ak}{a-h}\right)$ [using (1)]

or $\quad A = \frac{k}{2}\left[\frac{a^2}{a-h}\right]$

For min. value of A, $\frac{dA}{da} = 0$

$\Rightarrow \frac{k}{2}\left[\frac{2a(a-h)-a^2}{(a-h)^2}\right] = 0 \Rightarrow \frac{k}{2}\left[\frac{a^2-2ah}{(a-h)^2}\right] = 0 \Rightarrow a = 2h$

Also, $\frac{d^2A}{da^2} = \frac{(2a-2h)(a-h)^2 - 2(a-h)(-1)(a^2-2ah)}{(a-h)^4}$

$\therefore \quad \left.\frac{d^2A}{da^2}\right|_{a=2h} = \frac{(2h^3+2h)(0)}{h^4} = \frac{2}{h} > 0, [\because h > 0]$

$\therefore \quad A$ is min. when $a = 2h$

$\therefore \quad A_{\min} = \frac{k}{2}\left[\frac{4h^2}{h}\right] = 2kh$

25. The normal to the curve at P is

$a(y-1) + (x-1) = 0$

First we consider the case when $a \neq 0$

Slope of normal at $P(1, 1)$ is $= -\frac{1}{a}$

$\Rightarrow$ Slope of the tangent at (1, 1) is $= a$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = a$ ...(1)

But we are given that

$\frac{dy}{dx} \propto y \Rightarrow \frac{dy}{dx} = ky \Rightarrow \frac{dy}{y} = kdx$

$\Rightarrow \log|y| = kx + C \Rightarrow \quad |y| = e^{kx+c} = e^c.e^{kx}$

$\Rightarrow \quad y = \pm e^c e^{kx} \Rightarrow \quad y = Ae^{kx}$

Where A is constant. As the curve passes through (1, 1)

$\therefore \quad 1 = Ae^k \quad \Rightarrow \quad A = e^{-k}$

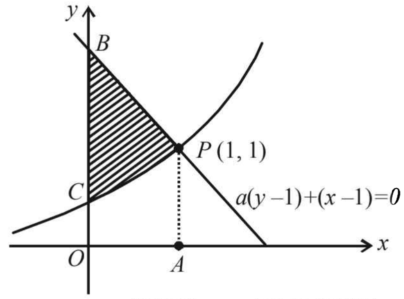
$\therefore \quad y = e^{k(x-1)} \quad \Rightarrow \quad \frac{dy}{dx} = ke^{k(x-1)}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = k$

From (1) and (2), $\left(\frac{dy}{dx}\right)_{1,1} = a = k$

$\therefore \quad y = e^{a(x-1)}$ which is the required curve.

Now the area bounded by the curve, y-axis and normal to curve at (1, 1) is as shown the shaded region in the fig.



$\therefore \quad$ Req. area $= ar(PBC) = ar(OAPBCO) - ar(OAPCO)$

$= \int_0^1 y_{\text{normal}} dx - \int_0^1 y_{\text{curve}} dx$

$= \int_0^1 \left(-\frac{1}{a}(x-1)+1\right) dx - \int_0^1 e^{a(x-1)} dx$

$= \left[-\frac{1}{2a}(x-1)^2 + x\right]_0^1 - \left[\frac{1}{a}e^{a(x-1)}\right]_0^1$

$= 1 + \frac{1}{2a} - \frac{1}{a} + \frac{1}{a}e^{-a} = 1 + \frac{1}{a}e^{-a} - \frac{1}{2a}$

Now we consider the case when $a = 0$. Then normal at (1, 1) becomes $x - 1 = 0$ which is parallel to y-axis, therefore tangent at (1, 1) should be parallel to x-axis. Thus

$\left(\frac{dy}{dx}\right)_{(1,1)} = 0$ ...(3)

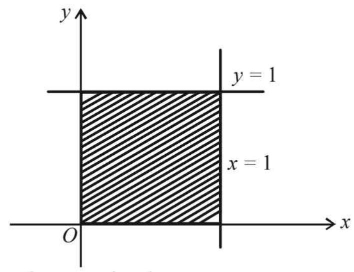
Since $\frac{dy}{dx} \propto y$ gives $y = e^{k(x-1)}$

(as in $a \neq 0$ case)

$\Rightarrow \quad \frac{dy}{dx} = ke^{k(x-1)}$

$\left(\frac{dy}{dx}\right)_{(1,1)} = k$ ...(4)

From (3) and (4), we get $k = 0$ and required curve becomes $y = 1$



In this case the required area

= shaded area in fig. = 1 sq. unit.

26. $f(x) = \frac{1}{8}\ell nx - bx + x^2, x > 0, b \geq 0$

$f'(x) = \frac{1}{8x} - b + 2x$ ...(1)

$f'(x) = 0 \Rightarrow 16x^2 - 8bx + 1 = 0$ (for max. or min.)

$\therefore \quad x = \frac{1}{4}\left[b \pm \sqrt{b^2-1}\right]$ ...(2)

Above will give real values of x if $b^2 - 1 \geq 0$ i.e. $b \geq 1$ or $b \leq -1$. But b is given to be +ve. Hence we choose $b \geq 1$

If $b = 1$ then $x = \frac{1}{4}$; If $b > 1$ then $x = \frac{1}{4}\left[b \pm \sqrt{b^2 - 1}\right]$

$$f''(x) = -\frac{1}{8x^2} + 2 = \frac{16x^2 - 1}{8x^2}$$

Its sign will depend on N^r, $16x^2 - 1$ as $8x^2$ is +ve. We shall consider its sign for $x = \frac{1}{4}$ and $x = \frac{1}{4}\left[b \pm \sqrt{b^2 - 1}\right]$

$f''(x) = 0$ at $x = 1/4$

$\therefore$ Neither max. nor min. as $f''(x) = 0$

N^r of $f''(x) = 16x^2 - 1 = [b + \sqrt{b^2 - 1}]^2 - 1$

= +ve for $b > 1$ $\therefore$ Minima

or N^r of $f''(x) = (b - \sqrt{b^2 - 1})^2 - 1$

= –ve for $b > 1$ $\therefore$ Maxima

27. Given that, $f(x) = \begin{cases} xe^{ax}, & x \leq 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$

Differentiating both sides, we have

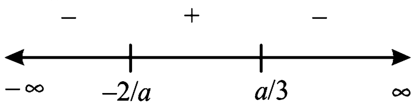
$$f'(x) = \begin{cases} axe^{ax} + e^{ax}, & x \leq 0 \\ 1 + 2ax - 3x^2, & x > 0 \end{cases}$$

Again differentiating both sides, we have

$$f''(x) = \begin{cases} 2ae^{ax} + a^2 xe^{ax}; & x \leq 0 \\ 2a - 6x; & x > 0 \end{cases}$$

For critical points, we put $f''(x) = 0$

$\Rightarrow x = -\frac{2}{a}$, if $x \leq 0$ $= \frac{a}{3}$, if $x > 0$



It is clear from number line that

$f''(x)$ is +ve on $\left(-\frac{2}{a}, \frac{a}{3}\right)$

$\Rightarrow f'(x)$ increases on $\left(-\frac{2}{a}, \frac{a}{3}\right)$

28. Let $b - a = t$, where $a + b = 4$

$\Rightarrow a = \frac{4-t}{2}$ and $b = \frac{t+4}{2}$

as given $a < 2$ and $b > 2 \Rightarrow t > 0$

Now $\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$

$$= \int_0^{\frac{4-t}{2}} g(x)\,dx + \int_0^{\frac{4+t}{2}} g(x)\,dx = \phi(t) \text{ [say]}$$

$\Rightarrow \phi'(t) = g\left(\frac{4-t}{2}\right)\left(-\frac{1}{2}\right) + g\left(\frac{4+t}{2}\right)\left(\frac{1}{2}\right)$ NOTE THIS STEP

$\left[\text{Using } \frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right] = f[v(x)].v'(x) - f[u(x)].u'(x)\right]$

$$= \frac{1}{2}\left[g\left(\frac{4+t}{2}\right) - g\left(\frac{4-t}{2}\right)\right]$$

Since $g(x)$ is an increasing function (given)

$\therefore$ for $x_1 > x_2 \Rightarrow g(x_1) > g(x_2)$

Here we have $\left(\frac{4+t}{2}\right) > \left(\frac{4-t}{2}\right)$

$\Rightarrow g\left(\frac{4+t}{2}\right) > g\left(\frac{4-t}{2}\right)$

$\Rightarrow \phi'(t) = \frac{1}{2}\left[g\frac{(4+t)}{2} - g\frac{(4-t)}{2}\right] > 0 \Rightarrow \phi'(t) > 0$

Hence $\phi(t)$ increase as t increases.

