3

Trigonometrical Functions

3.01 Introduction

Ancient Indian have knowledge of Trigonometry. Aaryabhatta (476 A. D), Bramhagupta (598 A.D.), Bhaskar prathma (600 A.D.) and Bhaskara Dvitiya (1114 A.D.) obtained the main results. This whole knowledge was spread from India to middle-east and again to Europe. Yunanians also started the study of trigonometry but their working was inapropriate whole world accepted the indian method.

In India, modern trigonometrical functions like sine of angle and total disciption of introduction of function is given into "Siddhant" (Astrological work written in sanskrit language) whose contribution is very important in the history of mathematics.

Bhaskara prathama (600 A.D.) gave the formula for value of sine of angles more than 90°. Work of 16th centuary of malyalam language is also a type of explanation of sin (A+B). Indian Jya and Cotijya have converted into sine and cosine of europian language. Two centuary ago from pythagoras, Indians were aware of pythagoras theorem Baudhayan and katyayan have proved this theorem. Also we get the detail of this application.

In this chapter we will study about the definition of trigonometrical functions their domain range and graph, angles related to angle θ, $(-\theta)$, $\left(\frac{\pi}{2}\pm\theta\right)$, $(\pi\pm\theta)$, $\left(\frac{3\pi}{2}\pm\theta\right)$, $(2\pi\pm\theta)$ etc. and representation of them in terms of θ representation of combine angles (A+B), (A–B), (A+B+C) etc. and trigonometrical functions in terms of A, B, C etc. and solution of trigonometrical equations. We will study about the proof and applications of formulae of sine and cosine.

3.02 Angle

Angle is a measure of rotation of a given ray about its initial point. The original ray is called the *initial side* and the final position of the ray after rotation is called the *terminal side* of the angle. The point of rotationi s called the *vertex*. If the direction of rotation is anticlockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is *negative*.

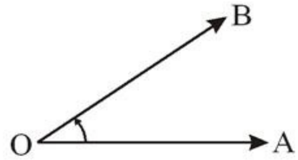

Fig. 3.01

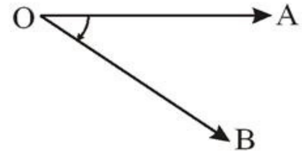

Fig. 3.02

In fig 3.01 the angle is measured positive whereas in fig. 3.02 angle is negative.

We shall describe two other units of measurement of an angle which are most commonly used, viz.

(i) Degree measure. (ii) Radian measure.

Degree measure : If revolving line is perpendicular to initial line then the angle is 90^0. Similarly the revolving line makes an angle two right angle from the initial line then both sides makes a straight angle. If revolving line is completing a round and comes in the initial position then the angle is 3600 i.e. four right angle. If a rotation from the initial side to terminal side is $\left(\frac{1}{360}\right)^{o}$ of a rvolution, the angle is said to have a measure of one *degree*, writen as 1°A degree is divided into 60 minutes, and a minute is divided into 60 seconds. One sixtieth of a degree is called a *minute*, written as 1', and one sixtieth of a minute is called a *second*, written as 1°. Thus, 1° = 60' and 1' = 60° Some of the angles whose measures are 90°, 180° and 360° are shown-

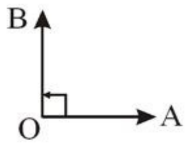

Fig. 3.03

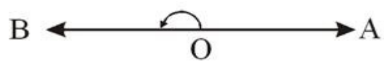

Fig. 3.04

Fig. 3.05

Radian Measure : There is another unit for measurement of an angle, called the *radian* measure. Angle subtended at the centre by an arc of length 1 unit in a unit cirlce (circle of radius 1 unit) is said to have a angles whose measure are 1 radian, 1.5 radian, –1 radian and –1.5 radian, 2 radian etc.

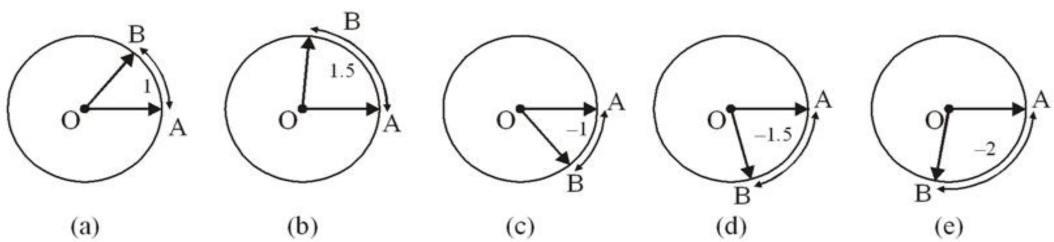

Fig. 3.07

We know that the circumference of a circle of radius 1 unit is 2π. Thus, one complete revolution of the initial side subtends an angle of 2π radian. More generally, in a circle of radius *r*, an arc of length *r* will subtend an angle of 1 radian. it is well-known that equal arcs of a circle subtend equal angle at the centre. Since in a circle of radius *r*, an arc of length *r* subtends an angle whsoe meaure is 1 radian, an arc of length *l* will subtend an angle whose measure is 1 / *r* radian. Thus, in a circle of radius *r*, an arc of length *l* subtends an angle θ radian at the centre, we have $\theta = \frac{1}{r} \Rightarrow \ell = r\theta$

Relation between degree and radian

Since a circle subtends and angle at the centre and angle whose radian measure is 2π and its degree measure is 360° . It follows that,

2π radian = 360 degree or π radian = 180 degree

If the degree measure is D and radian be R, then the relation between them be

$$\frac{D}{360^\circ} = \frac{R}{2\pi} \qquad \text{or} \qquad \frac{D}{90^\circ} = \frac{2R}{\pi}$$

If we take π = 22/7, then 1 radian = 57°16' approx.

$1° = \pi / 180$ radian = 0.01746 radian approx

Table 3.01

Degree	30°	45°	60°	90°	180°	270°	360°
Radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Illustrative Examples

Example 1. Convert $50°30'$ into radian measure.

Solution : We know that, $180° = \pi$ radian

$$\therefore\ 50°30' = 50\frac{1}{2}\text{ degree} = \left(\frac{\pi}{180}\right)\times\left(\frac{101}{2}\right)\text{radian}$$

$$= \frac{101\pi}{360}\text{radian.}$$

Example 2. Convert 6 radians into degree measure.

Solution : $\therefore$ π radian $= 180°$

$$\therefore\ 6\text{ radians} = \left(\frac{180}{\pi}\right)\times 6\text{ degree} = \frac{1080\times 7}{22} = 343\frac{7}{11}\text{degree}$$

$$= 343° + \frac{7\times 60}{11}\text{minutes} \qquad [\because\ 1° = 60']$$

$$= 343° + 38' + \frac{2}{11}\text{minutes}$$

$$= 343° + 38' + 10.9'' \qquad [\because\ 1' = 60'']$$

$$= 343°38'11''\ \text{approx}$$

Exercise 3.1

1 Find the radian masures corresponding to the following degree measures

(i) 25° (ii) –47°30' (iii) 520°

2. Find the degree measures corresponding to the following radian measure ($\pi = 22/7$)

(i) 11 / 16 (ii) –4 (iii) $5\pi/3$

3. A wheel makes 360 revolutions in one minute. Through how many raidans does it turn in one second?
4. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm ($\pi = 22/7$)
5. In a circle of diameter 40 cm, the length of chord is 20 cm. Find the length of minor arc of the chord.
6. If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii.
7. Find the angle in radian through which a pendulum swings if its length is 75 cm and the tip describes an arc of length

(i) 10 cm (ii) 21 cm

3.03 Signs of Trigonometric Functions

Consider a unit circle with centre O at origin of the coordinate axes.

