Chapter 7

MOTION IN A RESISTING MEDIUM: MOTION OF PARTICLES OF VARYING MASS

104. When a body moves in a medium like air, it experiences a resistance to its motion which increases as its velocity increases, and which may therefore be assumed to be equal to some function of the velocity, such as $k\rho f(v)$, where ρ is the density of the medium and k is some constant depending on the shape of the body.

Many efforts have been made to discover the law of resistance, but without much success. It appears, however, that for projectiles moving with velocities under about 244 metres per second the resistance approximately varies as the square of the velocity, that for velocities between this value and about 410 metres per second the resistance varies as the cube, or even a higher power, of the velocity, whilst for higher velocities the resistance seems to again follow the law of the square of the velocity.

For other motions it is found that other assumptions of the law for the resistance are more suitable. Thus in the case of the motion of an ordinary pendulum the assumption that the resistance varies as the velocity is the best approximation.

In any case the law assumed is more or less empiric, and its truth can only be tested by enquiring how far the results, which are theoretically obtained by its use, fit with the actually observed facts of the motion.

Whatever be the law of resistance, the forces are non-conservative, and the Principle of Conservation of Energy cannot be applied.

105. In the case of a particle falling under gravity in a resisting medium the velocity will never exceed some definite quantity.

For suppose the law of resistance to be $kv^n.m$. Then the downward acceleration $g - kv^n$ and this vanishes when $kv^n = g$, i.e. when the velocity $= \left(\frac{g}{k}\right)^{1/n}$. This therefore will be the maximum velocity possible, and it is called the limiting or terminal velocity.

It follows from this that we cannot tell the height from which drops of rain fall by observing their velocity on reaching the ground. For soon after they have started they will have approximately reached their terminal velocity, and will then continue to move with a velocity which is sensibly constant and very little differing from the terminal velocity.

In the case of a ship which is under steam there is a full speed beyond which it cannot travel. This full speed will depend on the dimensions of the ship and the size and power of its engines, etc.

But whatever the latter may be, there will be some velocity at which the work that must be done in overcoming the resistance of the water, which varies as some function of the velocity, will be just equivalent to the maximum amount of work that can be done by the engines of the ship, and then further increase of the speed of the ship is impossible.

106. *A particle falls under gravity (supposed constant) in a resisting medium whose resistance varies as the square of the velocity; to find the motion if the particle starts from rest.*

Let v be the velocity when the particle has fallen a distance x in time t from rest. The equation of motion is $\frac{d^2x}{dt^2} = g - \mu v^2$.

Let $\mu = \frac{g}{k^2}$, so that $\frac{d^2x}{dt^2} = g\left(1 - \frac{v^2}{k^2}\right)$ ...(1).

From (1) it follows that if v equalled k, the acceleration would be zero; the motion would then be unresisted and the velocity of the particle would continue to be k. For this reason k is called the "terminal velocity."

From (1), $v\frac{dv}{dx} = g\left(1 - \frac{v^2}{k^2}\right)$,

so that $\frac{2g}{k^2}x = \int \frac{2vdv}{k^2 - v^2} = -\log(k^2 - v^2) + A$,

since v and x are both zero initially, $\therefore\ A = \log k^2$.

$$\therefore \quad k^2 - v^2 = k^2 e^{-\frac{2gx}{k^2}} \qquad \therefore \quad v^2 = k^2\left(1 - e^{-\frac{2gx}{k^2}}\right) \qquad ...(2).$$

It follows that $x = \infty$ when $v = k$. Hence the particle would not actually acquire the "terminal velocity" until it had fallen an infinite distance.

Again (1) can be written $\frac{dv}{dt} = g\left(1 - \frac{v^2}{k^2}\right)$,

$$\therefore \frac{gt}{k^2} = \int \frac{dv}{k^2 - v^2} = \frac{1}{2k}\log\frac{k+v}{k-v} + B.$$

Since v and t were zero initially, $\therefore B = 0$.

Hence $\frac{k+v}{k-v} = e^{-\frac{2gt}{k}}$.

$$\therefore \quad v = k\frac{e^{-\frac{2gt}{k}} - 1}{e^{-\frac{2gt}{k}} + 1} = k\tanh\left(\frac{gt}{k}\right) \qquad ...(3).$$

From (2) and (3), we have $e^{-\frac{2gx}{k^2}} = 1 - \frac{v^2}{k^2} = 1 - \tanh^2 \frac{gt}{k} = \frac{1}{\cosh^2 \frac{gt}{k}}$,

$$\text{so that} \quad e^{\frac{gx}{k^2}} = \cosh \frac{gt}{k}, \text{ and } x = \frac{k^2}{g} \log \cosh \frac{gt}{k}. \qquad ...(4)$$

107. *If the particle were projected upwards instead of downwards, to find the motion.*

Let V be the velocity of projection.

The equation of motion now is

$$\frac{d^2x}{dt^2} = g - \mu v^2 = -g\left(1 + \frac{v^2}{k^2}\right) \qquad ...(5).$$

where x is measured upwards.

Hence $\qquad v\frac{dv}{dx} = -g\left(1 + \frac{v^2}{k^2}\right)$.

$\therefore \quad \frac{2g}{k^2}x = -\int \frac{2v\,dv}{v^2 + k^2} = -\log(v^2 + k^2) + A,$

where $0 = -\log(V^2 + k^2) + A$

$$\therefore \quad \frac{2gx}{k^2}x = \log \frac{V^2 + k^2}{v^2 + k^2} \qquad ...(6)$$

Again (5) gives $\qquad \frac{dv}{dt} = -g\left(1 + \frac{v^2}{k^2}\right)$.

$\therefore \quad -\frac{gt}{k^2} = \int \frac{dv}{k^2 + v^2} = \frac{1}{k}\tan^{-1}\frac{v}{k} + B$, where $0 = \frac{1}{k}\tan^{-1}\frac{V}{k} + B$.

$$\therefore \quad \frac{gt}{k} = \tan^{-1}\frac{V}{k} - \tan^{-1}\frac{v}{k} \qquad ...(7).$$

Equation (6) gives the velocity when the particle has described any distance, and (7) gives the velocity at the end of any time.

108. EX. *A person falls by means of a parachute from a height of 730 metres in $2\frac{1}{2}$ minutes. Assuming the resistance to vary as the square of the velocity, show that in a second and a half his velocity differs by less than one per cent, from its value when he reaches the ground and find an approximate value for the limiting velocity.*

When the parachute has fallen a space x in time t, we have, by Art. 106, if $\mu = \frac{g}{k^2}$,

$$v^2 = k^2\left[1 - e^{-\frac{2gx}{k^2}}\right] \qquad ...(1),$$

$$v = k\tanh\left(\frac{gt}{k}\right) \qquad ...(2),$$

and $$x = \frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right) \qquad ...(3).$$

Here $730\frac{g}{k^2} = \log\cosh\left(\frac{150g}{k}\right)$. $\quad \therefore\ e^{730\frac{g}{k^2}} = \frac{e^{\frac{150g}{k}} + e^{-\frac{150g}{k}}}{2}$ ...(4).

The second term on the right hand is very small, since k is positive. Hence (4) is approximately equivalent to

$$e^{730\frac{g}{k^2}} = \frac{1}{2}e^{\frac{150g}{k}} \quad \therefore 730\frac{g}{k^2} = \frac{150g}{k} - \log 2 = \frac{150g}{k} \text{ nearly.}$$

Hence $k = 4.9$ is a first approximation.

Putting $k = 4.9(1+y)$, (4) gives, for a second approximation,

$$e^{300(1-2y)} = \frac{e^{300(1-y)} + e^{-300(1-y)}}{2} = \frac{e^{300(1-y)}}{2}, \text{ very approx.}$$

$$\therefore\ e^{-300y} = \frac{1}{2}. \quad \text{i.e.,} \quad y = \frac{1}{300}\log_e 2 = \frac{.693}{300} = 0.0023.$$

Therefore a second approximation is $k = 4.9(1 + .0023)$, giving the terminal velocity.

Also the velocity v_1, when the particle reaches the ground, is, by (1), given by

$$v_1^2 = k^2\left[1 - e^{-\frac{2\times 9.81\times 730}{4.9^2}}\right]$$
$$= k^2[1-e^{-600}] = k^2, \text{ for all practical purposes.}$$

When v is 99% of the terminal velocity, (2) gives
$\tanh\dfrac{gt}{k} = \dfrac{99}{100} = .99.$
$\therefore\ e^{\frac{2gt}{k}} = 199 = e^{5.3}$, from the Tables.
$\therefore\ t = \dfrac{k}{2g}\times 5.3 = \dfrac{4.9}{2\times 9.81}\times 5.3 = 1.325$ approx.
i.e. t is less than $1\frac{1}{2}$ secs.

EXAMPLES

1. A particle, of mass m, is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle be released from rest, show that the distance fallen through in time t is

$$g\frac{m^2}{\mu^2}\left\{e^{-\frac{ut}{m}} - 1 + \frac{\mu t}{m}\right\}.$$

2. A particle, of mass m, is projected vertically under gravity, the resistance of the air being mk times the velocity; show that the greatest height attained by the particle is $\dfrac{V^2}{g}[\lambda - \log(1+\lambda)]$, where V is the terminal velocity of the particle and λV is its initial vertical velocity.
3. A heavy particle is projected vertically upwards with velocity u in a medium, the resistance of which is $gu^{-2}\tan^2\alpha$ times the square

of the velocity, α being a constant. Show that the particle will return to the point of projection with velocity $u\cos\alpha$, after a time

$$ug^{-1}\cot\alpha\left(\alpha+\log\frac{\cos\alpha}{1-\sin\alpha}\right).$$

4. A particle falls from rest under gravity through a distance x; in a medium whose resistance varies as the square of the velocity; if v be the velocity actually acquired by it, v_0 the velocity it would have acquired had there been no resisting medium, and V the terminal velocity, show that

$$\frac{v^2}{v_0^2}=1-\frac{1}{2}\frac{v_0^2}{V^2}+\frac{1}{2.3}\frac{v_0^4}{V^4}-\frac{1}{2.3.4}\frac{v_0^6}{V^6}+\cdots .$$

5. A particle is projected with velocity V along a smooth horizontal plane in a medium whose resistance per unit of mass is μ times the cube of the velocity. Show that the distance it has described in time t is $\frac{1}{\mu V}\left[\sqrt{1+2\mu V^2 t}-1\right]$, and that its velocity then is

$$\frac{V}{\sqrt{1+2\mu V^2 t}}.$$

6. A heavy particle is projected vertically upwards with a velocity u in a medium the resistance of which varies as the cube of the particle's velocity. Determine the height to which the particle will ascend.

7. If the resistance vary as the fourth power of the velocity, the energy of m kg at a depth x below the highest point when moving in a vertical line under gravity will be $E\tan\frac{mgx}{E}$ when rising, and, $E\tanh\frac{mgx}{E}$ when falling, where E is the terminal energy in the medium.