$\Rightarrow \int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases as $(b - a)$ increases.

29. Applying $R_3 \to R_3 - R_1 - 2R_2$ we get

$$f'(x) = \begin{vmatrix} 2ax & 2ax - a & 2ax + b + 1 \\ b & b + 1 & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2ax & 2ax - 1 \\ b & b + 1 \end{vmatrix} = \begin{vmatrix} 2ax & -1 \\ b & 1 \end{vmatrix} \quad [\text{Using } C_2 \to C_2 - C_1]$$

$\Rightarrow f'(x) = 2ax + b$

Integrating, we get, $f(x) = ax^2 + bx + C$

where C is an arbitrary constant. Since f has a maximum at $x = 5/2$,

$f'(5/2) = 0 \Rightarrow 5a + b = 0$ ...(1)

Also $f(0) = 2 \Rightarrow C = 2$

and $f(1) = 1 \Rightarrow a + b + c = 1$

$\therefore$ $a + b = -1$ ...(2)

Solving (1) and (2) for a, b we get,

$a = 1/4$, $b = -5/4$

Thus, $f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2.$

30. Equation of the tangent at point (x, y) on the curve is

$$Y - y = \frac{dy}{dx}(X - x)$$

This meets axes in

$A\left(x - y\frac{dx}{dy'}0\right)$ and $B\left(0, y - x\frac{dy}{dx}\right)$

Mid-point of AB is $\left(\frac{1}{2}\left(x - y\frac{dx}{dy}\right), \frac{1}{2}\left(y - x\frac{dy}{dx}\right)\right)$

We are given

$$\frac{1}{2}\left(x - y\frac{dx}{dy}\right) = x \text{ and } \frac{1}{2}\left(y - x\frac{dy}{dx}\right) = ya$$

$\Rightarrow x\frac{dy}{dx} = -y \Rightarrow \frac{dy}{y} = -\frac{dx}{x}$

Intergrating both sides,

$$\int \frac{dy}{y} = -\int \frac{dy}{x} \Rightarrow \log y = -\log x + c$$

Put $x = 1, y = 1$,

$\Rightarrow \log 1 = -\log 1 + c \Rightarrow c = 0$

$\Rightarrow \log y + \log x = 0 \Rightarrow \log yx = 0 \Rightarrow yx = e^0 = 1$

Which is a rectangular hyperbola.

31. Given that, $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ ...(1)
and $|p(x)| \le |e^{x-1} - 1|, \forall x \ge 0$ ...(2)
To prove that,
$|a_1 + 2a_2 + \ldots + n a_n| \le 1$
It can be clearly seen that in order to prove the result it is sufficient to prove that $|p'(1)| \le 1$
We know that,
$$|p'(1)| = \lim_{h \to 0} \left| \frac{p(1+h) - p(1)}{h} \right|$$
$$\le \lim_{h \to 0} \frac{|p(1+h)| + |p(1)|}{|h|}$$
[Using $|x - y| \le |x| + |y|$]
But $|p(1)| \le |e^0 - 1|$ [Using equation (2) for $x = 1$]
$\Rightarrow |p(1)| \le 0$
But being absolute value, $|p(1)| \ge 0$.
Thus we must have $|p(1)| = 0$
Also $|p(1+h)| \le |e^h - 1|$ (Using eq^n (2) for $x = 1 + h$)
Thus $|p'(1)| \le \lim_{h \to 0} \frac{|e^h - 1|}{|h|} = 1$
or $|p'(1)| \le 1 \Rightarrow |a_1 + 2a_2 + \ldots + na^n| \le 1$

32. Given that $-1 \le p \le 1$.
Consider $f(x) = 4x^3 - 3x - p = 0$
Now, $f(1/2) = \frac{1}{2} - \frac{3}{2} - p = -1 - p \le 0$ as $(-1 \le p)$
Also $f(1) = 4 - 3 - p = 1 - p \ge 0$ as $(p \le 1)$
$\therefore$ $f(x)$ has at least one real root between [1/2, 1].
Also $f'(x) = 12x^2 - 3 > 0$ on [1/2, 1]
$\Rightarrow$ f is increasing on [1/2, 1]
$\Rightarrow$ f has only one real root between [1/2, 1]
To find the root, we observe $f(x)$ contains $4x^3 - 3x$ which is multipe angle formula of $\cos 3\theta$ if we put $x = \cos\theta$.
$\therefore$ Let the req. root be $\cos\theta$ then,
$4\cos^3\theta - 3\cos\theta - p = 0$
$\Rightarrow \cos 3\theta = p \Rightarrow 3\theta = \cos^{-1} p \Rightarrow \theta = \frac{1}{3}\cos^{-1}(p)$
$\therefore$ Root is $\cos\left(\frac{1}{3}\cos^{-1}(p)\right)$.

33. The given curve is $\frac{x^2}{6} + \frac{y^2}{3} = 1$ (an ellipse)
Any parametric point on it is $P(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$.
Its distance from line $x + y = 7$ is given by
$$D = \frac{\sqrt{6}\cos\theta + \sqrt{3}\sin\theta - 7}{\sqrt{2}}$$
For min. value of D, $\frac{dD}{d\theta} = 0$
$\Rightarrow -\sqrt{6}\sin\theta + \sqrt{3}\cos\theta = 0 \Rightarrow \tan\theta = 1/\sqrt{2}$
$\Rightarrow \cos\theta = \frac{\sqrt{2}}{\sqrt{3}}$ and $\sin\theta = \frac{1}{\sqrt{3}}$
$\therefore$ Required point P is (2, 1)

34. Given that $2(1 - \cos x) < x^2, x \ne 0$
To prove $\sin(\tan x) \ge x, x \in [0, \pi/4)$.
Let us consider $f(x) = \sin(\tan x) - x$
$\Rightarrow f'(x) = \cos(\tan x)\sec^2 x - 1$
$$= \frac{\cos(\tan x) - \cos^2 x}{\cos^2 x}$$
As given $2(1 - \cos x) < x^2, x \ne 0$
$\Rightarrow \cos x > 1 - \frac{x^2}{2}$
Similary, $\cos(\tan x) > 1 - \frac{\tan^2 x}{2}$
$$\therefore \quad f'(x) > \frac{1 - \frac{1}{2}\tan^2 x - \cos^2 x}{\cos^2 x}$$
$$= \frac{\sin^2 x\left[1 - \frac{1}{2\cos^2 x}\right]}{\cos^2 x}$$
$$= \frac{\sin^2 x(\cos 2x)}{2\cos^4 x} > 0, \forall x \in [0, \pi/4)$$
$\therefore$ $f'(x) > 0 \Rightarrow f(x)$ is an increasing function.
$\therefore$ For $x \in [0, \pi/4)$,
$x \ge 0 \Rightarrow f(x) \ge f(0)$
$\Rightarrow \sin(\tan x) - x \ge \sin(\tan 0) - 0$
$\Rightarrow \sin(\tan x) - x \ge 0$
$\Rightarrow \sin(\tan x) \ge x$ Hence proved.

35. Given that f is a differentiable function on [0, 4]
$\therefore$ It will be continuous on [0, 4]
$\therefore$ By Lagrange's mean value theorem, we get
$$\frac{f(4) - f(0)}{4 - 0} = f'(a), \text{ for } a \in (0, 4) \quad \ldots(1)$$
Again since f is continuous on [0, 4] by intermediate mean value theorem, we get
$$\frac{f(4) + f(0)}{2} = f(b) \text{ for } b \in (0, 4) \quad \ldots(2)$$
[If $f(x)$ is continuous on $[\alpha, \beta]$ then $\exists\, \mu \in (\alpha, \beta)$ such that $f(\mu) = \frac{f(\alpha) + f(\beta)}{2}$]
Multiplying (1) and (2) we get
$$\frac{[f(4)]^2 - [f(0)]^2}{8} = f'(a) f(b);\ a, b \in (0, 4)$$
or $[f(4)]^2 - [f(0)]^2 = 8 f'(a) f(b)$

Hence Proved.

