Mathematics
Class XII
Sample Paper – 8 Solution

SECTION – A

1. The element '6' lies on 3rd row and 3rd column
So,
$a_{33} = 6$

2. $2x + 3y = \cos x$
Differentiating w.r.t. x, we get,

$$\frac{d}{dx}(2x+3y)=\frac{d}{dx}\cos x$$

$$2+3\frac{dy}{dx}=-\sin x$$

$$\frac{dy}{dx}=\frac{-\sin x-2}{3}$$

3. DE:

$$s^2\frac{d^2y}{dx^2}+sy\frac{dy}{dx}=s$$

It is nonlinear, since we have product of dependent variable and differential coefficient $y\frac{dy}{dx}$

4. $\frac{x-5}{3}=\frac{y-(-4)}{7}=\frac{z-6}{2}$

Clearly, it passes through (5, -4, 6) and has a direction ratios proportional to 3, 7, 2.
So its vector equation is

$$\vec{r}=\left(5\hat{i}-4\hat{j}+6\hat{k}\right)+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$$

OR

Let θ be the angles between, the given two lines
So, the angle between them given their direction cosines is given by

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{{a_1}^2+{b_1}^2+{c_1}^2}\sqrt{{a_2}^2+{b_2}^2+{c_2}^2}}$$

substituting we get

$$\theta=\cos^{1}\left(\frac{8}{5\sqrt{3}}\right)$$

SECTION – B

5. The binary operation * on the set {1, 2, 3, 4, 5} is defined by a * b = min {a, b}
The operation table for the given operation * on the given set is as follows

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

6. We have,

$$\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - 3\begin{bmatrix} x \\ 2y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} x^2 - 3x \\ y^2 - 6y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$$

$$x^2 - 3x = -2$$

$$y^2 - 6y = 9$$

$$x^2 - 3x + 2 = 0$$

$$(x-2)(x-1) = 0$$

$$x = 2 \text{ or } x = 1$$

$$y^2 - 6y - 9 = 0$$

$$y = \frac{6 \pm \sqrt{36+36}}{2} = 3 \pm 3\sqrt{2}$$

7. $\int \frac{5x-2}{1+2x+3x^2}dx$

$$=5\int \frac{x-\frac{2}{5}}{1+2x+3x^2}dx$$

$$=\frac{5}{6}\int \frac{6x-\frac{12}{5}}{1+2x+3x^2}dx$$

$$=\frac{5}{6}\int \frac{6x+2-\frac{12}{5}-2}{1+2x+3x^2}dx$$

$$=\frac{5}{6}\int \frac{6x+2-\frac{22}{5}}{1+2x+3x^2}dx$$

$$=\frac{5}{6}\int \frac{6x+2}{1+2x+3x^2}dx-\frac{5}{6}\times\frac{22}{5}\int \frac{1}{3\left\{\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right\}}dx$$

$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{9}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+\frac{2}{9}}dx$$

$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{9}\times\frac{3}{\sqrt{2}}\tan^{-1}\left[\frac{\left(x+\frac{1}{3}\right)}{\frac{\sqrt{2}}{3}}\right]+C$$

$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{3\sqrt{2}}\times\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$$

8. Let $x^2 = y$

$$\frac{x^2}{(x^2+4)(x^2+9)} = \frac{y}{(y+4)(y+9)} = \frac{A}{y+4} + \frac{B}{y+9}$$

$y = A(y+9) + B(y+4)$

Comparing both sides,

$A + B = 1$ and $9A + 4B = 0$

Solving, we get $A = \frac{-4}{5}$ and $B = \frac{9}{5}$

$$\therefore I = \int \left[\frac{-4}{5(x^2+4)} + \frac{9}{5(x^2+9)} \right] dx$$

$$= -\frac{4}{5} \times \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + \frac{9}{5} \times \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + C$$

$$= -\frac{2}{5} \tan^{-1}\frac{x}{2} + \frac{3}{5} \tan^{-1}\frac{x}{3} + C$$

OR

$$\int \frac{(x+3)e^x}{(x+5)^3} dx$$

$$= \int \frac{(x+5-2)e^x}{(x+5)^3} dx$$

$$= \int \left[\frac{(x+5)}{(x+5)^3} - \frac{2}{(x+5)^3} \right] e^x . dx$$

$$= \int \left[\frac{1}{(x+5)^2} - \frac{2}{(x+5)^3} \right] e^x . dx$$

This is of the form

$$\int e^x [f(x) + f'(x)] dx = e^x f(x) + C$$

$$\Rightarrow \int \left[\frac{1}{(x+5)^2} - \frac{2}{(x+5)^3} \right] e^x . dx$$

$$= \frac{e^x}{(x+5)^2} + C$$

9. We have to differentiate it w.r.t. x two times

differentiating

$$\frac{dy}{dx} = ab\cos(bx+c).......(1)$$

differentiating again

$$\frac{d^2y}{dx^2} = -ab^2\sin(bx+c).........(2)$$

$$\frac{d^2y}{dx^2} = -b^2y............\{\because y = a\sin(bx+c)\}$$

which is the required differential equation

10. ABCD is a parallelogram with,

$$\overrightarrow{AB} = 2\hat{i} - 4\hat{j} + 5\hat{k}; \overrightarrow{AD} = \hat{i} - 2\hat{j} - 3\hat{k}$$

Using the parallelogram law of vector addition, diagonal is given by

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD} = 2\hat{i} - 4\hat{j} + 5\hat{k} + \hat{i} - 2\hat{j} - 3\hat{k} = 3\hat{i} - 6\hat{j} + 2\hat{k}$$

Unit vector parallel to diagonal $\overrightarrow{AC}$

$$= \frac{\overrightarrow{AC}}{\left|\overrightarrow{AC}\right|} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\left|3\hat{i} - 6\hat{j} + 2\hat{k}\right|} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{3^2 + (-6)^2 + (2)^2}}$$

$$= \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{49}} = \frac{1}{7}\ 3\hat{i} - 6\hat{j} + 2\hat{k}$$

Area of the parallelogram ABCD= $\left|\overrightarrow{AB} \times \overrightarrow{AD}\right|$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix}$$

$$= \left|\hat{i}(12+10) - \hat{j}(-6-5) + \hat{k}(-4+4)\right| = \left|\hat{i}(22) - \hat{j}(-11) + \hat{k}(0)\right| = \left|\hat{i}(22) + \hat{j}(11)\right|$$

$$= \sqrt{(22)^2 + (11)^2 + 0^2} = 11\sqrt{5} \text{ sq units}$$

OR

We have $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$.

