Exercise 11.7

Q1E

Consider the series:

$$\sum_{n=1}^{\infty} \frac{1}{n+3^n}.$$

The objective is to test whether the series convergent or divergent.

To test whether the series is convergence and divergence follow the steps:

The comparison test state that:

Let $\sum a_n$ and $\sum b_n$ be series with positive terms, then

1) If $\sum b_n$ is convergent and $a_n \le b_n$ for all *n*, then $\sum a_n$ is also convergent.

2) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all *n*, then $\sum a_n$ is also divergent.

Here,

$$0 < 3^n < n+3^n$$

$$0 < \frac{1}{n+3^n} < \frac{1}{3^n}$$

And,

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{n-1}$$

The common ratio of the geometric series $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is $|r| = \frac{1}{3} < 1$.

So, $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is converges.

Hence, by the comparison test the series $\sum_{n=1}^{\infty} \frac{1}{n+3^n}$ is convergent.

Q2E

$$\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}} = \sum_{n=1}^{\infty} \frac{(2n+1)^n}{(n^2)^n}$$

So... Now use Root Test!

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{(2n+1)^n}{(n^2)^n}}$$

$$= \lim_{n\to\infty} \frac{2n+1}{n^2} = \lim_{n\to\infty} \frac{\frac{2}{n}+\frac{1}{n^2}}{1} = 0 < 1$$

By Root test the Series Converges absolutely and so converges .

Q3E

Consider the series,

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$$

The objective is to determine whether the series converges or diverges.

Alternating Series Test:

The alternating series $\sum_{n=1}^{\infty} (-1)^n b_n$ is said to be convergent if it satisfies the following two conditions,

$(i)\quad b_{n+1} \le b_n \qquad \text{for all } n$

$(ii)\quad \lim_{n\to\infty} b_n = 0$

Condition (i):

Show that, the sequence b_n is decreasing, that is, $b_{n+1} < b_n$.

For the given series, we have that

$$b_n = \frac{n}{n+2}$$

Consider $b_{n+1} - b_n$

$$\begin{aligned} b_{n+1} - b_n &= \frac{n+1}{n+3} - \frac{n}{n+2} \\ &= \frac{n^2+3n+2-\left(n^2+3n\right)}{(n+2)(n+3)} \\ &= \frac{2}{(n+2)(n+3)} > 0 \end{aligned}$$

Thus $b_{n+1} - b_n > 0$ for all $n \in N$

So, the condition (i) failed.

Condition (ii):

Show that the limit of the sequence b_n tends to zero, as n tends to infinity.

$$\begin{aligned} \lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{n}{n+2} \\ &= \lim_{n\to\infty} \frac{n}{n\left(1+\frac{2}{n}\right)} \\ &= \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}} \\ &= \frac{1}{1+0} \\ &= 1 \end{aligned}$$

Since $\lim_{n\to\infty} b_n = 1 \neq 0$,

So, the given series does not satisfy the two conditions of the Alternating Series Test.

Thus, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$ is divergent.

Q4E

Consider the following series:

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+2}$$

Test the absolute convergence of the given series:

Let, $a_n = (-1)^n \frac{n}{n^2+2}$

Then, the series of absolute values is computed as under:

$$\begin{aligned}\sum_{n=1}^{\infty}|a_n| &= \sum_{n=1}^{\infty}\left|(-1)^n \frac{n}{n^2+2}\right| \\ &= \sum_{n=1}^{\infty}\frac{n}{n^2+2}\end{aligned}$$

Verify the conditions of the Integral test.

Condition (i):

Check, whether the function $f(x)$ is decreasing or not.

Let, $f(x) = \frac{x}{x^2+2}$.

The first derivative of the function $f(x)$ is, $f'(x) = \frac{2-x^2}{(x^2+2)^2}$.

Clearly, $f'(x) < 0$ if $2-x^2 < 0$, that is, $x > \sqrt{2}$.

Thus, the function $f(x)$ is decreasing on the interval $(\sqrt{2}, \infty)$.

This means that, $f(n+1) < f(n)$.

So, the function $f(x) = \frac{x}{x^2+2}$ is decreasing.

Condition (ii):

Check the continuity of the function $f(x)$.

Clearly, the function $f(x)$ is continuous positive function, because x and x^2+2 are polynomial functions and always positive on the interval $[1, \infty)$.

The function, $f(x)=\frac{x}{x^2+2}$ is continuous, positive, and decreasing on $[1,\infty)$; so, use the Integral Test as shown below:

$$\begin{aligned}\int_1^\infty f(x)\,dx &= \int_1^\infty \frac{x}{x^2+2}\,dx \\ &= \left[\frac{1}{2}\ln\left(x^2+2\right)\right]_1^\infty \\ &= \left[\frac{1}{2}\ln\left(\infty^2+2\right)-\frac{1}{2}\ln\left(1^2+2\right)\right] \\ &= \infty\end{aligned}$$

Since the integral $\int_1^\infty f(x)\,dx$ is divergent, the series $\sum_{n=1}^{\infty}\frac{n}{n^2+2}$ is also divergent by the Integral Test.

Then, the series of absolute values $\sum_{n=1}^{\infty}|a_n|$ is also divergent.

Therefore, the series $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+2}$ is not absolutely convergent.

Since the given series is alternating, use the Alternating Series Test.

Let, $b_n=\frac{n}{n^2+2}$.

Then, the $(n+1)^{th}$ term is $b_{n+1}=\frac{n+1}{(n+1)^2+2}$.

Verify conditions (i) and (ii) of the Alternating Series Test:

Condition (i):

Show that, the sequence $b_n=\frac{n}{n^2+2}$ is decreasing, that is, $b_{n+1}<b_n$.