(ii) To prove
$$\int_0^4 f(t)\,dt = 2[\alpha f(\alpha^2) + \beta f(\beta^2)] \forall 0 < \alpha, \beta < 2$$
Let $I = \int_0^4 f(t)dt$
Let $t = u^2$ also $t \to 0 \Rightarrow u \to 0$
$\Rightarrow dt = 2u\,du$ as $t \to 4 \Rightarrow u \to 2$

$\therefore \quad \int_0^4 f(t)\,dt = \int_0^2 f(u^2).2u\,du \quad \ldots(1)$

Consider, $F(x) = \int_0^x f(u^2).2u\,du$

Then clearly $F(x)$ is differentiable and hence continuous on $[0, 2]$

By LMV theorem, we get some, $\mu \in (0, 2)$

such that $F'(\mu) = \frac{F(2)-F(0)}{2-0}$

$\Rightarrow \quad f(\mu^2).2\mu = \frac{\int_0^2 f(u^2)2u\,du}{2} \ldots(2)$

Again by intermediate mean value theorem,

$\exists\ \alpha, \beta$ such that $0 < \alpha < \mu < \beta < 2$

$\Rightarrow \quad F'(\mu) = \frac{F'(\alpha)+F'(\beta)}{2}$, as f is continuous on $[0, 2]$

$\Rightarrow$ F is continuous on $[0, 2]$

$\Rightarrow \quad f(\mu^2).2\mu = \frac{f(\alpha^2)2\alpha + f(\beta^2).2\beta}{2}$

$\Rightarrow \quad f(\mu^2).2\mu = \alpha f(\alpha^2) + \beta f(\beta^2) \quad \ldots(3)$

From (2) and (3), we get

$$\int_0^2 f(u^2)2u\,du = 2[\alpha\, f(\alpha^2) + \beta\, f(\beta^2)]$$

where $0 < \alpha, \beta < 2$

$$\int_0^4 f(t)\,dt = 2[\alpha\, f(\alpha^2) + \beta\, f(\beta^2)]$$

where $0 < \alpha, \beta < 2$ (Using eq^n (1))

Hence Proved.

36. We are given that,

$\frac{dP(x)}{dx} > P(x), \forall, x \geq 1$ and $P(1) = 0$

$\Rightarrow \quad \frac{dP(x)}{dx} - P(x) > 0$

Multiplying by e^{-x}, we get,

$e^{-x}\frac{dP(x)}{dx} - e^{-x}P(x) > 0$

$\Rightarrow \quad \frac{d}{dx}\left[e^{-x}P(x)\right] > 0$

$\Rightarrow \quad e^{-x}P(x)$ is an increasing function.

$\therefore \quad \forall\, x > 1, e^{-x}P(x) > e^{-1}P(1) = 0$ [Using $P(1) = 0$]

$\Rightarrow \quad e^{-x}P(x) > 0, \forall\, x > 1$

$\Rightarrow \quad P(x) > 0, \forall\, x > 1 \quad [\because e^{-x} > 0]$

37. We are given,

$P(x) = 51x^{101} - 2323x^{100} - 45x + 1035$

To show that at least one root of $P(x)$ lies in $(45^{1/100}, 46)$, using Rolle's theorem, we consider antiderivative of $P(x)$

i.e. $F(x) = \frac{x^{102}}{2} - \frac{2323x^{101}}{101} - \frac{45x^2}{2} + 1035x$

Then being a polynominal function $F(x)$ is continuous and differentiable.

Now, $F(45^{1/100}) = \frac{(45)^{\frac{102}{100}}}{2} - \frac{2323(45)^{\frac{101}{100}}}{101} - \frac{45.(45)^{\frac{2}{100}}}{2} + 1035(45)^{\frac{1}{100}}$

$= \frac{45}{2}(45)^{\frac{2}{100}} - 23\times 45(45)^{\frac{1}{100}} - \frac{45.(45)^{\frac{2}{100}}}{2} + 1035(45)^{\frac{1}{100}} = 0$

And $F(46) = \frac{(46)^{102}}{2} - \frac{2323(46)^{101}}{101} - \frac{45(46)^2}{2} + 1035(46)$

$= 23\,(46)^{101} - 23\,(46)^{101} - 23\times 45\times 46 + 1035\times 46 = 0$

$\therefore \quad F(45^{\frac{1}{100}}) = F(46) = 0$

$\therefore$ Rolle's theorem is applicable.

Hence, there must exist at least one root of $F'(x) = 0$

i.e. $P(x) = 0$ in the interval $\left(45^{\frac{1}{100}}, 46\right)$

38. Let us consider,

$f(x) = \sin x + 2x - \frac{3x(x+1)}{\pi}$

$\Rightarrow \quad f'(x) = \cos x + 2 - \frac{3}{\pi}(2x+1)$

$\Rightarrow \quad f''(x) = -\sin x - \frac{6}{\pi} < 0, \forall\, x \in [0, \pi/2]$

$\Rightarrow \quad f'(x)$ is a decreasing function. .... (1)

Also $f'(0) = 3 - \frac{3}{\pi} > 0$ ...(2)

and $f'(\pi/2) = 2 - \frac{3}{\pi}(\pi+1) = -1 - \frac{3}{\pi} < 0$ ...(3)

Equations (1), (2) and (3) shows that.

$\Rightarrow$ There exists a certain value of $x \in [0, \pi/2]$ for which $f'(x) = 0$ and this point must be a point of maximum for $f(x)$ since the sign of $f'(x)$ changes from +ve to –ve.

Also we can see that f(0) = 0 and

$f\left(\frac{\pi}{2}\right) = \pi + 1 - \frac{3}{2}\left(\frac{\pi}{2}+1\right) = \frac{\pi}{4} - \frac{1}{2} > 0$

Let $x = p$ be the point at which the max. of $f(x)$ occurs. There will be only one max. point in $[0, \pi/2]$. Since $f'(x) = 0$ is only once in the interval.

Consider, $x \in [0, p]$

$\Rightarrow \quad f'(x) > 0 \Rightarrow f(x)$ is an increasing function.

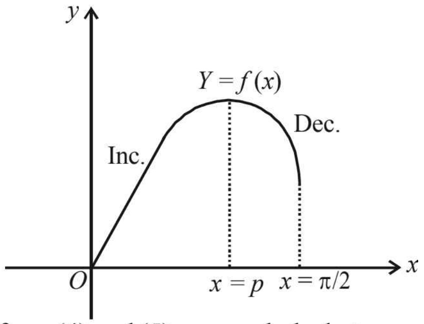
$\Rightarrow \quad f(0) \leq f(x)$ [as $0 \leq x$]

$\Rightarrow \quad f(x) \geq 0$ ..... (4)

Also for $x \in [p, \pi/2]$

$\Rightarrow \quad f'(x) < 0 \Rightarrow f(x)$ is decreasing function.

$\Rightarrow$ for $x < \pi/2, f(x) > f(\pi/2) > 0$ ..... (5)



Hence from (4) and (5) we conclude that

$f(x) \ge 0, \forall x \in [0, \pi/2]$.

39. Given that, $|f(x_1) - f(x_2)| < (x_1 - x_2)^2,\ x_1, x_2 \in R$

Let $x_1 = x + h$ and $x_2 = x$ then we get

$|f(x+h) - f(x)| < h^2 \Rightarrow |f(x+h) - f(x)| < |h|^2$

$\Rightarrow \left|\frac{f(x+h) - f(x)}{h}\right| < |h|$

Taking limit as $h \to 0$ on both sides, we get

$\lim_{h \to 0}\left|\frac{f(x+h) - f(x)}{h}\right| < \delta$ (a small +ve number)

$\Rightarrow |f'(x)| < \delta \Rightarrow f'(x) = 0$

$\Rightarrow f(x)$ is a constant function. Let $f(x) = k$ i.e., $y = k$

As $f(x)$ passes through (1, 2) $\Rightarrow y = 2$

$\therefore$ Equation of tangent at (1, 2) is,

$y - 2 = 0(x - 1)$ i.e. $y = 2$

40. Let $p(x) = ax^3 + bx^2 + cx + d$

$p(-1) = 10$

$\Rightarrow -a + b - c + d = 10$ ...... (i)

$p(1) = -6$

$\Rightarrow a + b + c + d = -6$ ...... (ii)

$p(x)$ has max. at $x = -1$

$\therefore p'(-1) = 0$

$\Rightarrow 3a - 2b + c = 0$ ...... (iii)

$p'(x)$ has min. at $x = 1$

$\therefore p''(1) = 0$

$\Rightarrow 6a + 2b = 0$ ...... (iv)

Solving (i), (ii), (iii) and (iv), we get

From (iv), $b = -3a$

From (iii), $3a + 6a + c = 0 \Rightarrow c = -9a$

From (ii), $a - 3a - 9a + d = -6 \Rightarrow d = 11a - 6$

From (i), $-a - 3a + 9a + 11a - 6 = 10$

$\Rightarrow 16a = 16 \Rightarrow a = 1 \Rightarrow b = -3, c = -9, d = 5$

$\therefore p(x) = x^3 - 3x^2 - 9x + 5 \Rightarrow p'(x) = 3x^2 - 6x - 9 = 0$

$\Rightarrow 3(x+1)(x-3) = 0$

$\Rightarrow x = -1$ is a point of max. (given)

and $x = 3$ is a point of min.