8. A particle is projected in a resisting medium whose resistance varies as (velocity)^n, and it comes to rest after describing a distance s in time t. Find the values of s and t and show that s is finite if $n < 2$, but infinite if n = or > 2, whilst t is finite if $n < 1$, but infinite if n = or > 1.
9. In the previous question if the resistance be k (velocity) and the initial velocity be V, show that $v = Ve^{-kt}$ and $s = \frac{V}{k}(1 - e^{-kt})$.
10. A heavy particle is projected vertically upwards in a medium the resistance of which varies as the square of the velocity. It has a kinetic energy K in its upward path at a given point; when it passes the same point on the way down, show that its loss of energy is $\frac{K^2}{K + K'}$, where K' is the limit to which its energy approaches in its downward course.
11. If the resistance to the motion of a railway train vary as its mass and the square of its velocity, and the engine work at constant H.P., show that full speed will never be attained, and that the distance traversed from rest when half-speed is attained is $\frac{1}{3\mu} \log_e \frac{8}{7}$, where μ is the resistance per unit mass per unit velocity.
   Find also the time of describing this distance.
12. A ship, with engines stopped, is gradually brought to rest by the resistance of the water. At one instant the velocity is 10 metres per sec. and one minute later the speed has fallen to 6 metre per sec. For speeds below 2 metres per sec. the resistance may be taken to vary as the speed, and for higher speeds to vary as the square of the speed. Show that, before coming to rest, the ship will move through $900[1 + \log_e 5]$ metres, from the point when the first velocity was observed.

13. A particle moves from rest at a distance a from a fixed point O under the action of a force to O equal to μ times the distance per unit of mass; if the resistance of the medium in which it moves be k times the square of the velocity per unit of mass, show that the square of the velocity when it is at a distance x from O is

$$\frac{\mu x}{k} - \frac{\mu a}{k} e^{2k(x-a)} + \frac{\mu}{2k^2}[1 - e^{2k(x-a)}].$$

Show also that when it first comes to rest it will be at a distance b given by

$$(1-2kb)e^{2kb} = (1+2ka)e^{-2ak}.$$

14. A particle falls from rest at a distance a from the centre of the Earth towards the Earth, the motion meeting with a small resistance proportional to the square of the velocity v and the retardation being μ for unit velocity; show that the kinetic energy at distance x from the centre is

$$mgr^2 \left\{ \frac{1}{x} - \frac{1}{a} + 2\mu\left(1 - \frac{x}{a}\right) - 2\mu \log_e \frac{a}{x} \right\},$$

the square of μ being neglected, and r being the radius of the Earth.

15. An attracting force, varying as the distance, acts on a particle initially at rest at a distance a. Show that, if V be the velocity when the particle is at a distance x, and V the velocity of the same particle when the resistance of the air is taken into account, then

$$V' = V\left[1 - \frac{1}{3}k\frac{(2a+x)(a-x)}{a+x}\right]$$

nearly, the resistance of the air being given to be k times the square of the velocity per unit of mass, where k is very small.

109. *A particle is projected under gravity and a resistance equal to mk(velocity) with a velocity u at an angle α to the horizon; to find the motion.*

Let the axes of x and y be respectively horizontal and vertical, and the origin at the point of projection. Then the equations of motion are

$$\ddot{x} = -k\frac{ds}{dt}.\frac{dx}{ds} = -k\frac{dx}{dt}, \text{ and } \ddot{y} = -k\frac{ds}{dt}.\frac{dy}{ds} - g = -k\frac{dy}{dt} - g.$$

Integrating, we have

$$\log \dot{x} = -kt + \text{ const.} = -kt + \log(u\cos\alpha),$$

$$\text{and } \log(k\dot{y}+g) = -kt + \text{ const. } = -kt + \log(ku\sin\alpha + g);$$

$$\therefore \quad \dot{x} = u\cos\alpha e^{-kt} \qquad ...(1),$$

$$\text{and} \qquad k\dot{y}+g = (ku\sin\alpha + g)e^{-kt} \qquad ...(2).$$

$$\therefore \quad x = -\frac{u\cos\alpha}{k}e^{-kt} + \text{ const. } = \frac{u\cos\alpha}{k}(1-e^{-kt}) \qquad ...(3),$$

$$\text{and } ky + gt = -\frac{ku\sin\alpha + g}{k}e^{-kt} + \text{ const. } = \frac{ku\sin\alpha + g}{k}(1-e^{-kt}) \qquad ...(4).$$

Eliminating t, we have

$$y = \frac{g}{k^2}\log\left(1 - \frac{kx}{u\cos\alpha}\right) + \frac{x}{u\cos\alpha}\left(u\sin\alpha + \frac{g}{k}\right) \qquad ...(5),$$

which is the equation to the path.

The greatest height is attained when $\dot{y} = 0$, *i.e.* when $e^{-kt} = \dfrac{g}{ku\sin\alpha + g}$,

i.e., at time $\dfrac{1}{k}\log\left(1 + \dfrac{ku\sin\alpha}{g}\right)$, and then $y = \frac{u\sin\alpha}{k} - \frac{g}{k^2}\log\left(1 + \frac{ku\sin\alpha}{g}\right)$.

It is clear from equations (3) and (4) that when $t = \infty, x = \frac{u\cos\alpha}{g}$ and $y = -\infty$.

Hence the path has a vertical asymptote at a horizontal distance $\frac{u\cos\alpha}{k}$ from the point of projection. Also, then, $\dot{x} = 0$ and $\dot{y} = -\frac{g}{k}$, *i.e.* the particle will then have just attained the limiting velocity.

COR. If the right-hand side of (5) be expanded in powers of k, it becomes

$$y = \frac{g}{k^2}\left[-\frac{kx}{u\cos\alpha} - \frac{1}{2}\frac{k^2x^2}{u^2\cos^2\alpha} - \frac{1}{3}\frac{k^3x^3}{u^3\cos^3\alpha} - \cdots\right]$$

$$+ \frac{x}{u\cos\alpha}\left(u\sin\alpha + \frac{g}{k}\right),$$

i.e. $$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} - \frac{1}{3}\frac{gkx^3}{u^3\cos^3\alpha} - \frac{1}{4}\frac{gk^2x^4}{u^4\cos^4\alpha} - \cdots .$$

On putting k equal to zero, we have the ordinary equation to the trajectory for unresisted motion.

110. *A particle is moving under gravity in a medium whose resistance = mμ (velocity)^2; to find the motion.*

When the particle has described a distance s, let its tangent make an angle ϕ with the upward drawn vertical, and let v be its velocity.

The equations of motion are then

$$v\frac{dv}{ds} = -g\cos\phi - \mu v^2 \qquad ...(1),$$

and $$\frac{v^2}{\rho} = g\sin\phi \qquad ...(2).$$

(1) gives $\frac{d.v^2}{d\phi}\frac{d\phi}{ds} = -2g\cos\phi - 2\mu v^2$,

i.e., from (2) $\frac{1}{\rho}\frac{d}{d\phi}(\rho\sin\phi) = -2\cos\phi - 2\mu\rho\sin\phi$.

$$\therefore \frac{1}{\rho}\frac{d\rho}{d\phi}\sin\phi + 3\cos\phi = -2\mu\rho\sin\phi.$$

$$\therefore \frac{d}{d\phi}\left(\frac{1}{\rho}\right).\frac{1}{\sin^3\phi} - \frac{3\cos\phi}{\sin^4\phi}.\frac{1}{\rho} = \frac{2\mu}{\sin^3\phi}.$$

$$\therefore \frac{1}{\rho\sin^3\phi} = 2\mu\int\frac{1}{\sin^3\phi}d\phi = -\mu\frac{\cos\phi}{\sin^2\phi} - \mu\log\frac{1+\cos\phi}{\sin\phi} + A \quad ...(3).$$

(2) then gives $v^2\left[A - \mu\frac{\cos\phi}{\sin^2\phi} - \mu\log\frac{1+\cos\phi}{\sin\phi}\right] = \frac{g}{\sin^2\phi}$.

Equation (3) gives the intrinsic equation of the path, but cannot be integrated further.

111. *A bead moves on a smooth wire in a vertical plane under a resistance* $\{= k\ (velocity)^2\}$*; to find the motion.*

When the bead has described an arcual distance s, let the velocity be v at an angle ϕ to the horizon (Fig., Art. 102), and let the reaction of the wire be R.

The equations of motion are

$$\frac{vdv}{ds} = g\sin\phi - kv^2 \quad ...(1),$$

$$\text{and} \quad \frac{v^2}{\rho} = g\cos\phi - R \quad ...(2).$$

Let the curve be $s = f(\phi)$.

Then (1) gives

$$\frac{d}{d\phi}\left(\frac{v^2}{2}\right) = f'(\phi)\ [g\sin\phi - kv^2],$$

i.e. $\dfrac{d}{d\phi}(v^2) + 2kf'(\phi).v^2 = 2g\sin\phi.f'(\phi),$

a linear equation to give v^2.

Particular case. Let the curve be a circle so that $s = a\phi$, if s and ϕ be measured from the highest point.

(1) then gives $\dfrac{d}{d\phi}(v^2) + 2akv^2 = 2ag\sin\phi.$

$$\therefore\ v^2 e^{2ak\phi} = 2ag\int \sin\phi.e^{2ak\phi}\,d\phi$$

$$= \frac{2ag}{1+4a^2k^2}e^{2ak\phi}(2ak\sin\phi - \cos\phi) + C.$$

$$\therefore\ v^2 = \frac{2ag}{1+4a^2k^2}(2ak\sin\phi - \cos\phi) + Ce^{-2ak\phi}.$$

EXAMPLES

1. A particle of unit mass is projected with velocity u at an inclination α above the horizon in a medium resistance is k times the velocity. Show that its direction will again make an angle α with after a time

$$\frac{1}{k}\log\left\{1 + \frac{2ku}{g}\sin\alpha\right\}.$$

2. If the resistance vary as the velocity and the range on the horizontal plane through the point of projection is a maximum, show that the angle α which the direction of projection makes with the vertical is given by

$$\frac{\lambda(1+\lambda\cos\alpha)}{\cos\alpha+\lambda} = \log[1+\lambda\sec\alpha],$$

where λ is the ratio of the velocity of projection to the terminal velocity.