Let, $f(x)=\frac{x}{x^2+2}$.

The first derivative of the function $f(x)$ is, $f'(x)=\frac{2-x^2}{\left(x^2+2\right)^2}$.

Clearly, $f'(x)<0$ if $2-x^2<0$, that is, $x>\sqrt{2}$.

Thus, the function $f(x)$ is decreasing on the interval $\left(\sqrt{2},\infty\right)$.

This means that, $f(n+1)<f(n)$ and therefore, $b_{n+1}<b_n$.

Condition (ii):

Show that the limit of the sequence b_n tends to zero, as n tends to infinity.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left[\frac{n}{n^2+2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{n}{n^2\left(1+\frac{2}{n^2}\right)}\right]$$

$$= \lim_{\frac{1}{n}\to 0}\left[\frac{1}{n\left(1+\frac{2}{n^2}\right)}\right]$$

$$= 0$$

Therefore, $\lim_{n\to\infty} b_n = 0.$

So, the given series satisfies the two conditions of the Alternating Series Test.

Hence, the given series is convergent by the Alternating Series Test.

Clearly, the given series is convergent but not absolutely convergent.

Therefore, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+2}$ is conditionally convergent.

Q5E

Consider the series

$$\sum_{n=1}^{\infty}\frac{n^2 2^{n-1}}{(-5)^n}$$

Its need to determine whether the series convergence or divergence.

Observe that, in the given series n^{th} power occurs, so it is better to apply the Root Test to decide the convergence of the given series rather than Ratio Test.

Root Test:

(i) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefor convergent).

(ii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent

(iii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive;

For the series $\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$, observe that $a_n = \frac{n^2 2^{n-1}}{(-5)^n}$

Consider $\lim_{n\to\infty} \sqrt[n]{|a_n|}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n^2 2^{n-1}}{(-5)^n}\right|}$$

$$= \lim_{n\to\infty} \sqrt[n]{\left|\frac{n^2 2^{n-1}}{(-1)^n 5^n}\right|}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{n^2 2^{n-1}}{5^n}}$$

$$= \lim_{n\to\infty} \left(\frac{n^2 2^{n-1}}{5^n}\right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{\left(\sqrt[n]{n}\right)^2 2^{\frac{n-1}{n}}}{\left(5^n\right)^{1/n}}$$

Continuation to the above

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{\left(\sqrt[n]{n}\right)^2 2^{1-\frac{1}{n}}}{5}$$

$$= \frac{1}{5}\left(\lim_{n\to\infty} \sqrt[n]{n}\right)^2 \left(\lim_{n\to\infty} 2^{1-\frac{1}{n}}\right)$$

$$= \frac{1}{5}\cdot(1)^2 \cdot 2^{1-0}$$

$$= \frac{2}{5}\ (<1)$$

Thus $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{2}{5}\ (<1)$

For the series $\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$, we observe that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{2}{5}(<1)$$

By the Root Test, the series $\sum a_n = \sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$ is $\boxed{\text{convergent}}$.

Q6E

Consider the series,

$$\sum_{n=1}^{\infty} \frac{1}{2n+1}$$

The objective is to find whether the series is convergent or divergent.

For large values of n, the denominator of each term is close to $2n$, because the $2n$ term will be very large compared to the constant term1.

Compute the limit of the ratios of the terms of $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ and $\sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{2}\cdot\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{2n+1} \cdot \frac{2n}{1}$$

$$= \lim_{n\to\infty} \frac{2n}{2n+1}$$

$$= \lim_{n\to\infty} \frac{(1/2n)}{(1/2n)} \frac{2n}{2n+1} \qquad \text{Divide and multiply by } (1/2n) \text{ for } n \neq 0.$$

$$= \lim_{n\to\infty} \frac{1}{2n\left(\frac{1+\frac{1}{2n}}{2n}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{1+(1/2n)} \qquad \text{Cancelling common factors,}$$

The term, $(1/2n) \to 0$ as $n \to \infty$

So, the limit above converges, $\lim_{n\to\infty} \frac{1}{1+(1/2n)} = 1$

That is, the limit of the ratios of terms of $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ and $\sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{n}$ converges, $\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{2}\cdot\frac{1}{n}} = 1$

The series $\sum_{n=1}^{\infty}\frac{1}{2n+1}$ and $\sum_{n=1}^{\infty}\frac{1}{2}\cdot\frac{1}{n}$ are series of positive terms, and the limit of the ratios of their terms $\lim_{n\to\infty}\frac{\frac{1}{2n+1}}{\frac{1}{2}\cdot\frac{1}{n}}$ converges to the finite positive number 1.

By the Limit Comparison Test,

The series $\sum_{n=1}^{\infty}\frac{1}{2n+1}$ converges if and only if $\sum_{n=1}^{\infty}\frac{1}{2}\cdot\frac{1}{n}$ converges.

The series $\sum_{n=1}^{\infty}\frac{1}{2}\cdot\frac{1}{n}$ is a constant multiple of the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$, and the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

So, $\sum_{n=1}^{\infty}\frac{1}{2}\cdot\frac{1}{n}$ diverges.

Therefore, by the Limit Comparison Test the series $\sum_{n=1}^{\infty}\frac{1}{2n+1}$ is divergent.