[$\because$ max. and min. occur alternatively]

$\therefore$ points of local max. is $(-1, 10)$ and local min. is $(3, -22)$.

And distance between them is

$= \sqrt{[3-(-1)^2] + (-22-10)^2}$

$= \sqrt{16 + 1024} = \sqrt{1040} = 4\sqrt{65}$

41. $g(x) = (f'(x))^2 + f''(x)f(x) = \frac{d}{dx}(f(x)f'(x))$

Let $h(x) = f(x)f'(x)$

Then, $f(x) = 0$ has four roots namely a, α, β, e

where $b < \alpha < c$ and $c < \beta < d$.

And $f'(x) = 0$ at three points k_1, k_2, k_3

where $a < k_1 < \alpha, \alpha < k_2 < \beta, \beta < k_3 < e$

[$\because$ Between any two roots of a polynomial function $f(x) = 0$ there lies atleast one root of $f'(x) = 0$]

$\therefore$ There are atleast 7 roots of $f(x) \cdot f'(x) = 0$

$\Rightarrow$ There are atleast 6 roots of $\frac{d}{dx}(f(x)f'(x)) = 0$

i.e. of $g(x) = 0$

F. Match the Following

1. (A) $f(x) = x + \sin x$ on $(-\pi/2, \pi/2)$

$f'(x) = 1 + \cos x$

As $0 \le \cos x \le 1$ for $x \in (-\pi/2, \pi/2)$

$\therefore f'(x) > 0$ on $(-\pi/2, \pi/2)$

(A) $\to$ p

(B) $f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$.

Clearly $f'(x) < 0$ in $(-\pi/2, 0)$ and $f'(x) > 0$ in $(0, \pi/2)$

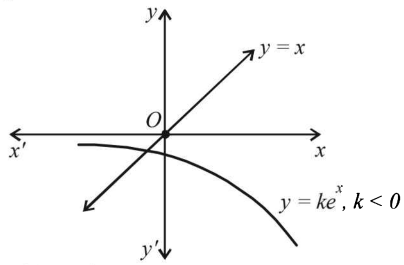
$\therefore$ On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $f(x)$ is neither increasing non decreasing.

(B) $\to$ r

G. Comprehension Based Questions

1. **(c)** For $k = 0$, line $y = x$ meets $y = 0$, i.e., x-axis only at one point.

For $k < 0$, $y = ke^x$ meets $y = x$ only once as shown in the graph.



2. **(a)** Let $f(x) = ke^x - x$

Now for $f(x) = 0$ to have only one root means the line $y = x$ must be tangential to the curve $y = ke^x$.

Let it be so at (x_1, y_1) then

$\left(\frac{dy}{dx}\right)_{\text{curve1}} = \left(\frac{dy}{dx}\right)_{\text{curve2}} \Rightarrow 1 = ke^{x_1}$

$\Rightarrow e^{x_1} = \frac{1}{k}$ also $y_1 = ke^{x_1}$ and $y_1 = x_1$

$\Rightarrow x_1 = 1 \Rightarrow 1 = ke \Rightarrow k = 1/e$

3. **(a)** $\because$ For $y = x$ to be tangent to the curve $y = ke^x, k = 1/e$

$\therefore$ For $y = ke^x$ to meet $y = x$ at two points we should have $k < \frac{1}{e} \Rightarrow k \in \left(0, \frac{1}{e}\right)$ as $k > 0$.

4. **(c)** For the statement P

$f(x)+2x=2(1+x^2)$

$\Rightarrow (1-x)^2\sin^2 x+x^2+2x=2(1+x^2)$

$\Rightarrow (1-x)^2\sin^2 x=x^2-2x+1+1$

$\Rightarrow (1-x)^2\sin^2 x=(1-x)^2+1$

$\Rightarrow (1-x)^2\cos^2 x=-1$

Which is not possible for any real value of x.

$\therefore P$ is not true.

Also let $H(x)=2f(x)+1-2x(1+x)$

$H(0)=2f(0)+1-0=1$

and $H(1)=2f(1)+1-4=-3$

$\Rightarrow H(x)$ has a solution in (0, 1)

$\therefore Q$ is true.

5. **(b)** We have $g(x)=\int_1^x\left(\frac{2(t-1)}{t+1}-\ln t\right)f(t)dt,\ x\in(1,\infty)$

$\therefore g'(x)=\left[\frac{2(x-1)}{x+1}-\ln x\right]f(x)$

$\Rightarrow$ Here $f(x)>0,\ \forall x\in(1,\infty)$

Also let $h(x)=\frac{2(x-1)}{x+1}-\ln x$

$$h'(x)=\frac{4}{(x+1)^2}-\frac{1}{x}=\frac{-(x-1)^2}{(x+1)^2x}<0, x\in(1,\infty)$$

$\therefore$ $h(x)$ is decreasing function.

$\therefore$ For $x>1$

$h(x)<h(1)\Rightarrow h(x)<0\ \forall\ x>1$

$\therefore g'(x)<0\ \forall\ x\in(1,\infty)$

$\therefore g(x)$ is decreasing on $(1,\infty)$.

6. **(d)** We have $f''(x)-2f'(x)+f(x)\ge e^x$

$\Rightarrow [f''(x)-f'(x)]-[f'(x)-f(x)]\ge e^x$

$\Rightarrow \left[e^{-x}f''(x)-e^{-x}f'(x)\right]-\left[e^{-x}f'(x)-e^{-x}f(x)\right]\ge 1$

$\Rightarrow \frac{d}{dx}\left[e^{-x}f'(x)\right]-\frac{d}{dx}\left[e^{-x}f(x)\right]\ge 1$

$\Rightarrow \frac{d}{dx}\left[e^{-x}f'(x)-e^{-x}f(x)\right]\ge 1$

$\Rightarrow \frac{d}{dx}\left[\frac{d}{dx}\left(e^{-x}f(x)\right)\right]\ge 1$

Let $g(x)=e^{-x}f(x)$

Then we have $g''(x)\ge 1>0$

So g is concave upward.

Also $g(0)=g(1)=0$

$g(x)<0,\ \forall\ x\in(0,1)$

$\Rightarrow e^{-x}f(x)<0\Rightarrow f(x)<0,\ \forall\ x\in(0,1)$

7. **(c)** $g(x)=e^{-x}f(x)$

$\Rightarrow g'(x)=e^{-x}f'(x)-e^{-x}f(x)=e^{-x}(f'(x)-f(x))$

As $x=\frac{1}{4}$ is point of local minima in [0, 1]

$\therefore g'(x)<0$ for $x\in\left(0,\frac{1}{4}\right)$

and $g'(x)>0$ for $x\in\left(\frac{1}{4},1\right)$

$\therefore$ In $\left(0,\frac{1}{4}\right)$, $g'(x)<0$

$\Rightarrow e^{-x}(f'(x)-f(x))<0\Rightarrow f'(x)<f(x)$

I. Integer Value Correct Type

1. **(7)** The given function is $f(x)=2x^3-15x^2+36x-48$

and $A=\{x\,|\,x^2+20\le 9x\}$

$\Rightarrow A=\left\{x\,|\,x^2-9x+20\le 0\right\}$

$\Rightarrow A=\left\{x\,|\,(x-4)(x-5)\le 0\right\}$

$\Rightarrow$ A = [4, 5]

Also $f'(x)=6x^2-30x+36=6(x^2-5x+6)$

$=6(x-2)(x-3)$

Clearly $\forall\ x\in A,\ f'(x)>0$

$\therefore$ f is strictly increasing function on A.