3. A particle acted on by gravity is projected in a medium of which the resistance varies as the velocity. Show that its acceleration retains a fixed direction and diminishes without limit to zero.
4. Show that in the motion of a heavy particle in a medium, the resistance of which varies as the velocity, the greatest height above the level of the point of projection is reached in less than half the total time of the flight above that level.
5. If a particle be moving in a medium whose resistance varies as the velocity of the particle, show that the equation of the trajectory can, by a proper choice of axes, be put into the form

$$y + ax = b \log x.$$

6. If the resistance of the air to a particle's motion be n times its weight, and the particle be projected horizontally with velocity V, show that the velocity of the particle, when it is moving at an inclination ϕ to the horizontal, is

$$V(1 - \sin\phi)^{\frac{n-1}{2}} (1 + \sin\phi)^{\frac{n+1}{2}}.$$

7. A heavy bead, of mass m, slides on a smooth wire in the shape of a cycloid, whose axis is vertical and vertex upwards, in a medium whose resistance is $m\dfrac{v^2}{2c}$ and the distance of the starting point from the vertex is c; show that the time of descent to the cusp is

$$\sqrt{\frac{8a(4a-c)}{gc}},$$

where $2a$ is the length of the axis of the cycloid.

8. A heavy bead slides down a smooth wire in the form of a cycloid, whose axis is vertical and vertex downwards, from rest at a cusp, and is acted on besides its weight by a tangential resistance proportional to the square of the velocity. Determine the velocity after a fall through the height x.
9. If a point travel on an equiangular spiral towards the pole with uniform angular velocity about the pole, show that the projection of the point on a straight line represents a resisted simple vibration.
10. A particle, moving in a resisting medium, is acted on by a central force $\frac{\mu}{r^n}$; if the path be an equiangular spiral of angle α, whose pole is at the centre of force, show that the resistance is
$$\frac{n-3}{2}\frac{\mu\cos\alpha}{r^n}.$$
11. A particle, of mass m, is projected in a medium whose resistance is
$mk(\text{velocity})$, and is acted on by a force to a fixed point (= $m.\mu$ distance). Find the equation to the path, and, in the case when $2k^2 = 9\mu$, show that it is a parabola and that the particle would ultimately come to rest at the origin, but that the time taken would be infinite.
12. If a high throw is made with a diabolo spool the vertical resistance may be neglected, but the spin and the vertical motion together account for a horizontal drifting force which may be taken as proportional to the vertical velocity. Show that if the spool is thrown so as to rise to the height h and return to the point of projection, the spool is at its greatest distance c from the vertical through that point when it is at a height $\frac{2h}{3}$; and show that the equation to the trajectory is of the form $4h^3x^2 = 27c^2y^2(h-y)$.

13. If a body move under a central force in a medium which exerts a resistance equal to k times the velocity per unit of mass, prove that

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}.e^{2kt},$$

where h is twice the initial moment of momentum about the centre of force.

14. A particle moves with a central acceleration P in a medium of which the resistance is $k.(\text{velocity})^2$; show that the equation to its path is

$$\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}e^{2ks},$$

where s is the length of the arc described, and h is twice the initial moment of momentum about the centre of force.

15. A particle moves in a resisting medium with a given central acceleration P; the path of the particle being given, show that the resistance is

$$-\frac{1}{2p^2}\frac{d}{ds}\left(p^3\frac{dr}{dp}P\right).$$

112. Motion where the mass moving varies.

The equation $P=mf$ is only true when the mass m is constant. Newton's second law in its more fundamental form is

$$P=\frac{d}{dt}(mv) \qquad ...(1)$$

Suppose that a particle gains in time δt an increment δm of mass and that this increment δm was moving with a velocity u.

Then in time δt the increment in the momentum of the particle $=m.\delta v+\delta m(v+\delta v-u)$, and the impulse of the force in this time is $P\delta t$.

Equating these we have, on proceeding to the limit,

$$m\frac{dv}{dt}+v\frac{dm}{dt}-u\frac{dm}{dt}=P, \quad i.e.\ \frac{d}{dt}(mv)=P+u\frac{dm}{dt} \qquad ...(2).$$

When u is zero we have the result (1).

113. EX. 1. *A spherical raindrop, falling freely, receives in each instant an increase of volume equal to λ times its surface at that instant; find the velocity at the end of time t, and the distance fallen through in that time.*

When the raindrop has fallen through a distance x in time t, let its radius be r and its mass M.

Then
$$\frac{d}{dt}\left[M\frac{dx}{dt}\right]=Mg \qquad ...(1)$$

Now $M=\frac{4}{3}\pi\rho r^3$,

so that $4\pi r^2\rho\frac{dr}{dt}=\frac{dM}{dt}=\rho.4\lambda\pi r^2$, by the question.

$\therefore\ \frac{dr}{dt}=\lambda$ and $r=a+\lambda t$, where a is the initial radius.

Hence (1) gives
$$\frac{d}{dt}\left[(a+\lambda t)^3\frac{dx}{dt}\right]=(a+\lambda t)^3 g.$$

$$\therefore\ (a+\lambda t)^3\frac{dx}{dt}=\frac{(a+\lambda t)^4}{4\lambda}g-\frac{a^4}{4\lambda}g,$$

since the velocity was zero to start with.

$$\therefore\ \frac{dx}{dt}=\frac{g}{4\lambda}\left[a+\lambda t-\frac{a^4}{(a+\lambda t)^3}\right], \text{ and}$$

$$x=\frac{g}{4\lambda^2}\left[\frac{(a+\lambda t)^2}{2}+\frac{a^4}{2(a+\lambda t)^2}\right]-\frac{g}{4\lambda^2}a^2,$$

since x and t vanish together.

$$\therefore\ x=\frac{g}{8\lambda^2}\left[(a+\lambda t)^2-2a^2+\frac{a^4}{(a+\lambda t)^2}\right]$$

$$= \frac{g}{8\lambda^2}\left[a+\lambda t - \frac{a^2}{a+\lambda t}\right]^2 = \frac{gt^2}{8}\left[\frac{2a+\lambda t}{a+\lambda t}\right]^2.$$

EX. 2. *A mass in the form of a solid cylinder, the area of whose cross-section is A, moves parallel to its axis, being acted on by a constant force F, through a uniform cloud of fine dust of volume density ρ which is moving in a direction opposite to that of the cylinder with constant velocity V. If all the dust that meets the cylinder clings to it, find the velocity and distance described in any time t, the cylinder being originally at rest, and its initial mass m.*

Let M be the mass at time t and v the velocity. Then

$M.\delta v + \delta M(v+\delta v+V)$

= increase in the momentum in time $\delta t = F\delta t$

$$\therefore M\frac{dv}{dt} + v\frac{dM}{dt} + V\frac{dM}{dt} = F \qquad ...(1)$$

in the limit.

$$\text{Also} \qquad \frac{dM}{dt} = A\rho(v+V) \qquad ...(2).$$

(1) gives $\quad Mv + MV = Ft + \text{ const.} = Ft + mV.$

Therefore (2) gives $M\dfrac{dM}{dt} = A\rho(Ft+mV).$

$\therefore \quad M^2 = A\rho(Ft^2 + 2mVt) + m^2.$

Therefore (2) gives

$$v = -V + \frac{Ft+mV}{M} = -V + \frac{Ft+mV}{\sqrt{m^2+2mA\rho Vt + AF\rho t^2}} \qquad ...(3).$$

Also if the hinder end of the cylinder has described a distance x from rest, so that

$$v = \frac{dx}{dt} \quad \text{then} \quad x = -Vt + \frac{1}{A\rho}\sqrt{m^2+2mA\rho Vt+AF\rho t^2} - \frac{m}{A\rho}.$$

From (3) we have that the acceleration

$$\frac{dv}{dt} = \frac{m^2(F - A\rho V^2)}{(m^2 + 2mA\rho Vt + AF\rho t^2)^{3/2}},$$

so that the motion is always in the direction of the force, or opposite, according $f \lesseqgtr A\rho V^2$.

EX. 3. *A uniform chain is coiled up on a horizontal plane and one end passes over a small light pulley at a height a above the plane; initially a length b, > a, hangs freely on the other side; find the motion.*

When the length b has increased to x, let v be the velocity; then in the time δt next ensuing the momentum of the part $(x+a)$ has increased by $m(x+a)\delta v$, where m is the mass per unit length. Also a length $m\delta x$ has been jerked into motion, and given a velocity $v+\delta v$. Hence

$$\begin{aligned} m(x+a)\delta v + m\delta x(v+\delta v) &= \text{change in the momentum} \\ &= \text{impulse of the acting force} \\ &= mg(x-a).\delta t. \end{aligned}$$

Hence, dividing by δt and proceeding to the limit, we have

$$(x+a)\frac{dv}{dt} + v^2 = (x-a)g.$$

$$\therefore v\frac{dv}{dx}.(x+a) + v^2 = (x-a)g.$$

$$\therefore \quad v^2(x+a)^2 = \int_b^x 2(x^2-a^2)g\,dx = 2\left\{\frac{x^3-b^3}{3} - a^2(x-b)\right\}g,$$

so that $v^2 = \dfrac{2g}{3}\,\dfrac{(x-b)(x^2+bx+b^2-3a^2)}{(x+a)^2}$.

This equation cannot be integrated further.

In the particular case when $b = 2a$, this gives $v^2 = \frac{2g}{3}(x - b)$, so that the end descends with constant acceleration $\frac{g}{3}$.

The tension T of the chain is clearly given by $T\delta t = m\delta x.v$, so that $T = mv^2$.

EXAMPLES

1. A spherical raindrop of radius a cms. falls from rest through a vertical height h, receiving throughout the motion an accumulation of condensed vapour at the rate of k grammes per square cm. per second, no vertical force but gravity acting; show that when it reaches the ground its radius will be

$$k\sqrt{\frac{2h}{g}}\left[1 + \sqrt{1 + \frac{ga^2}{2hk^2}}\right].$$

2. A mass in the form of a solid cylinder, of radius c, acted upon by no forces, moves parallel to its axis through a uniform cloud of fine dust, of volume density ρ, which is at rest. If the particles of dust which meet the mass adhere to it, and if M and u be the mass and velocity at the beginning of the motion, prove that the distance x traversed in time t is given by the equation

$$(M + \rho\pi c^2 x)^2 = M^2 + 2\rho\pi c^2 Mt.$$

3. A particle of mass M is at rest and begins to move under the action of a constant force F in a fixed direction. It encounters the resistance of a stream of fine dust moving in the opposite direction with velocity V, which deposits matter on it at a constant rate ρ. Show that its mass will be m when it has traveled a distance

$$\frac{k}{\rho^2}\left[m - M\left\{1+\log\frac{m}{M}\right\}\right] \text{ where } k = F - \rho V.$$

4. A spherical raindrop, whose radius is 0.1 cm., begins to fall from a height of 2000 metres, and during the fall its radius grows, by precipitation of moisture, at the rate of 2.5×10^{-4} cm. per second. If the motion be unresisted, show that its radius when it reaches the ground is 0.105 cm. and that it will have taken about 20 seconds to fall.
5. Snow slides off a roof clearing away a part of uniform breadth; show that, if it all slide at once, the time in which the roof will be cleared is $\sqrt{\frac{6\pi a}{g\sin\alpha}}\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{6}\right)}$, but that, if the top move first and gradually set the rest in motion, the acceleration is $\frac{1}{3}g\sin\alpha$ and the time will be $\sqrt{\frac{6a}{g\sin\alpha}}$, where α is the inclination of the roof and a the length originally covered with snow.
6. A ball, of mass m, is moving under gravity in a medium which deposits matter on the ball at a uniform rate μ. Show that the equation to the trajectory, referred to horizontal and vertical axes through a point on itself, may be written in the form $k^2uy = kx(g+kv)+gu\left(1-e^{\frac{kx}{u}}\right)$, where u,v are the horizontal and vertical velocities at the origin and $mk = 2\mu$.
7. A falling raindrop has its radius uniformly increased by access of moisture. If it have given to it a horizontal velocity, show that it will then describe a hyperbola, one of whose asymptotes is vertical.