Q7E

We have the series

$$\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$$

Let $f(x)=\frac{1}{x\sqrt{\ln x}}$

Since f(x) is an decreasing, positive, and continuous function, we can use integral test.

$$\int_2^{\infty} f(x)\,dx=\lim_{t\to\infty}\int_2^{t}\frac{1}{x\sqrt{\ln x}}dx$$

Let $u = \ln x \Rightarrow du=\frac{1}{x}dx$

When $x=2, u=\ln 2$ and when $x = t$, $u = \ln t$

Thus,

$$\begin{aligned}\int_2^{\infty} f(x)\,dx &= \lim_{t\to\infty}\int_{\ln 2}^{\ln t}u^{-1/2}du\\ &= \lim_{t\to\infty}\left[\frac{u^{1/2}}{1/2}\right]_{\ln 2}^{\ln t}\\ &= \lim_{t\to\infty}\left[2\sqrt{\ln t}-2\sqrt{\ln 2}\right]\\ &= \infty\end{aligned}$$

Therefore improper integral is divergent and so the series $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$ is divergent.

Q8E

We have the series $\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!} = \sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)(k+1)k!}$

$$= \sum_{k=1}^{\infty} \frac{2^k}{(k+2)(k+1)}$$

We use The Ratio Test with $a_k = \frac{2^k}{(k+2)(k+1)}$

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{2^{k+1}}{(k+3)(k+2)} \cdot \frac{(k+2)(k+1)}{2^k}\right|$$

$$= 2\frac{(k+1)}{k+3} = 2\left(\frac{1+1/k}{1+3/k}\right)$$

Then $\lim_{x\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \frac{2(1+0)}{1+0} = 2 > 1$

Thus by the Ratio test, the given series **diverges**.

Q9E

Here k^{th} term of the given series is,

$$a_k = k^2 e^{-k} = \frac{k^2}{e^k}$$

$(k+1)^{th}$ term of the given series is,

$$a_{k+1} = \frac{(k+1)^2}{e^{k+1}}$$

Now, $\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2}$

$$= \frac{\left(1+\frac{1}{k}\right)^2}{e}$$

Therefore, $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\left(1+\frac{1}{k}\right)^2}{e}$

$$= \frac{(1+0)^2}{e}$$

$$= \frac{1}{e}$$

$$< 1 \qquad \text{Since } e > 2$$

Thus, by ratio test, the given series $\sum_{k=1}^{\infty} k^2 e^{-k}$ is **convergent.**

Q10E

Given series is $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

Since the integral $\int_1^{\infty} x^2 e^{-x^3}\,dx$ can be easily evaluated, so we use integral test.

Let $f(x) = x^2 e^{-x^3}$, then f is continuous and positive for all x.

And $f'(x) = 2xe^{-x^3} + \left(-3x^2\right)x^2 e^{-x^3}$

$$= 2xe^{-x^3} - 3x^4 e^{-x^3}$$

$$= \left(2 - 3x^3\right)xe^{-x^3} < 0 \quad \text{for } x \geq 1$$

So the function f is a decreasing function

Let $x^3 = t$ then $3x^2 dx = dt$ and $t \to \infty$ as $x \to \infty$

Then $\int_1^{\infty} x^2 e^{-x^3}\,dx = \int_1^{\infty} e^{-t}\frac{dt}{3}$

$$= \frac{1}{3}\lim_{y\to\infty}\int_1^{y} e^{-t}\,dt$$

$$= \frac{1}{3}\lim_{y\to\infty}\left[-e^{-t}\right]_1^{y}$$

$$= \frac{1}{3}\lim_{y\to\infty}\left[-e^{-y} + e^{-1}\right]$$

$$= \frac{1}{3}\left[0 + e^{-1}\right] = \frac{1}{3e}$$

Thus $\int_1^{\infty} x^2 e^{-x^3}\,dx$ is convergent so by integral test the series $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ is

convergent

Q11E

Given series $\sum_{n=1}^{\infty}\left(\frac{1}{n^3} + \frac{1}{3^n}\right)$

Let $a_n = \frac{1}{n^3}$ and $b_n = \frac{1}{3^n}$

We have $\sum_{n=1}^{\infty} a_n$ is convergent since it is a p-series with p = 3>1.

We have $\sum_{n=1}^{\infty} b_n$ is convergent since it is a geometric series with $a = \frac{1}{3}$ and $r = \frac{1}{3} < 1$.

Hence $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{n^3} + \frac{1}{3^n}\right)$ is convergent.

Therefore given series is convergent.

Q12E

Given series $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$

We have

$$k^2+1 > k^2$$

$$\Rightarrow \sqrt{k^2+1} > \sqrt{k^2}$$

$$\Rightarrow \frac{1}{\sqrt{k^2+1}} < \frac{1}{k}$$

$$\Rightarrow \frac{1}{k\sqrt{k^2+1}} < \frac{1}{k^2}$$

We know that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent (p-series with p=2).

Thus given series $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$ is convergent by Comparison Test.

Q13E

Consider the series $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$.

Need to determine whether the series is absolutely convergent, conditionally convergent or divergent.

Here use the Ratio test.

Ratio Test:

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Let $a_n = \frac{3^n n^2}{n!}$

And

$$a_{n+1} = \frac{3^{n+1}(n+1)^2}{(n+1)!}$$

$$= \frac{3^n \cdot 3 \cdot (n+1)^2}{(n+1)n!}$$

$$= \frac{3^n \cdot 3 \cdot (n+1)}{n!}$$

Then the ratio is

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^n \cdot 3 \cdot (n+1)}{n!} \times \frac{n!}{3^n \cdot n^2}$$
$$= \frac{3(n+1)}{n^2}$$
$$= \frac{3n\left(1+\frac{1}{n}\right)}{n^2}$$
$$= \frac{3\left(1+\frac{1}{n}\right)}{n}$$

Take the limit

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3\left(1+\frac{1}{n}\right)}{n}$$
$$= \lim_{\frac{1}{n}\to 0}\left[\frac{1}{n}\cdot 3\left(1+\frac{1}{n}\right)\right]$$
$$= 0\cdot 3(1+0)$$
$$= 0$$

Therefore $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1.$

Hence by Ratio Test (Condition (i)), the given series is absolutely convergent.

