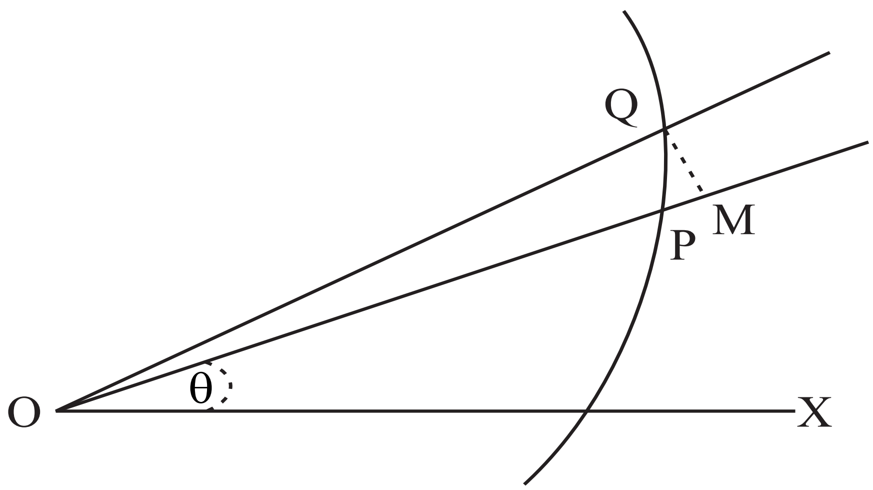
Chapter 4

UNIPLANAR MOTION REFERRED TO POLAR COORDINATES CENTRAL FORCES

48. In the present chapter we shall consider cases of motion which are most readily solved by the use of polar coordinates. We must first obtain the velocities and accelerations of a moving point along and perpendicular to the radius vector drawn from a fixed pole.

49. *Velocities and accelerations of a particle along and perpendicular to the radius vector to it from a fixed origin O.*

Let P be the position of the particle at time t, and Q its position at time $t + \triangle t$.



Let $XOP = \theta$, $XOQ = \theta + \triangle\theta$, $OP = r$, $OQ = r + \triangle r$, where OX is a fixed line.

Draw QM perpendicular to OP.

Let u, v be the velocities of the moving point along and perpendicular to OP. Then

$$u = \lim_{\triangle t=0} \left[\frac{\begin{array}{c}\text{Distance of particle measured along the line OP} \\ \text{at time } (t+\triangle t) - \text{ the similar distance at time } t\end{array}}{\triangle t} \right]$$

$$= \lim_{\triangle t=0} \frac{OM - OP}{\triangle t}$$

$$= \lim_{\triangle t=0} \frac{(r+\triangle r)\cos\triangle\theta - r}{\triangle t}$$

$$= \lim_{\triangle t=0} \frac{(r+\triangle r).1 - r}{\triangle t},$$

[small quantities above the first order being neglected.]

$$= \frac{dr}{dt} \qquad \ldots(1)$$

Also

$$v = \lim_{\triangle t=0} \left[\frac{\begin{array}{c}\text{Distance of particle measured perpendicular to the} \\ \text{line OP at time } (t+\triangle t) - \text{ the similar distance at time } t\end{array}}{\triangle t} \right]$$

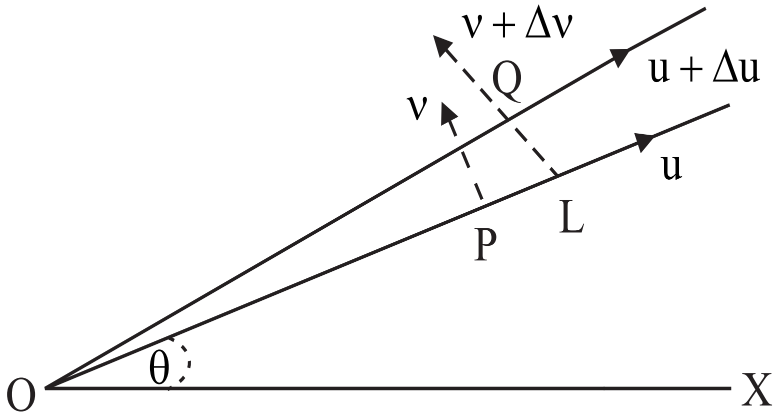
$$= \lim_{\triangle t=0} \frac{OM - 0}{\triangle t} = \lim_{\triangle t=0} \frac{(r+\triangle r)\sin\triangle\theta}{\triangle t}$$

$$= \lim_{\triangle t=0} \frac{(r+\triangle r).\triangle\theta}{\triangle t}, \text{on neglecting small quantities of the second order}$$

$$= r\frac{d\theta}{dt}, \text{ in the limit} \qquad ...(2)$$

The velocities along and perpendicular to OP being u and v, the velocities along and perpendicular to OQ are $u+\triangle u$ and $v+\triangle v$.

Let the perpendicular to OQ at Q be produced to meet OP at L.



Then the acceleration of the moving point along OP

$$= \lim_{\triangle t=0}\left[\frac{\begin{array}{c}\text{Its velocity along } OP \text{ at time } (t+\triangle t)\\ \text{– its similar velocity at time } t\end{array}}{\triangle t}\right]$$

$$= \lim_{\triangle t=0}\left[\frac{(u+\triangle u)\cos\theta-(v+\triangle v)\sin\triangle\theta-u}{\triangle t}\right]$$

$$= \lim_{\triangle t=0}\left[\frac{(u+\triangle u).1-(v+\triangle v).\triangle\theta-u}{\triangle t}\right],$$

on neglecting squares and higher powers of $\triangle\theta$,

$$= \lim_{\triangle t=0}\frac{\triangle u-v\triangle\theta}{\triangle\theta}=\frac{du}{dt}-v\frac{d\theta}{dt}, \text{ in the limit,}$$

$$= \frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2, \quad \text{by (1) and (2)}$$

$$...(3)$$

Also the acceleration of the moving point perpendicular to OP in the direction of θ increasing

$$= \lim_{\triangle t = 0} \left[\frac{\begin{array}{c}\text{Its velocity perpendicular to } OP \text{ at time } (t+\triangle t) \\ \text{– its similar velocity at time } t\end{array}}{\triangle t} \right]$$

$$= \lim_{\triangle t = 0} \left[\frac{(u+\triangle u)\sin\triangle\theta + (v+\triangle v)\cos\triangle\theta - v}{\triangle t} \right]$$

$$= \lim_{\triangle t = 0} \left[\frac{(u+\triangle u).\triangle\theta + (v+\triangle v).1 - v}{\triangle t} \right],$$

on neglecting squares and higher powers of $\triangle\theta$,

$$= u\frac{d\theta}{dt} + \frac{dv}{dt}, \text{ in the limit, } = \frac{dr}{dt}\frac{d\theta}{dt} + \frac{d}{dt}\left(r\frac{d\theta}{dt} \right), \text{by (1) and (2)}$$

$$= 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = \frac{1}{r}\frac{d}{dt}\left[r^2\frac{d\theta}{dt} \right] \qquad ...(4)$$

COR. If $r = a$, a constant quantity, so that the particle is describing a circle of centre O and radius a, the quantity (3) $= -a\theta^2$ and (4) $= a\ddot{\theta}$, so that the accelerations of P along the tangent PQ and the radius PO are $a\ddot{\theta}$ and $a\ddot{\theta}^2$.

50. The results of the previous article may also be obtained by resolving the velocities and accelerations along the axes of x and y in the directions of the radius vector and perpendicular to it.

For since $x = r\cos\theta$ and $y = r\sin\theta$,

$$\left. \begin{array}{ll} \therefore & \dfrac{dx}{dt} = \dfrac{dr}{dt}\cos\theta - r\sin\theta\dfrac{d\theta}{dt} \\ \text{and} & \dfrac{dy}{dt} = \dfrac{dr}{dt}\sin\theta + r\cos\theta\dfrac{d\theta}{dt} \end{array} \right\} \qquad ...(1)$$

Also

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= \frac{d^2r}{dt^2}\cos\theta - 2\frac{dr}{dt}\frac{d\theta}{dt}\sin\theta - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2} \\ \frac{d^2y}{dt^2} &= \frac{d^2r}{dt^2}\sin\theta + 2\frac{dr}{dt}\frac{d\theta}{dt}\cos\theta - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\frac{d^2\theta}{dt^2}\end{aligned}\right\} \quad ...(2)$$

The component velocity along *OP*

$$= \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta = \frac{d\theta}{dt}, \text{ by } (1),$$

and perpendicular to *OP* in the direction of θ increasing it

$$= \frac{dy}{dt}\cos\theta - \frac{dx}{dt}\sin\theta = r\frac{d\theta}{dt}, \text{ by } (1).$$

The component acceleration along OP

$$= \frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt^2}\sin\theta = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2, \text{ by } (2).$$

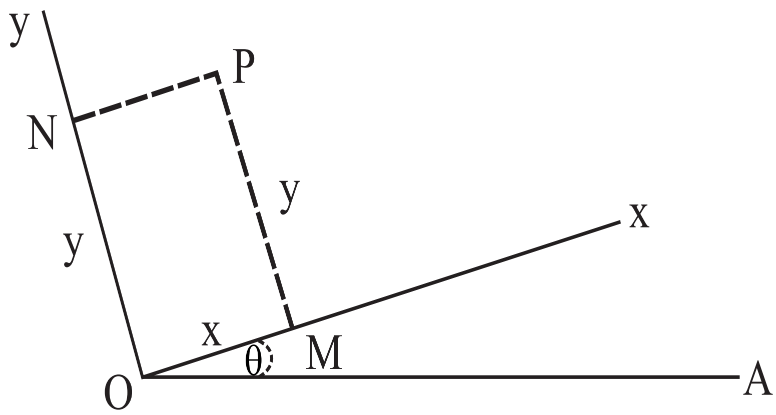
and perpendicular to *OP* it

$$= \frac{d^2y}{dt^2}\cos\theta - \frac{d^2x}{dt^2}\sin\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} \text{ by } (2),$$

$$= \frac{1}{r}\frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right].$$

51. By the use of Arts. 4 and 49 we can obtain the accelerations of a moving point referred to rectangular axes *Ox* and *Oy*, which are not fixed in space, but which revolve in any manner about the origin O in their own plane.

Let *OA* be a line fixed in space, and, at time t, let θ be the inclination of *Ox* to *OA*. Let *P* be the moving point; draw *PM* and *PN* perpendicular to *Ox* and *Oy*.



By Art. 49 the velocities of the point M are $\frac{dx}{dt}$ along OM and $x\frac{d\theta}{dt}$ along MP, and the velocities of N are $\frac{dy}{dt}$ along ON and $y\frac{d\theta}{dt}$ along PN produced.

$$\left[\text{for } \frac{d}{dt}(\angle AON) = \frac{d}{dt}(\angle AOM) = \frac{d\theta}{dt}.\right]$$

Hence the velocity of P parallel to Ox

$=$ the velocity of N parallel to Ox + the velocity of P relative to N.

$=$ vel. of N parallel to Ox + the vel. of M along OM

$$= -y\frac{d\theta}{dt} + \frac{dx}{dt} \qquad ...(1)$$

So the velocity of P parallel to Oy

$=$ vel. of M parallel to Oy + the vel. of P relative to M

$=$ vel. of M parallel to Oy + the vel. of N along to ON

$$= x\frac{d\theta}{dt} + \frac{dy}{dt} \qquad ...(2)$$

Again, the accelerations of M are, by Art. 49, $\frac{d^2x}{dt^2} - x\left(\frac{d\theta}{dt}\right)^2$ along

OM, and $\dfrac{1}{x}\dfrac{d}{dt}\left(x^2\dfrac{d\theta}{dt}\right)$ along MP, and the accelerations of N are $\dfrac{d^2y}{dt^2}-y\left(\dfrac{d\theta}{dt}\right)^2$ along ON, and $\dfrac{1}{y}\dfrac{d}{dt}\left(y^2\dfrac{d\theta}{dt}\right)$ along PN, produced.

Hence the acceleration of P parallel to Ox

$=$ acceleration of N parallel to Ox + acceleration of P relative to N

$=$ acceleration of N parallel to Ox + acceleration of M along OM

$$= -\frac{1}{y}\frac{d}{dt}\left(y^2\frac{d\theta}{dt}\right)+\frac{d^2x}{dt^2}-x\left(\frac{d\theta}{dt}\right)^2 \qquad ...(3)$$

Also the acceleration of P parallel to Oy

$=$ acceleration of M parallel to Oy + acceleration of P relative to M

$=$ acceleration of M parallel to Ox + acceleration of N along ON

$$= \frac{1}{x}\frac{d}{dt}\left(x^2\frac{d\theta}{dt}\right)+\frac{d^2y}{dt^2}-y\left(\frac{d\theta}{dt}\right)^2 \qquad ...(4)$$

COR. In the particular case when the axes are revolving with a constant angular velocity ω, so that $\dfrac{d\theta}{dt}=\omega$, these component velocities become

$$\frac{dx}{dt}-y\omega \text{ along } Ox, \text{ and } \frac{dy}{dt}+x\omega \text{ along } Oy;$$

also the component accelerations are

$$\frac{d^2x}{dt^2}-x\omega^2-2\omega\frac{dy}{dt} \text{ along } Ox, \text{ and } \frac{d^2y}{dt^2}-y\omega^2+2\omega\frac{dx}{dt} \text{ along } Oy.$$

52. EX. 1. *Show that the path of a point P which possesses two constant velocities u and v, the first of which is in a fixed direction and* *the second of which is perpendicular to the radius OP drawn from a fixed point O, is a conic whose focus is O and whose eccentricity is* $\frac{u}{v}$.

With the first figure of Art. 49, let u be the constant velocity along OX and v the constant velocity perpendicular to OP.

Then we have

$$\frac{dr}{dt} = u\cos\theta, \text{ and } \frac{rd\theta}{dt} = v - u\sin\theta. \qquad \therefore \frac{1}{r}\frac{dr}{d\theta} = \frac{u\cos\theta}{v-u\sin\theta}.$$

$$\therefore \quad \log r = -\log(v - u\sin\theta) + \text{const.},$$

$$i.e. \;\; r(v-u\sin\theta) = \text{ const.} = lv,$$

if the path cut the axis of x at a distance l. Therefore the path is

$$r = \frac{1}{1-\dfrac{u}{v}\sin\theta}, \quad i.e. \quad \text{a conic section whose eccentricity is } \frac{u}{v}.$$

EX. 2. *A smooth straight thin tube revolves with uniform angular velocity* ω *in a vertical plane about one extremity which is fixed; if at zero time the tube be horizontal, and a particle inside it be at a distance a from the fixed end, and be moving with velocity V along the tube, show that its distance at time t is*

$$a\cosh(\omega t) + \left(\frac{V}{\omega} - \frac{g}{2\omega^2}\right)\sinh(\omega t) + \frac{g}{2\omega^2}\sin\omega t.$$

At any time t let the tube have revolved round its fixed end through an angle ωt from the horizontal line OX in an upward direction; let P, where $OP = r$, be the position of the particle then.

By Art. 49,

$$\frac{d^2r}{dt^2} - r\omega^2 = \text{ acceleration of } P \text{ in the direction } OP$$
$$= -g\sin\omega t, \text{ since the tube is smooth.}$$

The solution of this equation is

$$r = Ae^{\omega t} + Be^{-\omega t} + \frac{1}{D^2-\omega^2}(-g\sin\omega t)$$
$$= L\cosh(\omega t) + M\sinh(\omega t) + \frac{g}{2\omega^2}\sin\omega t,$$

where A and B, and so L and M, are arbitrary constants.

The initial conditions are that $r = a$ and $\dot{r} = V$ when $t = 0$.

$$\therefore \quad a = L, \text{ and } V = M\omega + \frac{g}{2\omega}.$$
$$\therefore \quad r = a\cosh\omega t + \left[\frac{V}{\omega} - \frac{g}{2\omega^2}\right]\sinh(\omega t) + \frac{g}{2\omega^2}\sin\omega t.$$

If R be the normal reaction of the tube, then

$$\frac{R}{m} - g\cos\omega t = \text{ the acceleration perpendicular to } OP$$
$$= \frac{1}{r}\frac{d}{dt}(r^2\omega), \text{ by Art. 49,}$$
$$= 2\dot{r}\,\omega$$
$$= 2a\omega^2\sinh(\omega t) + (2V\omega - g)\cosh(\omega t) + g\cos\omega t.$$

EXAMPLES

1. A vessel steams at a constant speed v along a straight line whilst another vessel, steaming at a constant speed V, keeps the first al-

ways exactly abeam. Show that the path of either vessel relatively to the other is a conic section of eccentricity $\frac{v}{V}$.

2. A boat, which is rowed with constant velocity u, starts from a point A on the bank of a river which flows with a constant velocity nu; it points always towards a point B on the other bank exactly opposite to A; find the equation to the path of the boat.

   If n be unity, show that the path is a parabola whose focus is B.
3. An insect crawls at a constant rate u along the spoke of a cartwheel, of radius a, the cart moving with velocity v. Find the acceleration along and perpendicular to the spoke.
4. The velocities of a particle along and perpendicular to the radius from a fixed origin are λr and $\mu\theta$; find the path and show that the accelerations, along and perpendicular to the radius vector, are

$$\lambda^2 r - \frac{\mu^2\theta^2}{r} \quad \text{and} \quad \mu\theta\left[\lambda + \frac{\mu}{r}\right].$$

5. A point starts from the origin in the direction of the initial line with velocity $\frac{f}{\omega}$ and moves with constant angular velocity ω about the origin and with constant negative radial acceleration $-f$. Show that the rate of growth of the radial velocity is never positive, but tends to the limit zero, and prove that the equation of the path is

$$\omega^2 r = f(1 - e^{-\theta}).$$

6. A point P describes a curve with constant velocity and its angular velocity about a given fixed point O varies inversely as the distance from O; show that the curve is an equiangular spiral whose pole is O, and that the acceleration of the point is along the normal at P and varies inversely as OP.