Now, $\vec{b} + \vec{c} = \hat{i} + 2\hat{j} - 2\hat{k} + 2\hat{i} - \hat{j} + 4\hat{k}$

$= 3\hat{i} + \hat{j} + 2\hat{k}$

$|\vec{a}| = |2\hat{i} - 2\hat{j} + \hat{k}| = \sqrt{4+4+1} = 3$

Also, $\vec{a} \cdot (\vec{b} + \vec{c}) = (2\hat{i} - 2\hat{j} + \hat{k}) \cdot (3\hat{i} + \hat{j} + 2\hat{k})$

$= 6 - 2 + 2 = 6$

So projection of $(\vec{b} + \vec{c})$ on vector $\vec{a}$ = $\dfrac{\vec{a}.(\vec{b} + \vec{c})}{|\vec{a}|} = \dfrac{6}{3} = 2$ units

11. This is a case of Bernoulli trials.

p = P(Success) = P(getting a spade in a single draw) = $\dfrac{13}{52} = \dfrac{1}{4}$

$q = P(\text{Failure}) = 1 - p = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

(i)All the four cards are spades = P(X = 4) = ${}^4C_4 p^4 q^0 = \left(\dfrac{1}{4}\right)^4 = \dfrac{1}{256}$

(ii)Only 3 cards are spades= P(X = 3) = ${}^4C_3 p^3 q^1 = \dfrac{12}{256} = \dfrac{3}{64}$

(iii)None is a spade = P(X = 0) = ${}^4C_0 p^0 q^4 = \left(\dfrac{3}{4}\right)^4 = \dfrac{81}{256}$

12. (i) $\sum_{i=0}^{7} P(X_i) = 1$

$\Rightarrow \left[0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k\right] = 1$

$\Rightarrow 10k^2 + 9k = 1 \Rightarrow 10k^2 + 9k - 1 = 0 \Rightarrow 10k^2 + 10k - k - 1 = 0$

$\Rightarrow 10k(k+1) - (k+1) = 0$

$\Rightarrow (k+1)(10k-1) = 0$

$\Rightarrow k = -1, k = \frac{1}{10}$

k, also being a probability cannot be negative

$\Rightarrow k = \frac{1}{10}$

(ii) $P(X < 3) = P(0) + P(1) + P(2) = 0 + k + 2k = 3k = \frac{3}{10}$

(iii) $P(X > 6) = P(7) = 7k^2 + k = 7\left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right) = \frac{17}{100}$

(iv) $P(1 \le X < 3) = P(1) + P(2) = k + 2k = 3k = \frac{3}{10}$

OR

Let S denote the success (getting a ‘6’) and F denote the failure (not getting a ‘6’).

Thus, $P(S) = \frac{1}{6}$; $P(F) = \frac{5}{6}$

P(A wins in I throw) = P(S) = $\frac{1}{6}$

P(A wins in III throw) = P(FFS) = $\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$

P(A wins in v throw) = P(FFFFS) = $\left(\frac{5}{6}\right)^4 \times \frac{1}{6}$

P(A wins) = P(S) + P(FFS) + P(FFFFS) = $\frac{1}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + \left(\frac{5}{6}\right)^4 \times \frac{1}{6} + \ldots$

$= \frac{\frac{1}{6}}{1 - \frac{25}{36}} = \frac{6}{11}$

P(B wins) = $1 - \frac{6}{11} = \frac{5}{11}$

SECTION – C

13. Given that A = Q × Q and (a, b) * (c, d) = (ac, b + ad)

(i) We know that $a*e=a$ for identity element.

Let, $a=(a_1,a_2)$ and $e=(e_1,e_2)$

$\Rightarrow a*e=(a_1e_1,a_1e_2+a_2)$

It should be equals (a_1,a_2)

$(a_1e_1,a_1e_2+a_2)=(a_1,a_2)$ at $e=(1,0)$

Hence, $e=(1,0)$ satisfies condition.

(ii) Condition for invertible element $a*b=b*a=e$

Let, $a(a_1,a_2)$ and $b(x_1,x_2)$

$a*b=(a_1x_1,a_1x_2+a_2)=(1,0)$

This will satisfy when $x_1=\frac{1}{a_1}$ and $x_2=\frac{-a_2}{a_1}$

Hence, invertible element $=\left(\frac{1}{a_1},\frac{-a_2}{a_1}\right)$

OR

(a, b) * (c , d) = (ac, ad + b)

(c, d) * (a, b) = (ca, cb + d)

(ac, ad + b) $\neq$ (ca, cb + d)

So, '*' is not commutative

Let (a, b) (c, d), (e, f) $\in$ A, Then

((a, b)* (c, d)) * (e, f) = (ac , ad + b) * (e, f) = ((ac) e , (ac) f + (ad + b))

= (ace , acf + ad + b)

(a, b)* ((c, d)* (e, f)) = (a, b) * (ce , cf + d) =

(a (ce) , a (cf + d) +b) = (ace , acf + ad + b)

((a, b)* (c, d)) * (e, f) = (a, b) * ((c, d)* (e, f))

Hence, '*' is associative.