$\therefore$ Maximum value of f on A

$=f(5)=2\times 5^3-15\times 5^2+36\times 5-48$

$=250-375+180-48=430-423=7$

2. **(0)** Let $p(x)=ax^4+bx^3+cx^2+dx+e$

Now $\lim_{x\to 0}\left[1+\frac{p(x)}{x^2}\right]=2$

$\Rightarrow \lim_{x\to 0}\frac{p(x)}{x^2}=1$ ...(1)

$\Rightarrow p(0)=0\Rightarrow e=0$

Applying L 'Hospital's rule to eq^n (1), we get

$\lim_{x\to 0}\frac{p'(x)}{2x}=1\Rightarrow p'(0)=0$

$\Rightarrow d=0$

Again applying L 'Hospital's rule, we get

$\lim_{x\to 0}\frac{p''(x)}{2}=1\Rightarrow p''(0)=2$

$\Rightarrow 2c=2$ or $c=1$

$\therefore\ p(x)=ax^4+bx^3+x^2$

$\Rightarrow p'(x)=4ax^3+3bx^2+2x$

As $p(x)$ has extremum at $x=1$ and 2

$\therefore\ p'(1)=0$ and $p'(2)=0$

$\Rightarrow 4a+3b+2=0$ ...(i)

$\Rightarrow 32a+12b+4=0$ or $8a+3b+1=0$ ...(ii)

Solving eq's (i) and (ii) we get $a=\frac{1}{4}$ and $b=-1$

$\therefore\ p(x)=\frac{1}{4}x^4-x^3+x^2$

So, that $p(2)=\frac{16}{4}-8+4=0$

3. **(9)** The equation of tangent to the curve y = f(x) at the point P (x,y) is

$$\frac{Y-y}{X-x}=\frac{dy}{dx}\ or\ (X-x)\frac{dy}{dx}-(Y-y)=0$$

$\Rightarrow X\frac{dy}{dx}-Y=x\frac{dy}{dx}-y$

Its y-intercept $=y-x\frac{dy}{dx}=x^3\Rightarrow \frac{dy}{dx}-\frac{y}{x}=-x^2$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = \frac{1}{x}$$

$$\therefore\ y.\frac{1}{x} = \int -x^2\frac{1}{x}\,dx = \frac{-x^2}{2}+C,\ y = \frac{-x^3}{2}+Cx$$

As $f(1)=1 \Rightarrow At\ x=1, y=1$

$$\therefore\ 1 = \frac{-1}{2}+C \Rightarrow C = 3/2 \quad \therefore\ y = -\frac{x^3}{2}+\frac{3x}{2}$$

At $x=-3$, $y = \frac{27}{2}-\frac{9}{2} = 9$

$\therefore\ f(-3)=9.$

4. **(1)** We have ,

$$f'(x) = 2010\,(x-2009)(x-2010)^2\,(x-2011)^3(x-2012)^4$$

As $f(x) = ln\,g(x) \Rightarrow g(x) = e^{f(x)} \Rightarrow g'(x) = e^{f(x)}.f'(x)$

For max/min, $g'(x) = 0 \Rightarrow f'(x) = 0$

Out of two points one should be a point of maxima and other that of minima.

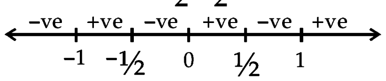
$\therefore$ There is only one point of local maxima.

5. **(5)** We have $f(x) = |x|+|x^2-1|$

$$= \begin{cases} -x+x^2-1, & x<-1 \\ -x-x^2+1, & -1\le x\le 0 \\ x-x^2+1, & 0<x<1 \\ x^2+x-1, & x\ge 1 \end{cases}$$

$$\text{We have } f'(x) = \begin{bmatrix} 2x-1 & , & x<-1 \\ -2x-1 & , & -1\le x\le 0 \\ -2x+1 & , & 0<x<1 \\ 2x+1 & , & x>1 \end{bmatrix}$$

Critical pts are $\frac{1}{2}, \frac{-1}{2}, -1, 0$ and 1



We observe at five points $f'(x)$ changes its sign

$\therefore$ There are 5 points of local maximum or local minimum.

6. **(9)** $\because$ $p(x)$ has a local maximum at $x=1$ and a local minimum at $x=3$ and $p(x)$ is a real polynomial of least degree

$\therefore$ Let $p'(x) = k(x-1)(x-3) = k(x^2-4x+3)$

$$\Rightarrow p(x) = k\left(\frac{x^3}{3}-2x^2+3x\right)+C$$

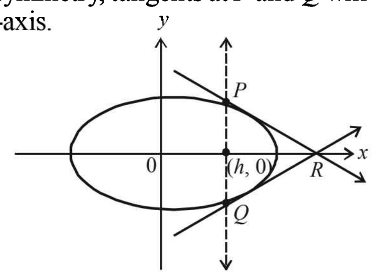
Given $p(1)=6$ and $p(3)=2$

$$\Rightarrow\ \frac{4}{3}k+C=6 \text{ and } 0+C=2 \Rightarrow k=3$$

$\therefore\ p'(x) = 3(x-1)(x-3) \Rightarrow p'(0) = 9$

7. **(9)** Vertical line $x=h$, meets the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at $P\left(h, \frac{\sqrt{3}}{2}\sqrt{4-h^2}\right)$ and $Q\left(h, \frac{-\sqrt{3}}{2}\sqrt{4-h^2}\right)$

By symmetry, tangents at P and Q will meet each other at x-axis.



Tangent at P is $\frac{xh}{4}+\frac{y\sqrt{3}}{6}\sqrt{4-h^2}=1$

which meets x-axis at $R\left(\frac{4}{h}, 0\right)$

Area of $\Delta PQR = \frac{1}{2}\times\sqrt{3}\sqrt{4-h^2}\times\left(\frac{4}{h}-h\right)$

i.e., $\Delta(h) = \frac{\sqrt{3}}{2}\frac{(4-h^2)^{3/2}}{h}$

$$\frac{d\Delta}{dh} = -\sqrt{3}\left[\frac{\sqrt{4-h^2}\,(h^2+2)}{h^2}\right]<0$$

$\therefore$ $\Delta(h)$ is a decreasing function.

$\therefore$ $\frac{1}{2}\le h\le 1 \Rightarrow \Delta_{max} = \Delta\left(\frac{1}{2}\right)$ and $\Delta_{min} = \Delta(1)$

$$\therefore\ \Delta_1 = \frac{\sqrt{3}}{2}\frac{\left(4-\frac{1}{4}\right)^{3/2}}{\frac{1}{2}} = \frac{45}{8}\sqrt{5}$$

$$\Delta_2 = \frac{\sqrt{3}}{2}\frac{3\sqrt{3}}{1} = \frac{9}{2}$$

$$\therefore\ \frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = 45-36 = 9$$

8. **(8)** $\left(y-x^5\right)^2 = x\left(1+x^2\right)^2$

$$2\left(y-x^5\right) = \left(\frac{dy}{dx}-5x^4\right) = \left(1+x^2\right)^2+2x\left(1+x^2\right).2x$$

At point (1, 3)

$$2(3-1)\left(\frac{dy}{dx}-5\right) = (1+1)^2+2(1+1).2 \Rightarrow \frac{dy}{dx} = 8$$

9. **(4)** Let r be the internal radius and R be the external radius. Let h be the internal height of the cylinder.

$\therefore$ $\pi r^2 h = V \Rightarrow h = \frac{V}{\pi r^2}$

Also Vol. of material $= M = \pi[(r+2)^2-r^2]h + \pi(r+2)^2\times 2$

or $M = 4\pi(r+1).\frac{V}{\pi r^2} + 2\pi(r+2)^2$

$$\Rightarrow\ M = 4V\left[\frac{1}{r}+\frac{1}{r^2}\right]+2\pi(r+2)^2$$

$$\frac{dM}{dr} = 4V\left[\frac{-1}{r^2}-\frac{2}{r^3}\right]+4\pi(r+2)$$

For min. value of M, $\frac{dM}{dr}=0$

$$\Rightarrow\ \frac{-4V}{r^3}(r+2)+4\pi(r+2)=0$$

$$\Rightarrow\ \frac{4V}{r^3} = 4\pi \text{ or } r^3 = \frac{V}{\pi} = 1000$$

$\therefore$ $V = 1000\pi$

$$\therefore\ \frac{V}{250\pi} = 4$$

Section-B JEE Main/ AIEEE

1. **(b)** Distance of origin from $(x, y) = \sqrt{x^2 + y^2}$

$= \sqrt{a^2 + b^2 - 2ab\cos\left(t - \frac{at}{b}\right)}$;

$\le \sqrt{a^2 + b^2 + 2ab} \quad \left[\left\{\cos\left(t - \frac{at}{b}\right)\right\}_{\min} = -1\right]$

$= a + b$

$\therefore$ Maximum distance from origin $= a + b$

2. **(a)** Let $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx \Rightarrow f(0) = 0$ and $f(1)$

$= \frac{a}{3} + \frac{b}{2} + c = \frac{2a + 3b + 6c}{6} = 0$

Also $f(x)$ is continuous and differentiable in [0, 1] and [0, 1[. So by Rolle's theorem, $f'(x) = 0$.

i.e $ax^2 + bx + c = 0$ has at least one root in [0, 1].