7. A point P describes an equiangular spiral with constant angular velocity about the pole; show that its acceleration varies as OP and is in a direction making with the tangent at P the same constant angle that OP makes.
8. A point moves in a given straight line on a plane with constant velocity V, and the plane moves with constant angular velocity ω about an axis perpendicular to itself through a given point O of the plane. If the distance of O from the given straight line be a, show that the path of the point in space is given by the equation $\frac{V\theta}{\omega} = \sqrt{r^2 - a^2} + \frac{V}{\omega}\cos^{-1}\frac{a}{r}$, referred to O as pole.
[If θ be measured from the line to which the given line is perpendicular at zero time, then $r^2 = a^2 + V^2.t^2$ and $\theta = \omega t + \cos^{-1}\frac{a}{r}$.]
9. A straight smooth tube revolves with angular velocity ω in a horizontal plane about one extremity which is fixed; if at zero time a particle inside it be at a distance a from the fixed end and moving with velocity V along the tube, show that its distance at time t is
$$a\cosh\omega t + \frac{V}{\omega}\sinh\omega t.$$
10. A thin straight smooth tube is made to revolve upwards with a constant angular velocity ω in a vertical plane about one extremity O; when it is in a horizontal position, a particle is at rest in it at a distance a from the fixed end O; if ω be very small, show that it will reach O in a time $\left(\frac{6a}{g\omega}\right)^{1/3}$ nearly.
11. A particle is at rest on a smooth horizontal plane which commences to turn about a straight line lying in itself with constant angular velocity ω downwards; if a be the distance of the particle from the axis of rotation at zero time, show that the body will leave

the plane at time t given by the equation

$$a \sinh \omega t + \frac{g}{2\omega^2} \cosh \omega t = \frac{g}{\omega^2} \cos \omega t.$$

12. A particle falls from rest within a straight smooth tube which is revolving with uniform angular velocity ω about a point O in its length, being acted on by a force equal to $m\mu$(distance) towards O. Show that the equation to its path in space is

$$r = a \cosh \left[\sqrt{\frac{\omega^2 - \mu}{\omega^2}} \theta \right]$$

$$\text{or } r = \cos \left\{ \sqrt{\frac{\mu - \omega^2}{\omega^2}} \theta \right\}, \quad \text{according as } \mu \lesseqgtr \omega^2.$$

If $\mu = \omega^2$, show that the path is a circle.

13. A particle is placed at rest in a rough tube at a distance a from one end, and the tube starts rotating with a uniform angular velocity ω about this end. Show that the distance of the particle at time t is

$$ae^{-\omega t}. \tan \varepsilon [\cosh(\omega t. \sec \varepsilon) + \sin \varepsilon \sinh(\omega t \sec \varepsilon)],$$

where $\tan \varepsilon$ is the coefficient of friction.

14. One end A of a rod is made to revolve with uniform angular velocity ω in the circumference of a circle of radius a, whilst the rod itself revolves in the opposite direction about that end with the same angular velocity. Initially the rod coincides with a diameter and a smooth ring capable of sliding freely along the rod is placed at the centre of the circle. Show that the distance of the ring from A at time t is

$$\frac{a}{5}[4 \cosh(\omega t) + \cos 2\omega t].$$

[If O be the centre of the circle and P, where $AP = r$, is the position of the ring at time t when both OA and AP have revolved through an angle θ, $(= \omega t)$, in opposite directions, the acceleration of A is $a\omega^2$ along OA and the acceleration of P relative to A is $\ddot{r} - r\dot{\theta}^2$, by Art. 49, *i.e.* $\ddot{r} - r\omega^2$. Hence the total acceleration of P along AP is $\ddot{r} - r\omega^2 + a\omega^2 \cos 2\omega t$, and this is zero since the ring is smooth.]

15. PQ is a tangent at Q to a circle of radius a; PQ is equal to ρ and makes an angle θ with a fixed tangent to the circle; show that the accelerations of P along and perpendicular to QP are respectively

$$\ddot{\rho} - \rho\,\dot{\theta}^2 + a\,\ddot{\theta}, \text{ and } \frac{1}{\rho}\frac{d}{dt}(\rho^2\,\dot{\theta}) + a\,\dot{\theta}^2.$$

[The accelerations of Q along and perpendicular to QP are $a\,\ddot{\theta}$ and $a\,\dot{\theta}^2$; the accelerations of P relative to Q in these same directions are $\ddot{\rho} - \rho\,\dot{\theta}^2$ and $\dfrac{1}{\rho}\dfrac{d}{dt}(\rho^2\,\dot{\theta})$.]

16. Two particles, of masses m and m', connected by an elastic string of natural length a, are placed in a smooth tube of small bore which is made to rotate about a fixed point in its length with angular velocity ω. The coefficient of elasticity of the string is $2mm'a\omega^2 \div (m+m')$. Show that, if the particles are initially just at rest relative to the tube and the string is just taut, their distance apart at time t is

$$2a - a\cos\omega t.$$

17. A weight can slide along the spoke of a horizontal wheel, whose mass may be neglected to the centre of the wheel by means of a light spring ; when the wheel is fixed, the period of oscillation of the weight is $2\pi/n$. If the wheel is started to rotate freely with

angular velocity $6n\sqrt{11}/55$, prove that the greatest extension of the spring is one-fifth of its original length.

18. A uniform chain AB is placed in a straight tube OAB which revolves in a horizontal plane, about the fixed point O, with uniform angular velocity ω. Show that the motion of the middle point of the chain is the same as would be the motion of a particle placed at this middle point, and that the tension of the chain at any point P is $\frac{1}{2}m\omega^2.AP.PB$, where m is the mass of a unit length of the chain.

53. *A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane; to obtain the differential equation of its path.*

Referred to O as origin and a fixed straight line OX through as initial line, let the polar coordinates of P be (r, θ). If P be the acceleration of the particle directed towards O, we have, by Art. 49,

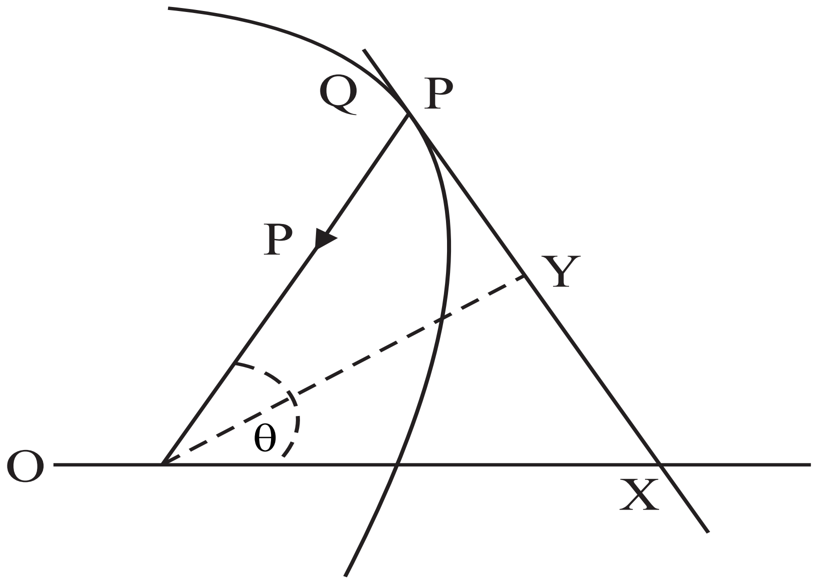
$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P. \qquad ...(1)$$

Also, since there is no acceleration perpendicular to OP, we have, by the same article,

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \qquad ...(2)$$

(2) gives $\quad r^2\frac{d\theta}{dt} = \text{ const.} = h \text{ (say)}. \qquad ...(3)$

$$\therefore \quad \frac{d\theta}{dt} = \frac{h}{r^2} = hu^2, \text{ if } u \text{ be equal to } \frac{1}{r}.$$



Then $\dfrac{dr}{dt}=\dfrac{d}{dt}\left(\dfrac{1}{u}\right)=-\dfrac{1}{u^2}\dfrac{du}{dt}=-\dfrac{1}{u^2}\dfrac{du}{d\theta}.\dfrac{d\theta}{dt}=-h\dfrac{du}{d\theta}$,

and $\dfrac{d^2r}{dt^2}=\dfrac{d}{dt}\left(-h\dfrac{du}{d\theta}\right)=-h\dfrac{d}{d\theta}\left(\dfrac{du}{d\theta}\right)\dfrac{d\theta}{dt}=-h^2u^2\dfrac{d^2u}{d\theta^2}$.

Hence equation (1) becomes

$$-h^2u^2\frac{d^2u}{d\theta^2}-\frac{1}{u}.h^2u^4=-P,\quad \text{i.e. } \frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2} \qquad ...(4).$$

Again, if p be the perpendicular from the origin O the tangent at P, we have

$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=u^2+\left(\frac{du}{d\theta}\right)^2.$$

Hence, differentiating with respect to θ, we have

$$-\frac{2}{p^3}\frac{dp}{d\theta}=2u\frac{du}{d\theta}+2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}.$$

$$\therefore\quad -\frac{1}{p^3}\frac{dp}{dr}=\left[u+\frac{d^2u}{d\theta^2}\right]\frac{du}{dr}=\left(u+\frac{du^2}{d\theta^2}\right)\left(-\frac{1}{r^2}\right).$$

$$\therefore\quad \frac{1}{u^2p^3}\frac{dp}{dr}=u+\frac{d^2u}{d\theta^2}$$

Hence (4) gives P

$$P = \frac{h^2}{p^3}\frac{dp}{dr} \qquad \text{...(5).}$$

Equation (4) gives the path in terms of r and θ, and (5) gives the (p, r) equation of the path.

54. *In every central orbit, the sectorial area traced out by the radius vector to the centre of force increases uniformly per unit of time, and the linear velocity varies inversely as the perpendicular from the centre upon the tangent to the path.*

Let Q be the position of the moving particle at time $t + \triangle t$, so that $\angle POQ = \triangle\theta$ and $OQ = r + \triangle r$.

$$\text{The area} \quad POQ = \frac{1}{2}OP.OQ.\sin POQ = \frac{1}{2}r(r+\triangle r)\sin\triangle\theta.$$

Hence the rate of description of sectorial area

$$= \lim_{\triangle t = 0} \frac{\frac{1}{2}r(r+\triangle r)\sin\triangle\theta}{\triangle t}$$

$$= \lim_{\triangle t = 0}\left[\frac{1}{2}r(r+\triangle r).\frac{\sin\triangle\theta}{\triangle\theta}.\frac{\triangle\theta}{\triangle t}\right]$$

$$= \frac{1}{2}r^2\frac{d\theta}{dt}, \text{ in the limit,}$$

$$= \text{ the constant } \frac{1}{2}h \text{ by equation (3) of the last article.}$$

The constant h is thus equal to twice the sectorial area described per unit of time.

Again, the sectorial area POQ = in the limit $\frac{1}{2}.PQ\times$ perpendicular from O on PQ, and

the rate of its description $= \lim\limits_{\triangle t=0} \frac{1}{2}.\frac{\triangle s}{\triangle t} \times$ Perpendicular from O on PQ.

Now, in the limit when Q is very close to P, $\frac{\triangle s}{\triangle t} =$ the velocity v, and the perpendicular from O on PQ

$=$ the perpendicular from O on the tangent at $P = p$.

$$\therefore \;\; h = v.p, \; i.e. \; v = \frac{h}{p}.$$

Hence, when a particle moves under a force to a fixed centre, its velocity at any point P of its path varies inversely as the perpendicular from the centre upon the tangent to the path at P.

Since $v = \frac{h}{p}$, and in any curve

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = u^2 + \left(\frac{du}{d\theta}\right)^2,$$

$$\therefore \;\; v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right].$$

55. *A particle moves in an ellipse under a force which is always directed towards its focus; to find the law of force, and the velocity at any point of its path.*

The equation to an ellipse referred to its focus is

$$r = \frac{l}{1+e\cos\theta}, \; i.e. \; u = \frac{1}{l} + \frac{e}{l}\cos\theta \qquad ...(1)$$

$$\therefore \;\; \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta.$$

Hence equation (4) of Art. 53 gives

$$P = h^2u^2\left[\frac{d^2u}{d\theta^2}+u\right] = \frac{h^2}{l}u^2 \qquad \text{...(2).}$$

The acceleration therefore varies inversely as the square of the distance of the moving particle from the focus and, if it be $\frac{\mu}{(\text{distance})^2}$, then (2) gives

$$h = \sqrt{\mu l} = \sqrt{\mu \times \text{ semi-latus-rectum}} \qquad \text{...(3).}$$

Also

$$v^2 = h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = h^2\left[\left(\frac{1}{l}+\frac{e}{l}\cos\theta\right)^2+\left(\frac{e}{l}\sin\theta\right)^2\right]$$

$$= \frac{\mu}{l}[1+2e\cos\theta+e^2] = \mu\left[2\frac{1+e\cos\theta}{l}-\frac{1-e^2}{l}\right]$$

$$= \mu\left[\frac{2}{r}-\frac{1}{a}\right], \text{ by (1)} \qquad \text{...(4),}$$

where $2a$ is the major axis of the ellipse.

It follows, since (4) depends only on the distance r, that the velocity at any point of the path depends only on the distance from the focus and that it is independent of the direction of the motion.

It also follows that the velocity V of projection from any point whose distance from the focus is r_o, must be less than $\frac{2\mu}{r_0}$, and that the a of the corresponding ellipse is given by

$$V^2 = \mu\left(\frac{2}{r_0}-\frac{1}{a}\right).$$

Periodic time. Since h is equal to twice the area described in a unit time, it follows, that if T be the time the particle takes to describe the whole arc of the ellipse, then

$$\frac{1}{2}h \times T = \text{Area of the ellipse } = \pi ab.$$

Also $\quad h = \sqrt{\mu \times \text{ Semi-latus-rectum }} = \sqrt{\mu \frac{b^2}{a}}.$

Hence $\quad T = \dfrac{2\pi ab}{h} = \dfrac{2\pi}{\sqrt{\mu}} a^{3/2}.$

56. EX. *Find the law of force towards the pole under which the curve* $r^n = a^n \cos n\theta$ *can be described.*

Here $\quad u^n a^n \cos n\theta = 1.$

Hence, taking the logarithmic differential, we obtain $\dfrac{du}{d\theta} = u \tan n\theta$.

$$\therefore \frac{d^2u}{d\theta^2} = \frac{du}{d\theta} \tan n\theta + nu \sec^2 n\theta = u[\tan^2 n\theta + n\sec^2 n\theta].$$

$$\therefore \frac{d^2u}{d\theta^2} + u = u(n+1)\sec^2 n\theta = (n+1)a^{2n}u^{2n+1}.$$

Hence equation (4) of Art. 53 gives $P = (n+1)h^2a^{2n}u^{2n+3}$, *i.e.* the curve can be described under a force to the pole varying inversely as the $(2n+3)$rd power of the distance.

Particular Cases **I.** Let $n = -\frac{1}{2}$, so that the equation to the curve is

$$r = \frac{a}{\cos^2 \dfrac{\theta}{2}} = \frac{2a}{1 + \cos\theta}$$

i.e. the curve is a parabola referred to its focus as pole.

Here $\quad P \propto \dfrac{1}{r^2}.$

II. Let $n = \frac{1}{2}$, so that the equation is $r = \frac{a}{2}(1 + \cos\theta)$, which is a cardioid.

Here $\quad P \propto \dfrac{1}{r^4}.$

III. Let $n = 1$, so that the equation to the curve is $r = a\cos\theta$, *i.e.* a circle with a point on its circumference as pole.

Here $\quad P \propto \dfrac{1}{r^5}.$

IV. Let $n = 2$, so that the curve is $r^2 = a^2\cos 2\theta$, *i.e.* a lemniscate of Bernouilli, and $P \propto \dfrac{1}{r^7}.$

V. Let $n = -2$, so that the curve is the rectangular hyperbola $a^2 = r^2\cos 2\theta$, the centre being pole, and $P \propto -r$, since in this case $(n+1)$ is negative. The force is therefore repulsive from the centre.

EXAMPLES

A particle describes the following curves under a force P to the pole, show that the force is as stated:

1. Equiangular spiral; $P \propto \dfrac{1}{r^3}.$
2. Lemniscate of Bernouilli; $P \propto \dfrac{1}{r^7}.$
3. Circle, pole on its circumference; $P \propto \dfrac{1}{r^5}.$
4. $\dfrac{a}{r} = e^{n\theta}, n\theta, \cosh n\theta,$ or $\sin n\theta$; $P \propto \dfrac{1}{r^3}.$
5. $r^n \cos n\theta = a^n$; $\qquad P \propto r^{2n-3}$
6. $r^n = A\cos n\theta + B\sin n\theta$; $P \propto \dfrac{1}{r^{2n+3}}.$
7. $r = a\sin n\theta$; $P \propto \dfrac{2n^2a^2}{r^5} - \dfrac{n-1}{r^3}.$
8. $au = \tanh\left(\dfrac{\theta}{\sqrt{2}}\right)$ or $\cosh\left(\dfrac{\theta}{\sqrt{2}}\right)$; $P \propto \dfrac{1}{r^5}$

9. $au = \dfrac{\cosh\theta - 2}{\cosh\theta + 1}$ or $\dfrac{\cosh\theta + 2}{\cosh\theta - 1}$; $P \propto \dfrac{1}{r^4}$

10. $a^2u^2 = \dfrac{\cosh 2\theta - 1}{\cosh 2\theta + 2}$ or $\dfrac{\cosh 2\theta + 1}{\cosh 2\theta - 2}$; $P \propto \dfrac{1}{r^7}$

11. Find the law of force to an internal point under which a body will describe a circle. Show that the hodograph of such motion is an ellipse.
[Use formula (5) of Art. 53. The hodograph of the path of a moving point P is obtained thus: From a fixed point O draw a straight line OQ parallel to, and proportional to, the velocity of P; the locus of the point Q, for the different positions of P, is the hodograph of the path of P.]