Let (x, y) $\in$A. Then (x, y) is an identity element, if and only if

(x, y) * (a, b) = (a, b) = (a, b) * (x, y), for every (a, b) $\in$A

Consider (x, y) * (a, b) = (xa, xb + y)

(a, b) * (x, y)= (ax, ay +b)

(xa , xb + y) = (a, b) = (ax, ay +b)

ax = x a = a $\Rightarrow$ x = 1

xb + y = b = ay +b $\Rightarrow$ b + y = b = ay + b $\Rightarrow$ y = 0 = ay $\Rightarrow$ y = 0

Therefore, (1, 0) is the identity element

14.

Let $y = \cot^{-1}\left(\sqrt{1+x^2} - x\right)$

Let $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$y = \cot^{-1}\left(\sqrt{1+\tan^2\theta} - \tan\theta\right)$$

$$y = \cot^{-1}\left(\sec\theta - \tan\theta\right)$$

$$y = \cot^{-1}\left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)$$

$$y = \cot^{-1}\left(\frac{1-\sin\theta}{\cos\theta}\right)$$

$$y = \cot^{-1}\left[\frac{1-\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)}\right]$$

$$y = \cot^{-1}\left[\frac{2\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right]$$

$$y = \cot^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

$$y = \cot^{-1}\left[\cot\left(\frac{\pi}{2}-\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$$

$$y = \frac{\pi}{4}+\frac{\theta}{2}$$

$$\therefore y = \frac{\pi}{4}+\frac{1}{2}\tan^{-1}x.$$

15. Let a, b, c be positive numbers not all are zero.

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}$$

$$\Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

$$= -(a+b+c)\left[a^2 + b^2 + c^2 - ab - bc - ca\right]$$

$$= -\frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$$

$$= -\frac{1}{2}(\text{A positive real number})[\text{At least one non zero positive real number}]$$

$$= -\frac{1}{2} \times \text{positive real number}$$

$= \text{ A negative real number}$

16. Differentiating both sides of the given relation with respect to x, we get

$$\frac{d}{dx}(\sin x)=\frac{d}{dx}\{x\sin(a+y)\}$$

$$\cos y\frac{dy}{dx}=1\times\sin(a+y)+x\cos(a+y)\frac{d}{dx}(a+y)$$

$$\cos y\frac{dy}{dx}=\sin(a+y)+x\cos(a+y)\frac{dy}{dx}$$

$$\{\cos y-x\cos(a+y)\}\frac{dy}{dx}=\sin(a+y)$$

$$\frac{dy}{dx}=\frac{\sin(a+y)}{\cos y-x\cos(a+y)}$$

$$\text{put } x=\frac{\sin y}{\sin(a+y)}$$

$$\frac{dy}{dx}=\frac{\sin(a+y)}{\cos y-\frac{\sin y}{\sin(a+y)}\times\cos(a+y)}$$

$$\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin(a+y)\cos y-\sin y\times\cos(a+y)}$$

$$\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin(a+y-y)}$$

$$\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin(a)}$$

OR

we have

$$y = b\tan^{-1}\left(\frac{x}{a} + \tan^{-1}\frac{y}{x}\right)$$

$$\frac{y}{b} = \tan^{-1}\left(\frac{x}{a} + \tan^{-1}\frac{y}{x}\right)$$

$$\tan\frac{y}{b} = \frac{x}{a} + \tan^{-1}\frac{y}{x}$$

differntiating w.r.t. x

$$\frac{1}{b}\sec^2\left(\frac{y}{b}\right)\frac{dy}{dx} = \frac{1}{a} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{x\frac{dy}{dx} - y}{x^2}$$

$$\frac{1}{b}\sec^2\left(\frac{y}{b}\right)\frac{dy}{dx} = \frac{1}{a} + \frac{x\frac{dy}{dx} - y}{x^2 + y^2}$$

$$\frac{d}{dx}\left\{\frac{1}{b}\sec^2\left(\frac{y}{b}\right) - \frac{x}{x^2 + y^2}\right\} = \frac{1}{a} - \frac{y}{x^2 + y^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{a} - \frac{y}{x^2 + y^2}}{\frac{1}{b}\sec^2\left(\frac{y}{b}\right) - \frac{x}{x^2 + y^2}}$$

17. We have

$$y = \cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$$

$$\cos y = \sqrt{\frac{\cos 3x}{\cos^3 x}}$$

$$\cos y = \sqrt{\frac{4\cos^3 x - 3\cos x}{\cos^3 x}}$$

$$\cos y = \sqrt{4 - 3\sec^2 x}$$

$$\cos^2 y = 4 - 3\sec^2 x$$

$$\cos^2 y = 4 - 3\left(1 + \tan^2 x\right)$$

$$\cos^2 y = 4 - 3 - 3\tan^2 x$$

$$1 - \cos^2 y = 3\tan^2 x$$

$$\sin^2 y = 3\tan^2 x$$

$$\sin y = \sqrt{3}\tan x$$

differentiating w.r.t. x

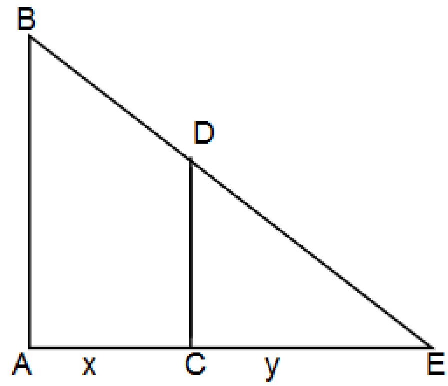
$$\cos y \frac{dy}{dx} = \sqrt{3}\sec^2 x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{3}}{\cos y \cos^2 x}$$

sub, $\cos y = \sqrt{\frac{\cos 3x}{\cos^3 x}}$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{3}{\cos x \cos 3x}}$$

18.



