Class 11

Important Formulas

Complex Numbers and Quadratic Equations

Complex Numbers:

1. $\sqrt{-1}$ is an imaginary quantity and is denoted by i which has the following properties:
$i^2 = -1,\ i^3 = -i,\ i^4 = 1$ and, $i^{\pm n} = i^{\pm k},\ n \in N$
where k is the remainder when n is denoted by 4.
2. For any positive real number a, $\sqrt{-a} = i\sqrt{a}$.
3. For any two real numbers a and b, we have
$$\sqrt{a}\,\sqrt{b} = \begin{cases} \sqrt{ab}, & \text{if at least one of a and b is positive} \\ -\sqrt{ab}, & \text{if } a<0, b<0. \end{cases}$$
4. If a, b are real numbers, then a number $z = a + ib$ is called a complex number, real number a is known as the real part of z and b is known as its imaginary part. We write $a = \text{Re}(z)$, $b = \text{Im}(z)$.
A complex number z is purely real iff $\text{Im}(z) = 0$ and z is purely imaginary iff $\text{Re}(z) = 0$
5. For any two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, we define
Addition: $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$
Subtraction: $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$
Multiplication: $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$
Reciprocal: $\dfrac{1}{z_1} = \dfrac{a_1}{a_1^2 + b_1^2} - i\dfrac{b_1}{a_1^2 + b_1^2}$
Division: $\dfrac{z_1}{z_2} = z_1\left(\dfrac{1}{z_2}\right) = (a_1 + ib_1)\left(\dfrac{a_2}{a_2^2 + b_2^2} - i\dfrac{b_2}{a_2^2 + b_2^2}\right) = \dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\dfrac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}$
Addition is commutative and associative. Complex number $0 = 0 + i0$ is the identity element for addition and every complex number $z = a + ib$ has its additive inverse $-z = -a - ib$.
Multiplication is also commutative and associative. Complex number $1 = 1 + 0i$ is the identity element for multiplication. Every non-zero complex number $z = a + ib$ has its multiplicative inverse $1/z$ (also known as reciprocal of z) such that $\dfrac{1}{z} = \dfrac{a - ib}{a^2 + b^2} = \dfrac{\bar{z}}{|z|^2}$.
6. The conjugate of a complex number $z = a + ib$ is denoted by $\bar{z}$ and is equal to $a - ib$.
For any three complex numbers z, z_1, z_2, we have
(i) $(\bar{\bar{z}}) = z$ (ii) $z + \bar{z} = 2\,Re(z)$
(iii) $z - \bar{z} = 2i\,Im(z)$ (iv) $z = \bar{z} \Leftrightarrow z$ is purely real
(v) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary (vi) $z\bar{z} = \{Re(z)\}^2 + \{Im(z)\}^2 = |z|^2$
(vii) $\overline{z_1 \pm z_2} = \bar{z_1} \pm \bar{z_2}$ (viii) $\overline{z_1 z_2} = \bar{z_1}\,\bar{z_2}$

(ix) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

7. The modulus of a complex number $z = a + ib$ is denoted by $|z|$ and is defined as

$$|z| = \sqrt{a^2 + b^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2}$$

If z, z_1, z_2 are three complex numbers, then

(i) $|z| = 0 \Leftrightarrow z = 0$ i.e. $\text{Re}(z) = \text{Im}(z) = 0$ (ii) $|z| = |\bar{z}| = |-z|$

(iii) $-|z| \leq \text{Re}(z) \leq |z|$; $-|z| \leq \text{Im}(z) \leq |z|$ (iv) $z\bar{z} = |z|^2$

(v) $|\text{Im}(z^n)| \leq n|\text{Im}(z)||z|^{n-1}, n \in N$ (vi) $|\text{Re}(z)| + |\text{Im}(z)| \leq \sqrt{2}|z|$

8. A complex number $z = x + iy$ can be represented by a point $P(x, y)$ (see Fig. 13.8) on the plane which is known as the Argand or Gaussian or Complex plane. The length of the line segment OP is called the modulus of z and is denoted by $|z|$.