12. A particle of unit mass describes an equiangular spiral, of angle α, under a force which is always in a direction perpendicular to the straight line joining the particle to the pole of the spiral; show that the force is $\mu r^{2\sec^2\alpha - 3}$, and that the rate of description of sectorial area about the pole is

$$\frac{1}{2}\sqrt{\mu \sin\alpha . \cos .\alpha} . r^{\sec^2\alpha}.$$

13. In an orbit described under a force to a centre the velocity at any point is inversely proportional to the distance of the point from the centre of force; show that the path is an equiangular spiral.

14. The velocity at any point of a central orbit is $\dfrac{1}{n}$th of what it would be for a circular orbit at the same distance; show that the central force varies as $\dfrac{1}{r^{2n^2+1}}$ and that the equation to the orbit is

$$r^{n^2-1} = a^{n^2-1}\cos\{(n^2-1)\theta\}.$$

57. **Apses.** An apse is a point in a central orbit at which the radius vector drawn from the centre of force to the moving particle has a maximum or minimum value.

By the principles of the Differential Calculus u is a maximum or a minimum if $\dfrac{du}{d\theta}$ is zero, and if the first differential coefficient of u that does not vanish is of an even order.

If p be the perpendicular from the centre of force upon the tangent to the path at any point whose distance is r from the origin, then

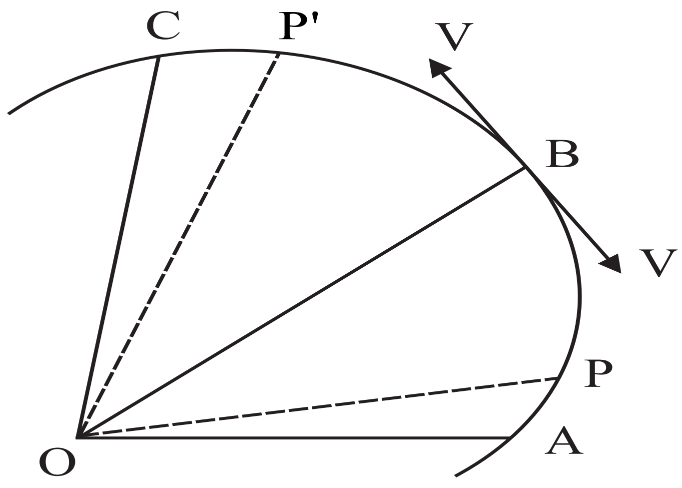
$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2.$$

When $\dfrac{du}{d\theta}$ is zero, $\dfrac{1}{p^2} = u^2 = \dfrac{1}{r^2}$, so that the perpendicular in the case of the apse is equal to the radius vector. Hence at an apse the particle is moving at right angles to the radius vector.

58. *When the central acceleration is a single-valued function of the distance (i.e. when the acceleration is a function of the distance only and is always the same at the same distance), every apse-line divides the orbit into two equal and similar portions and thus there can only be two apse-distances.*

Let ABC be a portion of the path having three consecutive apses A, B, and C and let O be the centre of force.

Let V be the velocity of the particle at B. Then, if the velocity of the particle were reversed at B, it would describe the path BPA. For, as the acceleration depends on the distance from O only, the velocity, by equations (1) and (3) of Art. 53, would depend only on the distance from O and not on the direction of the motion.



Again the original particle starting from B and the reversed particle, starting from B with equal velocity V, must describe similar paths. For the equations (1) and (3) of Art. 53, which do not depend on the direction of motion, show that the value of r and θ at any time t for the first particle (*i.e.* OP' and $\angle BOP'$) are equal to the same quantities at the same time t for the second particle (*i.e.* OP and $\angle BOP$).

Hence the curves $BP'C$ and BPA are exactly the same; either, by being rotated about the line OB, would give the other. Hence, since A and C are the points where the radius vector is perpendicular to the tangent, we have OA = OC.

Similarly, if D were the next apse after C, we should have OB and OD equal, and so on.

Thus there are only two different apse-distances.

The angle between any two consecutive apsidal distances is called the **apsidal angle**.

59. When the central acceleration varies as some integral power of the distance, say μu^n, it is easily seen analytically that there are at most two apsidal distances.

For the equation of motion is

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}=\frac{\mu}{h^2}u^{n-2}.$$

$$\therefore \quad \frac{h^2}{2}\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\frac{\mu}{n-1}u^{n-1}+\text{ Const.}$$

The particle is at an apse when $\frac{du}{d\theta}=0$ then this equation gives

$$u^{n-1}-\frac{n-1}{2}\frac{h^2}{\mu}u^2+C=0.$$

Whatever be the values of n or C this equation cannot have more than two changes of sign, and hence, by Descartes' Rule, it cannot have more than two positive roots.

60. *A particle moves with a central acceleration* $\frac{\mu}{(distance)^3}$*; to find the path and to distinguish the cases.*

The equation (4) of Art. 53 becomes

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2}u, \quad i.e. \quad \frac{d^2u}{d\theta^2}=\left(\frac{\mu}{h^2}-1\right)u \qquad ...(1).$$

Case I. Let $h^2<\mu$, so that $\frac{\mu}{h^2}-1$ is positive and equal to n^2, say.

The equation (1) is $\frac{d^2u}{d\theta^2}=n^2u$, the general solution of which is, as in Art. 29,

$$u=Ae^{n\theta}+Be^{-n\theta}=L\cosh n\theta+M\sinh n\theta,$$

where A,B or L,M are arbitrary constants.

This is a spiral curve with an infinite number of convolutions about the pole. In the particular case when A or B vanishes, it is an equiangular spiral.

Case II. Let $h^2 = \mu$, so that the equation (1) becomes

$$\frac{d^2u}{d\theta^2} = 0 \qquad \therefore\ u = A\theta + B = A(\theta - \alpha),$$ where A and α are arbitrary constants.

This represents a reciprocal spiral in general. In the particular case when A is zero, it is a circle.

Case III. Let $h^2 > \mu$ so that $\frac{\mu}{h^2} - 1$ is negative and equal to $-n^2$, say.

The equation (1) is therefore $\frac{d^2u}{d\theta^2} = -n^2u$, the solution of which is

$$u = A\cos(n\theta + B) = A\cos n(\theta - \alpha),$$

where A and α are arbitrary constants.

The apse is given $\theta = \alpha, u = A$.

61. The equations (4) or (5) of Art. 53 will give the path when P is given and also the initial conditions of projection.

EX. 1. *A particle moves with a central acceleration which varies inversely as the cube of the distance; if it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, show that the equation to its path is* $r\cos\frac{\theta}{\sqrt{2}} = a$.

Let the acceleration be μu^3.

If V_1 be the velocity in a circle of radius a with the same acceleration, then

$$\frac{V_1^2}{a} = \text{ normal acceleration } = \frac{\mu}{a^3}. \qquad \therefore\ V_1^2 = \frac{\mu}{a^2}.$$

Hence, if V be the velocity of projection in the required path,

$$V = \sqrt{2}V_1 = \frac{\sqrt{2\mu}}{a}.$$

The differential equation of the path is, from equation (4) of Art. 53,

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu u^3}{h^2u^2} = \frac{\mu}{h^2}u$$

Hence, multiplying by $\frac{du}{d\theta}$ and integrating, we have

$$\frac{1}{2}v^2 = \frac{1}{2}h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \frac{\mu}{2}u^2 + C \qquad ...(1).$$

The initial conditions give that when $u = \frac{1}{a}$, then $\frac{du}{d\theta} = 0$, and $v = \frac{\sqrt{2\mu}}{a}$.

Hence (1) gives

$$\frac{1}{2}.\frac{2\mu}{a^2} = \frac{1}{2}h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{2a^2} + C \qquad \therefore\ h^2 = 2\mu \text{ and } C = \frac{\mu}{2a^2}$$

$\therefore$ from equation (1) we have

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{u^2}{2} + \frac{1}{2a^2}. \qquad \therefore\ \frac{du}{d\theta} = \sqrt{\frac{1}{2}\left(\frac{1}{a^2} - u^2\right)} \qquad ...(2).$$

$$\therefore\ \frac{\theta}{\sqrt{2}} = \frac{a\,du}{\sqrt{1-a^2u^2}} = \sin^{-1} au + \gamma.$$

If θ be measured from the initial radius vector, then $\theta = 0$ when $u = \frac{1}{a}$, and therefore

$$\gamma = -\sin^{-1}(1) = -\frac{\pi}{2}.$$

$$\therefore\ au = \sin\left[\frac{\pi}{2} + \frac{\theta}{\sqrt{2}}\right] = \cos\frac{\theta}{\sqrt{2}}.$$

Hence the path is the curve $r\cos\dfrac{\theta}{\sqrt{2}}=a$.

If we take the negative sign on the right hand side of (2), we obtain the same result.

EX. 2. *A particle, subject to a force producing an acceleration* $\mu\dfrac{r+2a}{r^5}$ *towards the origin, is projected from the point* $(a,0)$ *with a velocity equal to the velocity from infinity at an angle* $\cot^{-1}2$ *with the initial line. Show that the equation to the path is* $r=a(1+2\sin\theta)$, *and find the apsidal angle and distances.*

The "velocity from infinity" means the velocity that would be acquired by the particle in falling with the given acceleration from infinity to the point under consideration. Hence if this velocity be V we have, as in Art. 22,

$$\frac{1}{2}V^2=\int_\infty^a -\mu\left[\frac{x+2a}{x^5}\right]dx=\mu\left[\frac{1}{3}\frac{1}{x^3}+\frac{1}{2}\frac{a}{x^4}\right]_\infty^a=\mu\left[\frac{1}{3a^3}+\frac{1}{2a^3}\right],$$

$$\text{so that } V^2=\frac{5\mu}{3a^3} \qquad \text{...(1)}.$$

The equation of motion of the particle is

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2u^2}[u^4+2au^5]=\frac{\mu}{h^2}[u^2+2au^3],$$

$$\therefore\ \frac{1}{2}v^2=\frac{h^2}{2}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left[\frac{u^3}{3}+\frac{1}{2}au^4\right]+C \qquad \text{...(2)}.$$

If p_0 be the perpendicular from the origin upon the initial direction of projection, we have $p_0=\sin\alpha$, where $\cot\alpha=2$, *i.e.* $p_0=\dfrac{a}{\sqrt{5}}$.

Hence, initially, we have

$$u^2+\left(\frac{du}{d\theta}\right)^2=\frac{1}{p_0^2}=\frac{5}{a^2} \qquad ...(3).$$

Hence (2) gives, initially, from (1) and (3)

$$\frac{5\mu}{6a^3}=\frac{h^2}{2}\times\frac{5}{a^2}=\mu\left[\frac{1}{3a^3}+\frac{1}{2a^3}\right]+C, \quad \text{so that } C=0 \text{ and } h^2=\frac{\mu}{3a}$$

From (2) we then have

$$\frac{\mu}{6a}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left[\frac{u^3}{3}+\frac{1}{2}au^4\right],$$

$$\text{i.e. } \left(\frac{du}{d\theta}\right)^2=u^2[2au+3a^2u^2-1]=u^2[au+1][3au-1].$$

On putting $u=\dfrac{1}{r}$, this equation gives

$$\left(\frac{dr}{d\theta}\right)^2=(a+r)(3a-r), \quad \text{and hence} \quad \theta=\int\frac{dr}{\sqrt{(a+r)(3a-r)}}.$$

Putting $r=a+y$, we have $\theta=\displaystyle\int\frac{dy}{\sqrt{4a^2-y^2}}=\sin^{-1}\frac{y}{2a}+\gamma.$

$$\therefore \quad \sin(\theta-\gamma)=\frac{y}{2a}=\frac{r-a}{2a}.$$

If we measure θ from the initial radius vector, then $\theta=0$ when $r=a$, and hence $\gamma=0$

Therefore the path is $r=a(1+2\sin\theta)$.

Clearly $\dfrac{dr}{d\theta}=0$, i.e. we have an apse, when $\theta=\dfrac{\pi}{2},\dfrac{3\pi}{2},\dfrac{5\pi}{2}$, etc.

Hence the apsidal angle is π and the apsidal distances are equal to $3a$ and a, and the apses are both on the positive directions of the axis of y at distances $3a$ and a from the origin. The path is a traced from its equation.

EXAMPLES

1. A particle moves under a central repulsive force $\left\{= \dfrac{m\mu}{(\text{ distance })^3}\right\}$, and is projected from an apse at a distance a with velocity V. Show that the equation to the path is $r \cos p\theta = a$, and that the angle θ described in time t is
$$\frac{1}{p}\tan^{-1}\left[\frac{pV}{a}t\right], \text{ where } p^2 = \frac{\mu + a^2V^2}{a^2V^2}.$$
2. A particle moves with a central acceleration, $\dfrac{\mu}{(\text{ distance })^5}$, and is projected from an apse at a distance a with a velocity equal to n times that which would be acquired in falling from infinity; show that the other apsidal distance is
$$\frac{a}{\sqrt{n^2-1}}.$$
If $n = 1$, and the particle be projected in any direction, show that the path is a circle passing through the centre of force.
3. A particle, moving with a central acceleration $\dfrac{\mu}{(\text{ distance })^3}$ is projected from an apse at a distance a with a velocity V; show that the path is
$$r\cosh\left[\frac{\sqrt{\mu - a^2V^2}}{aV}\theta\right] = a, \quad \text{or} \quad r\cos\left[\frac{\sqrt{a^2V^2-\mu}}{aV}\theta\right] = a,$$
according as V is $\lesseqgtr$ the velocity from infinity.
4. A particle moving under a constant force from a centre is projected in a direction perpendicular to the radius vector with the velocity acquired in falling to the point of projection from the centre. Show that its path is $\left(\dfrac{a}{r}\right)^3 = \cos^2\dfrac{3}{2}\theta$, and that the particle will ultimately

move in a straight line through the origin in the same way as if its path had always been this line.

If the velocity of projection be double that in the previous case, show that the path is

$$\frac{\theta}{2} = \tan^{-1}\sqrt{\frac{r-a}{a}} - \frac{1}{\sqrt{3}}\tan^{-1}\sqrt{\frac{r-a}{3a}}$$

5. A particle moves with a central acceleration $\mu\left(r+\frac{2a^3}{r^2}\right)$, being projected from an apse at a distance a with twice the velocity for a circle at that distance; find the other apsidal distance, and show that the equation to the path is

$$\frac{\theta}{2} = \tan^{-1}(t\sqrt{3}) - \frac{1}{\sqrt{5}}\tan^{-1}\left(\sqrt{\frac{5}{3}t}\right), \text{ where } t^2 = \frac{r-a}{3a-r}.$$

6. A particle moves with a central acceleration $\mu\left(r+\frac{a^4}{r^3}\right)$ being projected from an apse at distance a with a velocity $2\sqrt{\mu a}$; show that it describes the curve $r^2[2+\cos\sqrt{3}\theta] = 3a^2$.
7. A particle moves with a central acceleration $\mu(r^5 - c^4 r)$, being projected from an apse at distance c with a velocity $\sqrt{\frac{2\mu}{3}c^3}$, show that $x^4 + y^4 = c^4$.
8. A particle moves under a central force $m\lambda[3a^3u^4 + 8au^2]$; it is projected from an apse at a distance a from the centre of force with velocity $\sqrt{10}\lambda$; show that the second apsidal distance is half the first, and that the equation to the path is $2r = a\left[1+\sec h\frac{\theta}{\sqrt{5}}\right]$.
9. A particle describes an orbit with a central acceleration $\mu u^3 - \lambda\mu^5$, being projected from an apse at distance a with a velocity equal to

that from infinity; show that its path is

$$r = a\cosh\frac{\theta}{n}, \text{ where } n^2+1 = \frac{2\mu a^2}{\lambda}.$$

Prove also that it will be at distance r at the end of time

$$\sqrt{\frac{a^2}{2\lambda}}\left[a^2\log\frac{r+\sqrt{r^2-a^2}}{a}+r\sqrt{r^2-a^2}\right].$$

10. In a central orbit the force is $\mu u^3(3+2a^2u^2)$; if the particle be projected at a distance a with a velocity $\sqrt{\frac{5\mu}{a^2}}$ in a direction making $\tan^{-1}\frac{1}{2}$ with the radius, show that the equation to the path is $r = a\tan\theta$.
11. A particle moves under a force $m\mu[3au^4-2(a^2-b^2)u^5], a > b$, and is projected from an apse at a distance $a+b$ with velocity $\sqrt{\mu}\div(a+b)$; show that its orbit is
$$r = a+b\cos\theta.$$
12. A particle moves with a central acceleration $\lambda^2(8au^2+a^4u^5)$; it is projected with velocity 9λ from an apse at a distance $\frac{a}{3}$ from the origin; show that the equation to its path is
$$\frac{1}{\sqrt{3}}\sqrt{\frac{au+5}{au-3}} = \cot\frac{\theta}{\sqrt{6}}$$
13. A particle, subject to a central force per unit of mass equal to $\mu\{2(a^2+b^2)u^5-3a^2b^2u^7\}$, is projected at the distance a with a velocity $\frac{\sqrt{\mu}}{a}$ in a direction at right angles to the initial distance; show that the path is the curve $r^2 = a^2\cos^2\theta+b^2\sin^2\theta$.