Let AB – lamp post

CD – man

AC = x

CE = y

Given $\frac{dx}{dt} = 5 km/h$

To find $\frac{dy}{dt}$

We have ΔABE ~ ΔCDE (by AA)

$\Rightarrow \frac{AB}{CD} = \frac{AE}{CE}$

$\Rightarrow \frac{6}{2} = \frac{x+y}{y}$

$\Rightarrow x = 2y$

Differentiating $\frac{dx}{dt} = 2\frac{dy}{dt}$

$\frac{dy}{dt} = \frac{1}{2}\frac{dx}{dt} = \frac{5}{2}$ km/h

19. $\int \frac{dx}{\sin^4 x + \sin^2 x\cos^2 x + \cos^4 x}$

Let $I=\int \frac{dx}{\sin^4 x + \sin^2 x\cos^2 x + \cos^4 x}$

Multiply the numerator and the denominator by $\sec^4 x$, we have

$$I=\int \frac{\sec^4 x dx}{\tan^4 x + \tan^2 x + 1}$$

$$I=\int \frac{\sec^2 x \times \sec^2 x dx}{\tan^4 x + \tan^2 x + 1}$$

We know that $\sec^2 x = 1 + \tan^2 x$

Thus,

$$I=\int \frac{\left(1+\tan^2 x\right)\sec^2 x dx}{\tan^4 x + \tan^2 x + 1}$$

Now substitute t = tanx; dt = $\sec^2 x dx$

Therefore,

$$I=\int \frac{\left(1+t^2\right)dt}{1+t^2+t^4}$$

Let us rewrite the integrand as

$$\frac{\left(1+t^2\right)}{1+t^2+t^4} = \frac{\left(1+t^2\right)}{\left(t^2-t+1\right)\left(t^2+t+1\right)}$$

Using partial fractions, we have

$$\frac{\left(1+t^2\right)}{1+t^2+t^4} = \frac{At+B}{t^2-t+1} + \frac{Ct+D}{t^2+t+1}$$

$$\Rightarrow \frac{\left(1+t^2\right)}{1+t^2+t^4} = \frac{\left(At+B\right)\left(t^2+t+1\right)+\left(Ct+D\right)\left(t^2-t+1\right)}{\left(t^2-t+1\right)\left(t^2+t+1\right)}$$

$$\Rightarrow \frac{\left(1+t^2\right)}{1+t^2+t^4}$$

$$= \frac{At^3+At^2+At+Bt^2+Bt+B+Ct^3-Ct^2+Ct+Dt^2-Dt+D}{\left(t^2-t+1\right)\left(t^2+t+1\right)}$$

$$\Rightarrow \frac{(1+t^2)}{1+t^2+t^4}$$

$$=\frac{t^3(A+C)+t^2(A+B-C+D)+t(A+B+C-D)+(B+D)}{(t^2-t+1)(t^2+t+1)}$$

So we have,

$A+C=0; A+B-C+D=1; A+B+C-D=0; B+D=1$

Solving the above equations, we have

$A=C=0$ and $B=D=\frac{1}{2}$

$$I=\int \frac{(1+t^2)dt}{1+t^2+t^4}$$

$$=\int \left[\frac{1}{2(t^2-t+1)}+\frac{1}{2(t^2+t+1)}\right]dt$$

$$=\int \frac{dt}{2(t^2-t+1)}+\int \frac{dt}{2(t^2+t+1)}$$

$$=\frac{1}{2}\int \frac{dt}{t^2-t+1}+\frac{1}{2}\int \frac{dt}{t^2+t+1}$$

$$=I_1+I_2$$

where, $I_1=\frac{1}{2}\int \frac{dt}{t^2-t+1}$ and $I_2=\frac{1}{2}\int \frac{dt}{t^2+t+1}$

Consider I_1 :

$$I_1=\frac{1}{2}\int \frac{dt}{t^2-t+1}$$

$$=\frac{1}{2}\int \frac{dt}{t^2-t+\frac{1}{4}+1-\frac{1}{4}}$$

$$=\frac{1}{2}\int \frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$=\frac{1}{2}\times \frac{1}{\sqrt{\frac{3}{4}}}\tan^{-1}\left(\frac{t-\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\frac{2t-1}{\sqrt{3}}$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\frac{2\tan x-1}{\sqrt{3}}$$

Similarly,

Consider I_2 :

$$I_2 = \frac{1}{2}\int \frac{dt}{t^2 + t + 1}$$

$$= \frac{1}{2}\int \frac{dt}{t^2 + t + \frac{1}{4} + 1 - \frac{1}{4}}$$

$$= \frac{1}{2}\int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{\frac{3}{4}}} \tan^{-1}\left(\frac{t + \frac{1}{2}}{\sqrt{\frac{3}{4}}}\right)$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{2t + 1}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{2\tan x + 1}{\sqrt{3}}$$

Thus, $I = I_1 + I_2$

$$\Rightarrow I = \frac{1}{\sqrt{3}} \tan^{-1} \frac{2\tan x - 1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2\tan x + 1}{\sqrt{3}}$$

$$I = \frac{1}{\sqrt{3}}\left[\tan^{-1} \frac{2\tan x - 1}{\sqrt{3}} + \tan^{-1} \frac{2\tan x + 1}{\sqrt{3}}\right] + C$$

20. $\int_{-1}\ 7x-5\ \ dx;$

$a = -1$, $b = 2$; $h=\dfrac{2+1}{n}\Rightarrow nh=3, f(x)=7x-5$

$$\underset{h\to 0}{Lt}\ h\left[f(-1)+f(-1+h)+f(-1+2h)+.....+f\left(-1+\overline{n-1}h\right)\right] \qquad \text{(i)}$$

$$=f(-1)=-7-5=-12;\ f(-1+h)=7(-1+h)-5=7h-12$$

$$=f\left(-1+\overline{n-1}\ h\right)=7\left\{-1+\overline{n-1}\ h\right\}-5=7(n-1)h-12$$