8. If a rocket, originally of mass M, throw off every unit of time a mass eM with relative velocity V, and if M' be the mass of the case etc., show that it cannot rise at once unless $eV > g$, nor at all unless $\frac{eMV}{M'} > g$. If it rises vertically at once, show that its greatest velocity is $V \log \frac{M}{M'} - \frac{g}{e}\left(1 - \frac{M'}{M}\right)$, and that the greatest height it reaches is $\frac{V^2}{2g}\left(\log \frac{M}{M'}\right)^2 + \frac{V}{e}\left(1 - \frac{M'}{M} - \log \frac{M}{M'}\right)$.
9. A heavy chain, of length l, is held by its upper end so that its lower end is at a height l above a horizontal plane; if the upper end is let go, show that at the instant when half the chain is coiled up on the plane the pressure on the plane is to the weight of the chain in the ratio of 7 : 2.
10. A chain, of great length a, is suspended from the top of a tower so that its lower end touches the Earth; if it be then let fall, show that the square of its velocity, when its upper end has fallen a distance x, is $2gr \log \frac{a+r}{a+r-x}$, where r is the radius of the Earth.
11. A chain, of length l, is coiled at the edge of a table. One end is fastened to a particle, whose mass is equal to that of the whole chain, and the other end is put over the edge. Show that, immediately after leaving the table, the particle is moving with velocity $\frac{1}{2}\sqrt{\frac{5gl}{6}}$.
12. A uniform string, whose length is l and whose weight is W, rests over a small smooth pulley with its end just reaching to a horizontal plane; if the string be slightly displaced, show that when a length x has been deposited on the plane the pressure on it is

$W\left[2\log\frac{l}{l-x}-\frac{x}{l}\right]$, and that the resultant pressure on the pulley is $W\frac{l-2x}{l-x}$.

13. A mass M is attached to one end of a chain whose mass per unit of length is m. The whole is placed with the chain coiled up on a smooth table and M is projected horizontally with velocity V. When a length x of the chain has become straight, show that the velocity of M is $\frac{MV}{M+mx}$, and that its motion is the same as if there were no chain and it were acted on by a force varying inversely as the cube of its distance from a point in its line of motion.

    Show also that the rate at which kinetic energy is dissipated is at any instant proportional to the cube of the velocity of the mass.

14. A weightless string passes over a smooth pulley. One end is attached to a coil of chain lying on a horizontal table, and the other to a length l of the same chain hanging vertically with its lower end just touching the table. Show that after motion ensues the system will first be at rest when a length x of chain has been lifted from the table, such that $(l-x)e^{\frac{2x}{l}}$. Why cannot the Principle of Energy be directly applied to find the motion of such a system?

15. A ship's cable passes through a hole in the deck at a height a above the coil in which the cable is heaped, then passes along the deck for a distance b, and out at a hole in the side of the ship, immediately outside of which it is attached to the anchor. If the latter be loosed find the resulting motion, and, if the anchor be of weight equal to $2a+\frac{1}{2}b$ of the cable, show that it descends with uniform acceleration $\frac{1}{3}g$

16. A mass M is fastened to a chain of mass m per unit length coiled up on a rough horizontal plane (coefficient of friction = μ). The mass is projected from the coil with velocity V; show that it will be brought to rest in a distance $\dfrac{M}{m}\left\{\left(1+\dfrac{3mV^2}{2M\mu g}\right)^{1/3}-1\right\}$.
17. A uniform chain, of mass M and length l, is coiled up at the top of a rough plane inclined at an angle α to the horizon and has a mass M fastened to one end. This mass is projected down the plane with velocity V. If the system comes to rest when the whole of the chain is just straight, show that $V^2 = \dfrac{14gl}{3}\sec\varepsilon\ \sin(\varepsilon-\alpha)$, where ε is the angle of friction.
18. A uniform chain, of length l and mass ml, is coiled on the floor, and a mass mc is attached to one end and projected vertically upwards with velocity $\sqrt{2gh}$. Show that, according as the chain does or does not completely leave the floor, the velocity of the mass on finally reaching the floor again is the velocity due to a fall through a height $\dfrac{1}{3}\left[2l-c+\dfrac{a^3}{(l+c)^2}\right]$ or $a-c$, where $a^3 = c^2(c+3h)$.
19. A uniform chain is partly coiled on a table, one end of it being just carried over a smooth pulley at a height h immediately above the coil and attached there to a weight equal to that of a length $2h$ of the chain. Show that until the weight strikes the table, the chain uncoils with uniform acceleration $\dfrac{1}{3}g$, and that, after it strikes the table, the velocity at any moment is $\sqrt{\dfrac{2}{3}ghe^{-\frac{x-h}{2h}}}$, where x is the length of the chain uncoiled.
20. A string, of length l, hangs over a smooth peg so as to be at rest. One end is ignited and burns away at a uniform rate v. Show that

the other end will at time t be at a depth x below the peg, where x is given by the equation $(l-vt)\left(\frac{d^2x}{dt^2}+g\right)-v\frac{dx}{dt}-2gx=0$.

[At time t let x be the longer, and y the shorter part of the string, so that $x+y=l-vt$. Also let $V,(=\dot{x})$, be the velocity of the string then. On equating the change of momentum in the ensuing time δt to the impulse of the acting force, we have $(x+y-v\delta t)(V+\delta V)-(x+y)V=(x-y)g\delta t$, giving $(x+y)\frac{dY}{dt}-vV=(x-y)g=(2x-l+vt)g$, etc.].

21. A chain, of mass m and length $2l$, hangs in equilibrium over a smooth pulley when an insect of mass M alights gently at one end and begins crawling up with uniform velocity V relative to the chain; show that the velocity with which the chain leaves the pulley will be $\left[\frac{M^2}{(M+m)^2}V^2+\frac{2M+m}{M+m}gl\right]^{1/2}$.

    [Let V_0 be the velocity with which the chain starts, so that $V-V_0$ is the velocity with which the insect starts. Then $M(V-V_0)$ = the initial impulsive action between the insect and chain = mV_0, so that $V_0=\frac{M}{M+m}V$.

    At any subsequent time t let x be the longer, and y the shorter part of the chain, z the depth of the insect below the pulley, and P the force exerted by the insect on the chain. We then have $m\frac{d^2x}{dt^2}=P+\frac{mg}{2l}(x-y)$; $M\frac{d^2z}{dt^2}=Mg-P$; and $\frac{dx}{dt}-\frac{dz}{dt}=V$.

    Also $x+y=2l$. These equations give $(M+m)\dot{x}^2=2(M-m)gx+\frac{mg}{l}x^2+A$.

    Also, when $x=l, \dot{x}=V_0$, etc.]

22. A uniform cord, of length l, hangs over a smooth pulley and a monkey, whose weight is that of the length k of the cord, clings

to one end and the system remains in equilibrium. If he start suddenly, and continue to climb with uniform relative velocity along the curd, show that he will cease to ascend in space at the end of time $\left(\frac{l+k}{2g}\right)^{1/2}\cosh^{-1}\left(1+\frac{l}{k}\right)$.

23. One end of a heavy uniform chain, of length $5a$ and mass $5ma$, is fixed at a point O and the other passes over a small smooth peg at a distance a above O; the whole hangs in equilibrium with the free end at a depth $2a$ below the peg. The free end is slightly displaced downwards; prove that its velocity V, when the length of the free portion is x, is given by $V^2 = \frac{2(x-2a)^2(x+10a)}{(x+6a)^2}g$. and find the impulsive tension at O at the instant when the part of the chain between O and the peg becomes tight.

24. A machine gun, of mass M, stands on a horizontal plane and contains shot, of mass M'. The shot is fired at the rate of mass m per unit of time with velocity u relative to the ground. If the coefficient of sliding friction between the gun and the plane is μ, show that the velocity of the gun backward by the time the mass M' is fired is $\frac{M'}{M}u - \frac{(M+M')^2-M^2}{2mM}\mu g$.

ANSWERS WITH HINTS

Art. 111 EXAMPLES

8. $V^2 = \frac{g}{4\mu a}\left(2\sqrt{2ax}-\frac{1}{2\mu}\right)e^{4\mu\sqrt{2ax}} + \frac{g}{8\mu^2 a}$

Art. 113 EXAMPLES

23. Impulse tension $= 2ma\sqrt{7ga}$

Chapter 7

MOTION IN A RESISTING MEDIUM. MOTION OF PARTICLES OF VARYING MASS

End of Art 108
EXAMPLES

1. $\ddot{x}=g-\frac{\mu}{m}\dot{x}$, so that $\dot{x}e^{\frac{\mu}{m}t}=\int ge^{\frac{\mu}{m}t}=\frac{mg}{\mu}\left(e^{\frac{\mu t}{m}}-1\right)$.

$\therefore\ x=\frac{mg}{\mu}\left[t+\frac{m}{\mu}e^{-\frac{\mu t}{m}}-\frac{m}{\mu}\right]$, etc.

2. $\ddot{x}=-g-k\dot{x}$, where $kV=g$.

$$\therefore\ \dot{x}e^{kt}=-\int ge^{kt}=-Ve^{kt}+\lambda V+V.$$

$$\therefore\ x=-Vt-\frac{\lambda+1}{k}Ve^{-kt}+\frac{\lambda+1}{k}V.$$

Now $\dot{x}$ is zero when $e^{kt}=\lambda+1$, and then

$$x=-\frac{V}{k}\log(\lambda+1)-\frac{V}{k}+\frac{\lambda+1}{k}V, \text{ etc.}$$

3. From Art. 107, on putting $V=u$ and $k=u\cot\alpha$, we have the greatest height $x_1=\frac{u^2\cot^2\alpha}{2g}\log\frac{u^2+u^2\cot^2\alpha}{u^2\cot^2\alpha}=\frac{u^2\cot^2\alpha}{g}\log\sec\alpha$, and the corresponding time $t_1=\frac{\alpha u}{g}\cot\alpha$.