Q14E

$$\sum_{n=1}^{\infty} \frac{\sin(2n)}{1+2^n}$$

Try to go for absolute convergence since,

sin(x) goes from -1 to 1. By doing this the

domain is restricted to 0 to 1.

Apply the Direct Comparison Test

$$0 < \frac{\sin(2n)}{1+2^n} < \frac{1}{2^n}$$

Evaluate the compared series

$$\sum_{n=1}^{\infty} \frac{1}{2n}$$

this is a geometric series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$$

$$r = \frac{1}{2} \qquad -1 < \frac{1}{2} < 1$$

therefore the series is convergent by the

Geometric Series Test which makes the

original series convergent by the Direct

Comparison Test. This means the series

is Absolutely Convergent, because we took

the absolute value of the series originally.

Q15E

Consider the series

$$\sum_{k=1}^{\infty}\frac{2^{k-1}3^{k+1}}{k^k}.$$

To determine the given series is convergent or divergent, use the Ratio test.

Ratio test states that

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1,$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty,$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1,$ the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Use the Ratio Test with $a_k = \dfrac{2^{k-1}3^{k+1}}{k^k}$.

And

$$
\begin{aligned}
a_{k+1} &= \frac{2^{(k+1)-1}3^{(k+1)+1}}{(k+1)^{k+1}} \\
&= \frac{2^k 3^{k+2}}{(k+1)^k (k+1)}
\end{aligned}
$$

Now, the ratio $\dfrac{a_{k+1}}{a_k}$ is

$$
\begin{aligned}
\frac{a_{k+1}}{a_k} &= \frac{\dfrac{2^k 3^{k+2}}{(k+1)^k (k+1)}}{\dfrac{2^{k-1}3^{k+1}}{k^k}} \\
&= \frac{2^k 3^{k+2}}{(k+1)^k (k+1)} \cdot \frac{k^k}{2^{k-1}3^{k+1}}
\end{aligned}
$$

$$
= \frac{2^k 3^k \cdot 3^2}{k^k\left(1+\frac{1}{k}\right)^k k\left(1+\frac{1}{k}\right)} \cdot \frac{k^k}{2^k \cdot 2^{-1} \cdot 3^k \cdot 3}
$$

$$
= \frac{3}{\left(1+\frac{1}{k}\right)^k k\left(1+\frac{1}{k}\right)} \cdot \frac{1}{2^{-1}}
$$

$$
= \frac{6}{k\left(1+\frac{1}{k}\right)^k \left(1+\frac{1}{k}\right)}
$$

Then,

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{6}{k\left(1+\frac{1}{k}\right)^k\left(1+\frac{1}{k}\right)}\right|$$

$$= \frac{6}{k\left(1+\frac{1}{k}\right)^k\left(1+\frac{1}{k}\right)}$$

Then

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{6}{k\left(1+\frac{1}{k}\right)^k\left(1+\frac{1}{k}\right)}$$

$$= \lim_{k\to\infty}\left[\frac{1}{k}\cdot\frac{6}{\left(1+\frac{1}{k}\right)^k\left(1+\frac{1}{k}\right)}\right]$$

$$= 0\cdot\frac{6}{(1+0)^k(1+0)} \qquad \text{Since } \frac{1}{k}\to 0, \text{ as } k\to\infty$$

$$= 0$$

So, as $k\to\infty$, $\left|\frac{a_{k+1}}{a_k}\right| \to 0 < 1.$

Thus, by the Ratio Test, the given series $\sum_{k=1}^{\infty}\frac{2^{k-1}3^{k+1}}{k^k}$ converges.

Q16E

We have the series $\sum_{n=1}^{\infty}\frac{n^2+1}{n^3+1}$

We use the limit comparison test with $a_n = \frac{n^2+1}{n^3+1}$ and $b_n = \frac{n^2}{n^3} = \frac{1}{n}$

We have $\frac{a_n}{b_n} = \frac{n^2+1}{n^3+1}n = \frac{1+\frac{1}{n^2}}{1+\frac{1}{n^3}}$

Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1+1/n^2}{1+1/n^3} = 1 > 0$

But $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent p-series (p=1)

Thus the given series is also divergent.

Q17E

Consider the series $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1)}$.

Determine whether the given series is convergent or divergent.

Here, use the Ratio test.

Ratio test:

(i) If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

(ii) If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent

(iii) If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio test is inconclusive

Let $a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1)}$.

Then,

$$\begin{aligned} a_{n+1} &= \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) \cdot (2(n+1)-1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1) \cdot (3(n+1)-1)} \\ &= \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) \cdot (2n+1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1) \cdot (3n+2)} \end{aligned}$$

Find the ratio:

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) \cdot (2n+1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1) \cdot (3n+2)}}{\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1)}} \right| \\ &= \left| \left(\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) \cdot (2n+1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1) \cdot (3n+2)} \right) \cdot \left(\frac{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1)}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} \right) \right| \\ &= \left| \frac{(2n+1)}{(3n+2)} \right| \\ &= \left| \frac{n\left(2 + \frac{1}{n}\right)}{n\left(3 + \frac{2}{n}\right)} \right| \\ &= \left| \frac{\left(2 + \frac{1}{n}\right)}{\left(3 + \frac{2}{n}\right)} \right| \end{aligned}$$

Take limit on both sides.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty}\frac{\left(2+\frac{1}{n}\right)}{\left(3+\frac{2}{n}\right)}$$

$$= \frac{2}{3} \qquad \left[\text{As } n\to\infty, \frac{1}{n}\to 0\right]$$

$$\approx 0.6666666667 < 1$$

Since $L<1$, by the Ratio Test, the given series $\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{2\cdot5\cdot8\cdot\cdots\cdot(3n-1)}$ is absolutely convergent.