Substituting in (i)

$$\int_{-1}^{2}(7x-5)dx\ =\underset{h\to 0}{Lt}\ h\left[(-12)+(7h-12)+(14h-12)+.....+\left\{7(n-1)h-12\right\}\right]$$

$$=\underset{h\to 0}{Lt}\ h\left[7h\left(1+2+....+\overline{n-1}\right)-12n\right]=\underset{h\to 0}{Lt}\ h\left[7h\frac{(n-1)n}{2}-12n\right]$$

$$=\underset{h\to 0}{Lt}\left[\frac{7}{2}(nh)(nh-h)-12nh\right]=\underset{h\to 0}{Lt}\left[\frac{7}{2}(3)(3-h)-36\right]$$

$$=\frac{7}{2}\times 9-36=\frac{63}{2}-36=-\frac{9}{2}$$

21. The given D.E. can be written as

$$\frac{dy}{dx}=\frac{1}{\cos(x+y)}$$

let $x+y=v$. then,

$$1+\frac{dy}{dx}=\frac{dv}{dx}$$

D.E. becomes

$$\Rightarrow \frac{dv}{dx}-1=\frac{1}{\cos(v)}$$

$$\Rightarrow \frac{dv}{dx}=\frac{1+\cos v}{\cos v}$$

$$\Rightarrow \int \frac{\cos v}{1+\cos v}dv=\int dx$$

$$\Rightarrow \int \frac{\cos v(1-\cos v)}{1-\cos^2 v}dv=\int dx$$

$$\Rightarrow \int \frac{\cos v-\cos^2 v}{\sin^2 v}dv=\int dx$$

$$\Rightarrow \int \cot v \operatorname{cosec} v\, dv-\int \cot^2 v\, dv=x+c$$

$$\Rightarrow \int \cot v \operatorname{cosec} v\, dv-\int \operatorname{cosec}^2 v dv+\int dv=x+c$$

$$\Rightarrow -\operatorname{cosec} v+\cot v+v=x+c$$

$$\Rightarrow -\operatorname{cosec}(x+y)+\cot(x+y)+(x+y)=x+c$$

$$\Rightarrow -\frac{1-\cos(x+y)}{\sin(x+y)}+y=c$$

$$\Rightarrow -\frac{2\sin^2\frac{x+y}{2}}{2\sin\frac{x+y}{2}\cos\frac{x+y}{2}}+y=c$$

$$\Rightarrow -\tan\left(\frac{x+y}{2}\right)+y=c$$

$$\Rightarrow -\operatorname{cosec}\ x+y\ +\cot\ x+y\ +\ x+y\ = x+c$$

$$\Rightarrow -\frac{1-\cos(x+y)}{\sin(x+y)}+y=c$$

$$\Rightarrow -\tan\left(\frac{x+y}{2}\right)+y=c$$

given $y=0$ when $x=0$.

$$\Rightarrow 0=c$$

so,

$$y=\tan\left(\frac{x+y}{2}\right)$$

as required.

OR

$$(x+y)^2 \frac{dy}{dx} = a^2$$

let $x + y = v$

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

so,

$$v^2\left(\frac{dv}{dx} - 1\right) = a^2$$

$$\Rightarrow v^2 \frac{dv}{dx} - v^2 = a^2$$

$$\Rightarrow v^2 \frac{dv}{dx} = v^2 + a^2$$

$$\Rightarrow \frac{v^2}{v^2 + a} dv = dx$$

$$\Rightarrow \int \frac{v^2}{v^2 + a} dv = \int dx$$

$$\Rightarrow \int\left(1 - \frac{a^2}{v^2 + a}\right) dv = \int dx$$

$$\Rightarrow v - a\tan^{-1}\frac{v}{a} = x + c$$

$$\Rightarrow (x+y) - a\tan^{-1}\left(\frac{x+y}{a}\right) = x + c$$

as required

22.

a) Given $\vec{\alpha}=3\hat{i}-\hat{j},\ \vec{\beta}=2\hat{i}+\hat{j}-3\hat{k}$

Since $\vec{\beta}_1 \| \vec{\alpha}$ $\therefore$ let $\vec{\beta}_1=\lambda\vec{\alpha}=3\lambda\hat{i}-\lambda\hat{j}$

Now

$\vec{\beta}_2=\vec{\beta}-\vec{\beta}_1=(2\hat{i}+\hat{j}-3\hat{k})-((3\lambda)\hat{i}-\lambda\hat{j})=(2-3\lambda)\hat{i}+(1+\lambda)\hat{j}-3\hat{k}$

since $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$

we get $\vec{\alpha}.\vec{\beta}_2=0$

$\Rightarrow 3(2-3\lambda)-(1+\lambda)=0$

$\Rightarrow \lambda=\dfrac{1}{2}$

Hence $\vec{\beta}_1=\dfrac{3}{2}\hat{i}-\dfrac{1}{2}\hat{j}$ and $\vec{\beta}_2=\dfrac{1}{2}\hat{i}+\dfrac{3}{2}\hat{j}-3\hat{k}$

b) Since each of the vectors is perpendicular to the sum of other two

$\Rightarrow \vec{a}.(\vec{b}+\vec{c})=0, \quad \vec{b}.(\vec{c}+\vec{a})=0, \quad \vec{c}.(\vec{a}+\vec{b})=0$

Now $|\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c})$

$=\vec{a}.\vec{a}+\vec{a}.(\vec{b}+\vec{c})+\vec{b}.\vec{b}+\vec{b}.(\vec{c}+\vec{a})+\vec{c}.\vec{c}+\vec{c}.(\vec{a}+\vec{b})$

$=|\vec{a}|^2+0+|\vec{b}|^2+0+|\vec{c}|^2+0$

$=3^2+4^2+5^2=50$ Hence $|\vec{a}+\vec{b}+\vec{c}|=5\sqrt{2}$

23.

The equation of the given line is $\dfrac{x-2}{3}=\dfrac{y+1}{4}=\dfrac{z-2}{2}$ ....(1)

Any point on the given line is $(3\lambda+2, 4\lambda-1, 2\lambda+2)$.