P (a, b) be any point on the circle with angle AOP = x radian, i.e., length of arc AP = x

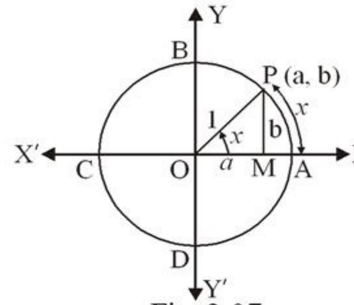

Fig. 3.07

therefore, $\sin x = b$ and $\cos x = a$

since ΔOMP is a right triangle, we have

$$OM^2 + MP^2 = OP^2$$

$$\Rightarrow \quad a^2 + b^2 = 1$$

$$\Rightarrow \quad \cos^2 x + \sin^2 x = 1$$

Since one complete revolution subtends an angle of 2π radian at the centre of the circle $\angle AOB = \frac{\pi}{2}$, $\angle AOC = \pi$ and $\angle AOD = \frac{3\pi}{2}$ All angles which are integral multiples of $\frac{\pi}{2}$ are called *quadrantal angles*. The coordinates of the points A, B, C, and D are respectively, (1, 0), (0, 1), (–1, 0) and (0, –1).

First Quadrant : In the first quadrant $\left(0 < x < \frac{\pi}{2}\right)$ a and b are both positive, in the second quadrant $\left(\frac{\pi}{2} < x < \pi\right)$ a is negative and b is positive, in the third quadrant $\left(\pi < x < \frac{3\pi}{2}\right)$ a and b are both negative and in the fourth qudrant $\left(\frac{3\pi}{2} < x < 2\pi\right)$ a is positive and b is negative. Therefore, sin x is positive for $(0 < x < \pi)$ and negative for $(\pi < x < 2\pi)$. Similarly, cos x is positive for $\left(0 < x < \frac{\pi}{2}\right)$, negative for $\left(\frac{\pi}{2} < x < \frac{3\pi}{2}\right)$ and also positive for $\left(\frac{3\pi}{2} < x < 2\pi\right)$ Likewise, we can find the signs of other trigonometric functions in different quadrants. In fact, we have the following table.

Table 3.02

	I quadrant	II quadrant	III quadrant	IV quadrant
$\sin x$	+	+	–	–
$\cos x$	+	–	–	+
$\tan x$	+	–	+	–
cosec x	+	+	–	–
$\sec x$	+	–	–	+
$\cot x$	+	–	+	–

3.04 Domain and Range of Trigonometric Functions

From the definition of sine and cosine fucntions, we observe that they are defined for all real numbers. Further, we observe that for each real nubmers x, $-1 \le \sin x \le 1$ and $-1 \le \cos x \le 1$

Thus, domain of $y = \sin x$ and $y = \cos x$ is the set of all real numbers and range is the interval [–1, 1] i.e. $-1 \le y \le 1$. Since $y = \operatorname{cosec} x = 1/\sin x$ the domain $\{x : x \in R, x \ne n\pi$ and $n \in Z\}$ and range is the set $\{y : y \in R, y \le -1 \text{ or } y \ge 1\}$

The domain of $y = \tan x = \dfrac{\sin x}{\cos x}$ is the set $\left\{x : x \in R, x \ne (2n+1)\dfrac{\pi}{2}; n \in Z\right\}$ and range is the set of all real numbers.

For $y = \cot x$ the domain set $\{x : x \in R, x \ne n\pi \, \forall \, n \in Z\}$ and range is the set of all real numbers

Table 3.03

	I quadrant	II quadrant	III quadrant	IV quadrant
sin	increases from 0 to 1	decreases from 1 to 0	decreases from 0 to -1	increases from -1 to 0
cos	decreases from 1 to 0	decreases from 0 to -1	increases from -1 to 0	increases from 0 to 1
tan	increases from 0 to ∞	increases from $-\infty$ to 0	increases from 0 to ∞	increases from $-\infty$ to 0
cot	decreases from ∞ to 0	decreases from 0 to $-\infty$	decreases from ∞ to 0	decreases from 0 to $-\infty$
sec	increases from 1 to ∞	increases from $-\infty$ to -1	decreases from -1 to $-\infty$	decreases from ∞ to 1
cosec	decreases from ∞ to 1	increases from 1 to ∞	increases from $-\infty$ to -1	decreases from -1 to $-\infty$

Note: In the above table, the statement tan x increases from 0 to ∞ (infinity) for $0 < x < \pi/2$ simply means that tan x increases as x increases for $0 < x < \pi/2$. Similarly, to say that cosec x decreases from –1 to $-\infty$ (minus infinity) in the fourth quadrant means that cosec x decreases for $x \in (3\pi/2, 2\pi)$ and assumes arbitarily large negative values as x approaches to 2π. The symbols ∞ and $-\infty$ simply specify certain types of behaviour of functiosn and variables.

3.05 Graph of Trignometrical Functions

We have already seen that values of sin x and cos x repeats after an interval of 2π. Hence, values of cosec x and sec x will also repeat after an interval of 2π. We shall see in the next section that $\tan(\pi + x) = \tan x$.