Q18E

Series is $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}-1}$

This is an alternating series with $b_n = \frac{1}{\sqrt{n}-1}$

We use test for convergence of Alternating series.

Since $\sqrt{n} < \sqrt{n+1}$

$$\Rightarrow \sqrt{n}-1 \le \sqrt{n+1}-1$$

$$\Rightarrow \frac{1}{\sqrt{n+1}-1} \le \frac{1}{\sqrt{n}-1}$$

$$\Rightarrow b_{n+1} \le b_n$$

Now $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}-1}$

$$= \lim_{n\to\infty}\frac{1/\sqrt{n}}{1-1/\sqrt{n}} = 0$$

$$\Rightarrow \lim_{n\to\infty} b_n = 0$$

So given series is $\boxed{\textit{convergent}}$

Q19E

Consider the series,

$$\sum_{n=1}^{\infty}(-1)^n \frac{\ln(n)}{\sqrt{n}}.$$

The object is to determine whether the series converges or diverges.

Rewrite the sequence as,

$$\sum_{n=1}^{\infty}(-1)^n \frac{\ln(n)}{\sqrt{n}} = (-1)^1\frac{\ln(1)}{\sqrt{1}} + (-1)^2\frac{\ln(2)}{\sqrt{2}} + (-1)^3\frac{\ln(3)}{\sqrt{3}} + \cdots$$
$$= -\frac{\ln(1)}{\sqrt{1}} + \frac{\ln(2)}{\sqrt{2}} - \frac{\ln(3)}{\sqrt{3}} + \cdots$$

This is an alternating series.

Use the alternating series test: suppose for a_n, there exist an N so that for all $n \geq N$

1. Here a_n is positive and monotone decreasing.

2. $\lim_{n\to\infty} a_n = 0$ Then the alternating series test a_n converges.

Let $a_n = \frac{\ln(n)}{\sqrt{n}}$.

Since a_n is positive and monotone decreasing from $n = 7$.

The limit of the series $a_n = \frac{\ln(n)}{\sqrt{n}}$ at $n \to \infty$ is

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln(n)}{\sqrt{n}}$$
$$= \lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} \qquad \text{Use } \lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$
$$= \lim_{n\to\infty}\frac{2\sqrt{n}}{n}$$
$$= \lim_{n\to\infty}\frac{2}{\sqrt{n}}$$
$$= \frac{2}{\lim_{n\to\infty}\sqrt{n}}$$
$$= 0$$

Hence, the series converges conditionally by the alternating series test.

Q20E

Given series $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}-1}{k\left(\sqrt{k}+1\right)}$

We have

$$\sqrt{k}+1>\sqrt{k}$$
$$\Rightarrow k\left(\sqrt{k}+1\right)>k\sqrt{k}$$
$$\Rightarrow \frac{1}{k\sqrt{k}+1}<\frac{1}{k\sqrt{k}}$$
$$\Rightarrow \frac{\sqrt[3]{k}-1}{k\sqrt{k}+1}<\frac{\sqrt[3]{k}}{k\sqrt{k}}$$
$$\Rightarrow \frac{\sqrt[3]{k}-1}{k\sqrt{k}+1}<\frac{1}{k^{7/6}}$$

We know that $\sum_{k=1}^{\infty} \frac{1}{k^{7/6}}$ is convergent $\left(p\text{-series with } p=\frac{7}{6}>1\right)$

Thus given series is convergent by Comparison Test.

Comparison test:

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent

Q21E

Consider the series

$$\sum_{n=1}^{\infty}(-1)^n\cos\left(\frac{1}{n^2}\right).$$

To determine the given series is convergent or divergent, use the Test for Divergence.

Test for Divergence:

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Take, $a_n = (-1)^n\cos\left(\frac{1}{n^2}\right)$.

Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^n\cos\left(\frac{1}{n^2}\right)$$

$$= \lim_{n\to\infty}(-1)^n \cdot \lim_{n\to\infty}\cos\left(\frac{1}{n^2}\right)$$

$$= \lim_{n\to\infty}(-1)^n \cdot \cos(0) \quad \text{Since } \frac{1}{n^2}\to 0 \text{ as } n\to\infty.$$

$$= \lim_{n\to\infty}(-1)^n \cdot (1) \quad \text{Since } \cos(0)=1$$

$$= \lim_{n\to\infty}(-1)^n$$

Clearly the limit $\lim_{n\to\infty}(-1)^n$ does not exist because the terms oscillate between 1 and -1 infinitely often, $(-1)^n$ does not approach any number.

Thus, $\lim_{n\to\infty}(-1)^n$ does not exist and therefore $\lim_{n\to\infty} a_n$ does not exist.

Hence the series $\sum_{n=1}^{\infty}(-1)^n\cos\left(\frac{1}{n^2}\right)$ diverges by the Test for Divergence.

Q22E

Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{2+\sin k}.$$

To determine the given series is convergent or divergent, use the Comparison test.

Comparison test states that

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all *n*, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all *n*, then $\sum a_n$ is also divergent.

Use the Comparison Test with $a_k = \dfrac{1}{2+\sin k}$.

Clearly, $-1 \le \sin k \le 1.$

Consider, $\sin k \le 1$

Add 2 on both sides of the inequality.

$$2+\sin k \le 3$$
$$\frac{1}{2+\sin k} \ge \frac{1}{3}$$

Take, $b_k = \dfrac{1}{3}$.

Therefore, $a_k \ge b_k$, for all *k*.