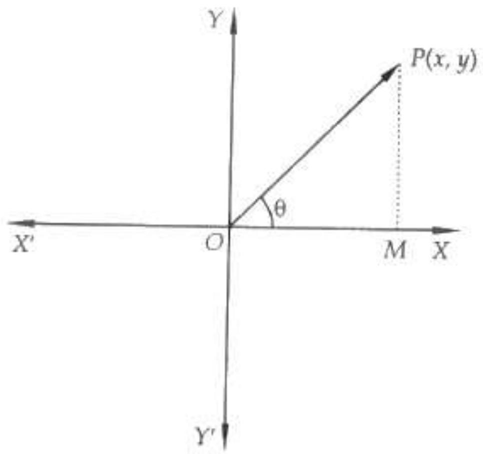

Fig. 13.8

Clearly, $|z| = \sqrt{x^2 + y^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2}$

The angle θ which OP makes with the positive direction of x-axis in anti-clockwise sense is called the argument or amplitude of z and is denoted by *arg* (z) or amp (z).

Clearly, $\tan\theta = \dfrac{\text{Im}(z)}{\text{Re}(z)}$.

Let $OP = r$ and $\angle XOP = \theta$. Then, $x = r\cos\theta$ and $y = r\sin\theta$

$\therefore \quad z = x + iy = r(\cos\theta + i\sin\theta)$

This is known as the polar form of complex number z. The Euler's notations are

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

$\therefore \quad z = r(\cos\theta + i\sin\theta)$

or, $\quad z = re^{i\theta}$, which is known as the Eulerian form of z.

Quadratic Equations:

1. Fundamental Theorem of Algebra: Every polynomial equation $f(x) = 0$ has at least one root, real or imaginary (complex).
2. Every polynomial equation $f(x) = 0$ of degree n has exactly n roots real or imaginary.
3. A quadratic equation cannot have more than two roots.
4. If $ax^2 + bx + c = 0, a \neq 0$ is a quadratic equation with real coefficients, then its roots α and β given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and, } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ or, } \alpha = \frac{-b + \sqrt{D}}{2a} \text{ and, } \beta = \frac{-b - \sqrt{D}}{2a}$$

where $D = b^2 - 4ac$ is as the discriminant of the equation.

(i) If $D = 0$, then $\alpha = \beta = -\frac{b}{2a}$

So, the equation has real and equal roots each equal to $-\frac{b}{2a}$.

(ii) If $a, b, c \in Q$ and D is positive and a perfect square, then roots are rational and unequal.

(iii) If $a, b, c \in R$ and D is positive and a perfect square, then the roots are real and distinct.

(iv) If $D > 0$ but it is not a perfect square, then roots are irrational and unequal.

(v) If $D < 0$, then the roots are imaginary and are given by

$$\alpha = \frac{-b + i\sqrt{4ac - b^2}}{2a} \text{ and } \beta = \frac{-b - i\sqrt{4ac - b^2}}{2a}$$

(vi) If $a = 1, b, c \in I$ and the roots are rational numbers, then these roots must be integers.

(vii) If a quadratic equation in x has more than two roots, then it is an identity in x that is $a = b = c = 0$.

(viii) Complex roots of an equation with real coefficients always occur in pairs. However, this may not be true in case of equations with complex coefficients. For example, $x^2 - 2ix - 1 = 0$ has both roots equal to i.

(ix) Surd root of an equation with rational coefficients always occur in pairs like $2 + \sqrt{3}$ and $2 - \sqrt{3}$. However, Surd roots of an equation with irrational coefficients may not occur in pairs. For example, $x^2 - 2\sqrt{3}\,x + 3 = 0$ has both roots equal to $\sqrt{3}$.