3. **(d)** $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$

$f'(x) = 6x^2 - 18ax + 12a^2$; $f''(x) = 12x - 18a$

For max. or min.

$6x^2 - 18ax + 12a^2 = 0 \Rightarrow x^2 - 3ax + 2a^2 = 0$

$\Rightarrow x = a$ or $x = 2a$. At $x = a$ max. and at $x = 2a$ min

$\therefore\ p = a$ and $q = 2a$

As per question $p^2 = q$

$\therefore a^2 = 2a \Rightarrow a = 2$ or $a = 0$

but $a > 0$, therefore, $a = 2$.

4. **(a)** $y^2 = 18x \Rightarrow 2y\frac{dy}{dx} = 18 \Rightarrow \frac{dy}{dx} = \frac{9}{y}$

Given $\frac{dy}{dx} = 2 \Rightarrow \frac{9}{y} = 2 \Rightarrow y = \frac{9}{2}$

Putting in $y^2 = 18x \Rightarrow x = \frac{9}{8}$

$\therefore$ Required point is $\left(\frac{9}{8}, \frac{9}{2}\right)$

5. **(b)** $f''(x) = 6(x - 1)$. Inegrating, we get

$f'(x) = 3x^2 - 6x + c$

Slope at (2, 1) $= f'(2) = c = 3$

[$\because$ slope of tangent at (2,1) is 3]

$\therefore f'(x) = 3x^2 - 6x + 3 = 3(x - 1)^2$

Inegrating again, we get $f(x) = (x - 1)^3 + D$

The curve passes through (2, 1)

$\Rightarrow 1 = (2 - 1)^3 + D \Rightarrow D = 0$

$\therefore f(x) = (x - 1)^3$

6. **(d)** $\frac{dx}{d\theta} = -a\sin\theta$ and $\frac{dy}{d\theta} = a\cos\theta$

$\therefore \frac{dy}{dx} = -\cot\theta.$

$\therefore$ The slope of the normal at $\theta = \tan\theta$

$\therefore$ The equation of the normal at θ is

$y - a\sin\theta = \tan\theta(x - a - a\cos\theta)$

$\Rightarrow y\cos\theta - a\sin\theta\cos\theta = x\sin\theta - a\sin\theta - a\sin\theta\cos\theta$

$\Rightarrow x\sin\theta - y\cos\theta = a\sin\theta$

$\Rightarrow y = (x - a)\tan\theta$

which always passes through $(a, 0)$

7. **(d)** Let us define a function

$f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$

Being polynomial, it is continuous and differentiable, also,

$f(0) = 0$ and $f(1) = \frac{a}{3} + \frac{b}{2} + c$

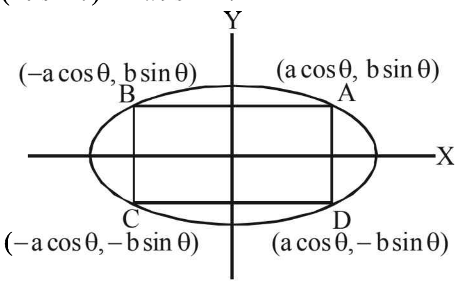
$\Rightarrow f(1) = \frac{2a + 3b + 6c}{6} = 0$ (given)

$\therefore f(0) = f(1)$

$\therefore f(x)$ satisfies all conditions of Rolle's theorem therefore $f'(x) = 0$ has a root in (0, 1)

i.e. $ax^2 + bx + c = 0$ has at lease one root in (0, 1)

8. **(a)** Area of rectangle $ABCD = 2a\cos\theta\,(2b\sin\theta) = 2ab\sin 2\theta$



$\Rightarrow$ Area of greatest rectangle is equal to $2ab$

When $\sin 2\theta = 1$.

9. **(d)** $x = a(\cos\theta + \theta\sin\theta)$

$\Rightarrow \quad \frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta)$

$\Rightarrow \quad \frac{dx}{d\theta} = a\theta\cos\theta$ .....(1)

$y = a(\sin\theta - \theta\cos\theta)$

$\frac{dy}{d\theta} = a[\cos\theta - \cos\theta + \theta\sin\theta]$

$\Rightarrow \frac{dy}{d\theta} = a\theta\sin\theta$ .....(2)

From equations (1) and (2), we get

$\frac{dy}{dx} = \tan\theta \Rightarrow$ Slope of normal $= -\cot\theta$

Equation of normal at 'θ' is $y - a(\sin\theta - \theta\cos\theta)$

$= -\cot\theta\ (x - a(\cos\theta + \theta\sin\theta))$

$\Rightarrow y\sin\theta - a\sin^2\theta + a\,\theta\cos\theta\sin\theta$

$= -x\cos\theta + a\cos^2\theta + a\,\theta\sin\theta\cos\theta$

$\Rightarrow \quad x\cos\theta + y\sin\theta = a$

Clearly this is an equation of straight line which is at a constant distance 'a' from origin.

10. **(b)** Given that

$\frac{dv}{dt} = 50\ \text{cm}^3/\text{min} \Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 50$

$\Rightarrow 4\pi r^2 \frac{dr}{dt} = 50$

$\Rightarrow \frac{dr}{dt} = \frac{50}{4\pi(15)^2} = \frac{1}{18\pi}$ cm/min (here $r = 10+5$)

11. **(b)** Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots\ldots + a_1 x = 0$

The other given equation,

$na_n x^{n-1} + (n-1)\,a_{n-1}x^{n-2} + \ldots + a_1 = 0 = f'(x)$

Given $a_1 \neq 0 \Rightarrow f(0) = 0$

Again $f(x)$ has root α, $\Rightarrow f(\alpha) = 0$

$\therefore \quad f(0) = f(\alpha)$

$\therefore$ By Roll's theorem $f'(x) = 0$ has root between $(0, \alpha)$

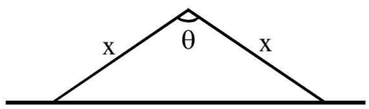
Hence $f'(x)$ has a positive root smaller than α.

12. **(a)** $f(x) = \frac{x}{2} + \frac{2}{x} \Rightarrow f'(x) = \frac{1}{2} - \frac{2}{x^2} = 0$

$\Rightarrow x^2 = 4$ or $x = 2, -2;$ $\quad f''(x) = \frac{4}{x^3}$

$f''(x)]_{x=2} = +ve \Rightarrow f(x)$ has local min at $x = 2$.

13. **(c)** Area $= \frac{1}{2}x^2\sin\theta$



Maximum value of $\sin\theta$ is 1 at $\theta = \frac{\pi}{2}$

$A_{\max} = \frac{1}{2}x^2$

14. **(c)** Using Lagrange's Mean Value Theorem
Let $f(x)$ be a function defined on $[a, b]$

then, $f'(c) = \frac{f(b) - f(a)}{b - a}$ ....(i)

$c \in [a, b]$

$\therefore$ Given $f(x) = \log_e x$ $\therefore$ $f'(x) = \frac{1}{x}$

$\therefore$ equation (i) become $\frac{1}{c} = \frac{f(3) - f(1)}{3 - 1}$

$\Rightarrow \frac{1}{c} = \frac{\log_e 3 - \log_e 1}{2} = \frac{\log_e 3}{2}$

$\Rightarrow c = \frac{2}{\log_e 3} \Rightarrow c = 2\log_3 e$

15. **(d)** Given $f(x) = \tan^{-1}(\sin x + \cos x)$

$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2}.(\cos x - \sin x)$

$= \frac{\sqrt{2}.\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)}{1 + (\sin x + \cos x)^2}$

$= \frac{\left(\cos\frac{\pi}{4}.\cos x - \sin\frac{\pi}{4}.\sin x\right)}{1 + (\sin x + \cos x)^2}$

$\therefore \quad f'(x) = \frac{\sqrt{2}\cos\left(x + \frac{\pi}{4}\right)}{1 + (\sin x + \cos x)^2}$

if $f'(x) > O$ then $f(x)$ is increasing function.