If this point lies on the given plane $x-y+z-5=0$, then

$3\lambda+2-(4\lambda-1)+2\lambda+2-5=0$

$\Rightarrow \lambda=0$

Putting $\lambda=0$ in $(3\lambda+2, 4\lambda-1, 2\lambda+2)$, we get the point of intersection of the given line and the plane is (2, -1, 2).

Let θ be the angle between the given line and the plane.

$$\therefore \sin\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}=\frac{(3\vec{i}+4\vec{j}+2\vec{k}).(\vec{i}-\vec{j}+\vec{k})}{\sqrt{3^2+4^2+2^2}\sqrt{1^2+1^2+1^2}}=\frac{3-4+2}{\sqrt{29}\sqrt{3}}=\frac{1}{\sqrt{87}}$$

$$\Rightarrow \theta=\sin^{-1}\left(\frac{1}{\sqrt{87}}\right)$$

Thus, the angle between the given line and the given plane is $\sin^{-1}\left(\dfrac{1}{\sqrt{87}}\right)$

SECTION – D

24.

We have $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 0 & 3 \end{bmatrix}$ and $f(x) = x^3 - 6x^2 + 7x + 2$

$\therefore f(A) = A^3 - 6A^2 + 7A + 2I$

Now,

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 0 & 3 \end{bmatrix}$$

$$A.A = A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 4 & 4 & 10 \\ 8 & 0 & 13 \end{bmatrix}$$

$$A^2.A = A^3 = \begin{bmatrix} 5 & 0 & 8 \\ 4 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 24 & 8 & 46 \\ 34 & 0 & 55 \end{bmatrix}$$

$$f(A) = A^3 - 6A^2 + 7A + 2I$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 24 & 8 & 46 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 4 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Hence, A is the root of the polynomials $f(x) = x^3 - 6x^2 + 7x + 2$.

OR

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & -3 & -3 \\ 1 & 0 & 0 \\ 6 & 11 & 6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 5 & 6 & 14 \\ 0 & -1 & 1 \\ -1 & -2 & 1 \end{bmatrix}$$

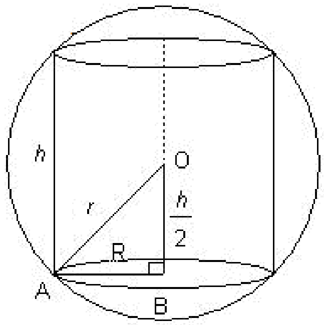
thus on comparing we get $AB \neq BA$

25. The given sphere is of radius r. Let *h* be the height and *R* be the radius of the cylinder inscribed in the sphere.

Volume of cylinder

$V=\pi R^2 h$ ...(1)

In right ΔOBA



$$AB^2 + OB^2 = OA^2$$

$$R^2 + \frac{h^2}{4} = r^2$$

$$\text{So}, R^2 = r^2 - \frac{h^2}{4}$$

Putting the value of R^2 in equation (1), we get

$$V = \pi\left(r^2 - \frac{h^2}{4}\right).h$$

$$V = \pi\left(r^2 h - \frac{h^3}{4}\right) \quad ...(3)$$

$$\therefore \quad \frac{dV}{dh} = \pi\left(r^2 - \frac{3h^2}{4}\right) \quad ...(4)$$

For stationary point, $\frac{dV}{dh} = 0$

$$\pi\left(r^2 - \frac{3h^2}{4}\right) = 0$$

$$r^2 = \frac{3h^2}{4} \Rightarrow h^2 = \frac{4r^2}{3} \qquad \Rightarrow h = \frac{2r}{\sqrt{3}}$$

Now $\frac{d^2V}{dh^2} = \pi\left(-\frac{6}{4}h\right)$

$\therefore$ $\left[\frac{d^2V}{dh^2}\right]_{\text{at } h=\frac{2r}{\sqrt{3}}} = \pi\left(-\frac{3}{2}.\frac{2r}{\sqrt{3}}\right) < 0$

$\therefore$ Volume is maximum at $h = \frac{2r}{\sqrt{3}}$

Maximum volume is

$= \pi\left(r^2.\frac{2r}{\sqrt{3}} - \frac{1}{4}.\frac{8r^3}{3\sqrt{3}}\right)$

$= \pi\left(\frac{2r^3}{\sqrt{3}} - \frac{2r^3}{3\sqrt{3}}\right)$

$= \pi\left(\frac{6r^3 - 2r^3}{3\sqrt{3}}\right)$

$= \frac{4\pi r^3}{3\sqrt{3}}$ cu. unit

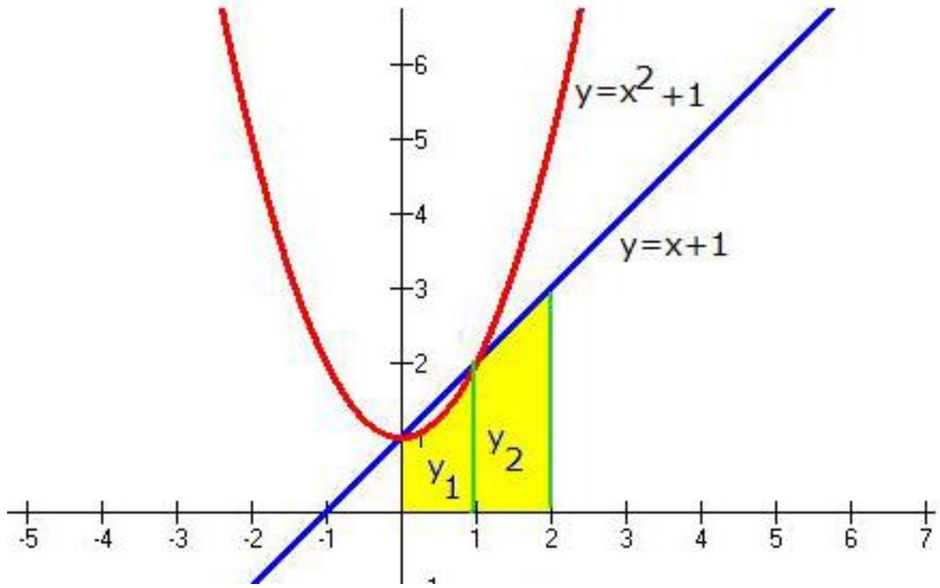
26. Points of intersection of $y = x^2 + 1$, $y = x + 1$