And the series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{3}$ is diverges as it is a constant series.

Since $\sum b_k$ diverges, so $\sum a_k$ is also diverges by condition (ii) of the Comparison Test.

Hence the given series $\sum_{k=1}^{\infty} \dfrac{1}{2+\sin k}$ is diverges by the Comparison Test.

Q23E

Consider the series $\sum_{n=1}^{\infty} \tan(1/n)$.

User the Limit Comparison Test with $a_n = \tan(1/n)$ and $b_n = 1/n$ as follows:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan(1/n)}{1/n}$$

Reason: The series $\sum a_n$ and $\sum b_n$ with positive terms. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = c$ where c is a finite number and $c > 0$.

Let $\frac{1}{n}=\theta$ then $\theta \to 0$ as $n \to \infty$ and obtain the series as follows:

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{\theta\to 0}\frac{\tan\theta}{\theta}$$

$$=\lim_{\theta\to 0}\frac{\sin\theta}{\theta.\cos\theta}$$

$$=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}.\lim_{\theta\to 0}\frac{1}{\cos\theta}$$

$$=1$$

So, $\lim_{n\to\infty}\frac{a_n}{b_n}=1>0$

Since this limit exists and $\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent *p*-series with $p=1$.

The given series $\sum_{n=1}^{\infty}\tan(1/n)$ divergence by the Limit Comparison Test.

Q24E

Consider the series $\sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$.

Determine whether the given series is convergent or divergent.

Here, use the Test for Divergence:

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Let $a_n = n\sin\left(\frac{1}{n}\right)$.

Find the limit of a_n as n tends to ∞.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[n\cdot\sin\left(\frac{1}{n}\right)\right]$$

$$=\infty\cdot\sin\left(\frac{1}{\infty}\right)$$

$$=\infty\cdot\sin(0)$$

$$=\infty\cdot 0$$

Indeterminate form $\infty\cdot 0$ is obtained by direct substitution.

To use the L'hospital's rule, it has to be changed as one of the indeterminate forms $\frac{0}{0},\frac{\infty}{\infty}$.

L'Hospital's Rule:

Suppose that f and g are differential functions and that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$. if $\lim_{x\to a} \frac{\frac{d}{dx}(f(x))}{\frac{d}{dx}(g(x))} = L$ then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$.

Rewrite a_n as follows:

$$a_n = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}.$$

Take limit on both sides.

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left[\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\right] \\ &= \frac{\sin\left(\frac{1}{\infty}\right)}{\frac{1}{\infty}} \\ &= \frac{\sin(0)}{0} \\ &= \frac{0}{0}\end{aligned}$$

Indeterminate form $\frac{0}{0}$ is obtained by direct substitution so L'hospital's rule can be applied:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\frac{d}{dn}\left[\sin\left(\frac{1}{n}\right)\right]}{\frac{d}{dn}\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(\cos\left(\frac{1}{n}\right)\right)\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty}\left(\cos\left(\frac{1}{n}\right)\right)$$

$$= \cos(0) \qquad \left[\text{Since as } n\to\infty, \frac{1}{n}\to 0.\right]$$

$$= 1$$

Therefore, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[n\sin\left(\frac{1}{n}\right)\right] = 1.$

Since $\lim_{n\to\infty} a_n = 1 \neq 0,$ by Test for divergence, conclude that the given series is diverges.

Hence, $\sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$ is diverges.

Q25E

Series is $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

Since series involves n! , so we use ratio test

Here $a_n = \frac{n!}{e^{n^2}}$, then $a_{n+1} = \frac{(n+1)!}{e^{(n+1)^2}}$

So $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!/e^{(n+1)^2}}{n!/e^{n^2}}$

$= \lim_{n\to\infty}\frac{n+1}{e^{(n+1)^2-n^2}}$

$= \lim_{n\to\infty}\frac{n+1}{e^{2n+1}}$

$= \lim_{x\to\infty}\frac{x+1}{e^{2x+1}}$

$= \lim_{x\to\infty}\frac{1}{2.e^{2x+1}}$ [Using L-hospital rule]

$= \frac{1}{2e}\lim_{x\to\infty} e^{-2x}$

$= \frac{1}{2e}.0 = 0 < 1$

So by ratio test given series is $\boxed{convergent}$

Q26E

We have the series $\sum_{n=1}^{\infty}\frac{n^2+1}{5^n}$

We use the Ratio test with $a_n = \frac{n^2+1}{5^n}$

We have $\frac{a_{n+1}}{a_n} = \frac{\left[(n+1)^2+1\right]}{5^{n+1}}\cdot\frac{5^n}{(n^2+1)}$

$= \frac{\left[(n+1)^2+1\right]}{5(n^2+1)}$

$= \frac{(1+1/n)^2+1/n^2}{5(1+1/n^2)} \to \frac{1}{5}$ as $n\to\infty$

So $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5} < 1$

Thus by the Ratio test the given series $\boxed{\text{converges}}$.