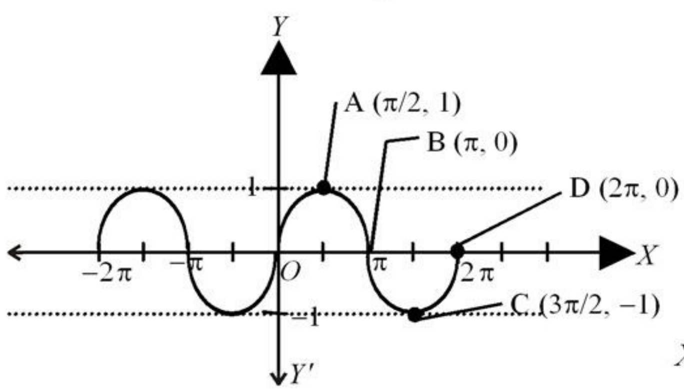

Fig. 3.08 $f(x) = \sin x$

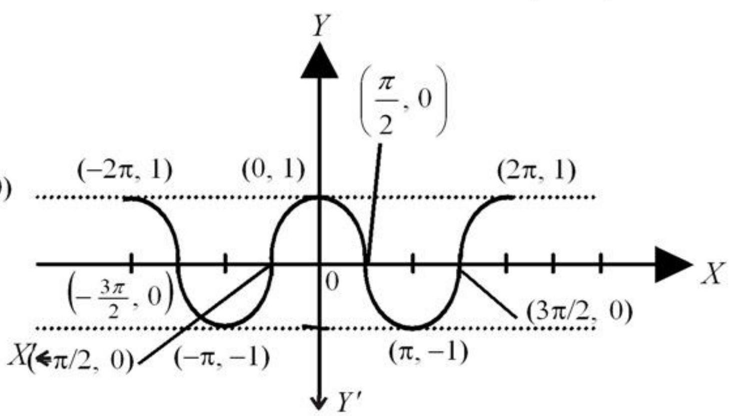

Fig. 3.09 $f(x) = \cos x$

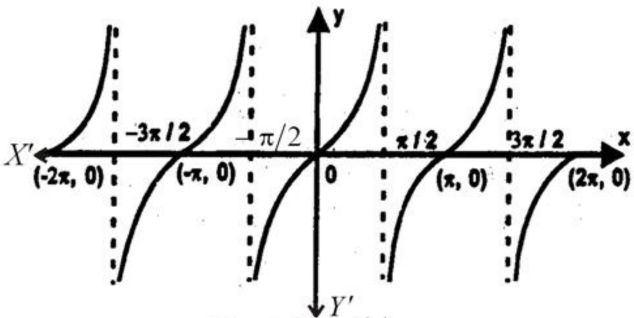

Fig. 3.10 $f(x) = \tan x$

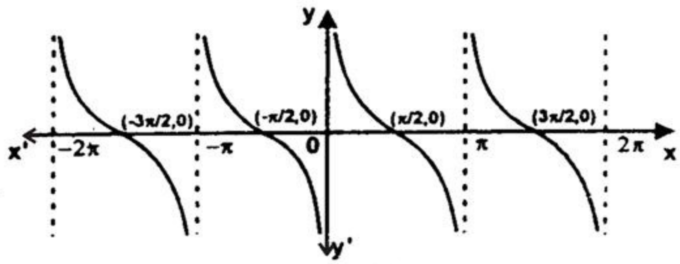

Fig. 3.11 $f(x) = \cot x$

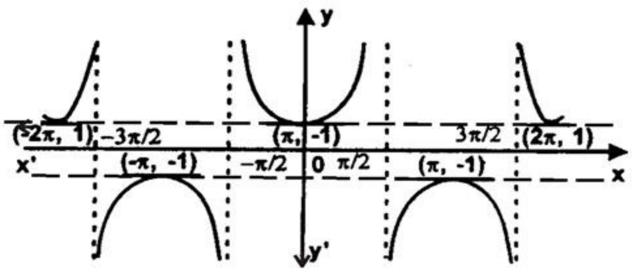

Fig. 3.12 $f(x)$ = sec x

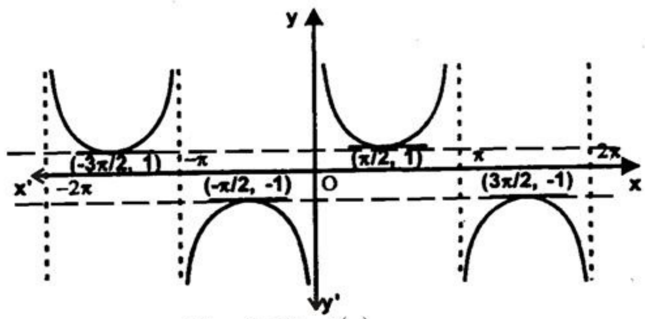

Fig. 3.13 $f(x)$ = cosec x

Hence, values of tan x will repeat after an interval of π. Since cot x is reciprocal of tan x, its values will also repeat after an interval of π. Using this knowledge and behaviour of trigonometric functions, we can sketch the graph of these functions. The graphs of these functions are given below

Example 3. If $\sin x = -4/5$, x lies in the third quadrant, find the values of other five trigonometric functions.

Solution : Since $\sin x = -4/5$, $\therefore$ $\operatorname{cosec} x = -5/4$

we have, $\sin^2 x + \cos^2 x = 1$ $\qquad \Rightarrow \qquad \cos^2 x = 1 - \sin^2 x = 1 - \frac{16}{25} = \frac{9}{25}$

$\therefore$ $\quad \cos x = \pm 3/5$ but x lies in the third quadrant, $\therefore$ $\cos x$ is negative. Therefore

$$\therefore \quad \cos x = -3/5 \Rightarrow \sec x = -5/3 \Rightarrow \tan x = \frac{\sin x}{\cos x} = \frac{4}{3} \text{ and } \cot x = \frac{3}{4}$$

Example 4. Find the value of $\sin \frac{31\pi}{3}$.

Solution : We know that values of sin x repea after an interval of 2π. Therefore

$$\sin \frac{31\pi}{3} = \sin\left(10\pi + \frac{\pi}{3}\right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Example 5. Find the vlaue of $\operatorname{cosec}(-1410°)$.

Solution : We know that values of $\operatorname{cosec} x$ repeat after an interval of 2π or $360°$

$$\therefore \quad \operatorname{cosec}(-1410°) = \operatorname{cosec}(-1410° + 4 \times 360°) = \operatorname{cosec}(-1410° + 1440°) = \operatorname{cosec}(30°) = 2.$$

Exericse 3.2

Find the values of other five trigonometric functions when

1. $\cos x = -1/2$, x lies in third quadrant.

2. $\cot x = 3/4$, x lies in third quadrant.

3. $\sec x = 13/5$, x lies in fourth qudrant.

Find the values of trigonometric functions.

4. $\sin 765°$ 5. $\tan 19\pi/3$ 6. $\sin\left(-\frac{11\pi}{3}\right)$ 7. $\cot\left(-\frac{15\pi}{4}\right)$

3.06 Trigonometrical Functions of Sum and Difference of Two Angles

In this Section, we shall derive expressions for trigonometric functions of the sum and difference of two angles and related expressions. The basic results in this connection are called *trigonometric identities*. We have seen that

1. $\sin(-x) = -\sin x$

2. $\cos(-x) = \cos x$

3. $\tan(-x) = -\tan x$

We shall now prove some more resutls:

4- $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

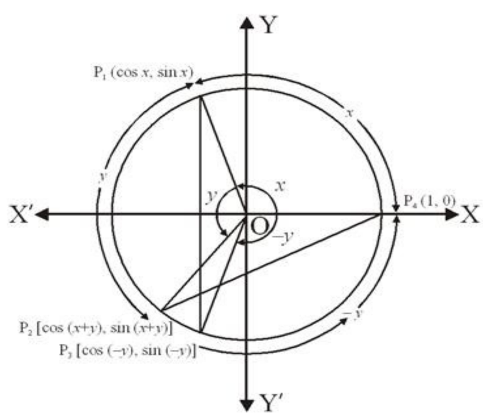

Fig. 3.14

Consider the unit circle with centre at the origin. Let x be the angle P_4OP_1, x and y be the angle P_1OP_2. Then $(x+y)$ is the angle P_4OP_2. Also let $(-y)$ be the angle P_4OP_3. Therefore, P_1, P_2, P_3 and P_4 will have the coordinates $P_1(\cos x, \sin x), P_2[\cos(x+y), \sin(x+y)]$, $P_3[\cos(-y), \sin(-y)]$ and $P_4(1,0)$ (Fig. 3.14).

$$P_1P_3^2 = \left[\cos x - \cos(-y)\right]^2 + \left[\sin x - \sin(-y)\right]^2$$

$$= (\cos x - \cos y)^2 + (\sin x + \sin y)^2$$

$$= \cos^2 x + \cos^2 y - 2\cos x \cos y + \sin^2 x + \sin^2 y + 2\sin x \sin y$$

$$= 2 - 2(\cos x \cos y - \sin x \sin y) \qquad [\sin^2 x + \cos^2 x = 1 \text{ etc}]$$

again $P_2P_4^2 = \left[1-\cos\left(x+y\right)\right]^2 + \left[0-\sin\left(x+y\right)\right]^2$

$$= 1 - 2\cos(x+y) + \cos^2(x+y) + \sin^2(x+y) = 2 - 2\cos(x+y)$$

again $P_1P_3 = P_2P_4 \Rightarrow P_1P_3^2 = P_2P_4^2$

$\therefore \quad 2 - 2(\cos x\cos y - \sin x\sin y) = 2 - 2\cos(x+y)$

$\therefore \quad \cos(x+y) = \cos x\cos y - \sin x\sin y$

5. $\cos(x-y) = \cos x\cos y - \sin x\sin y$

Replacing y by $-y$ in identity 4

$\cos(x+(-y)) = \cos x\cos(-y) - \sin x\sin(-y)$

or $\cos(x-y) = \cos x\cos y + \sin x\sin y$

6. $\cos\left(\frac{\pi}{2}-x\right) = \sin x$

If we replace x by $\pi/2$ and y by x in identity (5), we get

$$\cos\left(\frac{\pi}{2}-x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x = \sin x$$

7. $\sin\left(\frac{\pi}{2}-x\right) = \cos x$

Using the Identity 6, we have

$$\sin\left(\frac{\pi}{2}-x\right) = \cos\left[\frac{\pi}{2}-\left(\frac{\pi}{2}-x\right)\right] = \cos x.$$

8. $\sin(x+y) = \sin x\cos y + \cos x\sin y$

We know that $\sin(x+y) = \cos\left(\frac{\pi}{2}-(x+y)\right) = \cos\left(\left(\frac{\pi}{2}-x\right)-y\right)$

$$= \cos\left(\frac{\pi}{2}-x\right)\cos y + \sin\left(\frac{\pi}{2}-x\right)\sin y$$

$$= \sin x\cos y + \cos x\sin y$$

9. $\sin(x-y) = \sin x\cos y - \cos x\sin y$

If we replace y by $-y$ in the Identity 8, we get the result.

10. By taking suitable values of x and y in the identities 4, 5, 8 and 9, we get the following results

$\cos\left(\frac{\pi}{2}+x\right) = -\sin x \qquad \sin\left(\frac{\pi}{2}+x\right) = \cos x$

$\cos(\pi-x) = -\cos x \qquad \sin(\pi-x) = \sin x$

$\cos(\pi+x) = -\cos x \qquad \sin(\pi+x) = -\sin x$

$\cos(2\pi - x) = \cos x$ $\qquad\qquad$ $\sin(2\pi - x) = -\sin x$

Similar results for $\tan x, \cot x, \sec x$ and $\operatorname{cosec} x$ can be obtained from the results of $\sin x$ and $\cos x$.