$x^2 + 1 = x + 1$

$\Rightarrow x(x - 1) = 0$

$\Rightarrow x = 0, 1$

So points of intersection are P(0, 1) and Q(1, 2). The graph is represented as



Required area is given by

$A = \int_0^1 y_1\,dx + \int_1^2 y_2\,dx,$

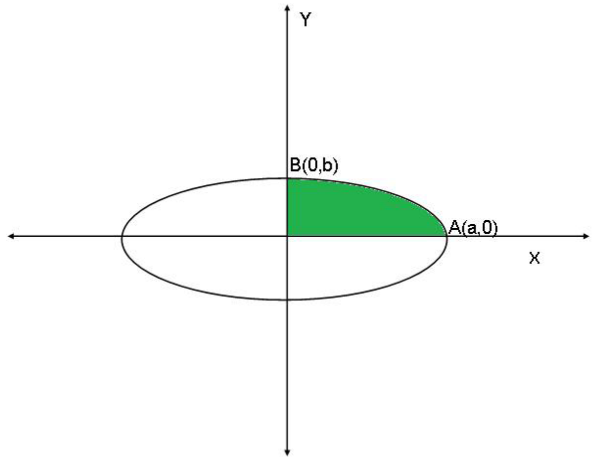
Where y_1 and y_2 represent the y co-ordinate of the parabola and straight line respectively.

$$\therefore A = \int_0^1 \left(x^2 + 1\right)dx + \int_1^2 (x + 1)\,dx$$

$$= \left(\frac{x^3}{3} + x\right)\Bigg]_0^1 + \left(\frac{x^2}{2} + x\right)_1^2$$

$$= \left[\left(\frac{1}{3} + 1\right) - 0\right] + \left[(2 + 2) - \left(\frac{1}{2} + 1\right)\right] = \frac{23}{6} \text{ sq. units}$$

OR



(i) between the curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the x-axis between x = 0 to x = a

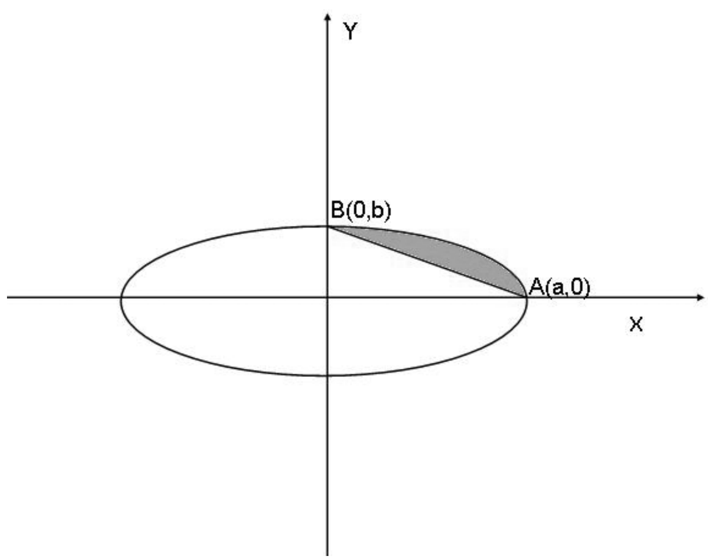
$$\int_0^a b\sqrt{1-\frac{x^2}{a^2}}dx=\frac{b}{a}\int_0^a\sqrt{a^2-x^2}dx$$

$$=\frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$=\frac{b}{2a}\left[\left(0+a^2\sin^{-1}(1)\right)-\left(0+a^2\sin^{-1}(0)\right)\right]$$

$$=\frac{b}{2a}\left[a^2\times\frac{\pi}{2}\right]$$

$$\therefore\int_0^a b\sqrt{1-\frac{x^2}{a^2}}dx=\frac{1}{4}\pi ab$$



(ii) Area of triangle AOB is in the first quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

where OA = a and OB = b.

=the area enclosed between the chord AB and the arc AB of the ellipse .

= Area of Ellipse (In quadrant I)- Area of ΔAOB=$\int_0^a b\sqrt{1-\frac{x^2}{a^2}}dx-\frac{1}{2}ab=\frac{1}{4}\pi ab-\frac{1}{2}ab$

$=\frac{(\pi-2)ab}{4}$

(iii) Ratio $=\frac{\frac{1}{4}\pi ab}{\frac{(\pi-2)}{4}ab}=\frac{\pi}{\pi-2}$

27. Let the equation of plane be ax + by + cz + d = 0 .... (1)

Since the plane passes through the point A (0, 0, 0) and B(3, -1, 2), we have

a × 0 + b × 0 + c × 0 + d = 0

$\Rightarrow$ d = 0 .... (2)

Similarly for point B (3, -1, 2), a × 3 + b × (-1) + c × 2 + d = 0

3a – b + 2c = 0 (Using, d = 0) ...(3)

Given equation of the line is $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$

We can also write the above equation as $\frac{x-4}{1}=\frac{y-(-3)}{-4}=\frac{x-(-1)}{7}$

The required plane is parallel to the above line.