Q27E

Series is $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$

Here $\quad a_k = \frac{k \ln k}{(k+1)^3} \quad$ we take $\quad b_k = \frac{k \ln k}{k^3} = \frac{\ln k}{k^2}$

We have $\qquad \frac{k \ln k}{(k+1)^3} \le \frac{\ln k}{k^2}$

Now we apply comparison test and for this we have to check that the series $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ is convergent or divergent

Let $f(x) = \frac{\ln x}{x^2}$, this function is positive and continuous for x >1

And $f'(x) = \frac{x^2(1/x) - (\ln x)(2x)}{x^4}$

$$\Rightarrow f'(x) = \frac{1 - 2\ln x}{x^3} < 0 \quad \text{for } 1 - 2\ln x < 0 \Rightarrow x > e^{1/2}$$

So the function f is decreasing

Then we use integral test.

$$\int_1^{\infty} \frac{\ln x}{x^2} dx = \lim_{t \to \infty} \int_1^t \frac{\ln x}{x^2} dx$$

We use integration by part

$$\int \frac{\ln x}{x^2} d = \ln x \int x^{-2} dx - \int \left(\frac{d}{dx} \ln x \right) \left(\int x^{-2} dx \right) dx$$

$$= -\frac{\ln x}{x} + \int \frac{1}{x} \cdot \frac{1}{x} dx$$

$$= -\frac{\ln x}{x} + \left(-\frac{1}{x} \right)$$

$$= -\frac{\ln x}{x} - \frac{1}{x}$$

So $\lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}dx = \lim_{t\to\infty}\left[-\frac{\ln t}{t}-\frac{1}{t}+0+1\right]$

$$= -\lim_{t\to\infty}\frac{\ln t}{t}+1$$

$$= -\lim_{t\to\infty}\frac{1/t}{1}+1 \qquad \text{[by L-hospital rule]}$$

$$\Rightarrow \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}dx = -0+1$$

$$\Rightarrow \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}dx = 1$$

So $\int_1^\infty \frac{\ln x}{x^2}dx$ converges. Therefore by integral test, the series $\sum_{k=1}^{\infty}\frac{\ln x}{k^2}$ is convergent. Then by comparison test, series $\sum_{k=1}^{\infty}\frac{k\ln k}{(k+1)^3}$ *is* $\boxed{\text{convergent}}$

Q28E

We have the series $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$

Since $\frac{e^{1/n}}{n^2} \le \frac{e}{n^2}$

And since $\sum_{n=1}^{\infty}\frac{e}{n^2} = e\sum_{n=1}^{\infty}\frac{1}{n^2}$ = constant multiple of convergent p-series with p = 2>1

Thus by comparison test, the given series $\boxed{\text{converges}}$.

Q29E

Given that $\sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh(n)}$

Substitute this in:

$$\cosh\left(n\right) = \frac{e^n + e^{-n}}{2}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\frac{e^n+e^{-n}}{2}} = \sum_{n=1}^{\infty} (-1)^n \frac{2}{e^n + e^{-n}}$$

Apply the alternating series test:

i) $b_{n+1} \leq b_n$, $b_n = \frac{2}{e^n + e^{-n}}$

$$\frac{2}{e^{n+1} + e^{-(n+1)}} \leq \frac{2}{e^n + e^{-n}}$$

ii) $\lim_{n \to \infty} b_n = 0$

$$\lim_{n \to \infty} \frac{2}{e^n + e^{-n}} = 0$$

Therefore, the series converges by the alternating series test.

Q30E

Since is $\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$

This is an alternating series so we apply the test for convergence of alternating series with $b_j = \frac{\sqrt{j}}{j+5}$

We take $f(x)=\dfrac{\sqrt{x}}{x+5}$

Then $f'(x)=\dfrac{(x+5)\dfrac{1}{2\sqrt{x}}-\sqrt{x}}{(x+5)^2}$

$$\Rightarrow f'(x)=\frac{\frac{1}{2\sqrt{x}}(x+5-2x)}{(x+5)^2}$$

$$\Rightarrow f'(x)=\frac{(5-x)}{2(x+5)^2\sqrt{x}}$$

Since $f'(x)<0$ when $x>5$, so the function f is decreasing

And then we have $b_{j+1}\le b_j$

Now $\lim\limits_{j\to\infty}\dfrac{\sqrt{j}}{j+5}=\lim\limits_{j\to\infty}\dfrac{1/\sqrt{j}}{1+5/j}=\dfrac{0}{1+0}=0$

So $\lim\limits_{j\to\infty}b_j=0$

Then the series $\sum\limits_{j=1}^{\infty}(-1)^j\dfrac{\sqrt{j}}{j+5}$ is $\boxed{\text{convergent}}$

Q31E

Consider the series,

$$\sum_{k=1}^{\infty}\frac{5^k}{3^k+4^k}.$$

The object is to test whether the series converges or diverges.

Let $a_k=\dfrac{5^k}{3^k+4^k}$.

Replace k by $k+1$, then the series becomes,

$$a_{k+1}=\frac{5^{k+1}}{3^{k+1}+4^{k+1}}$$

Recall that the ratio test, if there exist a N so that for all $n\ge N$, $a_k\ne 0$ and $\lim\limits_{n\to\infty}\left|\dfrac{a_{k+1}}{a_k}\right|=L$ then the following conditions are hold.

If $L<1$, then $\sum a_k$ converges.

If $L>1$, then $\sum a_k$ diverges.

If $L=1$, and then the test is inconclusive.

Use ratio test, then the limit value can be calculated as,

$$\lim_{n\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{n\to\infty}\left|\frac{\frac{5^{k+1}}{3^{k+1}+4^{k+1}}}{\frac{5^k}{3^k+4^k}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{5^{k+1}}{3^{k+1}+4^{k+1}} \times \frac{3^k+4^k}{5^k}\right|$$

$$= \lim_{n\to\infty}\left|\frac{5\left(3^k+4^k\right)}{3^{k+1}+4^{k+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{5\left(4^k\right)\left(\frac{3^k}{4^k}+1\right)}{\left(4^{k+1}\right)\left(\frac{3^{k+1}}{4^{k+1}}+1\right)}\right|$$

$$= \frac{5}{4}\lim_{n\to\infty}\frac{\left(\left(\frac{3}{4}\right)^k+1\right)}{\left(\left(\frac{3}{4}\right)^{k+1}+1\right)}$$

Since $\frac{3}{4} < 1$, so $\left(\frac{3}{4}\right)^k \to 0$ as $n \to \infty$.

$$= \frac{5}{4}\frac{\left(\lim_{n\to\infty}\left(\frac{3}{4}\right)^k+1\right)}{\left(\lim_{n\to\infty}\left(\frac{3}{4}\right)^{k+1}+1\right)}$$

$$= \frac{5}{4}$$

$$> 1$$

By ratio test, it confirms that $L > 1$ so the series diverges.