11. If none of the angles x, y and $(x + y)$ is an odd multiple of $\pi / 2$, then

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Since none of the x, y and $(x+y)$ is an odd multiple of $\pi/2$, it follows that $\cos x, \cos y$ and $\cos(x+y)$ are non-zero. Now,

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

Dividing numerator and denominator by $\cos x \cos y$, we have

$$\tan(x+y) = \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos y}} = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

12. $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

If we replace y by $-y$ in Identity 11 we get

$$\tan(x-y) = \tan[x + (-y)]$$

$$= \frac{\tan x + \tan -(y)}{1 - \tan x \tan(-y)} = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

13. If none of the angles x, y and $(x + y)$ is a multiple of π, then

$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

If none of the angles x, y and $(x + y)$ is multiple of π, we find that $\sin x \sin y$ and $\sin(x + y)$ are non-zero. Now

$$\cot(x+y) = \frac{\cos(x+y)}{\sin(x+y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y}$$

Dividing numerator and denominator by $\sin x \sin y$, we have

$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

14. $\cot(x-y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$, if none of angles x, y and $x - y$ is a multiple of π

If we replace y by $-y$ in identity 13, we get the result

15. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

Replacing y by x, we get

$$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1-\cos^2 x) = 2\cos^2 x - 1.$$

Again, $\cos 2x = \cos^2 x - \sin^2 x = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x.$

therefore, $\cos 2x = \cos^2 x - \sin^2 x = \dfrac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x}.$

Dividing each term by $\cos^2 x$

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}, \quad x \neq n\pi + \frac{\pi}{2}, \quad \forall n \in Z$$

16. $\sin 2x = 2\sin x \cos x = \dfrac{2\tan x}{1+\tan^2 x}$

We have $\sin(x+y) = \sin x \cos y + \cos x \sin y$

Replacing y by x, we get $\sin 2x = 2\sin x \cos x.$

Again, $$\sin 2x = \frac{2\sin x \cos x}{\cos^2 x + \sin^2 x}$$

Dividing each term by $\cos^2 x$

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

17. $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}, \quad 2x \neq n\pi + \dfrac{\pi}{2} \quad \forall n \in Z$

we have $$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Replacing y by x, we get $$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

18. $\sin 3x = 3\sin x - 4\sin^3 x$

$\because \quad \sin 3x = \sin(2x + x)$

$$= \sin 2x \cos x + \cos 2x \sin x$$
$$= 2\sin x \cos x \cos x + (1 - 2\sin^2 x)\sin x = 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x$$
$$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x = 3\sin x - 4\sin^3 x$$

19. $\cos 3x = 4\cos^3 x - 3\cos x$

$\because \quad \cos 3x = \cos(2x + x)$

$$= \cos 2x \cos x - \sin 2x \sin x$$

$$= (2\cos^2 x - 1)\cos x - 2\sin x \cos x \sin x = (2\cos^2 x - 1)\cos x - 2\cos x(1-\cos^2 x)$$

$$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x = 4\cos^3 x - 3\cos x$$

20. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$, $\quad 3x \neq n\pi + \dfrac{\pi}{2}\ \forall n \in Z$

$$\because \quad \tan 3x = \tan(2x + x) = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} = \frac{\dfrac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \dfrac{2\tan x . \tan x}{1 - \tan^2 x}}$$

$$= \frac{2\tan x + \tan x - \tan^3 x}{1 - \tan^2 x - 2\tan^2 x} = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

21. (i) $\cos x + \cos y = 2\cos\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$

(ii) $\cos x - \cos y = -2\sin\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$

(iii) $\sin x + \sin y = 2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$

(iv) $\sin x - \sin y = 2\cos\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$

we have $\quad \cos(x+y) = \cos x \cos y - \sin x \sin y \quad (1)$

and $\quad \cos(x-y) = \cos x \cos y + \sin x \sin y \quad (2)$

Adding and subtracting (1) and (2), we get

$$\cos(x+y) + \cos(x-y) = 2\cos x \cos y \quad (3)$$

and $\quad \cos(x+y) - \cos(x-y) = -2\sin x \sin y \quad (4)$

Further $\quad \sin(x+y) = \sin x \cos y + \cos x \sin y \quad (5)$

and $\quad \sin(x-y) = \sin x \cos y - \cos x \sin y \quad (6)$

Adding and subtracting (5) and (6), we get

$$\sin(x+y) + \sin(x-y) = 2\sin x \cos y \quad (7)$$

$$\sin(x+y) - \sin(x-y) = 2\cos x \sin y \quad (8)$$

Let $x + y = \theta$ and $x - y = \phi$, therefore

$$x = \left(\frac{\theta+\phi}{2}\right) \text{ and } y = \left(\frac{\theta-\phi}{2}\right)$$

Substituting the values of x and y in (3), (4), (7) and (8), we get

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

Since θ and ϕ can take any real values, we can replace θ by x and ϕ by y.

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2};\quad \cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2},$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2};\quad \sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$

Note: As a part of identities given in 21, we can prove the following results:

22. (i) $2\cos x\cos y = \cos(x+y) + \cos(x-y)$

 (ii) $-2\sin x\sin y = \cos(x+y) - \cos(x-y)$

 (iii) $2\sin x\cos y = \sin(x+y) + \sin(x-y)$

 (iv) $2\cos x\sin y = \sin(x+y) - \sin(x-y)$

23. **Sum of more than two angles:** Trigonometric functions containing more than sum of two angles can be found by pairing any two angles and using the formula for two angles.

 (i) $\sin(A+B+C) = \sin[(A+B)+C]$

$$= \sin(A+B)\cos C + \cos(A+B)\sin C$$
$$= [\sin A\cos B + \cos A\sin B]\cos C + [\cos A\cos B - \sin A\sin B]\sin C$$
$$= \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$$

 (ii) $\cos(A+B+C) = \cos[(A+B)+C]$

$$= \cos(A+B)\cos C - \sin(A+B)\sin C$$
$$= [\cos A\cos B - \sin A\sin B]\cos C - [\sin A\cos B + \cos A\sin B]\sin C$$
$$= \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$$

 (iii) similarly

$\tan(A+B+C) = \dfrac{\sin(A+B+C)}{\cos(A+B+C)}$, Dividing the numerator and denomiantor by $\cos A\cos B\cos C$

$$= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$$

Illustrative Examples

Example 6. Prove that $3\sin\frac{\pi}{6}\sec\frac{\pi}{3} - 4\sin\frac{5\pi}{6}\cot\frac{\pi}{4} = 1$.

Solution : L.H.S. $= 3\sin\frac{\pi}{6}\sec\frac{\pi}{3} - 4\sin\frac{5\pi}{6}\cot\frac{\pi}{4}$

$$= 3\times\frac{1}{2}\times 2 - 4\sin\left(\pi - \frac{\pi}{6}\right)\times 1 = 3 - 4\sin\frac{\pi}{6} = 3 - 4\times 1/2 = 1 = \text{R.H.S.}$$

Example 7. Evaluate $\cos 15°$.

Solution : $\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$

$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{1}{2\sqrt{2}}[\sqrt{3}+1].$$

Example 8. Prove that

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y}.$$

Solution : Here L.H.S. $= \frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x\cos y + \cos x \sin y}{\sin x \cos y - \cos x \sin y}$

Dividing the numerator and denominator by $\cos x \cos y$

$$= \frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x + \tan y}{\tan x - \tan y} = \text{ R.H.S.}$$

Example 9. Show that $\tan x \tan 2x \tan 3x = \tan 3x - \tan 2x - \tan x$.

Solution : We know that $3x = 2x + x$

i.e. $\tan 3x = \tan(2x + x)$

or $\tan 3x = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$

or $\tan 3x - \tan 3x \tan 2x \tan x = \tan 2x + \tan x$

or $\tan 3x - \tan 2x - \tan x = \tan 3x \tan 2x \tan x$

or $\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$.

Example 10. Prove that $\frac{\cos 7x + \cos 5x}{\sin 7x - \sin 5x} = \cot x$.