Hence $f(x)$ is increasing, if $-\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{\pi}{2}$

$\Rightarrow -\frac{3\pi}{4} < x < \frac{\pi}{4}$

Hence, $f(x)$ is increasing when $n \in \left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$

16. **(c)** Given that $p^2 + q^2 = 1$ $\quad\therefore\ p = \cos\theta$ and $q = \sin\theta$

Then $p + q = \cos\theta + \sin\theta$

We know that

$-\sqrt{a^2 + b^2} \leq a\cos\theta + b\sin\theta \leq \sqrt{a^2 + b^2}$

$\therefore -\sqrt{2} \leq \cos\theta + \sin\theta \leq \sqrt{2}$

Hence max. value of $p + q$ is $\sqrt{2}$

17. **(a)** Let $y = x^3 - px + q \Rightarrow \frac{dy}{dx} = 3x^2 - p$

For $\frac{dy}{dx} = 0 \Rightarrow 3x^2 - p = 0 \Rightarrow x = \pm\sqrt{\frac{p}{3}}$

$\frac{d^2y}{dx^2} = 6x$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\sqrt{\frac{p}{3}}} = +ve \text{ and } \left.\frac{d^2y}{dx^2}\right|_{x=-\sqrt{\frac{p}{3}}} = -ve$$

$\therefore$ y has minima at $x = \sqrt{\frac{p}{3}}$ and maxima at $x = -\sqrt{\frac{p}{3}}$

18. (b) Let $f(x) = x^7 + 14x^5 + 16x^3 + 30x - 560$

$\Rightarrow f'(x) = 7x^6 + 70x^4 + 48x^2 + 30 > 0, \forall\ x \in R$

$\Rightarrow f$ is an increasing function on R

Also $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$

$\Rightarrow$ The curve $y = f(x)$ crosses x-axis only once.

$\therefore f(x) = 0$ has exactly one real root.

19. (b) Given that $f(x) = x\,|x|$ and $g(x) = \sin x$

So that $gof(x) = g(f(x)) = g(x\,|x|) = \sin x\,|x|$

$$= \begin{cases} \sin(-x^2), & \text{if } x < 0 \\ \sin(x^2), & \text{if } x \ge 0 \end{cases}$$

$$= \begin{cases} -\sin x^2, & \text{if } x < 0 \\ \sin x^2, & \text{if } x \ge 0 \end{cases}$$

$$\therefore (gof)'(x) = \begin{cases} -2x\cos x^2, & \text{if } x < 0 \\ 2x\cos x^2, & \text{if } x \ge 0 \end{cases}$$

Here we observe

$L(gof)'(0) = 0 = R(gof)'(0)$

$\Rightarrow$ gof is differentiable at $x = 0$

and $(gof)'$ is continuous at $x = 0$

$$\text{Now } (gof)''(x) = \begin{cases} -2\cos x^2 + 4x^2\sin x^2, & x < 0 \\ 2\cos x^2 - 4x^2\sin x^2, & x \ge 0 \end{cases}$$

Here $L(gof)''(0) = -2$ and $R(gof)''(0) = 2$

$\because$ $L(gof)''(0) \ne R(gof)''(0)$

$\Rightarrow$ $gof(x)$ is not twice differentiable at $x = 0$.

$\therefore$ Statement - 1 is true but statement -2 is false.

20. (a) We have $P(x) = x^4 + ax^3 + bx^2 + cx + d$

$\Rightarrow$ $P'(x) = 4x^3 + 3ax^2 + 2bx + c$

But $P'(0) = 0 \Rightarrow c = 0$

$\therefore$ $P(x) = x^4 + ax^3 + bx^2 + d$

As given that $P(-1) < P(a)$

$\Rightarrow$ $1 - a + b + d < 1 + a + b + d \Rightarrow a > 0$

Now $P'(x) = 4x^3 + 3ax^2 + 2bx = x(4x^2 + 3ax + 2b)$

As $P'(x) = 0$, there is only one solution $x = 0$, therefore $4x^2 + 3ax + 2b = 0$ should not have any real roots i.e. $D < 0$

$\Rightarrow$ $9a^2 - 32b < 0 \quad \Rightarrow \quad b > \frac{9a^2}{32} > 0$

Hence $a, b > 0 \Rightarrow P'(x) = 4x^3 + 3ax^2 + 2bx > 0$ $\forall x > 0$

$\therefore$ $P(x)$ is an increasing function on $(0,1)$

$\therefore$ $P(0) < P(a)$

Similarly we can prove $P(x)$ is decreasing on $(-1, 0)$

$\therefore$ $P(-1) > P(0)$

So we can conclude that

Max $P(x) = P(1)$ and Min $P(x) = P(0)$

$\Rightarrow$ $P(-1)$ is not minimum but $P(1)$ is the maximum of P.

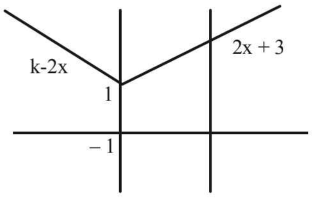
21. (c) Since tangent is parallel to x-axis,

$\therefore \frac{dy}{dx} = 0 \Rightarrow 1 - \frac{8}{x^3} = 0 \Rightarrow x = 2 \Rightarrow y = 3$

Equation of tangent is $y - 3 = 0\,(x - 2) \Rightarrow y = 3$

22. (c)

$$f(x) = \begin{cases} k - 2x, & \text{if } x \le -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$$



This is true where k = – 1

23. (d) $f(x) = \frac{1}{e^x + 2e^{-x}} = \frac{e^x}{e^{2x} + 2}$

$$f'(x) = \frac{(e^{2x} + 2)e^x - 2e^{2x}.e^x}{(e^{2x} + 2)^2}$$

$f'(x) = 0 \Rightarrow e^{2x} + 2 = 2e^{2x}$

$e^{2x} = 2 \Rightarrow e^x = \sqrt{2}$

maximum $f(x) = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}}$

$0 < f(x) \le \frac{1}{2\sqrt{2}} \quad \forall x \in R$

Since $0 < \frac{1}{3} < \frac{1}{2\sqrt{2}} \Rightarrow$ for some $c \in R$

$f(c) = \frac{1}{3}$

24. (a) Shortest distance between two curve occurred along the common normal

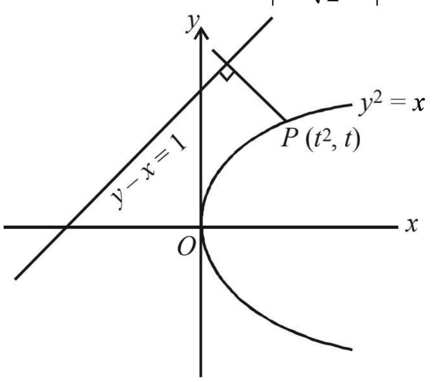
Slope of normal to $y^2 = x$ at point $P(t^2, t)$ is $-2t$ and slope of line $y - x = 1$ is 1.

