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Laplace Transform

Introduction

The main drawback of (continuous) Fourier transform is that Fourier transform (F.T) can be defined only for stable systems. Where as Laplace transform (L.T) can be defined for both stable and unstable systems.

Laplace Transform

Laplace transform of a general signal f(t)

- Bilateral (or two) sided) Laplace transform

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

- Unilateral (or one-sided) Laplace transform

$$F(s) = \int_{0}^{\infty} f(t)e^{-st}dt$$

Inverse Laplace Transform

$$f(t) = \int_{-\infty}^{\infty} F(s)e^{st}ds$$

Relation Between Laplace Transform and Fourier Transform

$$F(\sigma + j\omega) = \int_{-\infty}^{\infty} f(t)e^{-\sigma t}e^{-j\omega t}dt = e^{-\sigma t}F(\omega)$$

where, $s = \sigma + j\omega$

Note:

- At $\sigma = 0$, $s = j\omega$ then
  Laplace transform become equal to Fourier transform (or) L.T. calculated on $j\omega$ axis is F.T.
- Fourier transform used for signal analysis and Laplace transform used for designing of the system.

Region of Convergence (ROC)

The range of values of S for which F(s) i.e. $\int_{-\infty}^{\infty} \left|f(t)e^{-\sigma t}\right| < \infty$ is define is known as ROC.

Properties of ROC

(a) ROC does not contain any poles, it is bounded by the poles.
(b) ROC of F(s) contains of strips parallel to imaginary axis i.e. $j\omega$ axis.
(c) If f(t) is of finite duration and is absolutely integrable than the ROC is entire S-plane.
(d) If f(t) is a right sided and if the line Re{s} = σ_0 is in the ROC then all values of S for which Re{s} > σ_0 will also be in ROC.

Remember:

- For stability ROC must include imaginary axis (i.e. $j\omega$ axis)
- Solution of L.T. is unique only when the ROC is given

Properties of Laplace Transform

Linearity

$$f_1(t) \xrightarrow{L.T.} F_1(s) \text{ with ROC} = R_1$$

$$f_2(t) \xrightarrow{L.T.} F_2(s) \text{ with ROC} = R_2$$

$$af_1(t) + bf_2(t) \xrightarrow{L.T.} aF_1(s) + bF_2(s) \text{ ; ROC} = R_1 \cap R_2$$

Time-shifting

$$f(t) \xrightarrow{L.T.} F(s) \text{ with ROC} = R$$

$$f(t \pm t_0) \xrightarrow{L.T.} e^{\pm st_0}F(s) \text{ ; ROC} = R$$

Frequency shifting

$$f(t) \xrightarrow{L.T.} F(s) \text{ with ROC} = R$$

$$e^{\pm s_0 t}f(t) \xrightarrow{L.T.} F(s \mp s_0) \text{ ; ROC} = R + Re\{S_0\}$$

Time-reversal

$$f(t) \xrightarrow{L.T.} F(s)$$

$$\boxed{f(-t) \xrightarrow{L.T.} F(-s)}\ ;\ ROC = -R$$

Differentiation in S-domain

$$f(t) \xrightarrow{L.T.} F(s)\ ROC = R$$

$$\boxed{t\,f(t) \xrightarrow{L.T.} -\frac{d}{ds}F(s)}\ ;\ ROC = R$$

Convolution in Time

If $f(t) \xrightarrow{L.T.} F(s)$ with ROC = R_1

and $h(t) \xrightarrow{L.T.} H(s)$ with ROC = R_2

$$\boxed{f(t) * h(t) \xrightarrow{L.T.} F(s)H(s)}\ ;\ ROC = R_1 \cap R_2$$

Note: ....

L.T. of impulse response is known as system or Transfer function

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Frequency integration

$$\boxed{\frac{f(t)}{t} \xrightarrow{L.T.} \int_s^\infty F(s)dS}$$

Integration in time

$$\boxed{\int_0^t f(\tau)d\tau \xrightarrow{L.T.} \frac{F(s)}{s}}$$

Differentiation in time

$$\boxed{\frac{d}{dt}f(t) \xrightarrow{L.T.} sF(s) - f(0)}$$

$$\boxed{\frac{d^2}{dt^2}f(t) \xrightarrow{L.T.} s^2F(s) - sf(0) - f'(0)}$$

Initial value theorem

$$f(0) = \lim_{s\to\infty} sF(s)$$

Final value theorem

$$f(\infty) = \lim_{s\to 0} sF(s)$$

Note: ....

- Initial value theorem is applicable only for proper Laplace i.e. denominator power of function is more than numerator power.
- Final value theorem is not applicable if poles are conjugate or poles lie in right side of s-plane.

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Characterization of LTI Systems Using Laplace Transform

Causality

- ROC associated with the system function for a causal system is a right-half plane.
- For a system with a rational system function, causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole.
- A system is anticausal if its impulse response $h(t) = 0$ for $t > 0$.

Stability

- An LTI system is stable if and only if the ROC of its system function H(s) includes the $j\omega$-axis, [i.e. $Re\{s\} = 0$].
- A causal system with rational system functions H(s) is stable if and only if all of the poles of H(s) lie in the left-half of the s-plane-i.e. all of the poles have negative real parts.

Laplace Transforms of Elementary Functions

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$Re\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$Re\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$Re\{s\} < 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$Re\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$Re\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$Re\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$Re\{s\} > -\alpha$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$Re\{s\} < -\alpha$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$Re\{s\} > 0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$Re(s) > 0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha^2)+\omega_0^2}$	$Re\{s\} > -\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$Re\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$Re\{s\} > 0$

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