Solution : Using the identities $21(i)$ and $21(iv)$

$$\text{L.H.S.} = \frac{2\cos\frac{7x+5x}{2}\cos\frac{7x-5x}{2}}{2\cos\frac{7x+5x}{2}\sin\frac{7x-5x}{2}} = \frac{\cos x}{\sin x} = \cot x = \text{ R.H.S.}$$

Example 11. Prove that

$$\sin 20° \sin 40° \sin 60° \sin 80° = 3/16.$$

Solution : L.H.S. $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{1}{2}\sin 60°(2\sin 20° \sin 40°)\sin 80°$

$$= \frac{1}{2}\frac{\sqrt{3}}{2}[\cos 20° - \cos 60°]\sin 80°$$

$$[2\sin A \sin B = \cos(A-B) - \cos(A+B)]$$

$$= \frac{\sqrt{3}}{4}\left[\cos 20° \sin 80° - \left(\frac{1}{2}\right)\sin 80°\right]$$

$$= \frac{\sqrt{3}}{4}\left[\frac{1}{2}(2\cos 20° \sin 80°) - \left(\frac{1}{2}\right)\sin 80°\right]$$

$$= \frac{\sqrt{3}}{8}[\sin 100° + \sin 60° - \sin 80°]$$

$$= \frac{\sqrt{3}}{8}\left[\sin(180° - 80°) + \frac{\sqrt{3}}{2} - \sin 80°\right]$$

$$= \frac{\sqrt{3}}{8}\left[\sin 80° + \frac{\sqrt{3}}{2} - \sin 80°\right] = 3/16.$$

Example 12. Prove that

$$2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$$

Solution : L.H.S. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$

$$= \cos\left(\frac{\pi}{13} + \frac{9\pi}{13}\right) + \cos\left(\frac{\pi}{13} - \frac{9\pi}{13}\right) + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$$

$$= \cos\frac{10\pi}{13} + \cos\frac{8\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$$

$$[\cos(-\theta) = \cos\theta]$$

$$= \cos\left(\pi - \frac{3\pi}{13}\right) + \cos\left(\pi - \frac{5\pi}{13}\right) + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$$

$$= -\cos\frac{3\pi}{13} - \cos\frac{5\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$$

$$= 0.$$

Example 13. If $A + B + C = 180°$ then prove that

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Solution : $\sin A+\sin B+\sin C=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\sin\frac{C}{2}\cos\frac{C}{2}$

$$=2\sin\left(90^\circ-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\sin\frac{C}{2}\cos\frac{C}{2} \qquad [\because A+B+C=180^\circ]$$

$$=2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\sin\frac{C}{2}\right]$$

$$=2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\sin\left\{90^\circ-\left(\frac{A+B}{2}\right)\right\}\right]$$

$$=2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right]$$

$$=2\cos\frac{C}{2}\left[2\cos\frac{A}{2}\cos\frac{B}{2}\right]$$

$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Exercise 3.3

Prove that:

1. $\sin^2\frac{\pi}{6}+\cos^2\frac{\pi}{3}-\tan^2\frac{\pi}{4}=-\frac{1}{2}$.

2. $2\sin^2\frac{\pi}{6}+\text{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3}=\frac{3}{2}$.

3. $\cot^2\frac{\pi}{6}+\text{cosec}\frac{5\pi}{6}+3\tan^2\frac{\pi}{6}=6$.

4. $2\sin^2\frac{3\pi}{4}+2\cos^2\frac{\pi}{4}+2\sec^2\frac{\pi}{3}=10$.

5. Find the vlaue of – (i) $\sin 75^\circ$. (ii) $\tan 15^\circ$.

Prove that

6. $\tan 225^\circ\cot 405^\circ+\tan 765^\circ\cot 675^\circ=0$.

7. $\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$.

8. $\cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right)-\sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right)=\sin(x+y)$.

9. $\dfrac{\tan\left(\frac{\pi}{4}+x\right)}{\tan\left(\frac{\pi}{4}-x\right)}=\left(\dfrac{1+\tan x}{1-\tan x}\right)^2$.

10. $\dfrac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\dfrac{\pi}{2}+x\right)}=\cot^2 x$.

11. $\sin(n+1)x\sin(n+2)x+\cos(n+1)x\cos(n+2)x=\cos x$.

12. $\sin^2 6x-\sin^2 4x=\sin 2x\sin 10x$.

13. $\sin 2x+2\sin 4x+\sin 6x=4\cos^2 x\sin 4x$.

14. $\cot 4x(\sin 5x+\sin 3x)=\cot x(\sin 5x-\sin 3x)$.

15. $\dfrac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x}=\tan x$.

16. $\dfrac{\sin x-\sin y}{\cos x+\cos y}=\tan\dfrac{x-y}{2}$.

17. $\dfrac{\sin x+\sin 3x}{\cos x+\cos 3x}=\tan 2x$.

18. $\dfrac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x}=\cot 3x$.

19. $\tan 4x=\dfrac{4\tan x(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x}$.

20. $\cos 4x=1-8\sin^2 x\cos^2 x$.

21. $\cos 6x=32\cos^6 x-48\cos^4 x+18\cos^2 x-1$.

22. $[1+\cot\theta-\sec(\theta+\pi/2)][1+\cot\theta+\sec(\theta+\pi/2)]=2\cot\theta$.

3.07 Trigonometrical Equations

Equation involving trigonometric functions of a variable are called *trigonometric equations*. In this section, we shalls find the solutions of such equations. We have already learnt that the values of sin x and cos x repeat after an interval of 2 π and the values of tan x repeat after an interval of π. The solutions of a trigonometric equation for which $0\le x<2\pi$ are called *principal solutions*. The expression involving integer 'n' which gives all solutions of a trigonometric equation is called the *general solution*. We shall use **Z** to denote the set of integers.

The following examples will be helpful in solving trigonometric equations:

Example 14. Find the principal solutions of the equation $\sin x=\dfrac{\sqrt{3}}{2}$

Solution : We know that, $\sin\dfrac{\pi}{3}=\dfrac{\sqrt{3}}{2}$ and $\sin\dfrac{2\pi}{3}=\sin\left(\pi-\dfrac{\pi}{3}\right)=\sin\dfrac{\pi}{3}=\dfrac{\sqrt{3}}{2}$

Therefore, principal solutions are $x=\dfrac{\pi}{3}$ and $\dfrac{2\pi}{3}$.

Example 15. Find the principal solutions of the equation, $\tan x=-\dfrac{1}{\sqrt{3}}$

Solution : We know that, $\tan\dfrac{\pi}{6}=\dfrac{1}{\sqrt{3}}$. Thus, $\tan\left(\pi-\dfrac{\pi}{6}\right)=-\tan\dfrac{\pi}{6}=-\dfrac{1}{\sqrt{3}}$

and $\tan\left(2\pi - \frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$

$\therefore \quad \tan\frac{5\pi}{6} = \tan\frac{11\pi}{6} = -\frac{1}{\sqrt{3}}$

Therefore, principal solutions are $5\pi/6$ and $11\pi/6$

We will now find the general solutions of trigonometric equations. We have already seen that: $\sin x = 0$ gives $x = n\pi$, where $n \in \mathbf{Z}$

$\cos x = 0$ gives $x = (2n \pm 1)\frac{\pi}{2}$, where $n \in \mathbf{Z}$

Theorem 1 For any real numbers x and y

$\sin x = \sin y \Rightarrow x = n\pi + (-1)^n y$, where $n \in \mathbf{Z}$

Proof If $\sin x = \sin y$, then

$\sin x - \sin y = 0$ or $2\cos\frac{x+y}{2}\sin\frac{x-y}{2} = 0$

which gives, $\cos\frac{x+y}{2} = 0$ or $\sin\frac{x-y}{2} = 0$

$\Rightarrow \quad \frac{x+y}{2} = (2n+1)\frac{\pi}{2}$ or $\frac{x-y}{2} = n\pi$, where $n \in \mathbf{Z}$

$\Rightarrow \quad x = (2n+1)\pi - y$ or $x = 2n\pi + y$, where $n \in \mathbf{Z}$

or $\quad x = (2n+1)\pi + (-1)^{2n+1} y$ or $x = 2n\pi + (-1)^{2n} y$, where $n \in \mathbf{Z}$