14. A particle moves with a central acceleration $\mu\left(u^5 - \frac{a^2}{8}u^7\right)$; it is projected at a distance a with a velocity $\sqrt{\frac{25}{7}}$ times the velocity for a circle at that distance and at an inclination $\tan^{-1}\frac{4}{3}$ to the radius vector. Show that its path is the curve $4r^2 - a^2 = \frac{3a^2}{(1-\theta)^2}$.
15. A particle is acted on by a central repulsive force which varies as the nth power of the distance; if the velocity at any point of the path be equal to that which would be acquired in falling from the centre to the point, show that the equation to the path is of the form
$$r^{\frac{n+3}{2}} \cos\frac{n+3}{2}\theta = \text{ const.}$$
16. An elastic string, of natural length l, is tied to a particle at one end and is fixed at its other end to a point in a smooth horizontal table. The particle can move on the table and initially is at rest with the string straight and unstretched. A blow (which, if directed along the string would make the particle oscillate to a maximum distance $2l$ from the fixed end) is given to the particle in a direction inclined at an angle α to the string. Prove that the maximum length of the string during the ensuing motion is given by the greatest root of the equation
$$x^4 - 2lx^3 + l^4 \sin^2\alpha = 0.$$
17. A particle of mass m is attached to a fixed point by an elastic string of natural length a, the coefficient of elasticity being nmg; it is projected from an apse at a distance a with velocity $\sqrt{2pgh}$; show that the other apsidal distance is given by the equation
$$nr^2(r-a) - 2pha(r+a) = 0.$$

18. A particle acted on by a repulsive central force $\mu r \div (r^2 - 9c^2)^2$ is projected from an apse at a distance c with velocity $\sqrt{\dfrac{\mu}{8c^2}}$ show that it will describe a three-cusped hypocycloid and that the time to the cusp is $\dfrac{4}{3}\pi c^2\sqrt{\dfrac{2}{\mu}}$.
[Use equation (5) of Art. 53, and we have $8p^2 = 9c^2 - r^2$. Also $hdt = p.ds = pdr.\dfrac{r}{\sqrt{r^2 - p^2}}$, giving $ht = \int_c^{3c}\dfrac{rdr}{3}\sqrt{\dfrac{9c^2 - r^2}{r^2 - c^2}}$. To integrate, put $r^2 = c^2 + 8c^2\cos^2\phi$]
19. Find the path described about a fixed centre of force by a particle, when the acceleration toward the centre is of the form $\dfrac{\mu}{r^2} + \dfrac{\mu'}{r^3}$, in terms of the velocity V at an apse whose distance is a from the centre of force.
20. Show that the only law for a central attraction, for which the velocity in a circle at any distance is equal to the velocity acquired in falling from infinity to that distance, is that of the inverse cube.
21. A particle moves in a curve under a central attraction so that its velocity at any point is equal to that in a circle at the same distance and under the same attraction; show that the law of force is that of the inverse cube, and that the path is an equiangular spiral.
22. A particle moves under a central force $m\mu \div$ (distance)^n (where $n > 1$ but not = 3). If it be projected at a distance R in a direction making an angle β with the initial radius vector with a velocity equal to that due to a fall from infinity, show that the equation to the path is

$$r^{\frac{n-3}{2}}\sin\beta = R^{\frac{n-3}{2}}\sin\left(\frac{n-3}{2}\theta + \beta\right).$$

If $n > 3$ show that the maximum distance from the centre is $R co\sec^{\frac{2}{n-3}} \beta$, and if $n \overline{<} 3$ then the particle goes to infinity.

23. A particle moves with central acceleration $\mu u^2 + \nu u^3$ and the velocity of projection at distance R is V; show that the particle will ultimately go off to infinity if $V^2 > \dfrac{2\mu}{R} + \dfrac{\nu}{R^2}$.

24. A particle is projected from an apse at a distance a with a velocity $\dfrac{\sqrt{\mu+\lambda}}{a}$ and moves with a central attraction equal to $\dfrac{\mu}{2}(n-1)a^{n-3}r^{-n} + \lambda r^{-3}$, where $n > 3$, per unit of mass; show that it will arrive at the centre of force in time $\dfrac{a^2}{2}\sqrt{\dfrac{\pi}{\mu}}\Gamma\left(\dfrac{n+1}{2n-6}\right)/\Gamma\left(\dfrac{2}{n-3}\right)$.

25. In a central orbit if $P = \mu u^2 (cu + \cos\theta)^{-3}$, show that the path is one of the conics $(cu + \cos\theta)^2 = a + b\cos(2\theta + a)$.

26. A particle, of mass m, moves under an attractive force to the pole equal to $\dfrac{m\mu}{r^2}\sin^2\theta$. It is projected with velocity $\sqrt{\dfrac{2\mu}{3a}}$ from an apse at a distance a. Show that the equation to the orbit is $r(1+\cos^2\theta) = 2a$, and that the time of a complete revolution is $(3a)^{3/2} \times \dfrac{\pi}{\sqrt{\mu}}$.

27. If a particle move with a central acceleration $\dfrac{\mu}{r^2}(1+k^2\sin^2\theta)^{-3/2}$, find the orbit and interpret the result geometrically.

[Multiplying the equation of motion,
$h^2(\ddot{u}+u) = \mu(1+k^2\sin^2\theta)^{-3/2}$, by $\cos\theta$ and $\sin\theta$
in succession and integrating, we have

$$h^2(\dot{u}\cos\theta + u\sin\theta)$$
$$= \mu\sin\theta(1+k^2\sin^2\theta)^{-1/2} + A,$$

and

$$h^2(\dot{u}\sin\theta - u\cos\theta) = -\mu\cos\theta(1+k^2\sin^2\theta)^{-1/2} \div (1+k^2) + B.$$

Eliminating $\dot{u}$, we have
$h^2u = \mu(1+k^2\sin^2\theta)^{1/2} \div (1+k^2) + A\sin\theta - B\cos\theta$.]

28. A particle moves in a field of force whose potential is $\mu r^{-2}\cos\theta$ and it is projected at distance a perpendicular to the initial line with velocity $\frac{2}{a}\sqrt{\mu}$; show that the orbit described is

$$r = a\sec\left[\sqrt{2}\log\tan\frac{\pi+\theta}{4}\right].$$

29. A particle is describing a circle of radius a under the action of a constant force λ to the centre when suddenly the force is altered to $\lambda + \mu\sin nt$, where μ is small compared with λ and t is reckoned from the instant of change. Show that at any subsequent time t the distance of the particle from the centre of force is

$$a + \frac{\mu a}{3\lambda - an^2}\left[n\sqrt{\frac{a}{3\lambda}}\sin\left(t\sqrt{\frac{3\lambda}{a}}\right) - \sin nt\right].$$

What is the character of the motion if $3\lambda = an$?
[Use equations (1) and (2) of Art. 53; the second gives $r^2\dot{\theta} = \sqrt{\lambda a^3}$, and the first then becomes
$\ddot{r} - \frac{\lambda a^3}{r^3} = -\lambda - \mu\sin nt$. Put $r = a + \xi$ where ξ is small, and neglect squares of ξ.]

62. *A particle describes a path which is nearly a circle about a centre of force* $(= \mu u^n)$ *at its centre; find the condition that this may be a stable motion.*

The equation of motion is

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}u^{n-2} \qquad ...(1).$$

If the path is a circle of radius $\frac{1}{c}$, then $h^2 = \mu c^{n-3}$ ...(2).

Suppose the particle to be slightly displaced from the circular path in such a way that h remains unaltered (for example, suppose it is given a small additional velocity in a direction away from the centre of force by means of a blow, the perpendicular velocity being unaltered).

In (1) put $u = c + x$, where x is small; then it gives

$$\frac{d^2x}{d\theta^2} + c + x = \frac{(c+x)^{n-2}}{c^{n-3}} = c + (n-2)x + \cdots \qquad ...(3).$$

Neglecting squares and higher powers of x, *i.e.* assuming that x is always small, we have

$$\frac{d^2x}{d\theta^2} = -(3-n)x.$$

If n be < 3, so that $3-n$ is positive, this gives

$$x = A\cos[\sqrt{3-n}\theta + B].$$

If n be > 3, so that $n-3$ is positive, the solution is

$$x = A_1 e^{\sqrt{n-3}\theta} + B_1 e^{-\sqrt{n-3}\theta},$$

so that x continually increases as θ increases ; hence x is not always small and the orbit does not continue to be nearly circular.

If $n < 3$, the approximation to the path is

$u = c + A\cos[\sqrt{3-n}\theta + B]$ ...(4).

The apsidal distances are given by the equation $\frac{du}{d\theta} = 0$, *i.e.* by $0 = \sin[\sqrt{3-n}\theta + B]$.

The solutions of this equation are a series of angles, the difference between their successive values being $\dfrac{\pi}{\sqrt{3-n}}$. This is therefore the apsidal angle of the path.

If $n=3$, this apsidal angle is infinite. In this case it would be found that the motion is unstable, the particle departing from the circular path altogether and describing a spiral curve.

The maximum and minimum values of u, in the case $n<3$, are $c+A$ and $c-A$, so that the motion is included between these values.

63. The general case may be considered in the same manner. Let the central acceleration be $\phi(u)$.

The equations (1) and (2) then become

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2}.\frac{\phi(u)}{u^2} \qquad \text{...(5)},$$

$$\text{and} \quad h^2c^3=\mu\phi(c) \qquad \text{...(6)}.$$

Also (3) is now

$$\begin{aligned}\frac{d^2x}{d\theta^2}+c+x&=\frac{c^3}{\phi(c)}.\frac{\phi(c+x)}{(c+x)^2}\\&=\frac{c}{\phi(c)}[\phi(c)+x\phi'(c)+\cdots]\left[1-\frac{2x}{c}+\cdots\right]\\&=c-2x+x\frac{c\phi'(c)}{\phi(c)}, \text{ neglecting squares of } x.\\ \therefore \quad \frac{d^2x}{d\theta^2}&=-\left\{3-\frac{c\phi'(c)}{\phi(c)}\right\}x,\end{aligned}$$

and the motion is stable only if $\dfrac{c\phi'(c)}{\phi(c)}<3$.

In this case the apsidal angle is $\pi\div\left\{3-\dfrac{c\phi'(c)}{\phi(c)}\right\}^{1/2}$

64. If, in addition to the central acceleration P, we have an acceleration T perpendicular to P, the equations of motion are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P \qquad ...(1),$$

$$\text{and} \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = T \qquad ...(2).$$

Let $r^2\dfrac{d\theta}{dt} = h$. In this case h is not a constant.

$$\text{Then (2) gives } T = u\frac{dh}{dt} = u\frac{dh}{d\theta}.\frac{d\theta}{dt} = hu^3\frac{dh}{d\theta} \qquad ...(3),$$

$$\therefore \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} - \frac{1}{u^2}\frac{du}{d\theta}.hu^2 = -h\frac{du}{d\theta}$$

and $$\frac{d^2r}{dt^2} = -\frac{d}{d\theta}\left(h\frac{du}{d\theta}\right)\frac{d\theta}{dt} = -hu^2\left[h\frac{d^2u}{d\theta^2} + \frac{dh}{d\theta}\frac{du}{d\theta}\right]$$

$$= -h^2u^2\frac{d^2u}{d\theta^2} - \frac{T}{u}\frac{du}{d\theta}, \text{ by equation (3).}$$

Therefore (1) gives

$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{T}{u}\frac{du}{d\theta} - h^2u^3 = -P,$$

$$\text{i.e.} \quad \frac{d^2u}{d\theta^2} + u = \frac{P - \dfrac{T}{u}\dfrac{du}{d\theta}}{h^2u^2} \qquad ...(4).$$

This may also be written in the form

$$\frac{d}{d\theta}\left[\frac{\dfrac{P}{u^2} - \dfrac{T}{u^3}\dfrac{du}{d\theta}}{\dfrac{d^2u}{d\theta^2} + u}\right] = \frac{d}{d\theta}(h^2) = 2h\frac{dh}{d\theta} = \frac{2T}{u^3},$$

from equation (3).

EXAMPLES

1. One end of an elastic string, of unstretched length a, is tied to a point on the top of a smooth table, and a particle attached to the other end can move freely on the table. If the path be nearly a circle of radius b, show that its apsidal angle is approximately
$$\pi\sqrt{\frac{b-a}{4b-3a}}.$$
2. If the nearly circular orbit of a particle be $p^2(a^{m-2}-r^{m-2})=b^m$, show that the apsidal angle is $\dfrac{\pi}{\sqrt{m}}$ nearly.
[Using equation (5) of Art. 53 we see that P varies as r^{m-3}; the result then follows from Art. 62.]
3. A particle moves with a central acceleration $\dfrac{\mu}{r^2}-\dfrac{\lambda}{r^3}$; show that the apsidal angle is $\pi\div\sqrt{1+\dfrac{\lambda}{h^2}}$, where $\dfrac{h}{2}$ is the constant areal velocity.
4. Find the apsidal angle in a nearly circular orbit under the central force ar^m+br^n.
5. Assuming that the moon is acted on by a force $\dfrac{\mu}{(\text{distance})^2}$ to the earth and that the effect of the sun's disturbing force is to cause a force $m^2\times$ distance from the earth to the moon, show that, the orbit being nearly circular, the apsidal angle is $\pi\left(1+\dfrac{3}{2}\dfrac{m^2}{n^2}\right)$ nearly, where $\dfrac{2\pi}{n}$ is a mean lunar month, and cubes of m are neglected.
6. A particle is moving in an approximately circular orbit under the action of a central force $\dfrac{\mu}{r^2}$ and a small constant tangential retarda-

tion f; show that, if the mean distance be a, then $\theta = nt + \frac{3}{2}\frac{f}{a}t^2$, where $\mu = a^3n^2$ and the squares of f are neglected.

7. *Two particles of masses M and m are attached to the ends of an inextensible string which passes through a smooth fixed ring, the whole resting on a horizontal table. The particle m being projected at right angles to the string, show that its path is*

$$a = r\cos\left[\sqrt{\frac{m}{m+M}}\theta\right]$$

The tension of the string being T, the equations of motion are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{T}{M} \quad \text{...(1)},$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \quad \text{...(2)},$$

$$\text{and} \quad \frac{d^2}{dt^2}(l-r) = -\frac{T}{M} \quad \text{...(3)}.$$

(2) gives $$r^2\dot{\theta} = h \quad \text{...(4)},$$

and then (1) and (3) give $$\left(1+\frac{M}{m}\right)\ddot{r} = \frac{h^2}{r^3}.$$

$$\therefore \quad \left(1+\frac{M}{m}\right)\dot{r}^2 = -\frac{h^2}{r^2} + A = h^2\left(\frac{1}{a^2} - \frac{1}{r^2}\right),$$

since $\dot{r}$ is zero initially, when $r = a$.

This equation and (4) gives

$$\left(1+\frac{M}{m}\right)\left(\frac{dr}{d\theta}\right)^2 = \left(1+\frac{M}{m}\right)\dot{r}^2 \div \dot{\theta}^2 = \frac{r^2-a^2}{a^2}r^2.$$

$$\therefore \quad \theta \times \sqrt{\frac{M}{m}} = \int \frac{a\,dr}{r\sqrt{r^2-a^2}} = \cos^{-1}\frac{a}{r} + C,$$

and C vanishes if θ be measured from the initial radius vector.

$$\therefore \quad a = r\cos\left[\sqrt{\frac{m}{m+M}}\theta\right] \text{ is the path}$$

8. Two masses M, m are connected by a string which passes through a hole in a smooth horizontal plane, the mass m hanging vertically. Show that M describes on the plane a curve whose differential equation is $\left(1+\frac{M}{m}\right)\frac{d^2u}{d\theta^2}+u = \frac{mg}{M}\frac{1}{h^2u^2}$.
   Prove also that the tension of the string is $\frac{Mm}{M+m}(g+h^2u^3)$.
9. In the previous question if $m = M$, and the latter be projected on the plane with velocity $\sqrt{\frac{8ag}{3}}$ from an apse at a distance a, show that the former will rise through a distance a.
10. Two particles, of masses M and m, are connected by a light string; the string passes through a small hole in the table, m hangs vertically, and M describes a curve on the table which is very nearly a circle whose centre is the hole; show that the apsidal angle of the orbit of M is $\pi\sqrt{\frac{M+m}{3M}}$.
11. A particle of mass m can move on a smooth horizontal table. It is attached to a string which passes through a smooth hole in the table, goes under a small smooth pulley of mass M and is attached to a point in the under side of the table so that the parts of the string hang vertically. If the motion be slightly disturbed, when the mass m is describing a circle uniformly, so that the angular momentum is unchanged, show that the apsidal angle is $\pi\sqrt{\frac{M+4m}{12m}}$.
12. Two particles on a smooth horizontal table are attached by an elastic string, of natural length a, and are initially at rest at a distance a

apart. One particle is projected at right angles to the string. Show that if the greatest length of the string during the subsequent motion be $2a$, then the velocity of projection is $\sqrt{\dfrac{8a\lambda}{3m}}$, where m is the harmonic mean between the masses of the particles and λ is the modulus of elasticity of the string.

[Let the two particles be A and B of masses M and M', of which B is the one that is projected. When the connecting string is of length r and therefore of tension T, such that $T = \lambda \dfrac{r-a}{a}$, the acceleration of A is $\dfrac{T}{M}$ along AB, and that of B is $\dfrac{T}{M'}$, along BA. To get the relative motion we give to both B and A an acceleration equal and opposite to that of A. The latter is then "reduced to rest" and the acceleration of B relative to A is along BA and

$$= \frac{T}{M} + \frac{T}{M'} = \frac{2}{m}\lambda \frac{r-a}{a} = \frac{2\lambda}{ma} \frac{1-au}{u}.$$

The equation to the relative path of B is now $\dfrac{d^2u}{d\theta^2} + u = \dfrac{2\lambda}{mah^2} \dfrac{1-au}{u^3}$. Integrate and introduce the conditions that the particle is projected from an apse at a distance a with velocity V. The fact that there is another apse at a distance $2a$ determines V].

13. A particle is moving in a circular orbit, of radius a, under a force of intensity $\mu u^3(2a^2u^2 - 1)$ towards the centre. Show that the orbit is unstable and that if a slight disturbance takes place, inward or outward, the path may be represented by either $r = a \tanh\theta$ or $r = a \coth\theta$.