Therefore, a × 1 + b × (-4) + c × 7= 0

$\Rightarrow$ a – 4b + 7c = 0 ... (4)

Cross multiplying equations (3) and (4), we obtain:

$$\frac{a}{(-1)\times7-(-4)\times2}=\frac{b}{2\times1-3\times7}=\frac{c}{3\times(-4)-1\times(-1)}$$

$$\Rightarrow\frac{a}{-7+8}=\frac{b}{2-21}=\frac{c}{-12+1}$$

$$\Rightarrow\frac{a}{1}=\frac{b}{-19}=\frac{c}{-11}=k$$

$$\Rightarrow a=k, b=-19k, c=-11k$$

Substituting the values of a, b and c in equation (1), we obtain the equation of plane as:

$kx-19ky-11kz+d=0$

$\Rightarrow k(x-19y-11z)=0$ (From equation (2))

$\Rightarrow x-19y-11z=0$

So, the equation of the required plane is x – 19y - 11z = 0

OR

We know that, equation of a line passing through (x_1, y_1, z_1) with direction ratios a, b, c is given by $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

So, the required equation of a line passing through $(-1,3,-2)$ is:

$$\frac{x+1}{a} = \frac{y-3}{b} = \frac{z+2}{c} \quad -----(1)$$

Given that line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is perpendicular to line (1), so

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$(a)(1) + (b)(2) + (c)(3) = 0$$

$$a + 2b + 3c = 0 \quad ---(2)$$

And line $\frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$ is perpendicular to line (1), so

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$(a)(-3) + (b)(2) + (c)(5) = 0$$

$$-3a + 2b + 5c = 0 \quad ---(3)$$

Solving equation (2) and (3) by cross multiplication,

$$\frac{a}{(2)(5)-(2)(3)} = \frac{b}{(-3)(3)-(1)(5)} = \frac{c}{(1)(2)-(2)(-3)}$$

$$\Rightarrow \frac{a}{10-6} = \frac{b}{-9-5} = \frac{c}{2+6}$$

$$\Rightarrow \frac{a}{4} = \frac{b}{-14} = \frac{c}{8}$$

$$\Rightarrow \frac{a}{2} = \frac{b}{-7} = \frac{c}{4} = \lambda \text{ (Say)}$$

$$\Rightarrow a = 2\lambda,\ b = -7\lambda,\ c = 4\lambda$$

Putting the value of a, b, and c in (1) gives

$$\frac{x+1}{2\lambda} = \frac{y-3}{-7\lambda} = \frac{z+2}{4\lambda}$$

$$\Rightarrow \frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$$

28. Suppose x is the number of pieces of Model A and y is the number of pieces of Model B. Then

Total profit (in Rs.) = $8000x + 12000y$

Let $Z = 8000x + 12000y$

Mathematical model for the given problem is as follows:

Maximise $Z = 8000x + 12000y$ ... (1)

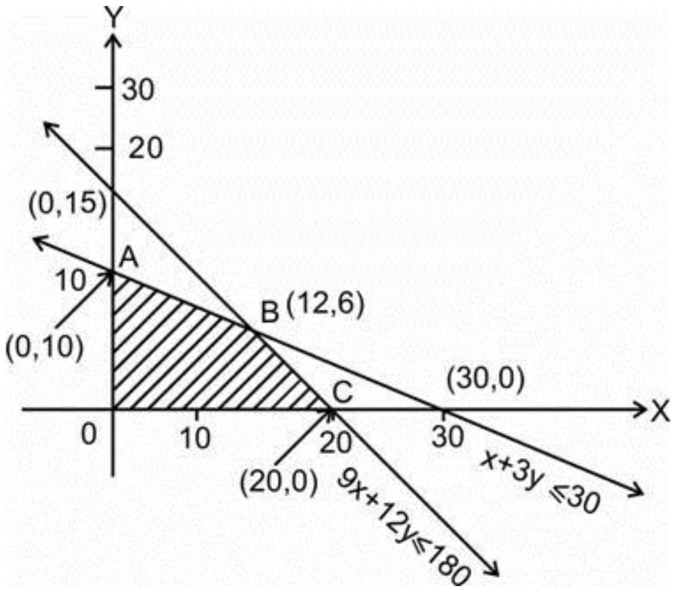
subject to the constraints,

$9x + 12y \le 180$ (Fabrication constraint) i.e. $3x + 4y \le 60$ .... (2)

$x + 3y \le 30$ (Finishing constraint) .... (3)

$x \ge 0, y \ge 0$ .... (4)

The feasible region (shaded) OABC determined by the linear inequalities (2) to (4) is shown below.



Corner Point	$Z = 8000x + 12000y$
Corner Point	$Z = 8000x + 12000y$
A(0, 10)	120000
B(12, 6)	168000 ← Maximum
C(20, 0)	16000

The company should produce 12 pieces of Model A and 6 pieces of Model B to realise maximum profit and maximum profit then will be Rs. 1,68,000.

29.

Let E_1 be the event that a red ball is transferred from bag A to bag B

Let E_2 be the event that a black ball is transferred from bag A to bag B

$\therefore E_1$ and E_2 are mutually exclusive and exhaustive.

$P(E_1) = 3/7$; $P(E_2) = 4/7$

Let E be the event that a red ball is drawn from bag B

$$P(E|E_1) = \frac{4+1}{(4+1)+5} = \frac{5}{10} = \frac{1}{2}$$

$$P(E|E_2) = \frac{3+1}{(5+1)+4} = \frac{4}{10} = \frac{2}{5}$$

$$\therefore \text{ Required probability } = P(E_2|E) = \frac{P(E|E_2)P(E_2)}{P(E|E_1)P(E_1) + P(E|E_2)P(E_2)}$$

$$= \frac{\frac{4}{10} \times \frac{4}{7}}{\frac{1}{2} \times \frac{3}{7} + \frac{4}{10} \times \frac{4}{7}} = \frac{\frac{16}{70}}{\frac{3}{14} + \frac{16}{70}} = \frac{\frac{16}{70}}{\frac{31}{70}} = \frac{16}{31}$$

$$\therefore \text{ Required probability} = P(E_1|E) = \frac{P(E|E_1)P(E_1)}{P(E|E_1)P(E_1) + P(E|E_2)P(E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{3}{7}}{\frac{1}{2} \times \frac{3}{7} + \frac{4}{10} \times \frac{4}{7}} = \frac{\frac{3}{14}}{\frac{3}{14} + \frac{16}{70}} = \frac{\frac{3}{14}}{\frac{31}{70}} = \frac{15}{31}$$