Combining these two results, we get $x = n\pi + (-1)^n y$, where $n \in \mathbf{Z}$

Theorem 2: For any real number x and y, $\cos x = \cos y$ implies $x = 2n\pi \pm y$, where $n \in \mathbf{Z}$

Proof : If $\cos x = \cos y$, then

$\cos x - \cos y = 0$ or $-2\sin\frac{x+y}{2}\sin\frac{x-y}{2} = 0$

which gives $\sin\frac{x+y}{2} = 0$ or $\sin\frac{x-y}{2} = 0$

$\Rightarrow \quad \frac{x+y}{2} = n\pi$ or $\frac{x-y}{2} = n\pi$, where $n \in \mathbf{Z}$

$\Rightarrow \quad x = 2n\pi - y$ or $x = 2n\pi + y$, where $n \in \mathbf{Z}$

$\therefore \quad x = 2n\pi \pm y$, where $n \in \mathbf{Z}$

Theorem 3: Prove that, if x and y are not odd multiple of $\pi/2$ then $\tan x = \tan y$ implies $x = n\pi + y$, where $n \in \mathbf{Z}$

Proof If $\tan x = \tan y$, then $\tan x - \tan y = 0$

or $$\frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y} = 0$$

or $$\sin(x - y) = 0$$

$\Rightarrow$ $x - y = n\pi$ i.e. $x = n\pi + y$, where $n \in \mathbf{Z}$.

Example 16. Solve the equation $2\sin\theta + 1 = 0$.

Solution : Given $$2\sin\theta + 1 = 0$$

$\Rightarrow$ $$\sin\theta = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin(-\frac{\pi}{6})$$

Hence $$\theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right), \text{ where } n \in Z$$

Example 17. Find the solution of $\sin x = -\frac{\sqrt{3}}{2}$

Solution : Given $\sin x = -\frac{\sqrt{3}}{2} = -\sin\frac{\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = \sin\frac{4\pi}{3}$

$\therefore$ $$\sin x = \sin\frac{4\pi}{3}$$

$\Rightarrow$ $$x = n\pi + (-1)^n\frac{4\pi}{3}, \text{ where } n \in \mathbf{Z}.$$ [by 1]

Note: $4\pi/3$, x is one such value of x for which $\sin x = -\frac{\sqrt{3}}{2}$. One may take any other value of x for which $\sin x = -\frac{\sqrt{3}}{2}$. The solutions obtained will be the same although these may apparently look different.

Example 18: Solve $\cos x = 1/2$.

Solution : Given, $\cos x = \frac{1}{2} = \cos\frac{\pi}{3}$

$\Rightarrow$ $$x = 2n\pi \pm \frac{\pi}{3}, \text{ where } n \in \mathbf{Z}.$$ [Theorem 2]

Example 19. Solve $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right)$.

Solution : Given $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{2} + x + \frac{\pi}{3}\right)$

or $$\tan 2x = \tan\left(x + \frac{5\pi}{6}\right)$$

$\Rightarrow \quad 2x = n\pi + x + \frac{5\pi}{6}, \quad$ where $n \in \mathbf{Z}$ [Theorem 3]

$\Rightarrow \quad x = n\pi + \frac{5\pi}{6}, \quad$ where $n \in \mathbf{Z}$.

Example 20. Solve: $\sin 2x - \sin 4x + \sin 6x = 0$

Solution : Equation can be written as,

$$\sin 6x + \sin 2x - \sin 4x = 0$$

$\Rightarrow \quad 2\sin 4x \cos 2x - \sin 4x = 0$

$\Rightarrow \quad \sin 4x(2\cos 2x - 1) = 0$

$\Rightarrow \quad \sin 4x = 0 \quad$ or $\quad \cos 2x = 1/2$

$\Rightarrow \quad \sin 4x = \sin 0 \quad$ or $\quad \cos 2x = \cos\frac{\pi}{3}$

$\Rightarrow \quad 4x = n\pi \quad$ or $\quad 2x = 2n\pi \pm \frac{\pi}{3}, \quad$ where $n \in \mathbf{Z}$

$\Rightarrow \quad x = \frac{n\pi}{4} \quad$ or $\quad x = n\pi \pm \frac{\pi}{6}, \quad$ where $n \in \mathbf{Z}$

Example 21. Solve $2\cos^2 x + 3\sin x = 0$.

Solution : Equation can be written as:

$$2\left(1 - \sin^2 x\right) + 3\sin x = 0$$

or $\quad 2\sin^2 x - 3\sin x - 2 = 0$

or $\quad (2\sin x + 1)(\sin x - 2) = 0$

$\therefore \quad \sin x = -1/2 \quad$ or $\quad \sin x = 2$

but $\sin x \neq 2 \quad [\because -1 \leq \sin x \leq 1]$

therefore $\sin x = -\frac{1}{2} = \sin\frac{7\pi}{6}$

$\therefore \quad x = n\pi + (-1)^n \left(7\pi/6\right)$ where $n \in \mathbf{Z}$.

Example 22. Solve: $\sin^2\theta - \cos\theta = 1/4$

Solution : Given, $4\sin^2\theta - 4\cos\theta - 1 = 0$

$\Rightarrow \quad 4\left(1 - \cos^2\theta\right) - 4\cos\theta - 1 = 0$

$\Rightarrow \quad 4\cos^2\theta + 4\cos\theta - 3 = 0$

$\Rightarrow \quad \left(2\cos\theta + 3\right)\left(2\cos\theta - 1\right) = 0$

If $2\cos\theta + 3 = 0 \Rightarrow \cos\theta = -3/2$ which is not true, therefore

$2\cos\theta - 1 = 0 \quad \Rightarrow \cos\theta = 1/2 = \cos\pi/3$

$\therefore \quad \theta = 2n\pi \pm \pi/3,$ where $n \in \mathbf{Z}$.

Example 23. Solve: $\cos 3\theta - \sin\theta = \cos 5\theta$

Solution : Given,

$$\cos 3\theta - \cos 5\theta = \sin\theta$$

$\Rightarrow \quad -2\sin\left(\frac{3\theta+5\theta}{2}\right)\sin\left(\frac{3\theta-5\theta}{2}\right) - \sin\theta = 0 \qquad [\because \sin(-\theta) = -\sin\theta]$

$\Rightarrow \quad 2\sin 4\theta \sin\theta - \sin\theta = 0$

$\Rightarrow \quad (2\sin 4\theta - 1)\sin\theta = 0$

$\Rightarrow \quad 2\sin 4\theta - 1 = 0 \quad$ or $\quad \sin\theta = 0$

$\Rightarrow \quad \sin 4\theta = 1/2 = \sin\pi/6 \quad$ or $\quad \sin\theta = \sin 0$

$\Rightarrow \quad 4\theta = n\pi + (-1)^n \pi/6 \quad$ or $\quad \theta = n\pi$

$\therefore \quad \theta = n\pi; \frac{n\pi}{4} + (-1)^n \frac{\pi}{24},$ where $n \in \mathbf{Z}$.

Exercise 3.4

Find the principal and general solutions of the following euqations

1. $\tan x = \sqrt{3}$
2. $\sec x = 2$
3. $\cot x = -\sqrt{3}$
4. $\operatorname{cosec} x = -2$

Find the general solution for each of the following euqations:

5. $\cos 4x = \cos 2x$
6. $\cos 3x + \cos x - \cos 2x = 0$
7. $\sin 2x + \cos x = 0$
8. $\sec^2 2x = 1 - \tan 2x$
9. $\sin x + \sin 3x + \sin 5x = 0$

Miscellaneous Examples

Example 24. If $\sin x = \frac{3}{5}, \cos y = -\frac{12}{13}$ where x and y both lie in second quadrant, find the value of $\sin(x+y)$

Solution : We know that $\sin(x+y) = \sin x\cos y + \cos x \sin y$ (1)

Now $\cos^2 x = 1 - \sin^2 x = 1 - \frac{9}{25} = \frac{16}{25} \quad \Rightarrow \quad \cos x = \pm\frac{4}{5}$

Since x lies in second quadrant, $\cos x$ is negative $\cos x = -4/5$

Hence $\sin^2 y = 1 - \cos^2 y = 1 - \frac{144}{169} = \frac{25}{169} \quad \Rightarrow \quad \sin y = \pm 5/13$

Since y lies in second quadrant, hence $\sin y$ is positive.