From Art. 106, since the particle falls from a height x_1, we have

$x_1=\frac{k^2}{g}\log\cosh\frac{gt_2}{k}$, so that $\cosh\frac{gt_2}{k}=\sec\alpha$, and $\therefore\ \sinh\frac{gt_2}{u\cot\alpha}=\tan\alpha$.

$$\therefore\ e^{\frac{gt_2}{u}\tan\alpha}=\sec\alpha+\tan\alpha=\frac{\cos\alpha}{1-\sin\alpha}.$$

$$\therefore\ t_2=\frac{u}{g}\cot\alpha\log\frac{\cos\alpha}{1-\sin\alpha}.\quad \text{Hence } t_1+t_2=\text{as given.}$$

Also, from the same article, the final velocity

$$=u\cot\alpha\tanh\frac{gt_2}{u\cot\alpha}=u\cot\alpha\,.\sin\alpha=u\cos\alpha.$$

4. On putting $k=V$ we have, from Art. 106,

$$v^2=V^2\left(1-e^{-\frac{2gx}{V^2}}\right),\ \text{and } v_0^2=2gx.$$

$$\therefore\ \frac{v^2}{V^2}=1-e^{-\frac{v_0^2}{V^2}}=1-\left[1-\frac{v_0^2}{V^2}+\frac{1}{1.2}\frac{v_0^4}{V^4}-\frac{1}{1.2.3}\frac{v_0^6}{V^6}+\dots\right].$$

$$\therefore\ \frac{v^2}{v_0^2}=1-\frac{1}{1.2}\frac{v_0^2}{V^2}+\frac{1}{1.2.3}\frac{v_0^4}{V^4}-\dots.$$

5. $v\dfrac{dv}{dx}=-\mu v^3$. $\quad\therefore\ -\mu x=\displaystyle\int\frac{dv}{v^2}=\frac{1}{V}-\frac{1}{v}$.

$$\therefore\ \frac{dx}{dt}=v=\frac{V}{1+\mu Vx}. \qquad \therefore\ Vt=x+\frac{\mu Vx^2}{2},$$

since x and t vanish together. Hence, etc.

6. Let the resistance be $mg\,.\,\dfrac{v^3}{a^3}$, so that $v\dfrac{dv}{dx}=-g-\dfrac{g}{a^3}v^3$.

$$\therefore\ \frac{gx}{a^3}=-\int_u^0\frac{v\,dv}{v^3+a^3}=\int_0^u\left(\frac{A}{v+a}+\frac{B\left(v-\frac{a}{2}\right)}{v^2-va+a^2}+\frac{C}{v^2-va+a^2}\right)dv,$$

where $\qquad A=-\dfrac{1}{3a},\ B=\dfrac{1}{3a}$, and $C=\dfrac{1}{2}$.

$$\therefore\ \frac{gx}{a^3}=A\log\left(\frac{u+a}{a}\right)+\frac{B}{2}\log\left(\frac{u^2-ua+a^2}{a^2}\right)+\frac{2C}{\sqrt{3}a}\left[\tan^{-1}\frac{2u-a}{\sqrt{3}a}+\frac{\pi}{6}\right].$$

7. *Downward motion.* $mv\dfrac{dv}{dx}=mg-\lambda v^4$, where $mg=\lambda V^4$, and $E=\dfrac{1}{2}mV^2$.

$$\therefore\ \frac{gx}{4E^2}=\int\frac{v\,dv}{4E^2-m^2v^4}=\frac{1}{8Em}\log\frac{2E+mv^2}{2E-mv^2}.$$

$$\therefore\ \frac{2E+mv^2}{2E-mv^2}=e^{\frac{2mgx}{E}}, \text{ and } \therefore\ \tfrac{1}{2}mv^2=E\tanh\frac{mgx}{E}.$$

Upward motion. Here $v\dfrac{dv}{dy}=-g\left[1+\dfrac{m^2v^4}{4E^2}\right]$.

$$\therefore\ -\frac{gy}{4E^2}=\int\frac{v\,dv}{4E^2+m^2v^4}=\frac{1}{4mE}\tan^{-1}\frac{mv^2}{2E}-\frac{gh}{4E^2}.$$

$$\therefore\ \frac{mv^2}{2E}=\tan\left[\frac{mg}{E}(h-y)\right]=\tan\left(\frac{mg}{E}x\right).$$

8. $v\dfrac{dv}{dx}=-\mu v^n$, so that $\mu x=-\displaystyle\int\frac{dv}{v^{n-1}}=\frac{1}{n-2}\left[\frac{1}{v^{n-2}}-\frac{1}{V^{n-2}}\right]$.

If $n>2$, then x is infinite when $v=0$.

If $n=2$, then $\mu x=-\displaystyle\int\frac{dv}{v}=\log\frac{V}{v}$, so that x is infinite when $v=0$.

If $n<2$, let it $=2-\lambda$, so that $\mu x=\dfrac{1}{\lambda}(V^\lambda-v^\lambda)$, and hence

$$s=[x]_{v=0}=\frac{1}{\mu\lambda}V^\lambda=\text{finite}.$$

Again $\qquad \dfrac{dv}{dt}=-\mu v^n$, so that $\mu t=\dfrac{1}{n-1}\left[\dfrac{1}{v^{n-1}}-\dfrac{1}{V^{n-1}}\right]$.

Let $n<1$, and $=1-p$; then $\mu t=\dfrac{1}{p}[V^p-v^p]$, so that, when $v=0$,

$$\mu t=\frac{1}{p}V^p=\frac{1}{1-n}V^{1-n}=\text{a finite quantity}.$$

Let $n>1$, and $=1+p$; then $\mu t=\dfrac{1}{p}\left[\dfrac{1}{v^p}-\dfrac{1}{V^p}\right]$; hence, when $v=0$, $t=\infty$.

Let $n=1$; then $\mu t=-\displaystyle\int\frac{dv}{v}=-\log v+\log V=\log\frac{V}{v}$; hence, when $v=0$, $t=\infty$.

9. $\frac{dv}{dt} = -kv$; $\therefore kt = -\log v + C = \log \frac{V}{v}$. $\therefore v = Ve^{-kt}$.

$$\therefore s = \int Ve^{-kt}\,dt = -\frac{V}{k}e^{-kt} + A = \frac{V}{k}(1 - e^{-kt}).$$

10. By Art. 107 at height y, $v^2 + k^2 = (V^2 + k^2)e^{-\frac{2gy}{k^2}}$, and, if h is the greatest height, $k^2 = (V^2 + k^2)e^{-\frac{2gh}{k^2}}$.

By Art. 106, when the particle has fallen through $h - y$,

$$v_1^2 = k^2\left[1 - e^{-\frac{2g}{k^2}(h-y)}\right], \text{ and } K' = \tfrac{1}{2}mk^2.$$

$$\therefore v_1^2 = k^2\left[1 - \frac{k^2}{V^2 + k^2}\cdot\frac{V^2 + k^2}{v^2 + k^2}\right] = \frac{k^2v^2}{v^2 + k^2}. \quad \therefore \text{new energy} = \frac{KK'}{K + K'}.$$

11. $\frac{v\,dv}{dx} = \frac{C}{v} - v^2$, since the force exerted by the engine $\times$ the velocity $=$ the constant horse-power.

$$\therefore x = \int \frac{v^2\,dv}{C - \mu v^3} = -\frac{1}{3\mu}\log\frac{C - \mu v^3}{C}, \text{ since } v = 0 \text{ when } x = 0.$$

$\therefore C - \mu v^3 = Ce^{-3\mu x}$; the full speed V is given by $C - \mu V^3 = 0$, so that $v^3 = V^3(1 - e^{-3\mu x})$.

Hence $\quad v = \frac{1}{2}V$ when $e^{-3\mu x} = \frac{7}{8}$, and $\therefore x = \frac{1}{3\mu}\log_e\frac{8}{7}$.

Again $\quad \frac{dv}{dt} = \frac{C}{v} - \mu v^2 = \mu\frac{V^3 - v^3}{v}$, so that

$$\mu t = \int_0^v \frac{v\,dv}{V^3 - v^3} = \frac{1}{3V}\int_0^v\left[\frac{1}{V - v} + \frac{v + \frac{V}{2}}{v^2 + Vv + V^2} - \frac{\frac{3V}{2}}{v^2 + Vv + V^2}\right]dv$$

$$= \frac{1}{3V}\left[-\log\left(\frac{V - v}{V}\right) + \frac{1}{2}\log\left(\frac{v^2 + Vv + V^2}{V^2}\right) - \sqrt{3}\tan^{-1}\frac{2v + V}{\sqrt{3}V} + \sqrt{3}\frac{\pi}{6}\right].$$

When $v = \frac{V}{2}$, we then obtain the time to half-speed. When $v = V$, $t = \infty$.

12. When $v \leq 2$, the resistance is λv, and, when $v \geqq 2$, the resistance is μv^2, so that $\lambda . 2 = \mu . 2^2$, *i.e.* $\lambda = 2\mu$.

Whilst $v > 2$, we have $v\frac{dv}{dx} = \frac{dv}{dt} = -\mu v^2$.

Hence $\quad \mu t = \frac{1}{v} + A$, where $0 = \frac{1}{10} + A$ and $\mu . 60 = \frac{1}{6} + A$.

$$\therefore A = -\frac{1}{10} \text{ and } \mu = \frac{1}{900}, \text{ so that } v\frac{dv}{dx} = -\frac{v^2}{900}.$$

Hence $\quad -\log_e v = \frac{x}{900} + B = \frac{x}{900} - \log_e 10$, so that, when $v = 2$, then

$$x_1 = 900\log_e 5.$$

The equation of motion then becomes

$$\frac{v\,dv}{dx} = -\lambda v = -2\mu v = -\frac{1}{450}v, \text{ so that } v = -\frac{x}{450} + 2.$$

Hence when $v = 0$, $x_2 = 900$. Hence $x_1 + x_2 =$ etc.

13. $v\dfrac{dv}{dx} = -\mu x + kv^2$. $\therefore \dfrac{dv^2}{dx} - 2k \cdot v^2 = -2\mu x$.

$$\therefore v^2 e^{-2kx} = -2\mu \int_a^x x e^{-2kx} = \frac{\mu}{k} e^{-2kx}\left(x + \frac{1}{2k}\right) - \frac{\mu}{k} e^{-2ka}\left(a + \frac{1}{2k}\right).$$

$$\therefore v^2 = \frac{\mu x}{k} - \frac{\mu a}{k} e^{2k(x-a)} + \frac{\mu}{2k^2}\left[1 - e^{2k(x-a)}\right].$$

The particle passes through the origin O and comes to rest at the point $x = -b$, such that $0 = -b - ae^{-2k(b+a)} + \dfrac{1}{2k}\left[1 - e^{-2k(b+a)}\right]$, etc.