14. Einstein's discussion of planetary motion suggests the following problem:

    A particle moves in one place subject to an acceleration to a fixed

centre of magnitude $\mu\left(\frac{1}{r^2}+\frac{3h^2}{c^2r^4}\right)$, h being the moment of the velocity of the particle about the centre of acceleration, and c the velocity of light. Show that the angle between successive apse-lines is $\pi\left(1+\frac{3h^2}{c^2l^2}\right)$, $\frac{h}{cl}$ begin small, and l being the latus rectum of ellipse which the particle would describe with the same moment of momentum, if the law were $\frac{\mu}{r^2}$.
Supposing the planet Mercury to be subject to an acceleration of this type directed towards the Sun, show that its apse-line progresses at the rate of $42.9''$ per century, given that $l = 5.55\times 10^7$ kilometres, and $\frac{\mu}{c^2} = 1.47$ kilometres, and that the periodic time of Mercury is 87.97 days.

ANSWERS WITH HINTS

Art. 52 EXAMPLES.

2. The path is $r\cos\theta\left\{\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}^{1/n} = \text{const.}$ When $n = 1$ curve is $r = \frac{a}{1+\sin\theta}$

3. $-r\frac{v^2}{a^2}, \frac{2uv}{a},$

4. Path: $\frac{1}{r}+\frac{\lambda}{\mu}\log\theta = \text{const.}$

Art. 56 EXAMPLES.

11. A conic section of focus at the internal point.

Art. 61 EXAMPLES.

5. $v^2 = h^2(u^2+\dot{u}^2) = 2\mu\int\left(\frac{1}{u^3}+2a^3\right)$

Chapter 4

UNIPLANAR MOTION REFERRED TO POLAR COORDINATES CENTRAL FORCES

End of Art 47
EXAMPLES

1. Since the second ship, B, keeps the first ship, A, always abeam the velocity V must be perpendicular to AB. Hence, if AB be r and make an angle θ with the direction of v, we have

$$v \sin\theta - r\dot{\theta} = V, \text{ and } v\cos\theta + \dot{r} = 0.$$

$$\therefore \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{v\cos\theta}{V - v\sin\theta}; \quad \therefore \log r + \log(V - v\sin\theta) = \text{const.}$$

$\therefore r = \dfrac{C}{V - v\sin\theta}$, so that the path required is a conic section of focus A and eccentricity $\dfrac{v}{V}$.

2. Let $BP = r$ and $\angle ABP = \theta$. Then we have

$$\dot{r} = -u + nu\sin\theta \text{ and } r\dot{\theta} = nu\cos\theta.$$

$$\therefore \frac{dr}{r} = d\theta\,\frac{n\sin\theta - 1}{n\cos\theta}; \quad \therefore \log r = -\log\cos\theta - \frac{1}{n}\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \text{const.},$$

i.e. the path is $r\cos\theta\left\{\tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)\right\}^{\frac{1}{n}} = \text{const.}$

If $n = 1$, the curve is $r\cos\theta\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} = \text{const.}$, *i.e.* $r = \dfrac{a}{1+\sin\theta}$ etc.

3. Let P be the insect and C the centre of the wheel. Let CP be r and make an angle θ with the vertical, so that $v = a\dot{\theta}$.

The acceleration of C is zero, and $\dfrac{dr}{dt} = u$.

The acceleration of P along $CP = \ddot{r} - r\dot{\theta}^2 = -r\cdot\dfrac{v^2}{a^2}$.

The acceleration of P perpendicular to $CP = \dfrac{1}{r}\dfrac{d}{dt}(r^2\dot{\theta}) = 2\dot{r}\dfrac{v}{a} = \dfrac{2uv}{a}$.

4. $\dot{r} = \lambda r$, and $r\dot{\theta} = \mu\theta$; $\therefore \dfrac{dr}{r^2} = \dfrac{\lambda}{\mu}\dfrac{d\theta}{\theta}$. $\therefore \dfrac{1}{r} + \dfrac{\lambda}{\mu}\log\theta = \text{const.}$ is the path.

Also $$\ddot{r} - r\dot{\theta}^2 = \lambda\dot{r} - \frac{1}{r}\mu^2\theta^2 = \lambda^2 r - \frac{\mu^2}{r}\theta^2,$$

and $$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}\frac{d}{dt}(\mu r\theta) = \mu\left[\frac{\dot{r}}{r}\theta + \dot{\theta}\right] = \mu\theta\left(\lambda + \frac{\mu}{r}\right).$$

5. $\dot{\theta} = \omega$, and $\ddot{r} - r\dot{\theta}^2 = -f$; $\therefore \ddot{r} = \omega^2\left(r - \dfrac{f}{\omega^2}\right)$.

$$\therefore r - \frac{f}{\omega^2} = Ae^{\omega t} + Be^{-\omega t}, \text{ where } \frac{-f}{\omega^2} = A + B \text{ and } \frac{f}{\omega} = A\omega - B\omega.$$

$$\therefore r = \frac{f}{\omega^2}[1 - e^{-\omega t}]. \quad \text{Also } \theta = \omega t. \quad \text{Hence } r = \frac{f}{\omega^2}(1 - e^{-\theta}).$$

Also $\ddot{r} = -fe^{-\omega t}$, which is always negative but tends to the limit zero.

6. $\dot{\theta}=\frac{\lambda}{r}$. $\therefore\ \lambda=r\frac{d\theta}{dt}=V\sin\phi$. Hence ϕ is constant, and the curve is therefore an equiangular spiral whose pole is O.

The normal acceleration $=\frac{V^2}{\rho}=\frac{V^2}{r\frac{dr}{dp}}=\frac{V^2\sin\alpha}{r}$, i.e. $\propto\frac{1}{OP}$.

7. $r=ae^{\theta\cot\alpha}$, and $\dot{\theta}=\text{const.}=\omega$. $\therefore\ \dot{r}=ae^{\theta\cot\alpha}\times\dot{\theta}\cot\alpha=\omega r\cot\alpha$.

$$\therefore\ \ddot{r}-r\dot{\theta}^2=\omega\dot{r}\cot\alpha-\omega^2 r=\omega^2 r(\cot^2\alpha-1),$$

and
$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})=2\dot{r}\omega=2\omega^2 r\cot\alpha.$$

Hence, if PF be the direction of the resultant acceleration, and OP be produced to C, then

$$\tan FPC=\frac{2\cot\alpha}{\cot^2\alpha-1}=\tan 2\alpha,\ \text{i.e.}\ \angle FPC=2\angle TPC,\ \text{i.e.}\ \angle FPT=\angle TPC.$$

9. $\ddot{r}-r\omega^2=0$. $\therefore\ r=C\cosh\omega t+D\sinh\omega t$,

where
$$a=C\ \text{and}\ V=[\dot{r}]_{t=0}=D\omega.\ \therefore\ \text{etc.}$$

10. $\dot{\theta}=\omega$, and $\ddot{r}-r\omega^2=-g\sin\theta=-g\sin\omega t$.

$$\therefore\ r=A\cosh\omega t+B\sinh\omega t+\frac{g}{2\omega^2}\sin\omega t,$$

where $a=A$, and $0=[\dot{r}]_{t=0}=B\omega+\frac{g}{2\omega}$.

The particle therefore reaches O when

$$0=a\cosh\omega t-\frac{g}{2\omega^2}\sinh\omega t+\frac{g}{2\omega^2}\sin\omega t$$

$$=a\left(1+\frac{\omega^2t^2}{2}+\dots\right)-\frac{g}{2\omega^2}\left[\omega t+\frac{\omega^3t^3}{6}+\dots\right]+\frac{g}{2\omega^2}\left(\omega t-\frac{\omega^3t^3}{6}+\dots\right)$$

$$=a-\frac{g\omega t^3}{6},\ \text{on neglecting squares of}\ \omega,\ \text{i.e. when}\ t=\left(\frac{6a}{g\omega}\right)^{\frac{1}{3}}.$$

11. $\dot{\theta}=\omega$, and $\ddot{r}-r\omega^2=g\sin\theta=g\sin\omega t$.

$\therefore\ r=A\cosh\omega t+B\sinh\omega t-\frac{g}{2\omega^2}\sin\omega t$

$$=a\cosh\omega t+\frac{g}{2\omega^2}(\sinh\omega t-\sin\omega t),\ \text{since}\ r=a\ \text{and}\ \dot{r}=0\ \text{when}\ t=0.$$

Also
$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})=g\cos\theta-\frac{R}{m},\ \text{i.e.}\ \frac{R}{m}=g\cos\omega t-2\dot{r}\omega$$
$$=2g\cos\omega t-2a\omega^2\sinh\omega t-g\cosh\omega t.$$

Hence R vanishes, and the particle therefore leaves the plane, when

$$2g\cos\omega t=2a\omega^2\sinh\omega t+g\cosh\omega t.$$

12. $\theta=\omega t$, and $\ddot{r}-r\omega^2=-\mu r$.

If $\omega^2>\mu$, then $r=A\cosh\sqrt{\omega^2-\mu}\,t+B\sinh\sqrt{\omega^2-\mu}\,t$, where $a=A$, and $0=[\dot{r}]_{t=0}=B$, etc.

If $\omega^2<\mu$, then $r=C\cos\sqrt{\mu-\omega^2}\,t+D\sin\sqrt{\mu-\omega^2}\,t$, where $a=C$ and $0=[\dot{r}]_{t=0}=D$, etc.

If $\omega^2=\mu$, then $\ddot{r}=0$, $\dot{r}=\text{const.}=0$, and $r=a$, so that the path is a circle.

13. $\theta = \omega t$, $\ddot{r} - r\dot{\theta}^2 = -\dfrac{\mu R}{m}$, and $\dfrac{1}{r}\dfrac{d}{dt}(r^2\dot{\theta}) = \dfrac{R}{m}$. $\therefore\ \ddot{r} - r\omega^2 = -\mu \,.\, 2\omega\dot{r}$.

Put $r = Ae^{pt}$, so that $p^2 + 2\mu\omega p = \omega^2$, and hence

$$p = -\mu\omega \pm \omega\sqrt{1+\mu^2} = \omega\left[-\tan\epsilon \pm \sec\epsilon\right].$$

$$\therefore\ r = e^{-\omega\tan\epsilon \,.\, t}\left[A\cosh(\omega\sec\epsilon \,.\, t) + B\sinh(\omega\sec\epsilon \,.\, t)\right],$$

where $\qquad a = A$, and $0 = [\dot{r}]_{t=0} = -A\omega\tan\epsilon + B\omega\sec\epsilon$.

Hence $\quad r = ae^{-\omega\tan\epsilon \,.\, t}\left[\cosh(\omega\sec\epsilon \,.\, t) + \sin\epsilon\sinh(\omega\sec\epsilon \,.\, t)\right]$.

14. $\ddot{r} - r\omega^2 = -a\omega^2\cos 2\omega t$. $\quad\therefore\ r = A\cosh\omega t + B\sinh\omega t + \dfrac{a}{5}\cos 2\omega t$,

where $a = A + \dfrac{a}{5}$, and $0 = [\dot{r}]_{t=0} = B\omega$. Hence, etc.

16. At time t, if ξ and η are the distances of m and m' from the fixed point, we have

$$m(\ddot{\xi} - \xi\omega^2) = T, \text{ and } m'(\ddot{\eta} - \eta\omega^2) = -T.$$

Hence, if $\eta - \xi = x$, we have

$$\ddot{x} - x\omega^2 = -T\frac{m+m'}{mm'} = -\lambda\frac{m+m'}{mm'}\,\frac{x-a}{a} = -2\omega^2(x-a).$$

$$\therefore\ \ddot{x} = -\omega^2(x-2a), \text{ so that } x - 2a = A\cos\omega t + B\sin\omega t,$$

where $a - 2a = A$, and $0 = [\dot{x}]_{t=0} = B\omega$, etc.

17. The force to the centre $= n^2(r-l)$, since the time for free motion is $\dfrac{2\pi}{n}$.

$$\therefore\ \ddot{r} - r\dot{\theta}^2 = -n^2(r-l). \qquad \text{.......................(1)}$$

The reaction between the spoke and the weight is zero; for otherwise the angular acceleration of the wheel would be infinite, since its mass is zero. Hence $\dfrac{1}{r}\dfrac{d}{dt}(r^2\dot{\theta}) = 0$.

$$\therefore\ r^2\dot{\theta} = \text{const.} = l^2\omega = l^2 \,.\, 6n\frac{\sqrt{11}}{55}.$$

(1) then gives $\qquad \ddot{r} - \dfrac{l^4\omega^2}{r^3} = -n^2(r-l)$.

$\therefore\ \dot{r}^2 = -\dfrac{l^4\omega^2}{r^2} - n^2(r-l)^2 + l^2\omega^2$, and $\dot{r}$ is zero, *i.e.* r is a maximum, when

$$n^2(r-l)^2 = \omega^2 l^2\frac{r^2-l^2}{r^2},$$

i.e. when $\qquad n^2r^2(r-l) = \dfrac{36n^2\times 11}{55\times 55}l^2(r+l) = \dfrac{36n^2}{275}l^2(r+l)$,

and thus $r = \dfrac{6l}{5}$, the other roots being imaginary.

18. At time t, let $OA = x$ and let ξ be the distance of any element from A. Since $\theta = \omega t$, we have

$$m\,d\xi\left[\frac{d^2}{dt^2}(x+\xi) - (x+\xi)\omega^2\right] = T + dT - T = dT. \qquad \text{.........(1)}$$

Integrating this from $\xi = 0$ to $\xi = 2l$, we have

$$\ddot{x} \,.\, 2l - 2lx\omega^2 - \omega^2\frac{4l^2}{2} = \left[\frac{T}{m}\right]_{\xi=0}^{\xi=2l} = 0, \text{ since } \ddot{\xi} = 0,$$

i.e. $\ddot{x} = (x+l)\omega^2$, which is the equation of motion of a mass placed at the middle point of the string.

Also (1) gives $\qquad \dfrac{1}{m}\dfrac{dT}{d\xi} = \ddot{x} - x\omega^2 - \xi\omega^2 = \omega^2(l-\xi)$.

$$\therefore\ T = \omega^2 m\left(l\xi - \frac{\xi^2}{2}\right) = \frac{m\omega^2}{2} \,.\, AP \,.\, PB.$$

End of Art 56
EXAMPLES

1. $p = r \sin \alpha$. $\quad \therefore P = \dfrac{h^2}{p^3}\dfrac{dp}{dr} = \dfrac{h^2}{\sin^2\alpha}\cdot\dfrac{1}{r^3}$.

2. $r^2 = a^2 \cos 2\theta$. $\quad \therefore au = (\cos 2\theta)^{-\frac{1}{2}}$, $a\dot{u} = (\cos 2\theta)^{-\frac{3}{2}} \sin 2\theta$,

and
$$a\ddot{u} = 3(\cos 2\theta)^{-\frac{5}{2}} - (\cos 2\theta)^{-\frac{1}{2}} = 3a^5u^5 - au.$$
$$\therefore P = h^2u^2[\ddot{u} + u] = 3h^2a^4u^7, \text{ i.e. } P \propto \frac{1}{r^7}.$$

3. $r = a \cos\theta$. $\quad \therefore au = (\cos\theta)^{-1}$, $a\dot{u} = (\cos\theta)^{-2}\sin\theta$.
$$\therefore a\ddot{u} + au = 2(\cos\theta)^{-3}\sin^2\theta + (\cos\theta)^{-1} + (\cos\theta)^{-1} = 2(\cos\theta)^{-3} = 2a^3u^3.$$
$$\therefore P = h^2u^2(\ddot{u} + u) = 2a^2h^2u^5, \text{ i.e. } P \propto \frac{1}{r^5}.$$

4. $au = e^{n\theta}$, $n\theta$, $\cosh n\theta$, or $\sin n\theta$.
$$a\ddot{u} = n^2e^{n\theta},\ 0,\ n^2\cosh n\theta, \text{ or } -n^2\sin n\theta$$
$$= n^2au,\ 0,\ n^2au, \text{ or } -n^2au.$$
$$\therefore \frac{P}{h^2u^2} = (1+n^2)u,\ u,\ (1+n^2)u, \text{ or } (1-n^2)u. \text{ Hence, etc.}$$

5. $au = (\cos n\theta)^{\frac{1}{n}}$; $\quad a\ddot{u} = (\cos n\theta)^{\frac{1}{n}-2} \times (1-n)\sin^2 n\theta - n(\cos n\theta)^{\frac{1}{n}}$
$$= (1-n)(\cos n\theta)^{\frac{1}{n}-2} - (\cos n\theta)^{\frac{1}{n}}.$$
$$\therefore \frac{P}{h^2u^2} = u + \ddot{u} = (1-n)(au)^{1-2n}.$$
$$\therefore P \propto u^{3-2n}, \text{ i.e. } \propto r^{2n-3}.$$

6. $r^n = C\cos(n\theta - \alpha)$; $\quad \therefore \dfrac{n}{r}\dfrac{dr}{d\theta} = -n\tan(n\theta - \alpha)$.
$$\therefore \tan\phi = \frac{r\,d\theta}{dr} = -\cot(n\theta - \alpha), \text{ so that } \phi = \frac{\pi}{2} + n\theta - \alpha.$$
$$\therefore p = r\sin\phi = r\cos(n\theta - \alpha) = \frac{r^{n+1}}{C}.$$
$$\therefore P = \frac{h^2}{p^3}\frac{dp}{dr}, \text{ i.e. } \propto \frac{1}{r^{2n+3}}.$$

7. $r = a\sin n\theta$; $\quad \therefore \dot{u} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta} = -\dfrac{n\cos n\theta}{a\sin^2 n\theta}$.
$$\therefore \ddot{u} = \frac{n^2}{a}\left[\frac{2}{\sin^3 n\theta} - \frac{1}{\sin n\theta}\right] = n^2[2a^2u^3 - u].$$
$$\therefore \frac{P}{h^2u^2} = 2n^2a^2u^3 + (1-n^2)u, \text{ i.e. } P \propto \frac{2n^2a^2}{r^5} - \frac{n^2-1}{r^3}.$$

8. $au=\tanh\frac{\theta}{\sqrt{2}}$ or $\coth\frac{\theta}{\sqrt{2}}$. $\therefore\ a\dot{u}=\frac{1}{\sqrt{2}}\operatorname{sech}^2\frac{\theta}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}\operatorname{cosech}^2\frac{\theta}{\sqrt{2}}$.