$\therefore \quad \sin y = 5/13$ Substituting the values of $\sin x, \sin y, \cos x$ and $\cos y$ in eq. (1) we have

$$\sin(x+y)=\frac{3}{5}\times\left(-\frac{12}{13}\right)+\left(-\frac{4}{5}\right)\times\frac{5}{13}=-\frac{36}{65}-\frac{20}{65}=-\frac{56}{65}$$

Example 25. Prove that: $\cos 2x\cos\frac{x}{2}-\cos 3x\cos\frac{9x}{2}=\sin 5x\sin\frac{5x}{2}$.

Solution : L.H.S. $=\frac{1}{2}\left[2\cos 2x\cos\frac{x}{2}-2\cos\frac{9x}{2}\cos 3x\right]$

$$=\frac{1}{2}\left[\cos\left(2x+\frac{x}{2}\right)+\cos\left(2x-\frac{x}{2}\right)-\cos\left(\frac{9x}{2}+3x\right)-\cos\left(\frac{9x}{2}-3x\right)\right]$$

$$=\frac{1}{2}\left[\cos\frac{5x}{2}+\cos\frac{3x}{2}-\cos\frac{15x}{2}-\cos\frac{3x}{2}\right]=\frac{1}{2}\left[\cos\frac{5x}{2}-\cos\frac{15x}{2}\right]$$

$$=\frac{1}{2}\left[-2\sin\left\{\frac{\frac{5x}{2}+\frac{15x}{2}}{2}\right\}\sin\left\{\frac{\frac{5x}{2}-\frac{15x}{2}}{2}\right\}\right]$$

$$=-\sin 5x\sin\left(-\frac{5x}{2}\right)=\sin 5x\sin\frac{5x}{2}=\text{R.H.S.}$$

Example 26. Find the value of $\tan\frac{\pi}{8}$.

Solution : Let $x=\pi/8$ now $2x=\pi/4$

now $\tan 2x=\frac{2\tan x}{1-\tan^2 x}$ or $\tan\frac{\pi}{4}=\frac{2\tan(\pi/8)}{1-\tan^2(\pi/8)}$

Let $y=\tan\frac{\pi}{8}$ then $1=\frac{2y}{1-y^2}$

or $y^2+2y-1=0$

$\therefore$ $y=\frac{-2\pm 2\sqrt{2}}{2}=-1\pm\sqrt{2}$

Since $\pi/8$ lies in the first quadrant $y=\tan(\pi/8)$ is positive. Hence $\tan\frac{\pi}{8}=\sqrt{2}-1$.

Example 27. If $\tan x=3/4, \pi<x<3\pi/4,$ then, Evaluate $\sin(x/2), \cos(x/2)$ and $\tan(x/2)$.

Solution : Since $\pi<x<3\pi/2$ $\therefore$ $\cos x$ is negative. Again,

$$\frac{\pi}{2}<\frac{x}{2}<\frac{3\pi}{4}.$$

$\therefore \quad \sin\frac{x}{2}$ is positive and $\cos\frac{x}{2}$ is negative

Now, $\sec^2 x = 1+\tan^2 x = 1+(9/16) = 25/16$

$\therefore \quad \cos^2 x = 16/25$ or $\cos x = -4/5$ $\left[\pi < x < \frac{3\pi}{2}\right]$

Now, $2\sin^2\frac{x}{2} = 1-\cos x = 1+\frac{4}{5} = \frac{9}{5}$ $\left[\cos x = 1-2\sin^2\frac{\pi}{2}\right]$

$\therefore \quad \sin^2\frac{x}{2} = \frac{9}{10}$ or $\sin\frac{x}{2} = \frac{3}{\sqrt{10}}$ $\left[\frac{\pi}{2} < \frac{\pi}{2} < \frac{3\pi}{4}\right]$

Again, $2\cos^2\frac{x}{2} = 1+\cos x = 1-\frac{4}{5} = \frac{1}{5}$

$\therefore \quad \cos^2\frac{x}{2} = \frac{1}{10}$ या $\cos\frac{x}{2} = -\frac{1}{\sqrt{10}}$ $\left[\frac{\pi}{2} < \frac{\pi}{2} < \frac{3\pi}{4}\right]$

$$\therefore \quad \tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{3}{\sqrt{10}} \times \left(\frac{-\sqrt{10}}{1}\right) = -3$$

Example 28. Prove that: $\cos^2 x + \cos^2\left(x+\frac{\pi}{3}\right) + \cos^2\left(x-\frac{\pi}{3}\right) = \frac{3}{2}$.

Solution : We have,

$$\text{L.H.S.} = \frac{1+\cos 2x}{2} + \frac{1+\cos\left(2x+\frac{2\pi}{3}\right)}{2} + \frac{1+\cos\left(2x-\frac{2\pi}{3}\right)}{2}$$

$$= \frac{1}{2}\left[3+\cos 2x+\cos\left(2x+\frac{2\pi}{3}\right)+\cos\left(2x-\frac{2\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\frac{2\pi}{3}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\left(\pi-\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x-2\cos 2x\cos\frac{\pi}{3}\right]$$

$$=\frac{1}{2}[3+\cos 2x-\cos 2x]=\frac{3}{2}=\text{R.H.S.}$$

Miscellaneous Exercise

1. A right angle is

(A) equal to one radian (B) equal to 90 deg (C) equal to 1 deg (D) equal to 90 radian

2. Which of the following is positive in the third quadrant

(A) $\sin\theta$ (B) $\tan\theta$ (C) $\cos\theta$ (D) $\sec\theta$

3. $\operatorname{cosec}(-\theta)$ is equal to

(A) $\sin\theta$ (B) $\tan\theta$ (C) $\cos\theta$ (D) $-\operatorname{cosec}\theta$

4. $\tan(90^\circ-\theta)$ is equal to

(A) $-\tan\theta$ (B) $\cot\theta$ (C) $\tan\theta$ (D) $-\cot\theta$

5. If $\cos\theta=-\frac{1}{2}$ then the value of θ is

(A) $\frac{2\pi}{3}$ (B) $\frac{\pi}{3}$ (C) $-\frac{2\pi}{3}$ (D) $\frac{3\pi}{4}$

6. If n is a positive integer, then the vlaue of $\sin(2n\pi\pm\theta)$ will be

(A) $\pm\cos\theta$ (B) $\pm\tan\theta$ (C) $\pm\sin\theta$ (D) $\pm\cot\theta$

7. The value of $\cot 15^\circ$ is

(A) $2+\sqrt{3}$ (B) $-2+\sqrt{3}$ (C) $2-\sqrt{3}$ (D) $-2-\sqrt{3}$

8. The value of $\cos 15^\circ$ is

(A) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (B) $\frac{\sqrt{3}-1}{2}$ (C) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (D) $\frac{\sqrt{3}+1}{2}$

9. The value of $2\sin\frac{5\pi}{12}\cos\frac{\pi}{12}$

(A) 1 (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{\sqrt{3}}{2}-1$ (D) $\frac{\sqrt{3}}{2}+1$

10. The value of $\cos\frac{\pi}{12}-\sin\frac{\pi}{12}$

(A) $\frac{1}{2\sqrt{2}}$ (B) 0 (C) $-\frac{1}{2\sqrt{2}}$ (D) $\frac{1}{\sqrt{2}}$

11. If $\sin A=\frac{3}{5}$, then the value of $\sin 2A$ will be

(A) 4 / 25 (B) 5 / 25 (C) 24 / 25 (D) 4 / 5

12. If $\sin A = 3/4$, then the value of $\sin 3A$ will be
(A) 9 /16 (B) –9 / 16 (C) 9 / 32 (D) 7 / 16

13. If $\tan A = 1/5$, then the vlaue of $\tan 3A$ will be
(A) 47 / 25 (B) 37 / 55 (C) 37 / 11 (D) 47 / 55

14. If $A+B=\pi/4$, then the value of $(1+\tan A)(1+\tan B)$ will be
(A) 3 (B) 2 (C) 4 (D) 1

15. The principal value of the equation $\sec^2\theta = 2$ will be
(A) $2n\pi \pm \frac{\pi}{4}$ (B) $n\pi \pm \frac{\pi}{2}$ (C) $n\pi \pm (-1)^n \frac{\pi}{4}$ (D) $n\pi \pm \frac{\pi}{4}$