14. The acceleration at distance $x = \dfrac{\mu}{x^2} = \dfrac{gr^2}{x^2}$.

$$\therefore \frac{d}{dx}\left(\frac{v^2}{2}\right) = v\frac{dv}{dx} = \ddot{x} = -\frac{gr^2}{x^2} + \mu v^2.$$

$$\therefore v^2 e^{-2\mu x} = -2gr^2 \int_a^x \frac{e^{-2\mu x}}{x^2} = -2gr^2 \int_a^x \frac{1 - 2\mu x + \text{sqs. of } \mu}{x^2}\,dx$$

$$= -2gr^2\left[-\frac{1}{x} + \frac{1}{a} - 2\mu \log\frac{x}{a}\right].$$

$$\therefore v^2 = 2gr^2\left[\frac{1}{x} - \frac{1}{a} + 2\mu \log\frac{x}{a}\right]\left[1 + 2\mu x + \text{sqs. of } \mu\right].$$

$$\therefore \frac{1}{2}mv^2 = mgr^2\left[\frac{1}{x} - \frac{1}{a} + 2\mu\left(1 - \frac{x}{a}\right) - 2\mu \log\frac{a}{x}\right], \text{ neglecting squares of } \mu.$$

15. $v\dfrac{dv}{dx} = -\mu x + kv^2$.

$$\therefore \frac{v^2}{\mu} e^{-2kx} = -\int_a^x 2x e^{-2kx}\,dx = \frac{1 + 2kx}{2k^2} e^{-2kx} - \frac{1 + 2ka}{2k^2} e^{-2ka}.$$

$$\therefore \frac{V'^2}{\mu} = \frac{1 + 2kx}{2k^2} - \frac{1 + 2ka}{2k^2}\left[1 + 2k(x-a) + \frac{4k^2(x-a)^2}{2!} + \frac{8k^3(x-a)^3}{3!} + \dots\right]$$

$$= a^2 - x^2 - k(x-a)^2 \cdot \tfrac{2}{3} \cdot (x + 2a), \text{ on reduction.}$$

Also $$\frac{V^2}{\mu} = a^2 - x^2. \quad \therefore \frac{V'^2}{V^2} = 1 - \frac{2k}{3}\frac{(a-x)(2a+x)}{a+x},$$

and hence $$V' = V\left[1 - \frac{1}{3}k\frac{(a-x)(2a+x)}{a+x}\right].$$

End of Art 111
EXAMPLES

1. With the results of Art. 109, we have

$$-\tan\alpha = \frac{\dot{y}}{\dot{x}} = -\frac{g}{ku\cos\alpha\, e^{-kt}} + \frac{ku\sin\alpha + g}{ku\cos\alpha}. \quad \therefore\ ge^{kt} = 2ku\sin\alpha + g, \text{ etc.}$$

2. By Art. 109, y is zero when $kgt = (g + ku\sin\alpha)(1 - e^{-kt})$, ...(1)

and then $$x = \frac{u\cos\alpha}{k}(1 - e^{-kt}), \quad \text{..........(2)}$$

x is thus a maximum when $\frac{dx}{d\alpha} = 0$, *i.e.* when $\frac{dt}{d\alpha}\cos\alpha\, ke^{-kt} = \sin\alpha\,(1 - e^{-kt})$.

But from (1) $\frac{dt}{d\alpha}[g(1 - e^{-kt}) - ku\sin\alpha\, e^{-kt}] = u\cos\alpha\,(1 - e^{-kt})$.

Eliminating t, we have $e^{-kt} = \dfrac{g\sin\alpha}{ku + g\sin\alpha}$.

Hence (1) gives $g\log\dfrac{ku + g\sin\alpha}{g\sin\alpha} = (g + ku\sin\alpha)\cdot\dfrac{ku}{ku + g\sin\alpha}$.

Also $u \div \dfrac{g}{k} = \lambda$, so that $\log[1 + \lambda\operatorname{cosec}\alpha] = \dfrac{\lambda(1 + \lambda\sin\alpha)}{\lambda + \sin\alpha}$.

Changing α into $90° - \alpha$, we have the result stated.

3. By Art. 109, $\ddot{x} = -ku\cos\alpha\, e^{-kt}$ and $\ddot{y} = -(ku\sin\alpha + g)\, e^{-kt}$.

$\therefore\ \dfrac{\ddot{y}}{\ddot{x}} = \text{const.}$, and total acc. $= e^{-kt}\sqrt{g^2 + 2kug\sin\alpha + k^2u^2\cos^2\alpha}$,

which continually diminishes as t increases and is ultimately zero.

4. By Art. 109, the time t_1 to the highest point, where $\dot{y}$ is zero, is given by $t_1 = \frac{1}{k}\log\mu$, where $ku\sin\alpha = g(\mu - 1)$.

Also the height y_2 at time $2t_1$ is, by equation (4), given by

$$\frac{k^2}{g}y_2 = -2\log\mu + \mu - \frac{1}{\mu} = \phi(\mu) \text{ say.}$$

Now $$\phi'(\mu) = -\frac{2}{\mu} + 1 + \frac{1}{\mu^2} = \left(1 - \frac{1}{\mu}\right)^2 = \text{positive.}$$

$\therefore\ \phi(\mu)$ is always increasing as μ increases, and it is zero when $\mu = 1$, so that $\phi(\mu)$ is always positive.

Hence y_2 is positive, and so at time $2t_1$ the particle has not yet reached the ground. Hence, etc.

5. We have, as in Art. 109,

$$x = \frac{V\cos\alpha}{\mu}(1 - e^{-\mu t}) \text{ and } y = -\frac{gt}{\mu} + \left(\frac{V\sin\alpha}{\mu} + \frac{g}{\mu^2}\right)(1 - e^{-\mu t}).$$

$$\therefore\ e^{-\mu t} = 1 - \frac{\mu x}{V\cos\alpha} = \frac{\mu X}{V\cos\alpha} \text{ (say).}$$

$$\therefore\ y = \frac{g}{\mu^2}\log\frac{\mu X}{V\cos\alpha} + \frac{\mu V\sin\alpha + g}{\mu^2}\left(1 - \frac{\mu X}{V\cos\alpha}\right).$$

$$\therefore\ \frac{g}{\mu^2}\log X - \frac{\mu V\sin\alpha + g}{\mu V\cos\alpha}X = y - \frac{g}{\mu^2}\log\frac{\mu}{V\cos\alpha} - \frac{\mu V\sin\alpha + g}{\mu^2} = Y \text{ (say).}$$

Hence, etc.

6. Here $$v\frac{dv}{ds} = -ng + g\sin\phi, \text{ and } \frac{v^2}{\rho} = g\cos\phi.$$

$\therefore\ \dfrac{dv}{v} = \dfrac{\sin\phi - n}{\cos\phi}\,d\phi$, so that $\log v = -\log\cos\phi - n\log\tan\left(\dfrac{\pi}{4} + \dfrac{\phi}{2}\right) + \log V$,
since $v = V$ when $\phi = 0$.

$$\therefore\ v = V\left(\frac{1-\sin\phi}{1+\sin\phi}\right)^{\frac{n}{2}} \frac{1}{\sqrt{1-\sin^2\phi}} = \text{as stated.}$$

7. $s = 4a\sin\theta$, and $\rho = \dfrac{ds}{d\theta} = 4a\cos\theta$.

Hence $$\frac{v\,dv}{ds} = -\frac{v^2}{2c} + g\sin\theta = -\frac{v^2}{2c} + \frac{g}{4a}s.$$

$$\therefore\ v^2 e^{\frac{s}{c}} = \frac{g}{2a}\int s e^{\frac{s}{c}}\,ds = \frac{gc}{2a}(s-c)\,.\,e^{\frac{s}{c}}, \text{ the constant vanishing.}$$

$$\therefore\ t\sqrt{\frac{gc}{2a}} = \int_c^{4a} \frac{ds}{\sqrt{s-c}} = 2\sqrt{4a-c}.$$

8. Here $$v\frac{dv}{ds} = -g\sin\theta - \mu v^2 = -g\frac{s}{4a} - \mu v^2.$$

$$\therefore\ v^2 e^{2\mu s} = -\frac{g}{2a}\int s e^{2\mu s} = -\frac{g}{4\mu a}\left(s - \frac{1}{2\mu}\right)e^{2\mu s} + A.$$

Now $v = 0$ when $s = 2\sqrt{2ax}$, and $v = V$ when $s = 0$.

$$\therefore\ V^2 = \frac{g}{4\mu a}\left[2\sqrt{2ax} - \frac{1}{2\mu}\right]e^{4\mu\sqrt{2ax}} + \frac{g}{8\mu^2 a}.$$

9. Take the given straight line as the initial line, so that
$$r = ae^{\lambda\theta} \text{ and } \dot{\theta} = -\omega.$$
Let ξ be the projection of r, so that $\xi = r\cos\theta = ae^{\lambda\theta}\,.\,\cos\theta$.

$\therefore\ \dot{\xi} = -a\omega e^{\lambda\theta}(\lambda\cos\theta - \sin\theta)$, and $\ddot{\xi} = a\omega^2 e^{\lambda\theta}[(\lambda^2 - 1)\cos\theta - 2\lambda\sin\theta]$.

$$\therefore\ \ddot{\xi} = -(1+\lambda^2)\omega^2\xi + 2\lambda\omega\dot{\xi},$$

giving a resisted simple harmonic motion.

10. $\dfrac{v\,dv}{ds} = -\dfrac{\mu}{r^n}\cos\alpha - P$, and $\dfrac{v^2}{\rho} = \dfrac{\mu}{r^n}\sin\alpha$,

where $$p = r\sin\alpha,\ \rho = r\frac{dr}{dp} = \frac{r}{\sin\alpha} \text{ and } \therefore\ v^2 = \frac{\mu}{r^{n-1}}.$$

$$\therefore\ P = -\frac{\mu}{r^n}\cos\alpha - \frac{v\,dv}{dr}\cos\alpha = \frac{n-3}{2}\cdot\frac{\mu\cos\alpha}{r^n}.$$

11. $\ddot{x} = -\mu x - k\dot{x}$, and $\ddot{y} = -\mu y - k\dot{y}$.

$$\therefore\ x = Ae^{pt} + Be^{qt}, \text{ and } y = Ce^{pt} + De^{qt},$$

where $$p = \tfrac{1}{2}\left[-k + \sqrt{k^2 - 4\mu}\right] \text{ and } q = \tfrac{1}{2}\left[-k - \sqrt{k^2 - 4\mu}\right].$$

The initial conditions determine the constants and we thus have the path. If $2k^2 = 9\mu$, then $p = -\sqrt{\frac{\mu}{2}}$ and $q = -2\sqrt{\frac{\mu}{2}}$.

$$\therefore\ x = Ae^{-p_1 t} + Be^{-2p_1 t}, \text{ and } y = Ce^{-p_1 t} + De^{-2p_1 t}, \text{ where } p_1 = \sqrt{\frac{\mu}{2}}.$$

$$\therefore\ (Dx - By)^2 = (AD - BC)(Ay - Cx), \text{ i.e. a parabola.}$$

The particle will be at rest at the origin at time t given by

$$0 = e^{-p_1 t}[A + Be^{-p_1 t}], \qquad 0 = e^{-p_1 t}[A + 2Be^{-p_1 t}],$$

$$0 = e^{-p_1 t}[C + De^{-p_1 t}], \text{ and } 0 = e^{-p_1 t}[C + 2De^{-p_1 t}],$$

which are satisfied only by $e^{-p_1 t} = 0$, and then t is infinite.