$\therefore\ a\ddot{u}=\operatorname{sech}\frac{\theta}{\sqrt{2}}\left(-\operatorname{sech}\frac{\theta}{\sqrt{2}}\tanh\frac{\theta}{\sqrt{2}}\right)$, or $-\operatorname{cosech}\frac{\theta}{\sqrt{2}}\left(-\operatorname{cosech}\frac{\theta}{\sqrt{2}}\coth\frac{\theta}{\sqrt{2}}\right)$

$$=-\operatorname{sech}^2\frac{\theta}{\sqrt{2}}\tanh\frac{\theta}{\sqrt{2}},\ \text{or}\ \operatorname{cosech}^2\frac{\theta}{\sqrt{2}}\coth\frac{\theta}{\sqrt{2}}.$$

$$\therefore\ a(\ddot{u}+u)=\tanh^3\frac{\theta}{\sqrt{2}},\ \text{or}\ \coth^3\frac{\theta}{\sqrt{2}}=a^3u^3,\ \text{in each case.}$$

$$\therefore\ P\propto u^5.$$

9. $au=1-\frac{3}{1+\cosh\theta}$; $a\dot{u}=\frac{3\sinh\theta}{(1+\cosh\theta)^2}$;

$$a\ddot{u}=\frac{3\cosh\theta}{(1+\cosh\theta)^2}-\frac{6\sinh^2\theta}{(1+\cosh\theta)^3}=\frac{3\cosh\theta}{(1+\cosh\theta)^2}-6\frac{\cosh\theta-1}{(1+\cosh\theta)^2}.$$

$$\therefore\ a(\ddot{u}+u)=\frac{6-3\cosh\theta}{(1+\cosh\theta)^2}+\frac{\cosh\theta-2}{\cosh\theta+1}=\left(\frac{\cosh\theta-2}{\cosh\theta+1}\right)^2=a^2u^2.$$

$\therefore\ P\propto u^4$. Similarly for the second case.

10. $a^2u^2=\frac{\cosh 2\theta-1}{\cosh 2\theta+2}=1-\frac{3}{\cosh 2\theta+2}$.

$$\therefore\ 2a^2u\dot{u}=2\cdot\frac{3}{(\cosh 2\theta+2)^2}\sinh 2\theta.$$

$$\therefore\ a\dot{u}=\frac{3\sqrt{\cosh^2 2\theta-1}}{(\cosh 2\theta+2)^2}\cdot\sqrt{\frac{\cosh 2\theta+2}{\cosh 2\theta-1}}=3\frac{\sqrt{\cosh 2\theta+1}}{(\cosh 2\theta+2)^{\frac{3}{2}}}.$$

$$\therefore\ a^2(u^2+\dot{u}^2)=\frac{9(\cosh 2\theta+1)}{(\cosh 2\theta+2)^3}+\frac{\cosh 2\theta-1}{\cosh 2\theta+2}$$

$$=\frac{(\cosh 2\theta-1)^3+2(\cosh 2\theta+2)^3}{3(\cosh 2\theta+2)^3}=\frac{2}{3}+\frac{1}{3}a^6u^6.$$

$$\therefore\ 2a^2(u+\ddot{u})=2a^6u^5.\quad \therefore\ P=h^2u^2(u+\ddot{u})\propto u^7.$$

Similarly for the second case.

11. O the internal point, C the centre, $OC=c$, $OP=r$, p the perpendicular on the tangent at P and $\angle PCX=\theta$. Then

$$p=a+c\cos\theta,\ \text{and}\ r^2=a^2+c^2+2ac\cos\theta=c^2-a^2+2ap.$$

$$\therefore\ P=\frac{h^2}{p^3}\frac{dp}{dr}=\frac{h^2}{p^3}\cdot\frac{r}{a}=\frac{8h^2a^2\cdot r}{(r^2+a^2-c^2)^3}.$$

The locus of Q is given by

$$r=\lambda V=\lambda\frac{h}{p}=\frac{K}{a+c\cos\theta}=\frac{K}{a+c\sin XOQ},$$

i.e. a conic section of focus O.

12. $r = ae^{\theta \cot \alpha}$; $\ddot{r} - r\dot{\theta}^2 = 0$; and $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = P$.

These give $\qquad \dot{\theta} \cot \alpha = \frac{\dot{r}}{r}$, and $\therefore \ddot{r} = \frac{\dot{r}^2}{r} \tan^2 \alpha$.

$$\therefore \log \dot{r} = \tan^2 \alpha \,.\, \log r + \text{const., } i.e. \ \dot{r} = Ar^{\tan^2 \alpha}.$$

$$\therefore r^2\dot{\theta} = r\dot{r} \tan \alpha = A \tan \alpha \, r^{\sec^2 \alpha}. \quad \text{..................(1)}$$

$$P = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{A \tan \alpha}{r}\frac{d}{dt}(r^{\sec^2 \alpha})$$

$$= A \tan \alpha \sec^2 \alpha r^{\sec^2 \alpha - 2} \,.\, \dot{r} = A^2 \tan \alpha \sec^2 \alpha \,.\, r^{2\sec^2 \alpha - 3}$$

$$= \mu r^{2\sec^2 \alpha - 3}, \text{ where } A^2 = \mu \cos^2 \alpha \cot \alpha,$$

and hence $\qquad A \tan \alpha = \sqrt{\mu \sin \alpha \cos \alpha}$.

13. $v = \frac{\mu}{r}$, and always $v = \frac{h}{p}$.

$\therefore p \propto r$, so that the path is an equiangular spiral.

14. Given $P = \frac{n^2 v^2}{r}$, so that

$$u^2 h^2 [u^2 + \dot{u}^2] = \frac{P}{u}. \quad \text{.........................(1)}$$

$$\therefore 2u^2 h^2 [u + \ddot{u}] = \frac{1}{u}\frac{dP}{du} - \frac{P}{u^2},$$

i.e. $\qquad 2n^2 \,.\, \frac{P}{u^2} = \frac{1}{u}\frac{dP}{du} - \frac{P}{u^2}$, so that $\frac{dP}{du} = \frac{P}{u}(1 + 2n^2)$.

$$\therefore \log P = (1 + 2n^2) \log u + \text{const., } i.e. \ P \propto u^{2n^2+1}.$$

(1) then gives $\qquad \left(\frac{du}{d\theta}\right)^2 = Au^{2n^2} - u^2$.

$$\therefore \theta = \int \frac{du}{\sqrt{Au^{2n^2} - u^2}} = -\int \frac{r^{n^2-2}\,dr}{\sqrt{A - r^{2n^2-2}}} = -\frac{1}{(n^2-1)} \cos^{-1} \frac{r^{n^2-1}}{\sqrt{A}} + \text{const.},$$

i.e. $r^{n^2-1} = a^{n^2-1} \cos \{(n^2-1)\theta\}$, if the initial line is properly chosen.

End of Art 61

EXAMPLES

1. $v^2 = h^2[u^2 + \dot{u}^2] = -2\int \frac{P}{u^2}\,du = -\mu u^2 + C$,

where $\qquad V^2 = \frac{h^2}{a^2} = -\frac{\mu}{a^2} + C$, i.e. $C = p^2 V^2$.

$$\frac{du}{d\theta} = -\frac{p}{a}\sqrt{1 - a^2u^2}, \quad \therefore p\theta = \cos^{-1} au, \ i.e. \ r \cos p\theta = a.$$

Also $\qquad r^2 \frac{d\theta}{dt} = h = Va. \quad \therefore t = \frac{a}{V}\int_0^\theta \frac{d\theta}{\cos^2 p\theta} = \frac{a}{pV} \tan p\theta$, etc.

2. $v^2 = h^2[u^2+\dot{u}^2] = \int \mu u^5\,du = \frac{\mu}{2}u^4 + A,$ ....(1)

where $$\frac{n^2\mu}{2a^4} = \frac{h^2}{a^2} = \frac{\mu}{2a^4} + A,$$

$\left[\text{since the square of the velocity in falling from infinity} = \int_\infty^a \frac{-2\mu}{x^5}\,dx = \frac{\mu}{2a^4}\right].$

Hence (1) gives $$\frac{n^2}{a^2}\dot{u}^2 = \frac{(a^2u^2-1)(a^2u^2+1-n^2)}{a^4}.$$ ....(2)

The other apsidal distance is therefore given by

$$a^2u^2 = n^2-1, \text{ i.e. it is } \frac{a}{\sqrt{n^2-1}}.$$

When $n=1$, and $A=0$, then (1) gives

$$\left(\frac{du}{d\theta}\right)^2 = u^2[b^2u^2-1], \text{ where } \frac{\mu}{2h^2} = b^2.$$

$$\therefore \theta = \int \frac{du}{u\sqrt{b^2u^2-1}} = -\int\frac{dr}{\sqrt{b^2-r^2}} = -\sin^{-1}\frac{r}{b} + \text{const.},$$

i.e. $r = -a\sin(\theta-\gamma)$, a circle passing through the origin.

3. $v^2 = h^2(u^2+\dot{u}^2) = 2\int \mu u\,du = \mu u^2 + C$, where $V^2 = \frac{h^2}{a^2} = \frac{\mu}{a^2} + C.$

$$\therefore V^2a^2\dot{u}^2 = \left(\frac{\mu}{a^2} - V^2\right)(a^2u^2-1).$$

(1) Let $V < \sqrt{\frac{\mu}{a^2}}$, the velocity from infinity,

$$\therefore \frac{\sqrt{\mu - a^2V^2}}{Va^2}\theta = \int\frac{du}{\sqrt{a^2u^2-1}} = \frac{1}{a}\cosh^{-1}(au), \text{ since } u = \frac{1}{a} \text{ when } \theta = 0.$$

$$\therefore r\cosh\left(\frac{\sqrt{\mu-a^2V^2}}{aV}\theta\right) = a.$$

(2) Let $V > \sqrt{\frac{\mu}{a^2}}$. We then have

$$\frac{\sqrt{a^2V^2-\mu}}{Va}\theta = \int\frac{a\,du}{\sqrt{1-a^2u^2}} = \sin^{-1}au + D,$$

where $0 = \frac{\pi}{2} + D$, so that $\frac{a}{r} = \cos\left[\frac{\sqrt{a^2V^2-\mu}}{Va}\theta\right].$

4. $v^2 = h^2(u^2+\dot{u}^2) = -2\int\frac{\mu}{u^2} = \frac{2\mu}{u} + C$, where $2\mu a = \frac{h^2}{a^2} = 2\mu a + C.$

$$\therefore \frac{du}{d\theta} = \sqrt{\frac{1-a^3u^3}{a^3u}}.$$

$$\therefore \theta = \int\frac{du\,.\,a^{\frac{3}{2}}u^{\frac{1}{2}}}{\sqrt{1-a^3u^3}} = \frac{2}{3}\cos^{-1}(a^{\frac{3}{2}}u^{\frac{3}{2}}), \text{ i.e. } a^3u^3 = \cos^2\frac{3\theta}{2}.$$

In the second case, $V^2=8\mu a$, so that $h^2=8\mu a^3$ and $C=6\mu a$. Hence

$$8a^3(u^2+\dot{u}^2)=\frac{2}{u}+6a.$$

$$\therefore\ 8a^3\left[\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right]=2r+6a,\ \text{i.e.}\ 4a^3\left(\frac{dr}{d\theta}\right)^2=r^3(r-a)(r+2a)^2.$$

$$\therefore\ 16\left(\frac{dt}{d\theta}\right)^2=(1+t^2)^2(3+t^2)^2,\ \text{where}\ r=a(1+t^2).$$

$$\therefore\ \frac{\theta}{2}=\int\frac{2dt}{(1+t^2)(3+t^2)}=\int\left(\frac{1}{1+t^2}-\frac{1}{3+t^2}\right)dt$$

$$=\tan^{-1}t-\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}=\tan^{-1}\sqrt{\frac{r-a}{a}}-\frac{1}{\sqrt{3}}\tan^{-1}\sqrt{\frac{r-a}{3a}},$$

since $r=a$ when $\theta=0$.

5. $v^2=h^2(u^2+\dot{u}^2)=2\mu\int\left(\frac{1}{u^3}+2a^5\right)=\mu\left[-\frac{1}{u^2}+4a^5u\right]+C,$

where $$4\cdot 3\mu a\cdot a=\frac{h^2}{a^2}=\mu\cdot 3a^2+C.$$

$$\therefore\ 12a^4\dot{u}^2=\frac{1}{u^2}\left[-12a^4u^4+4a^3u^3+9a^2u^2-1\right]=\frac{1}{u^2}(1-au)(3au-1)(1+2au)^2.$$

$$\therefore\ 12a^4\left(\frac{dr}{d\theta}\right)^2=r^2(r-a)(3a-r)(r+2a)^2.\quad\ldots\ldots\ldots\ldots(1)$$

Put $r-a=(3a-r)t^2$, and hence $dr(1+t^2)=2(3a-r)t\,dt$, and (1) becomes

$$\frac{\theta}{2\sqrt{3}}=\int\frac{2(1+t^2)}{(1+3t^2)(3+5t^2)}dt=\int\left[\frac{1}{1+3t^2}-\frac{1}{3+5t^2}\right]dt$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}t)-\frac{1}{\sqrt{15}}\tan^{-1}\left(\sqrt{\frac{5}{3}}\,t\right),\ \text{etc.}$$

6. $v^2=h^2(u^2+\dot{u}^2)=\int 2\mu\left(\frac{1}{u^3}+a^4u\right)=\mu\left(-\frac{1}{u^2}+a^4u^2\right)+A,$

where $$4\mu a^3=\frac{h^2}{a^2}=A.$$

Hence $$4a^4\dot{u}^2=4a^2-\frac{1}{u^2}-3a^4u^2=\frac{(3a^2u^2-1)(1-a^2u^2)}{u^2}.$$

$$\therefore\ \frac{\theta}{2}=\int\frac{a^2u\,du}{\sqrt{(3a^2u^2-1)(1-a^2u^2)}}\qquad\left[\text{Put } a^2u^2=\cos^2\phi+\tfrac{1}{3}\sin^2\phi=\tfrac{1}{3}(2+\cos 2\phi).\right]$$

$$=-\int\frac{1}{\sqrt{3}}d\phi=-\frac{\phi}{\sqrt{3}},\ \text{since}\ \phi=0\ \text{when}\ u=\frac{1}{a}\ \text{and}\ \theta=0.$$

$$\therefore\ \cos\sqrt{3}\theta=\cos 2\phi=3a^2u^2-2.$$

7. $v^2=h^2(u^2+\dot{u}^2)=\int 2\mu\left(\frac{1}{u^4}-\frac{c^4}{u^2}\right)du=\mu\left[-\frac{1}{3u^3}+\frac{c^4}{u^3}\right]+C,$

where $$\frac{2\mu}{3}c^3=\frac{h^2}{c^2}=\mu\left[-\frac{1}{3}c^6+c^6\right]+C.$$

$$\therefore\ \dot{u}^2=\frac{-1+3c^4u^4-2c^8u^8}{2c^8u^6}=\frac{(1-c^4u^4)(2c^4u^4-1)}{2c^8u^6}.$$

$$\therefore\ 4\theta=\int\frac{4\sqrt{2}\,c^4u^3\,du}{\sqrt{(1-c^4u^4)(2c^4u^4-1)}}\qquad\left[\text{Put } c^4u^4=\cos^2\phi+\tfrac{1}{2}\sin^2\phi=\frac{3+\cos 2\phi}{4}.\right]$$

$$=-\int 2\,d\phi=-2\phi,\ \text{since } \phi=0 \text{ when } u=\frac{1}{c} \text{ and } \theta=0.$$

$$\therefore\ \cos 4\theta=\cos 2\phi=4c^4u^4-3.$$

$$\therefore\ c^4u^4=\frac{3+\cos 4\theta}{4}=\cos^4\theta+\sin^4\theta.\quad \therefore\ x^4+y^4=c^4.$$

8. $v^2=h^2(u^2+\dot{u}^2)=\int 2\lambda\,(3a^2u^2+8a)\,du=\lambda\,(2a^2u^3+16au)+C,$

where
$$10\lambda=\frac{h^2}{a^2}=18\lambda+C.$$

$$\therefore\ 5a^2\dot{u}^2=a^3u^3-5a^2u^2+8au-4=(au-1)\,(2-au)^2.$$

$$\therefore\ \frac{\theta}{\sqrt{5}}=\int\frac{a\,.\,du}{(2-au)\sqrt{au-1}}\qquad[\text{Put } au=1+t^2.]$$

$$=\int\frac{2\,dt}{1-t^2}=2\tanh^{-1}t.$$

$$\therefore\ au=1+\tanh^2\frac{\theta}{2\sqrt{5}}=1+\frac{\cosh\frac{\theta}{\sqrt{5}}-1}{\cosh\frac{\theta}{\sqrt{5}}+1}=\frac{2\cosh\frac{\theta}{\sqrt{5}}}{1+\cosh\frac{\theta}{\sqrt{5}}},\ \text{etc.}$$

9. $v^2=h^2(u^2+\dot{u}^2)=2\int(\mu u-\lambda u^3)\,du=\mu u^2-\frac{\lambda}{2}u^4+C,$

where
$$\frac{\mu}{a^2}-\frac{\lambda}{2a^4}=\frac{h^2}{a^2}=\frac{\mu}{a^2}-\frac{\lambda}{2a^4}+C.$$

$$\therefore\ (2a^2\mu-\lambda)\,\dot{u}^2=\lambda u^2-\lambda a^2u^4.$$

$$\therefore\ \frac{\theta}{n}=-\int\frac{du}{u\sqrt{1-a^2u^2}}=\int\frac{dr}{\sqrt{r^2-a^2}}=\cosh^{-1}\frac{r}{a}+B.$$

$$\therefore\ r=a\cosh\frac{\theta}{n},\ \text{since } r=a \text{ when } \theta=0.$$

Also
$$r^2\frac{d\theta}{dt}=h=\sqrt{\frac{2\mu a^2-\lambda}{2a^2}}=\sqrt{\frac{n^2\lambda}{2a^2}}.$$

$$\therefore\ t\sqrt{\frac{\lambda}{2a^2}}=\int\frac{r^2\,dr}{\sqrt{r^2-a^2}}=\int_a^r\left(\sqrt{r^2-a^2}+\frac{a^2}{\sqrt{r^2-a^2}}\right)dr$$

$$=\frac{r\sqrt{r^2-a^2}}{2}+\frac{a^2}{2}\log\left(r+\sqrt{r^2-a^2}\right)-\frac{a^2}{2}\log a,\ \text{etc.}$$

10. $v^2=\frac{h^2}{p^2}=h^2(u^2+\dot{u}^2)=\int 2\mu\,(3u+2a^2u^3)\,du=\mu\left[3u^2+a^2u^4\right]+C,$

where
$$\frac{5\mu}{a^2}=\frac{h^2}{a^4\cdot\frac{1}{5}}=\mu\left[\frac{3}{a^2}+\frac{1}{a^2}\right]+C.$$

$$\therefore\ a^2\dot{u}^2=a^4u^4+2a^2u^2+1=(a^2u^2+1)^2.$$

$$\therefore\ \frac{\theta}{a}=-\int\frac{du}{1+a^2u^2}=-\frac{1}{a}\tan^{-1}au+\text{const.}$$

$$\therefore \frac{a}{r} = \tan(\gamma - \theta) = \tan\left(\frac{\pi}{4} - \theta\right), \text{ since } r = a \text{ when } \theta = 0.$$

$\therefore r = a\tan\left(\frac{\pi}{4} + \theta\right)$, *i.e.* the given result if the initial line is turned through $\frac{\pi}{4}$.