As they are perpendicular to each other

$\therefore (-2t) = -1 \Rightarrow t = \frac{1}{2}$

$\therefore P\left(\frac{1}{4}, \frac{1}{2}\right)$

and shortest distance $= \left|\frac{\frac{1}{2} - \frac{1}{4} - 1}{\sqrt{2}}\right|$



So shortest distance between them is $\frac{3\sqrt{2}}{8}$

25. **(c)** $f'(x)=\sqrt{x}\sin x,\ f'(x)=0$

$\Rightarrow x=0$ or $\sin x=0$

$\Rightarrow x=2\pi,\ \pi\in\left(0,\frac{5\pi}{2}\right)$

$f''(x)=\sqrt{x}\cos x+\frac{1}{2\sqrt{x}}\sin x$

$=\frac{1}{2\sqrt{x}}(2x\cos x+\sin x)$

At $x=\pi,\ f''(x)<0$

Hence, local maxima at $x=\pi$

At $x=2\pi,\ f''(x)>0$

Hence local minima at $x=2\pi$

26. **(c)** Volume of spherical balloon $=V=\frac{4}{3}\pi r^3$

$\Rightarrow\ 4500\pi=\frac{4\pi r^3}{3}$ ($\because$ Given, volume $=4500\pi m^3$)

Differentiating both the sides, w.r.t 't' we get,

$$\frac{dV}{dt}=4\pi r^2\left(\frac{dr}{dt}\right)$$

Now, it is given that $\frac{dV}{dt}=72\pi$

$\therefore$ After 49 min, Volume $=(4500-49\times72)\pi$
$=(4500-3528)\pi$
$=972\pi m^3$

$\Rightarrow\ V=972\pi m^3 \qquad \therefore\ 972\pi=\frac{4}{3}\pi r^3$

$\Rightarrow\ r^3=3\times243=3\times3^5=3^6=(3^2)^3\Rightarrow r=9$

Also, we have $\frac{dV}{dt}=72\pi$

$\therefore\ 72\pi=4\pi\times9\times9\left(\frac{dr}{dt}\right)\ \Rightarrow\ \frac{dr}{dt}=\left(\frac{2}{9}\right)$

27. **(b)** Given, $f(x)=ln|x|+bx^2+ax$

$\therefore\ f'(x)=\frac{1}{x}+2bx+a$

At $x=-1,\ f'(x)=-1-2b+a=0$

$\Rightarrow\ a-2b=1$ ...(i)

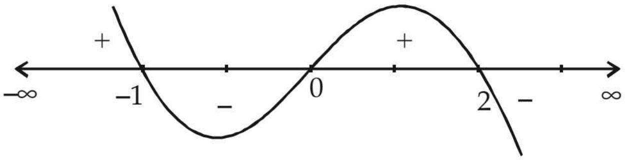
At $x=2,\ f'(x)=\frac{1}{2}+4b+a=0$

$\Rightarrow\ a+4b=-\frac{1}{2}$ ...(ii)

On solving (i) and (ii) we get $a=\frac{1}{2},b=-\frac{1}{4}$

Thus, $f'(x)=\frac{1}{x}-\frac{x}{2}+\frac{1}{2}=\frac{2-x^2+x}{2x}$

$=\frac{-x^2+x+2}{2x}=\frac{-(x^2-x-2)}{2x}=\frac{-(x+1)(x-2)}{2x}$



So maxima at $x=-1,2$

Hence both the statements are true and statement 2 is a correct explanation for 1.

28. **(c)** Equation of a line passing through (x_1,y_1) having slope m is given by $y-y_1=m(x-x_1)$

Since the line PQ is passing through (1,2) therefore its equation is

$(y-2)=m(x-1)$

where m is the slope of the line PQ.

Now, point $P(x,0)$ will also satisfy the equation of PQ

$\therefore\ y-2=m(x-1)$

$\Rightarrow\ 0-2=m(x-1)$

$\Rightarrow\ -2=m(x-1)\ \Rightarrow\ x-1=\frac{-2}{m}$

$\Rightarrow\ x=\frac{-2}{m}+1$

Also, $OP=\sqrt{(x-0)^2+(0-0)^2}=x=\frac{-2}{m}+1$

Similarly, point $Q(0,y)$ will satisfy equation of PQ

$\therefore\ y-2=m(x-1)$

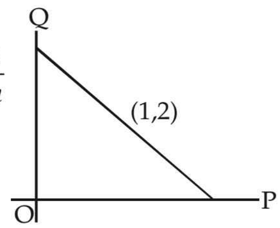
$\Rightarrow\ y-2=m(-1)\Rightarrow\ y=2-m$ and $OQ=y=2-m$

Area of $\Delta POQ=\frac{1}{2}(OP)(OQ)=\frac{1}{2}\left(1-\frac{2}{m}\right)(2-m)$

($\because$ Area of $\Delta=\frac{1}{2}\times$base$\times$height)

$=\frac{1}{2}\left[2-m-\frac{4}{m}+2\right]=\frac{1}{2}\left[4-\left(m+\frac{4}{m}\right)\right]$

$=2-\frac{m}{2}-\frac{2}{m}$



Let Area $=f(m)=2-\frac{m}{2}-\frac{2}{m}$

Now, $f'(m)=\frac{-1}{2}+\frac{2}{m^2}$

Put $f'(m)=0$

$\Rightarrow m^2=4\Rightarrow m=\pm2$

Now, $f''(m)=\frac{-4}{m^3}$

$f''(m)\big|_{m=2}=-\frac{1}{2}<0$

$f''(m)\big|_{m=-2} = \frac{1}{2} > 0$

Area will be least at $m = -2$

Hence, slope of PQ is -2.

29. (a) Since, $y = \int_0^x |t|\,dt,\ x \in R$

therefore $\frac{dy}{dx} = |x|$

But from $y = 2x$, $\frac{dy}{dx} = 2$

$\Rightarrow |x| = 2 \Rightarrow x = \pm 2$

Points $y = \int_0^{\pm 2} |t|\,dt = \pm 2$

$\therefore$ equation of tangent is

$y - 2 = 2(x-2)$ or $y + 2 = 2(x+2)$

$\Rightarrow$ x-intercept $= \pm 1$.

30. (b) Since, f and g both are continuous functions on [0, 1] and differentiable on (0, 1) then $\exists c \in (0,1)$ such that

$$f'(c) = \frac{f(1)-f(0)}{1} = \frac{6-2}{1} = 4$$

and $g'(c) = \frac{g(1)-g(0)}{1} = \frac{2-0}{1} = 2$

Thus, we get $f'(c) = 2g'(c)$

31. (a) $\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right] = 3 \Rightarrow \lim_{x\to 0}\frac{f(x)}{x^2} = 2$

So, f(x) contains terms in x^2, x^3 and x^4.

Let $f(x) = a_1x^2 + a_2x^3 + a_3x^4$

Since $\lim_{x\to 0}\frac{f(x)}{x^2} = 2 \Rightarrow a_1 = 2$

Hence, $f(x) = 2x^2 + a_2x^3 + a_3x^4$

$f'(x) = 4x + 3a_2x^2 + 4a_3x^3$

As given : $f'(1) = 0$ and $f'(2) = 0$

Hence, $4 + 3a_2 + 4a_3 = 0$ ...(1)

and $8 + 12a_2 + 32a_3 = 0$ ...(2)

By 4x (eq1) – eq (2), we get

$16 + 12a_2 + 16a_3 - (8 + 12a_2 + 32a_3) = 0$

$\Rightarrow 8 - 16a_3 = 0 \Rightarrow a_3 = 1/2$

and by eqn. (1), $4 + 3a_2 + 4/2 = 0 \Rightarrow a_2 = -2$

$\Rightarrow f(x) = 2x^2 - 2x^3 + \frac{1}{2}x^4$

$f(2) = 2\times 4 - 2\times 8 + \frac{1}{2}\times 16 = 0$

32. (d) $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$

$$= \tan^{-1}\left(\sqrt{\frac{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{\left(\sin\frac{x}{x}-\cos\frac{x}{2}\right)^2}}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)$$

$\Rightarrow y = \frac{\pi}{4} + \frac{x}{2}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}$

Slope of normal $= \frac{-1}{\left(\frac{dy}{dx}\right)} = -2$

At $\left(\frac{\pi}{6}, \frac{\pi}{4}+\frac{\pi}{12}\right)$

$y - \left(\frac{\pi}{4}+\frac{\pi}{12}\right) = -2\left(x - \frac{\pi}{6}\right)$

$y - \frac{4\pi}{12} = -2x + \frac{2\pi}{6}$

$y - \frac{\pi}{3} = -2x + \frac{\pi}{3}$

$y = -2x + \frac{2\pi}{3}$

This equation is satisfied only by the point $\left(0, \frac{2\pi}{3}\right)$

33. (a) $4x + 2\pi r = 2 \Rightarrow 2x + \pi r = 1$

$S = x^2 + \pi r^2$

$S = \left(\frac{1-\pi r}{2}\right)^2 + \pi r^2$

$\frac{dS}{dr} = 2\left(\frac{1-\pi r}{2}\right)\left(\frac{-\pi}{2}\right) + 2\pi r$

$\Rightarrow \frac{-\pi}{2} + \frac{\pi^2 r}{2} + 2\pi r = 0 \Rightarrow r = \frac{1}{\pi+4}$

$\Rightarrow x = \frac{2}{\pi+4} \Rightarrow x = 2r$