12. $\ddot{y} = -g$, so that $\dot{y} = v - gt$, $y = vt - \frac{1}{2}gt^2$, and $v^2 = 2gh$.

Then $\ddot{x} = -\mu\dot{y}$, so that $\dot{x} = u - \mu vt + \frac{1}{2}\mu g t^2$, and $x = ut - \frac{\mu v t^2}{2} + \frac{\mu g}{6}t^3$.

Now $y = 0$ when $t = \frac{2v}{g}$, and then x is given to be zero, so that

$$u = \frac{\mu v}{2}t - \frac{\mu g}{6}t^2 = \frac{1}{3}\frac{\mu v^2}{g}.$$

$$\therefore\ x = \frac{g\mu}{6}\left(t^3 - \frac{3v}{g}t^2 + \frac{2v^2}{g^2}t\right) = \frac{\mu}{3}\left(\frac{v}{g} - t\right)\left(vt - \frac{1}{2}gt^2\right).$$

Also $$h - y = \frac{v^2}{2g} - vt + \frac{1}{2}gt^2 = \frac{g}{2}\left(\frac{v}{g} - t\right)^2,$$

so that $$x = \frac{\mu}{3}y\sqrt{\frac{2}{g}(h-y)}, \text{ and hence } 9gx^2 = 2\mu^2 y^2 (h-y).$$

Now $$\frac{dx}{dy} = 0, \text{ when } y = \frac{2h}{3}, \text{ and thus } 9gc^2 = \frac{8\mu^2 h^3}{27},$$

so that $$4h^3 x^2 = 27c^2 y^2 (h-y).$$

13. $\ddot{r} - r\dot{\theta}^2 = -P - kv\frac{dr}{ds}$, and $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = -kv \cdot \frac{rd\theta}{ds}$.

$$\therefore\ \frac{d}{dt}(r^2\dot{\theta}) = -kr^2\frac{d\theta}{dt}, \text{ so that } \log(r^2\dot{\theta}) = -kt + \text{const.}$$

$$\therefore\ r^2\dot{\theta} = he^{-kt},\ \frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}e^{-kt},$$

and $$\frac{d^2r}{dt^2} = hk\frac{du}{d\theta}e^{-kt} - h \cdot e^{-kt} \cdot \frac{d^2u}{d\theta^2} \cdot hu^2 e^{-kt}.$$

Substituting in the first equation, we have the given result.

14. $\ddot{r} - r\dot{\theta}^2 = -P - kv^2\frac{dr}{ds}$, and $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -kv^2 \cdot \frac{rd\theta}{ds} = -kr\frac{d\theta}{dt} \cdot \frac{ds}{dt}$.

$$\therefore\ \log\left(r^2\frac{d\theta}{dt}\right) = -ks + \text{const.}$$

$$\therefore\ r^2\dot{\theta} = he^{-ks},\ \dot{r} = -\frac{du}{d\theta} \cdot he^{-ks}, \text{ and } \ddot{r} = -\frac{d^2u}{d\theta^2} \cdot h^2u^2e^{-2ks} + hk\frac{du}{d\theta}e^{-ks}\frac{ds}{dt}.$$

Substitute in the first equation, etc.

15. $v\frac{dv}{ds} = -P\frac{dr}{ds} - Q$, and $\frac{v^2}{\rho} = P \cdot \frac{p}{r}$, where Q is the resistance.

$$\therefore -2p^3 Q = p^2 \frac{d}{ds}\left[Pp\frac{dr}{dp}\right] + 2Pp^2\frac{dr}{ds} = \frac{d}{ds}\left[Pp\frac{dr}{dp} \cdot p^2\right] = \text{etc.}$$

End of Art 113

EXAMPLES

1. $\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{dM}{dt} = k \cdot 4\pi r^2$, and hence $r = kt + a$.

Also $\frac{d}{dt}(M\dot{x}) = Mg$, so that $\frac{d}{dt}(r^3\dot{x}) = gr^3 = \frac{g}{k} \cdot r^3\dot{r}$.

$$\therefore \dot{x}r^3 = \frac{g}{4k}(r^4 - a^4), \text{ i.e. } \dot{x} = \frac{g}{4k^2}\left(r - \frac{a^4}{r^3}\right)\dot{r}.$$

$$\therefore x = \frac{g}{8k^2}\left(r^2 + \frac{a^4}{r^2} - 2a^2\right) = \frac{g}{8k^2}\left(\frac{r^2 - a^2}{r}\right)^2.$$

Hence, when $x = h$, r is as stated.

2. The mass, when a distance x has been described, is $M + \pi c^2 x\rho$. Hence $\frac{d}{dt}[(M + \pi c^2 x\rho)\dot{x}] = 0$.

$$\therefore (M + \pi c^2 x\rho)\dot{x} = \text{const.} = Mu.$$

$\therefore (M + \pi c^2 x\rho)^2 = 2M\pi c^2 \rho u t + M^2$, since x and t vanished together.

3. $m\delta v + \delta m(v + \delta v + V) =$ increase in the momentum in time $t = F\delta t$, so that

$$m\frac{dv}{dt} + (v + V)\frac{dm}{dt} = F.$$

Also $m = M + \rho t$. $\therefore (M + \rho t)v = \int(F - V\rho)\,dt = kt$.

$$\therefore s = \int\frac{kt}{M+\rho t} = \frac{k}{\rho}\left[t - \frac{M}{\rho}\log(M + \rho t)\right] + \text{const.} = \frac{k}{\rho}\left[t - \frac{M}{\rho}\log\frac{M+\rho t}{M}\right]$$

$$= \frac{k}{\rho^2}\left[m - M\left(1 + \log\frac{m}{M}\right)\right].$$

4. $r = \frac{400 + t}{12 \times 10^4}$. Also $\frac{d}{dt}\left(\frac{4}{3}\pi r^3\dot{x}\right) = \frac{4\pi}{3}r^3 g$,

so that $\frac{d}{dt}[(400+t)^3\dot{x}] = g(400+t)^3$. $\therefore (400+t)^3\dot{x} = \frac{g}{4}[(400+t)^4 - 400^4]$.

$$\therefore x = \frac{g}{8}\left[(400+t)^2 + \frac{400^4}{(400+t)^2} - 2 \cdot 400^2\right] = \frac{g}{8}\left[400 + t - \frac{400^2}{400+t}\right]^2.$$

Hence when $x = 6400$, then $(400+t)^2 - 40(400+t) = 400^2$.

$$\therefore (t + 400) - 20 = \sqrt{160400} = 400 \cdot 5.$$

$$\therefore t = 20 \cdot 5, \text{ and } r = \frac{420 \cdot 5}{12 \times 10^4} \text{ ft.} = \cdot 0420 \text{ inch.}$$

5. (i) If y be the length on the roof at time t, then

$$\frac{d}{dt}[y(-\dot{y})]=yg\sin\alpha, \text{ so that } y^2\dot{y}^2=\frac{2g\sin\alpha}{3}[a^3-y^3].$$

$$\therefore \sqrt{\frac{2g\sin\alpha}{3}}\,T=\int_0^a \frac{y\,dy}{\sqrt{a^3-y^3}} \qquad [\text{Put } y=aY^{\frac{1}{3}}.]$$

$$=\int_0^1 \frac{\sqrt{a}}{3}\,Y^{-\frac{1}{3}}(1-Y)^{-\frac{1}{2}}\,dY=\frac{\sqrt{a}}{3}\,B\left(\frac{2}{3},\frac{1}{2}\right).$$

$$\therefore \sqrt{\frac{6g\sin\alpha}{a}}\,T=\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{3}+\frac{1}{2}\right)}=\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{6}\right)}=6\sqrt{\pi}\,\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{6}\right)}.$$

(ii) When a length x is in motion, we have

$$\frac{d}{dt}\left(x\frac{dx}{dt}\right)=g\sin\alpha\,.\,x, \text{ so that } x^2\dot{x}^2=\frac{2g\sin\alpha}{3}\,x^3.$$

$$\therefore \sqrt{\frac{2g\sin\alpha}{3}}\,.\,T=\int_0^a\frac{dx}{\sqrt{x}}=\left(2x^{\frac{1}{2}}\right)_0^a=2a^{\frac{1}{2}}.$$

6. $\frac{d}{dt}\left[(m+\mu t)\frac{dx}{dt}\right]=0$, so that $(m+\mu t)\frac{dx}{dt}=mu$, and hence

$$x=\frac{mu}{\mu}\log\frac{m+\mu t}{m}, \text{ and } \frac{dx}{dt}=ue^{-\frac{\mu x}{mu}}.$$

Also $$\frac{d}{dt}\left[(m+\mu t)\frac{dy}{dt}\right]=-(m+\mu t)g, \text{ so that}$$

$$\frac{d}{dx}\left[mu\frac{dy}{dx}\right]\frac{dx}{dt}=-(m+\mu t)\,g, \text{ and } \therefore u^2\frac{d^2y}{dx^2}=-ge^{\frac{2\mu x}{mu}}=-ge^{\frac{kx}{u}}.$$

$$\therefore u\frac{dy}{dx}=-\frac{g}{k}e^{\frac{kx}{u}}+\left[v+\frac{g}{k}\right]. \quad \therefore k^2uy=-gu\,e^{\frac{kx}{u}}+k\,(g+kv)\,x+gu.$$

7. The mass at time $t=m\,(1+\lambda t)^3$.

Then $\frac{d}{dt}[m(1+\lambda t)^3\,\dot{x}]=0$, so that $\dot{x}(1+\lambda t)^3=$ given horizontal velocity $=2\lambda a$ (say).

$$\therefore x=a\left[1-\frac{1}{(1+\lambda t)^2}\right]. \quad \text{Also } \frac{d}{dt}[m\,(1+\lambda t)^3\,\dot{y}]=m\,(1+\lambda t)^3\,g.$$

$\therefore \dot{y}\,(1+\lambda t)^3=\frac{g}{4\lambda}\left[(1+\lambda t)^4-1\right]$, and hence, if y and t vanish together,

$$y=\frac{g}{8\lambda^2}\left[(1+\lambda t)^2+\frac{1}{(1+\lambda t)^2}\right]+C=\frac{g}{8\lambda^2}\left[1+\lambda t-\frac{1}{1+\lambda t}\right]^2.$$

Eliminating t, we have $\quad x^2+\frac{8\lambda^2 a}{g}\,xy=\frac{8\lambda^2 a^2 y}{g}$,

which is a hyperbola with one asymptote parallel to $x=0$.