16. Prove that
(i) $\cos\theta + \sin(270^\circ + \theta) - \sin(270^\circ - \theta) + \cos(180^\circ + \theta) = 0$
(ii) $\sec\left(\frac{3\pi}{2} - \theta\right)\sec\left(\theta - \frac{5\pi}{2}\right) + \tan\left(\frac{5\pi}{2} + \theta\right)\tan\left(\theta - \frac{3\pi}{2}\right) = -1$

17. Find the value of $\sin\left\{n\pi + (-1)^n \frac{\pi}{4}\right\}$ where n is an integer.

18. If $\sin A + \sin B = a$ and $\cos A + \cos B = b$, then prove that
(i) $\sin(A+B) = (2ab)/(a^2+b^2)$ (ii) $\cos(A+B) = (b^2-a^2)/(a^2+b^2)$

19. If $A+B+C=180^\circ$, then prove that
(i) $\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A \sin B \cos C$
(ii) $\sin A - \sin B + \sin C = 4\sin\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

20. If $A+B+C=2\pi$, then Prove that
$\cos^2 B + \cos^2 C - \sin^2 A = 2\cos A \cos B \cos C$

21. Solve the following equation:
$2\tan\theta - \cot\theta + 1 = 0$

Important Points

1. If in a circle of radius r, an arc of length l subtends an angle of θ radians, then $\ell = r\theta$
2. Radian measure is the ratio of arc length to the radius of the ricle.
3. Radian measure $= \frac{\pi}{180} \times$ Degree measure
4. Degree meaasure $= \frac{180}{\pi} \times$ Radian measure
5. $\cos^2 x + \sin^2 x = 1$
6. $1 + \tan^2 x = \sec^2 x$

7. $1+\cot^2 x = \operatorname{cosec}^2 x$

8. $\cos(2n\pi + x) = \cos x$

9. $\sin(2n\pi + x) = \cos x$

10. $\sin(-x) = -\sin x$

11. $\cos(-x) = \cos x$

12. $\cos(x+y) = \cos x \cos y - \sin x \sin y$

13. $\cos(x-y) = \cos x \cos y + \sin x \sin y$

14. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

15. $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

16. $\sin(x+y) = \sin x \cos y + \cos x \sin y$

17. $\sin(x-y) = \sin x \cos y - \cos x \sin y$

18. $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$ $\qquad \sin\left(\frac{\pi}{2} + x\right) = \cos x$

$\cos(\pi - x) = -\cos x$ $\qquad \sin(\pi - x) = \sin x$

$\cos(\pi + x) = -\cos x$ $\qquad \sin(\pi + x) = -\sin x$

$\cos(2\pi - x) = \cos x$ $\qquad \sin(2\pi - x) = -\sin x$

19. If none of the angles x, y and $(x \pm y)$ is an odd multiple of $\pi/2$, then

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \text{ and } \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

20. If none of the angles x, y and $(x \pm y)$ is a multiple of π, then

$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x} \text{ and } \cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$$

21. $\sin 2x = 2\sin x \cos x = \dfrac{2\tan x}{1+\tan^2 x}$, $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

22. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

23. $\sin 3x = 3\sin x - 4\sin^3 x$, $\cos 3x = 4\cos^3 x - 3\cos x$

24. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

25. (i) $\cos x + \cos y = 2\cos\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$

(ii) $\cos x - \cos y = -2\sin\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$

(iii) $\sin x + \sin y = 2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$

(iv) $\sin x - \sin y = 2\cos\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$

26. (i) $2\cos x \cos y = \cos(x+y) + \cos(x-y)$ (ii) $-2\sin x \sin y = \cos(x+y) - \cos(x-y)$

(iii) $2\sin x \cos y = \sin(x+y) + \sin(x-y)$ (iv) $2\cos x \sin y = \sin(x+y) - \sin(x-y)$

27. $\sin x = 0$, then $x = n\pi$, where $n \in \boldsymbol{Z}$

28. $\cos x = 0$, then $x = (2n \pm 1)\frac{\pi}{2}$] where $n \in \boldsymbol{Z}$

29. $\sin x = \sin y$, then $x = n\pi + (-1)^n y$] where $n \in \boldsymbol{Z}$

30. $\cos x = \cos y$, then $x = 2n\pi \pm y$] where $n \in \boldsymbol{Z}$

31. $\tan x = \tan y$, then $x = n\pi + y$, where $n \in \boldsymbol{Z}$

Answers

Exercise 3.1

1. (i) $\frac{5\pi}{36}$; (ii) $-\frac{19\pi}{72}$; (iii) $\frac{26\pi}{9}$ 2. (i) $39^\circ 22' 30''$; (ii) $-229^\circ 5' 29''$; (iii) 300°

3. 12π 4. $12^\circ 36'$ 5. $\frac{20\pi}{3}$ 6. $5 : 4$ 7. (i) $\frac{2}{15}$; (ii) $\frac{7}{25}$

Exercise 3.2

1. $\sin x = -\frac{\sqrt{3}}{2}, \text{cosec} x = -\frac{2}{\sqrt{3}}, \sec x = -2, \tan x = \sqrt{3}, \cot x = \frac{1}{\sqrt{3}}$

2. $\sin x = -\frac{4}{5}, \text{cosec} x = -\frac{5}{4}, \cos x = -\frac{3}{5}, \sec x = -\frac{5}{3}, \sec x = -\frac{5}{3}, \tan x = \frac{4}{3}$

3. $\sin x = -\frac{12}{13}, \text{cosec} x = -\frac{13}{12}, \cos x = \frac{5}{13}, \tan x = -\frac{12}{5}, \cot x = -\frac{5}{12}$

4. $\frac{1}{\sqrt{2}}$ 5. $\sqrt{3}$ 6. $\frac{\sqrt{3}}{2}$ 7. 1

Exercise 3.3

5. (i) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (ii) $2-\sqrt{3}$

Exercise 3.4

1. $\frac{\pi}{3}, \frac{4\pi}{3}, n\pi + \frac{\pi}{3}, n \in \boldsymbol{Z}$ 2. $\frac{\pi}{3}, \frac{5\pi}{3}, 2n\pi \pm \frac{\pi}{3}, n \in \boldsymbol{Z}$

3. $\frac{5\pi}{6}, \frac{11\pi}{6}, n\pi +, \frac{5\pi}{6}, n \in \boldsymbol{Z}$ 4. $\frac{7\pi}{6}, \frac{11\pi}{6}, n\pi + (-1)^n \frac{7\pi}{6}, n \in \boldsymbol{Z}$

5. $x=\frac{n\pi}{3}$ or $x=n\pi,\ n\in \boldsymbol{Z}$

6. $x=(2n+1)\frac{\pi}{4}$ or $2n\pi\pm\frac{\pi}{3},\ n\in \boldsymbol{Z}$

7. $x=n\pi+(-1)^n\frac{7\pi}{6}$ or $(2n+1)\frac{\pi}{2},\ n\in \boldsymbol{Z}$

8. $x=\frac{n\pi}{2}$ or $\frac{n\pi}{2}+\frac{3\pi}{8},\ n\in \boldsymbol{Z}$

9. $x=\frac{n\pi}{3}$ or $n\pi\pm\frac{\pi}{3},\ n\in \boldsymbol{Z}$

Miscellaneous Exercise 3.

1. B	**2.** B	**3.** D	**4.** B	**5.** A	**6.** C
7. A	**8.** A	**9.** D	**10.** D	**11.** C	**12.** A
13. B	**14.** B	**15.** A	**17.** $1/\sqrt{2}$		

21. $n\pi+\tan^{-1}\frac{1}{2},\ n\pi+\frac{3\pi}{4}$ where $n\in \mathbf{Z}$