11. $v^2 = h^2(u^2 + \dot{u}^2) = \int 2\mu\,[3au^2 - 2(a^2 - b^2)u^3]\,du$

$$= \mu\,[2au^3 - (a^2 - b^2)u^4] + C,$$

where
$$\frac{\mu}{(a+b)^2} = \frac{h^2}{(a+b)^2} = \mu\left[\frac{2a}{(a+b)^3} - \frac{a-b}{(a+b)^3}\right] + C.$$

$$\therefore \dot{u}^2 = -u^2 + 2au^3 - (a^2 - b^2)u^4.$$

$$\therefore \left(\frac{dr}{d\theta}\right)^2 = -r^2 + 2ar - (a^2 - b^2) = b^2 - (r-a)^2.$$

$$\therefore \theta = \int \frac{dr}{\sqrt{b^2 - (r-a)^2}} = \sin^{-1}\frac{r-a}{b} + D,$$

where $0 = \sin^{-1} 1 + D$, and $\therefore r = a + b\cos\theta$.

12. $v^2 = h^2(u^2 + \dot{u}^2) = \int 2\lambda^2(8a + a^4u^3)\,du = \lambda^2\left[16au + \frac{a^4u^4}{2}\right] + C,$

where
$$81\lambda^2 = \frac{h^2 \times 9}{a^2} = \lambda^2\left[48 + \frac{81}{2}\right] + C.$$

$$\therefore a^2\dot{u}^2 = \frac{a^4u^4 - 18a^2u^2 + 32au - 15}{18} = \frac{(au-1)^2(au+5)(au-3)}{18}.$$

$$\therefore \frac{\theta}{\sqrt{18}} = \int \frac{a\,du}{(au-1)\sqrt{(au+5)(au-3)}} \qquad [\text{Put } au + 5 = (au-3)y^2.]$$

$$= -\int \frac{dy}{y^2+3} = -\frac{1}{\sqrt{3}}\tan^{-1}\frac{y}{\sqrt{3}} + \gamma.$$

$$\therefore \sqrt{\frac{au+5}{au-3}} = y = \sqrt{3}\tan\left(\gamma - \frac{\theta}{\sqrt{6}}\right) = \sqrt{3}\cot\frac{\theta}{\sqrt{6}},$$

since $u = \frac{3}{a}$ when $\theta = 0$, and hence $\gamma = \frac{\pi}{2}$.

13. $v^2 = h^2(u^2 + \dot{u}^2) = 2\mu\int[2(a^2 + b^2)u^3 - 3a^2b^2u^5]\,du$

$$= \mu\,[(a^2 + b^2)u^4 - a^2b^2u^6] + A,$$

where
$$\frac{\mu}{a^2} = \frac{h^2}{a^2} = \mu\left[\frac{1}{a^2}\right] + A.$$

$$\therefore \dot{u}^2 = (a^2 + b^2)u^4 - a^2b^2u^6 - u^2.$$

$$\therefore \left(\frac{dr}{d\theta}\right)^2 = -\frac{(a^2 - r^2)(b^2 - r^2)}{r^2}.$$

$$\therefore \theta = \int \frac{r\,dr}{\sqrt{(a^2 - r^2)(r^2 - b^2)}} \qquad [\text{Put } r^2 = a^2\cos^2\phi + b^2\sin^2\phi.]$$

$$= \int \frac{(b^2 - a^2)\sin\phi\cos\phi\,d\phi}{(a^2 - b^2)\sin\phi\cos\phi} = -\phi + B.$$

But $\theta = 0$ when $r = a$, *i.e.* when $\phi = 0$, and $\therefore B = 0$, and

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta.$$

14. $v^2 = h^2(u^2+\dot{u}^2) = \int 2\mu\left(u^2 - \frac{a^2}{8}u^5\right) - \mu\left(\frac{u^4}{2} - \frac{a^2u^6}{24}\right) + C,$

where $$\frac{25}{7} \times \frac{7}{8}\frac{\mu}{a^4} = \frac{h^2}{a^2}\left(\frac{5}{4}\right)^2 - \frac{11\mu}{24a^4} + C.$$

$$\therefore \dot{u}^2 = \frac{1}{48a^2}\left[-a^6u^6 + 12a^4u^4 - 48a^2u^2 + 64\right] = -\frac{1}{48a^2}(a^2u^2-4)^3.$$

$$\therefore \left(\frac{dr}{d\theta}\right)^2 = \frac{(4r^2-a^2)^3}{48a^2r^2}, \text{ and } \frac{\theta}{\sqrt{3}} = \int \frac{4ar\,dr}{(4r^2-a^2)^{\frac{3}{2}}} = -\frac{a}{\sqrt{4r^2-a^2}} + \frac{1}{\sqrt{3}}, \text{ etc.}$$

15. $v^2 = h^2(u^2+\dot{u}^2) = -\int 2\mu u^{-n-2}\,du = \frac{2\mu}{n+1}u^{-n-1} + C,$

where $$\frac{2\mu}{n+1}a^{n+1} = \frac{2\mu}{n+1}a^{n+1} + C.$$

$$\therefore \dot{u}^2 = -u^2 + \lambda^2 u^{-n-1}, \text{ where } \lambda^2 = 2\mu/(n+1)h^2.$$

$$\therefore \theta = -\int \frac{u^{\frac{n+1}{2}}\,du}{\sqrt{\lambda^2 - u^{n+3}}} = \frac{2}{n+3}\cos^{-1}\frac{u^{\frac{n+3}{2}}}{\lambda} + B,$$

so that the equation is as given.

16. $P = mV$, where $\frac{1}{2}mV^2 = (2l-l)\cdot\frac{1}{2}\lambda\frac{2l-l}{l} = \frac{l\lambda}{2}$.

$$\therefore P = \sqrt{ml\lambda} \text{ and } V = \sqrt{\frac{l\lambda}{m}}.$$

The equations of motion are $\ddot{r} - r\dot{\theta}^2 = -\frac{T}{m} = -\frac{\lambda}{m}\frac{r-l}{l}$ and $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$, *i.e.* $r^2\dot{\theta} = \text{const.} = l \cdot V\sin\alpha$.

$$\therefore \ddot{r} - \frac{l^2V^2\sin^2\alpha}{r^3} = -\frac{\lambda}{m}\frac{r-l}{l} = -\frac{V^2}{l^2}(r-l).$$

$$\therefore \dot{r}^2 + \frac{l^2V^2\sin^2\alpha}{r^2} = -\frac{V^2}{l^2}(r-l)^2 + A,$$

where $$V^2\cos^2\alpha + V^2\sin^2\alpha = 0 + A, \text{ i.e. } A = V^2.$$

Hence $\dot{r}$ is zero again, when

$$l^4\sin^2\alpha = -r^2(r-l)^2 + r^2l^2 = -r^4 + 2lr^3.$$

17. $v^2 = H^2(u^2+\dot{u}^2) = \int \frac{2}{u^2}\cdot\frac{\lambda}{m}\cdot\frac{r-a}{a}\,du = \int \frac{2ng}{a}\frac{1-au}{u^3}$

$$= \frac{ng}{a}\left[-\frac{1}{u^2} + \frac{2a}{u}\right] + C,$$

where $$2pgh = \frac{H^2}{a^2} = \frac{ng}{a}[a^2] + C.$$

Hence $\dot{u}$ is again zero when

$$2pa^2gh\cdot u^2 = \frac{ng}{a}\left[-\frac{1}{u^2} + \frac{2a}{u}\right] + 2pgh - nga,$$

i.e. when $$2pgh(a^2-r^2) = -\frac{ng}{a}[r^2-ar]^2, \text{ etc.}$$

18. $\dfrac{h^2}{p^3}\dfrac{dp}{dr} = P = -\dfrac{\mu r}{(r^2-9c^2)^2}$.

$$v^2 = \frac{h^2}{p^2} = \frac{-\mu}{r^2-9c^2} + A, \text{ where } \frac{\mu}{8c^2} = \frac{h^2}{c^2} = \frac{\mu}{8c^2} + A. \quad \therefore 8p^2 = 9c^2 - r^2.$$

This is the equation to a hypocycloid formed by the rolling of a circle of radius c upon a circle of radius $3c$.

19. $v^2 = h^2(u^2 + \dot{u}^2) = \int 2(\mu + \mu' u)\,du = 2\mu u + \mu' u^2 + C$,

where
$$V^2 = \frac{h^2}{a^2} = \frac{2\mu}{a} + \frac{\mu'}{a^2} + C.$$

$$\therefore V^2 a^4 \dot{u}^2 = (\mu' - a^2V^2)(a^2u^2 - 1) + 2\mu a(au - 1).$$

First let $\mu' - a^2V^2$ be positive, and $= \dfrac{a\mu}{p^2}$.

$$\therefore \frac{p^2V^2a^2}{\mu}\dot{u}^2 = (au+p^2)^2 - (p^2+1)^2.$$

$$\therefore \frac{\theta}{pV}\sqrt{\frac{\mu}{a}} = \int \frac{a\,du}{\sqrt{(au+p^2)^2 - (p^2+1)^2}} = \cosh^{-1}\frac{au+p^2}{1+p^2},$$

since $au = 1$ when $\theta = 0$.

$$\therefore au + p^2 = (1+p^2)\cosh\left[\frac{\theta}{pV}\sqrt{\frac{\mu}{a}}\right].$$

Secondly, let $\mu' - a^2V^2$ be negative, and equal to $-\dfrac{a\mu}{q^2}$. Then

$$\frac{q^2V^2a^2}{\mu}\dot{u}^2 = (1-q^2)^2 - (au-q^2)^2,$$

and hence $\dfrac{\theta}{qV}\sqrt{\dfrac{\mu}{a}} = \cos^{-1}\dfrac{au-q^2}{1-q^2}$, *i.e.* $au - q^2 = (1-q^2)\cos\left[\dfrac{\theta}{qV}\sqrt{\dfrac{\mu}{a}}\right]$.

20. Let the law be $\phi'(r)$.

$$\therefore r\phi'(r) = (\text{vel. in a circle})^2 = -\int_\infty^r 2\phi'(r)\,dr = -2\phi(r) + A.$$

$$\therefore 2r\phi(r) + r^2\phi'(r) = Ar. \quad \therefore r^2\phi(r) = \frac{Ar^2}{2} + C.$$

$$\therefore \phi(r) = \frac{A}{2} + \frac{C}{r^2}, \text{ and } \phi'(r) = -\frac{C}{r^3}.$$

21. If the law of force is $u^2\phi'(u)$, then

$$v^2 = h^2(u^2 + \dot{u}^2) = 2\phi(u) + A, \quad\text{.......................(1)}$$

where always
$$\frac{v^2}{r} = u^2\phi'(u), \text{ i.e. } v^2 = u\phi'(u).$$

$$\therefore u\phi'(u) = 2\phi(u) + A, \text{ so that } \frac{1}{u^2}\frac{d\phi}{du} - \frac{2}{u^3}\phi(u) = \frac{A}{u^3}.$$

$$\therefore \frac{1}{u^2}\phi(u) = -\frac{A}{2u^2} + \text{const., and hence } \phi(u) = Bu^2 - \frac{A}{2}.$$

$$\therefore u^2\phi'(u) = 2Bu^3 = \text{the law of force.}$$

(1) gives $\dfrac{h^2}{p^2} = \dfrac{2B}{r^2}$, *i.e.* $p \propto r$, so that the path is an equiangular spiral.

22. Let $n = 3 + 2\lambda$, so that $v^2 = h^2(u^2 + \dot{u}^2) = \int 2\mu u^{1+2\lambda} = \dfrac{\mu}{\lambda+1} u^{2+2\lambda} + A$,

where $$\frac{\mu}{\lambda+1}\,\frac{1}{R^{2+2\lambda}} = \frac{h^2}{R^2 \sin^2\beta} = \frac{\mu}{\lambda+1}\,\frac{1}{R^{2+2\lambda}} + A.$$

$$\therefore\ \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{R^{2\lambda} - r^{2\lambda}\sin^2\beta}{\sin^2\beta\, r^{2+2\lambda}},$$

and hence $$\theta = \int \frac{r^{\lambda-1}\sin\beta\, dr}{\sqrt{R^{2\lambda} - r^{2\lambda}\sin^2\beta}} = \frac{1}{\lambda}\sin^{-1}\left[\left(\frac{r}{R}\right)^{\lambda}\sin\beta\right] - \frac{\beta}{\lambda},$$

since $r = R$, when $\theta = 0$.

$$\therefore\ r^{\lambda}\sin\beta = R^{\lambda}\sin(\lambda\theta + \beta).$$

The greatest value of r is when $\dot{r} = 0$, and then $\cos(\lambda\theta + \beta) = 0$, and $r = R(\operatorname{cosec}\beta)^{\frac{1}{\lambda}}$, if λ is positive and $\therefore n > 3$.

If $n < 3$, λ is negative $(= -\mu^2$ say$)$. Then $r^{\mu^2}\sin(\beta - \mu^2\theta) = R^{\mu^2}\sin\beta$, and r is infinite when $\theta = \dfrac{\beta}{\mu^2}$.

If $n = 3$, $\lambda = 0$, and the original equation gives $\dot{u}^2 = u^2\cot^2\beta$. Hence $\log u = -\theta\cot\beta + \text{const.}$, *i.e.* the path is the equiangular spiral $r = Re^{\theta\cot\beta}$.

23. $v^2 = h^2(u^2 + \dot{u}^2) = \int(2\mu + 2\nu u)\,du = 2\mu u + \nu u^2 + A$,

where $$V^2 = \frac{2\mu}{R} + \frac{\nu}{R^2} + A.$$

If $\nu > h^2$, $(= h^2 + p^2)$, this gives

$$\frac{p\theta}{h} = \int \frac{p\,du}{\sqrt{p^2u^2 + 2\mu u + A}} = \log\left[pu + \frac{\mu}{p} + \sqrt{p^2u^2 + 2\mu u + A}\right] + \text{const.}$$

Hence $u = 0$, *i.e.* the particle is at infinity, when

$$\frac{p(\theta - \gamma)}{h} = \log\left[\frac{\mu}{p} + \sqrt{A}\right],$$

which always gives a real value for θ, if A be positive, *i.e.* if $V^2 > \dfrac{2\mu}{R} + \dfrac{\nu}{R^2}$.

If $\nu < h^2$, $(= h^2 - q^2)$, we have

$$\frac{q\theta}{h} = \int \frac{q\,du}{\sqrt{A + 2\mu u - q^2u^2}} = \sin^{-1}\frac{q^2u - \mu}{\sqrt{Aq^2 + \mu^2}} + \text{const.}$$

Hence $u = 0$, when $\quad -\mu = \sqrt{Aq^2 + \mu^2}\,\sin\dfrac{q(\theta - \gamma)}{h}$,

which again gives a real value for θ, if A is positive.

24. $v^2 = h^2(u^2 + \dot{u}^2) = \int[\mu(n-1)a^{n-3}u^{n-2} + 2\lambda u]\,du = \mu a^{n-3}u^{n-1} + \lambda u^2 + C$,

where $$\frac{\lambda + \mu}{a^2} = \frac{h^2}{a^2} = \frac{\mu + \lambda}{a^2} + C.$$

$$\therefore\ (\lambda + \mu)\,\dot{u}^2 = \mu(a^{n-3}u^{n-1} - u^2).$$

Now $$\sqrt{\lambda+\mu}=h=r^2\frac{d\theta}{dt}=\frac{1}{u^2}\frac{d\theta}{dt},$$

so that $$\sqrt{\mu}\,.\,T=\int_{\frac{1}{a}}^{0}\frac{\sqrt{\mu}}{\sqrt{\lambda+\mu}}\cdot\frac{d\theta}{u^2}=\int_{\frac{1}{a}}^{\infty}\frac{du}{u^2\sqrt{a^{n-3}u^{n-1}-u^2}}\quad\left[\text{Put } au=\eta^{-\frac{1}{n-3}}.\right]$$

$$=\int_0^1\frac{a^2}{n-3}\frac{\eta^{\frac{7-n}{2(n-3)}}}{(1-\eta)^{\frac{1}{2}}}\,d\eta=\frac{a^2}{n-3}\,B\left[\frac{n+1}{2n-6},\,\frac{1}{2}\right]$$

$$=\frac{a^2}{n-3}\,\frac{\Gamma\left(\frac{n+1}{2n-6}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{n-3}\right)}=\frac{a^2\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2n-6}\right)}{\Gamma\left(\frac{2}{n-3}\right)}.$$

25. $h^2(\ddot{u}+u)=\mu(cu+\cos\theta)^{-3}$.