Hence, the series $\sum_{k=1}^{\infty}\frac{5^k}{3^k+4^k}$ diverges.

Q32E

given series $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

apply root test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(n!)^n}{n^{4n}}\right|}$$

$$= \lim_{n\to\infty} \left(\frac{(n!)^n}{n^{4n}}\right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{n!}{n^4}$$

$=\infty > 1$ [since for a large value of n, n! will beat out n4]

so the given series diverges by the Root Test.

Q33E

We have the series $\sum_{n=1}\left(\frac{n}{n+1}\right)^{n^2}$

We use the Root test with $a_n = \left(\frac{n}{n+1}\right)^{n^2}$

We have $(a_n)^{1/n} = \left(\frac{n}{n+1}\right)^n$

Then $\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$

$$= \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^n$$

$$= \frac{1}{\lim_{n\to\infty}(1+1/n)^n} = \frac{1}{e} < 1$$

Thus by the Root test the given series **converges**.

Q34E

Series is $\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$

We have $\cos n \le 1$

$\Rightarrow \cos^2 n \le 1$

$\Rightarrow 1+\cos^2 n \le 2$

$\Rightarrow n+n\cos^2 n \le 2n$

$\Rightarrow \frac{1}{n+n\cos^2 n} \ge \frac{1}{2n}$

If $a_n = \frac{1}{n+n\cos^2 n}$ and $b_n = \frac{1}{2n}$ then $a_n \ge b_n$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2n}$

And $\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ is a constant multiple of divergent (harmonic) series.

Then by comparison test, the series $\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$ *is* $\boxed{divergent}$

Q35E

Given series $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$

Let $a_n = \frac{1}{n^{1+\frac{1}{n}}}, b_n = \frac{1}{n}$

Then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{1}{\frac{1}{n^n}}$$

$$= 1 > 0$$

We know that the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is diverges.

Hence the given series is diverges by the Limit Comparison Test.

Limit Comparison Test:

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty}\frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Q36E

Series is $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

Since $(\ln n)^{\ln n} = \left(e^{\ln \ln n}\right)^{\ln n}$

$= \left(e^{\ln n}\right)^{\ln \ln n}$

$= n^{\ln \ln n}$

Since $\ln \ln n \to \infty$ *as* $n \to \infty$

So $\ln \ln n > 2$ *for* large n $(n > 1700)$

Thus $(\ln n)^{\ln n} > n^2$

Then $\frac{1}{(\ln n)^{\ln n}} < \frac{1}{n^2}$

Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is a convergent p-series with $p = 2 > 1$. Then by comparison test

The series $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$ is also $\boxed{\text{convergent}}$

Q37E

Series is $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$

This is the form of $\sum a_n^n$ so we use root test

We have $\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \left(\sqrt[n]{2} - 1\right)$

$= \lim_{n \to \infty} \left(2^{1/n} - 1\right)$

$= 2^0 - 1$

$= 0$

So $\lim_{n \to \infty} \sqrt[n]{|a_n|} < 1$

Then series $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$ is $\boxed{\text{convergent}}$

Q38E

Let $a_n = \sqrt[n]{2} - 1$ and $b_n = \frac{1}{n}$.

Clearly $\sum b_n = \sum \frac{1}{n}$ is a divergent harmonic series.

Then

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{2^{\frac{1}{n}} - 1}{\frac{1}{n}}.$$

Use L'Hospital's rule to evaluate this limit.

L'Hospital's rule: Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a. Suppose that

$\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$

Or that $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

In other words an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

The limit of the numerator is

$$\begin{aligned}\lim_{n \to \infty}(a_n) &= \lim_{n \to \infty}\left(2^{\frac{1}{n}} - 1\right)\\ &= \lim_{\frac{1}{n} \to 0}\left(2^{\frac{1}{n}} - 1\right)\\ &= 2^0 - 1\end{aligned}$$

$$= 1 - 1$$

$$= 0$$

And the limit of the denominator is

$$\begin{aligned}\lim_{n \to \infty} b_n &= \lim_{n \to \infty} \frac{1}{n}\\ &= 0\end{aligned}$$

So we get the indeterminate form $\frac{0}{0}$, L'Hospital's rule gives

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{2^{\frac{1}{n}}-1}{\frac{1}{n}}$$

$$=\lim_{n\to\infty}\frac{\frac{d}{dn}\left(2^{\frac{1}{n}}-1\right)}{\frac{d}{dn}\left(\frac{1}{n}\right)}$$

$$=\lim_{n\to\infty}\frac{2^{\frac{1}{n}}\ln 2\left(-\frac{1}{n^2}\right)}{\left(-\frac{1}{n^2}\right)}$$

$$=\lim_{n\to\infty}\left[2^{\frac{1}{n}}\ln 2\right]$$

$$=\lim_{\frac{1}{n}\to 0}\left[2^{\frac{1}{n}}\ln 2\right]$$

$$=2^0(\ln 2)$$

$$=(1)(\ln 2)$$

$$=\ln 2$$

So $\lim_{n\to\infty}\frac{a_n}{b_n}=\ln 2>0.$

Since this limit exists and $\sum b_n=\sum\frac{1}{n}$ is a divergent harmonic series, the given series diverges by the Limit Comparison Test.