Put $cu+\cos\theta=\xi$, so that $c\ddot{u}-\cos\theta=\ddot{\xi}$, and hence $h^2(\ddot{\xi}+\xi)=\frac{\mu c}{\xi^3}$.

$\therefore\ \dot{\xi}^2+\xi^2=-\frac{\lambda}{\xi^2}+2A$, where $\lambda=\frac{\mu c}{h^2}$.

$$\therefore\ \theta=\int\frac{\xi\,d\xi}{\sqrt{A^2-\lambda-(\xi^2-A)^2}}=\frac{1}{2}\sin^{-1}\frac{\xi^2-A}{\sqrt{A^2-\lambda}}+\text{const.}$$

$$\therefore\ (cu+\cos\theta)^2=\xi^2=A+\sqrt{A^2-\lambda}\,\sin(2\theta+\text{const.}),\ \text{etc.}$$

26. $h^2(\ddot{u}+u)=\mu\sin^2\theta$.

$$\therefore\ u=A\cos(\theta+B)+\frac{\mu}{2h^2}\frac{1}{D^2+1}(1-\cos 2\theta)$$
$$=A\cos(\theta+B)+\frac{\mu}{2h^2}\left(1+\frac{1}{3}\cos 2\theta\right),$$

where $$\frac{1}{a}=A\cos B+\frac{2\mu}{3h^2};\quad 0=\left(\frac{du}{d\theta}\right)_0=-A\sin B,$$

and $$\frac{2\mu}{3a}=\left(\frac{h^2}{p^2}\right)_{\theta=0}=\frac{h^2}{a^2};\ \text{so that } h^2=\frac{2\mu a}{3},\ A=0,\ \text{and } B=0.$$

Hence $$u=\frac{1}{2a}(1+\cos^2\theta),\ i.e.\ r(1+\cos^2\theta)=2a.$$

Also $$\sqrt{\frac{2\mu a}{3}}=h=r^2\frac{d\theta}{dt}=\frac{4a^2}{(1+\cos^2\theta)^2}\frac{d\theta}{dt}.$$

$$\therefore\ \frac{T}{4a^2}\sqrt{\frac{2\mu a}{3}}=\int_0^{2\pi}\frac{d\theta}{(1+\cos^2\theta)^2}\qquad[\text{Put }\tan\theta=\sqrt{2}\tan\phi.]$$

$$=\frac{\sqrt{2}}{4}\int_0^{2\pi}(1+\sin^2\phi)\,d\phi=\sqrt{2}\left[\frac{\pi}{2}+\frac{1}{2}\cdot\frac{\pi}{2}\right]=\frac{3\pi\sqrt{2}}{4},\ \text{etc.}$$

27. The curve is $h^2=\mu\cos\alpha\sqrt{y^2+x^2\cos^2\alpha}+Ay-Bx$, where $\tan\alpha=k$.

Put $y=Y\cos\alpha$, *i.e.* let this curve be the projection of a second curve in a plane through the axis of x inclined at α to the plane of xy. This second curve is then

$$h^2=\mu\cos^2\alpha\sqrt{x^2+Y^2}+AY\cos\alpha-Bx=(\mu\cos^2\alpha+A\cos\alpha\sin\theta-B\cos\theta)\,r,$$

i.e. it is of the form $$r=\frac{l}{1-e\cos(\theta-\beta)}.$$

The required curve is thus the projection of the orbit of a particle moving in an ellipse in a plane inclined at $\tan^{-1}k$ to the plane of reference.

28. $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{d}{dr}\left[\frac{\mu\cos\theta}{r^2}\right] = -\frac{2\mu\cos\theta}{r^3}$, ...(1)

and $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - \frac{1}{r}\frac{d}{d\theta}\left(\frac{\mu\cos\theta}{r^2}\right) = -\frac{\mu\sin\theta}{r^3}$ ....(2)

(2) gives $\left(r^2\frac{d\theta}{dt}\right)^2 = 2\mu\cos\theta + A = 2\mu(1+\cos\theta)$. $\therefore r^2\frac{d\theta}{dt} = 2\sqrt{\mu}\cos\frac{\theta}{2}$.

Now $\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -2\sqrt{\mu}\frac{du}{d\theta}\cos\frac{\theta}{2}$,

and $\frac{d^2r}{dt^2} = -2\sqrt{\mu}\frac{d}{d\theta}\left[\frac{du}{d\theta}\cos\frac{\theta}{2}\right]\cdot\frac{d\theta}{dt} = -4\mu\cdot u^2\cos\frac{\theta}{2}\left[\frac{d^2u}{d\theta^2}\cos\frac{\theta}{2} - \frac{1}{2}\sin\frac{\theta}{2}\frac{du}{d\theta}\right]$.

Hence (1) becomes

$2(1+\cos\theta)\frac{d^2u}{d\theta^2} - \frac{du}{d\theta}\cdot\sin\theta + 2u = 0$. $\therefore \left(\frac{du}{d\theta}\right)^2(1+\cos\theta) + u^2 = \text{const.} = \frac{1}{a^2}$.

$\therefore -\frac{a\,du}{\sqrt{1-a^2u^2}} = \frac{d\theta}{\sqrt{2}\cos\frac{\theta}{2}}$, $\therefore \sin^{-1}(au) - \frac{\pi}{2} = \sqrt{2}\log\tan\left(\frac{\pi}{4}+\frac{\theta}{4}\right)$, etc.

29. The equation becomes $\ddot{\xi} = -\frac{3\lambda\xi}{a} - \mu\sin nt$. ....(1)

$\therefore r = a + \xi = a + A\sin\sqrt{\frac{3\lambda}{a}}\,t + B\cos\sqrt{\frac{3\lambda}{a}}\,t - \frac{\mu a}{3\lambda - an^2}\sin nt$

= the given result, since $r = a$ and $\dot{r} = 0$, when $t = 0$.

If $3\lambda = an^2$ equation (1) becomes $(D^2+n^2)\xi = -\mu\sin nt$. Hence, as in the Appendix, page v,

$\xi = A\sin nt + B\cos nt + \frac{\mu}{2n}t\cos nt$. Hence $r = a - \frac{\mu}{2n^2}\sin nt + \frac{\mu}{2n}t\cos nt$,

from the initial conditions, so that the amplitude of the oscillations continually increases.

End of Art 64 EXAMPLES

1. $h^2(\ddot{u}+u) = \frac{\lambda}{mu^2}\frac{r-a}{a} = \frac{\lambda}{ma}\left[\frac{1}{u^3} - \frac{a}{u^2}\right]$. ....(1)

Let b be the undisturbed radius of the path, so that $\frac{h^2}{b} = \frac{\lambda}{ma}(b^3 - ab^2)$.

On putting $u = \frac{1}{b} + x$, where x is small, we have

$(b-a)\ddot{x} = -(b-a)\frac{1+xb}{b} + \frac{1}{b^2}\left[b^3(1+xb)^{-3} - ab^2(1+xb)^{-2}\right] = -x(4b-3a)$

so that the apsidal angle is $\pi\sqrt{\frac{b-a}{4b-3a}}$.

3. $h^2(\ddot{u}+u) = \mu - \lambda u$. Put $u = x + \frac{\mu}{h^2+\lambda}$, and we have

$\ddot{x} = -\left(1+\frac{\lambda}{h^2}\right)x$, so that $x = A\cos\left[\sqrt{1+\frac{\lambda}{h}}\,t + B\right]$,

and the apsidal angle is as given.

4. With the notation of Art. 63, $\phi(u)=au^{-m}+bu^{-n}$, and

$$\phi'(u)=-ma\,u^{-m-1}-nb\,u^{-n-1}.$$

Hence, if R is the radius of the circle described with this force,

$$\phi(c)=aR^m+bR^n,\text{ and } c\phi'(c)=-maR^m-nbR^n,$$

and the apsidal angle $=\pi\div\left[3+\dfrac{maR^m+nbR^n}{aR^m+bR^n}\right]^{\frac{1}{2}}$, as in Art. 63.

5. If d is the distance of the moon, then $\dfrac{2\pi}{\sqrt{\mu}}d^{\frac{3}{2}}=\dfrac{2\pi}{n}$, so that $\mu=n^2d^3$.

Hence, since $h^2=v^2d^2=\mu d=n^2d^4$,

$$\frac{d^2u}{d\theta^2}+u=\frac{1}{h^2}\left(\mu-\frac{m^2}{u^3}\right)=\frac{1}{d}-\frac{m^2}{n^2d^4u^3}.$$

Put $u=p+x$, where $p=\dfrac{1}{d}-\dfrac{m^2}{n^2d^4p^3}$, and x is small.

$$\therefore\ \frac{d^2x}{d\theta^2}+x=\frac{3m^2}{n^2d^4p^4}x=\frac{3m^2}{n^2}x,\text{ on neglecting powers of } m^2.$$

$$\therefore\ x=A\cos\left[\sqrt{1-\frac{3m^2}{n^2}}\,\theta+B\right],$$

and the apsidal angle $=\pi\div\sqrt{1-\dfrac{3m^2}{n^2}}=\pi\left(1+\dfrac{3m^2}{2n^2}\right)$ approx.

6. $\ddot{r}-r\dot{\theta}^2=-\dfrac{\mu}{r^2}=-\dfrac{a^3n^2}{r^2}$, and $\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)=-f$.

Let $\theta=nt+\dfrac{\lambda f}{2a}t^2$, so that $\dot{\theta}=n+\dfrac{\lambda f}{a}t$, and $\ddot{\theta}=\dfrac{\lambda f}{a}$.

Then the second equation gives

$$2\dot{r}\,(na+\lambda ft)+\left(r+\frac{a}{\lambda}\right)f\lambda=0.$$

$$\therefore\ \left(r+\frac{a}{\lambda}\right)\sqrt{na+\lambda ft}=\text{const.}=\left(a+\frac{a}{\lambda}\right)\sqrt{na}.$$

$$\therefore\ r=-\frac{a}{\lambda}+a\left(1+\frac{1}{\lambda}\right)\left[1-\frac{\lambda ft}{2na}+\frac{3}{8}\frac{\lambda^2f^2t^2}{n^2a^2}+\dots\right]=a-(1+\lambda)\frac{ft}{2n},$$

on neglecting squares of f.

$$\therefore\ \dot{r}=-(1+\lambda)\frac{f}{2n},\text{ and }\ddot{r}=0,\text{ to this approximation.}$$

Hence the first equation gives

$$a^3n^2=r^3\dot{\theta}^2=\left[a^3-\frac{3a^2}{2n}ft\,(1+\lambda)\right]\left[n^2+\frac{2n\lambda f}{a}t\right]=a^3n^2+\frac{a^2nft}{2}(\lambda-3),$$

so that $\lambda=3$ and $\therefore\ \theta=nt+\dfrac{3f}{2a}t^2$.

8. $\ddot{r}-r\dot{\theta}^2=-\dfrac{T}{M}$; $\dfrac{1}{r}\dfrac{d}{dt}(r^2\dot{\theta})=0$; and $\dfrac{d^2}{dt^2}(l-r)=g-\dfrac{T}{m}$.

$$\therefore\ (M+m)\frac{d^2r}{dt^2}-Mr\dot{\theta}^2=-mg,\text{ and }\dot{\theta}=hu^2.$$

Now $\qquad \dot{r}=\frac{dr}{d\theta}\cdot\dot{\theta}=-h\frac{du}{d\theta}$, and $\ddot{r}=-h^2\frac{d^2u}{d\theta^2}u^2$.

Hence $$\frac{d^2u}{d\theta^2}\left(1+\frac{m}{M}\right)+u=\frac{mg}{M}\cdot\frac{1}{h^2u^2}. \quad \text{......................(1)}$$

Also $$g-T\left(\frac{1}{M}+\frac{1}{m}\right)=-r\dot{\theta}^2=-h^2u^3, \text{ etc.}$$

9. If $M=m$, the equation (1) becomes

$$2\frac{d^2u}{d\theta^2}+u=\frac{g}{h^2u^2}\,;\ \therefore\ \left(\frac{du}{d\theta}\right)^2+\frac{1}{2}u^2=-\frac{g}{h^2u}+C.$$

Initially, $h=a\sqrt{\frac{8ag}{3}}$, and hence $C=\frac{7}{8a^2}$.

Hence the maximum value of u is when $\frac{du}{d\theta}=0$, and then

$$\frac{u^2}{2}=\frac{7}{8a^2}-\frac{3}{8a^3u},$$

i.e. $$(au-1)(2au-1)(2au+3)=0,\ \textit{i.e.}\ r=2a.$$

10. Here $$\frac{d^2u}{d\theta^2}(M+m)+Mu=\frac{mg}{h^2u^2}.$$

For a circle, $\frac{M}{a}=\frac{mga^2}{h^2}$, and we have $(M+m)\frac{d^2u}{d\theta^2}+Mu=\frac{M}{a^3u^2}$.

Put $u=\frac{1}{a}+x$, where x is small, and we obtain $(M+m)\frac{d^2x}{d\theta^2}=-3Mx$, so that the apsidal angle is as given.

11. $h^2\left(\frac{d^2u}{d\theta^2}+u\right)=\frac{T}{mu^2}$, and $Mg-2T=M\frac{d^2}{dt^2}\left(\frac{l-r}{2}\right)$. $\quad$ ....(1)

Now $\qquad \frac{dr}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}=-h\frac{du}{d\theta}$, and $\frac{d^2r}{dt^2}=-h^2u^2\frac{d^2u}{d\theta^2}$.

Hence (1) gives $\ddot{u}\frac{M+4m}{4m}+u=\frac{Mg}{2mh^2u^2}$, where $\frac{1}{a}=\frac{Mga^3}{2mh^2}$. Hence, putting $u=\frac{1}{a}+x$, we have $\frac{d^2x}{d\theta^2}(M+4m)=-12mx$. Hence the apsidal angle is as stated.

12. Integrating, $v^2=h^2(u^2+\dot{u}^2)=\frac{2\lambda}{ma}\left(-\frac{1}{u^2}+\frac{2a}{u}\right)+C$,

where $$V^2=\frac{h^2}{a^2}=\frac{2\lambda}{ma}\cdot a^2+C.$$

Also there is another apse when $u=\frac{1}{2a}$.

$$\therefore\ \frac{h^2}{4a^2}=\frac{2\lambda}{ma}(-4a^2+4a^2)+C.\ \text{ Hence } V^2=\frac{2\lambda a}{m}+\frac{V^2}{4}, \text{ etc.}$$

13. $v^2 = h^2(u^2 + \dot{u}^2) = \int 2\mu\,(2a^2u^3 - u)\,du = \mu\,(a^2u^4 - u^2) + C,$

where
$$\frac{\mu}{a^2} = \frac{h^2}{a^2} = 0 + C,$$

so that
$$\dot{u}^2 = \frac{(1 - a^2u^2)^2}{a^2}, \text{ i.e. } \frac{a\,dr}{r^2 - a^2} = \pm d\theta.$$

First, let $r > a$, so that $\dfrac{a\,dr}{r^2 - a^2} = d\theta$.

$$\therefore\ -\coth^{-1}\frac{r}{a} = \theta + A, \text{ i.e. } r = a\coth(A - \theta).$$

Secondly, let $r < a$, so that $\dfrac{a\,dr}{a^2 - r^2} = -d\theta$.

$$\therefore\ \tanh^{-1}\frac{r}{a} = -\theta + \text{const., i.e. } r = a\tanh(A - \theta).$$

Hence the result if the initial line is properly chosen.

14. $h^2 = \mu l$, and $\dfrac{d^2u}{d\theta^2} + u = \dfrac{\mu}{h^2u^2}\left[u^2 + \dfrac{3h^2}{c^2}u^4\right] - \dfrac{1}{l}\left[1 + \dfrac{3h^2u^2}{c^2}\right]$.

Put $u = \dfrac{1}{l}\left(1 + \dfrac{3h^2}{c^2l^2}\right) + x$, where x is small,

$$\therefore\ \frac{d^2x}{d\theta^2} + x = \frac{6xh^2}{l^2c^2}, \text{ on neglecting higher powers of } \frac{h}{l}.$$

$$\therefore\ x = A\cos\left[\sqrt{1 - \frac{6h^2}{l^2c^2}}\,\theta + B\right],$$

and the apsidal angle $= \pi \div \sqrt{1 - \dfrac{6h^2}{l^2c^2}} = \pi\left(1 + \dfrac{3h^2}{c^2l^2}\right)$ approx.

In the case of Mercury $\dfrac{3h^2\pi}{c^2l^2} = \dfrac{3\mu\pi}{c^2l} = \dfrac{3 \times 1{\cdot}47\pi}{5{\cdot}55 \times 10^7} = \dfrac{441}{555}\cdot\dfrac{\pi}{10^7}$, and it progresses this in $\dfrac{87{\cdot}97}{2}$ days.

Hence the amount it progresses in a century

$$= \frac{100 \times 365 \times 2}{87{\cdot}97} \times \frac{441}{555} \times \frac{180 \times 60 \times 60}{10^7} \text{ secs. of angle}$$

$$= \frac{73}{88} \times \frac{441}{555} \times \frac{18 \times 36}{10} \text{ secs. approx.}$$

$$= \frac{73}{11} \times \frac{4}{5} \times \frac{81}{10} \text{ secs.} = 43 \text{ secs. nearly.}$$