8. If u, u' are the velocities communicated in unit time by the explosion then, at the start, $(M-eM)\,u=eMu'$, and $u'+u=V$, so that $u=eV$. Also gravity in unit time gives a velocity g. Hence the rocket will not rise at once, unless $eV>g$.

When all the powder is burnt, except the last element, similarly $M'u=eMu'$, and $u'+u=V$, so that $u=\frac{eMV}{M'+eM}$. Hence it will not start at all unless $eMV>(M'+eM)\,g>M'g$.

When the rocket has risen a distance x in time t, then

$$\frac{d}{dt}[(M-eMt)\,\dot{x}]=\text{change in the momentum}=-eM(\dot{x}-V)-(M-eMt)\,g.$$

$$\therefore\ \ddot{x}=\frac{eV}{1-et}-g\ ;\ \text{hence}\ \dot{x}=-V\log(1-et)-gt,$$

and
$$x=\frac{V}{e}\left[(1-et)\{\log(1-et)-1\}\right]-\frac{gt^2}{2}+\frac{V}{e}.$$

The greatest velocity is attained when the powder is all burnt, *i.e.* when

$$M-eMt=M',\ \text{i.e. when}\ \dot{x}=V\log\frac{M}{M'}-\frac{g}{e}\left(1-\frac{M'}{M}\right)=V_1,$$

and then
$$x=\frac{V}{e}\left(1-\frac{M'}{M}\right)-\frac{M'V}{eM}\log\frac{M}{M'}-\frac{g}{2e^2}\left(1-\frac{M'}{M}\right)^2=x_1,$$

and total height attained $=x_1+\frac{V_1^2}{2g}=$ the result given.

9. The moving part of the chain falls with acceleration g, so that, when a length x has been deposited, $v^2=2g(l+x)$.

The part of the reaction, R, due to the impact is given by $R\delta t=m\delta x\,.\,v$, so that $R=mv^2=2mg(l+x)$. Hence total pressure $=R+mgx=mg(2l+3x)$, so that, when $x=\frac{l}{2}$, this $=\frac{7}{2}mg$.

10. When the chain has fallen a distance x, a mass $m\dot{x}$ of the chain is deposited per unit of time.

$$\therefore\ \frac{d}{dt}[m(a-x)\,\dot{x}]=\text{change in the momentum per unit of time}$$

$$=-m\dot{x}\,.\,\dot{x}+\int_0^{a-x}\frac{\gamma E\,.\,m\,dy}{(r+a-x-y)^2},\ \text{where}\ \frac{\gamma E}{r^2}=g.$$

$$\therefore\ (a-x)\,\ddot{x}=gr^2\left[\frac{1}{r+a-x-y}\right]_0^{a-x}=\frac{gr(a-x)}{r+a-x}.$$

$$\therefore\ \dot{x}^2=-2gr\log(r+a-x)+2gr\log(r+a)=\text{etc.}$$

11. $\frac{d}{dt}\left(\frac{Mx}{l}\dot{x}\right)=\frac{Mx}{l}g$, so that $\dot{x}^2=\frac{2g}{3}x$. Hence, just before the mass is jerked, the velocity V of the chain $=\sqrt{\frac{2gl}{3}}$. The horizontal velocity communicated to the mass $= \frac{m}{m+m} V = \frac{V}{2}$. When the mass is just off the plane, the jerk gives it a vertical velocity of $\frac{m}{m+m} \frac{V}{2}$, i.e. $\frac{V}{4}$.

$\therefore$ total velocity of the particle then $= \sqrt{\frac{V^2}{4} + \frac{V^2}{16}} = \frac{V}{4} . \sqrt{5} = \frac{1}{2} \sqrt{\frac{5gl}{6}}$.

12. $\ddot{x} = \dfrac{\frac{l}{2} - \left(\frac{l}{2} - x\right)}{\frac{l}{2} + \left(\frac{l}{2} - x\right)} g = \frac{x}{l-x} g = g\left[\frac{l}{l-x} - 1\right]$.

$$\therefore \dot{x}^2 = 2g\left[-l \log (l-x) - x + l \log l\right] = 2g\left[l \log \frac{l}{l-x} - x\right].$$

Hence, as in Ex. 9, the pressure $= \frac{M}{l} \dot{x}^2 + \frac{W}{l} x = W\left[2 \log \frac{l}{l-x} - \frac{x}{l}\right]$.

Also $\quad T - \frac{Mg}{l}\left(\frac{l}{2} - x\right) - \frac{M}{l}\left(\frac{l}{2} - x\right) \ddot{x} = \frac{Mg}{2l} \frac{l-2x}{l-x} . x$;

and the pressure on the peg $= 2T$.

13. The momentum of the moving parts is constant since no forces act on the system.

$$\therefore (M+mx) \dot{x} = \text{const.} = MV. \quad \therefore \ddot{x} = -\frac{Mm V \dot{x}}{(M+mx)^2} = -\frac{M^2 m V^2}{(M+mx)^3},$$

so that M moves as if it were acted upon as stated.

Also $\quad \frac{d}{dt}\left[(M+mx)\frac{\dot{x}^2}{2}\right] = \frac{d}{dt}\left(\frac{MV}{2} \dot{x}\right) = \frac{1}{2} MV\ddot{x} = -\frac{m}{2} . \dot{x}^3$. Hence, etc.

14. A mass ml is always in motion and, $m\dot{x}$ being deposited per unit of time,

$$\therefore ml\ddot{x} = mg\left[(l-x) - x\right] - m\dot{x} . \dot{x}.$$

$$\therefore l\ddot{x} + \dot{x}^2 = g(l-2x), \text{ i.e. } \dot{x}^2 e^{\frac{2x}{l}} = 2\frac{g}{l}\int (l-2x) e^{\frac{2x}{l}} dx.$$

$$\therefore \dot{x}^2 e^{\frac{2x}{l}} = 2g(l-x) e^{\frac{2x}{l}} - 2gl, \text{ since } \dot{x} = 0 \text{ initially.}$$

Also $\dot{x} = 0$ again when $(l-x) e^{\frac{2x}{l}} = l$.

Since energy is lost by the deposit of the portions of the chain on the table the Principle of Energy cannot be directly applied.

15. $\frac{d}{dt}\left[\{M + m(x+a+b)\} \dot{x}\right] = \left[M + m(x-a)\right] g$.

If $M = m\left(2a + \frac{b}{2}\right)$, this gives

$$\left(x + 3a + \frac{3b}{2}\right)^2 \dot{x}^2 = 2g \int \left(x + a + \frac{b}{2}\right)\left(x + 3a + \frac{3b}{2}\right) dx$$

$$= \frac{2gx}{3}\left[x^2 + 6\left(a + \frac{b}{2}\right) x + 9\left(a + \frac{b}{2}\right)^2\right], \text{ so that } \dot{x}^2 = \frac{2gx}{3}, \text{ etc.}$$

16. $\frac{d}{dt}[(M+mx)\dot{x}] = -\mu(M+mx)g$, so that

$$(M+mx)^2\dot{x}^2 = -\frac{2\mu g}{3m}(M+mx)^3 + M^2V^2 + \frac{2\mu M^3}{3m}g.$$

$\dot{x}$ is zero when $(M+mx)^3 = M^3 + \frac{3M^2m}{2\mu g}V^2$, etc.

17. $\frac{d}{dt}\left[\left(M+\frac{Mx}{l}\right)\dot{x}\right] = \left(M+\frac{Mx}{l}\right)g(\sin\alpha - \mu\cos\alpha)$.

$$\therefore \dot{x}^2(x+l)^2 = 2g\sec\epsilon\sin(\alpha-\epsilon)\left[\frac{x^3}{3} + lx^2 + l^2x\right] + l^2V^2.$$

$\dot{x}$ is zero, when $x=l$, if, etc.

18. $\frac{d}{dt}[(mc+mx)\dot{x}] = -m(c+x)g$, so that

$$(c+x)^2\dot{x}^2 = -\frac{2g}{3}(x^3+3cx^2+3c^2x) + c^2 . 2gh.$$

Let the chain leave the floor. Then when $x=l$, $(c+l)^2V^2 = \frac{2g}{3}[a^3-(l+c)^3]$, and the total height $= l + \frac{V^2}{2g} =$ etc.

Let the chain not leave the floor. Then $\dot{x}=0$, when $(x+c)^3 = a^3$, *i.e.* when $x = a - c$.

Also the mass falls freely from this height x.

19. $\frac{d}{dt}[m(2h+x+h)\dot{x}] = mg(2h+x) - mgh$.

$$\therefore (x+3h)^2\dot{x}^2 = 2g\int(x+h)(x+3h) = \frac{2g}{3}[x^3+6hx^2+9h^2x].$$

$\therefore \dot{x}^2 = \frac{2g}{3}x$, and the acceleration $= \frac{g}{3}$. When $x=h$, then $V^2 = \frac{2gh}{3}$.

After the impact, if v is the velocity, then

$$2h . v - v . \delta x = 2h(v+\delta v), \text{ so that } \frac{\delta v}{v} = -\frac{\delta x}{2h}.$$

$$\therefore \log v = -\frac{x-h}{2h} + \log V, \text{ etc.}$$

21. $A = V_0^2(M+m) - (2M-m)gl = \frac{M^2V^2}{M+m} - (2M-m)gl$.

Also, when $x=2l$, $(M+m)\dot{x}^2 = 4Mgl + A =$ etc.

22. If V is the given relative velocity and V_1 the starting velocity of the cord, then $mk(V-V_1) =$ pull of the monkey on the cord $= mlV_1$, so that

$$V_1 = \frac{k}{k+l}V.$$

Initially the lengths of the cord on the two sides of the pulley are

$$\frac{l-k}{2} \text{ and } \frac{l+k}{2}.$$

At time t, let a length x of the cord have gone over the pulley, and the monkey have ascended a distance y along the cord, so that $\dot{y}=V$ and $\ddot{y}=0$.

Then $$ml\ddot{x}=\left\{\left(\frac{l-k}{2}+x\right)-\left(\frac{l+k}{2}-x\right)\right\}mg+P=P+(2x-k)mg,$$

and $$mkg-P=mk(\ddot{x}-\ddot{y})=mk\ddot{x}.$$

$$\therefore\ (k+l)\ddot{x}=2xg, \text{ so that } (k+l)\dot{x}^2=2gx^2+\frac{k^2V^2}{k+l}.$$

$$\therefore\ t\sqrt{\frac{2g}{k+l}}=\sinh^{-1}\left[\frac{x}{kV}\sqrt{2g(k+l)}\right], \text{ the constant vanishing.}$$

The monkey stops ascending in space when $\dot{x}=V$, *i.e.* when

$$2gx^2=V^2\frac{l^2+2kl}{k+l}, \text{ and then } t\sqrt{\frac{2g}{k+l}}=\sinh^{-1}\sqrt{\frac{2l}{k}+\frac{l^2}{k^2}}.$$

$$\therefore\ \cosh^2\left(t\sqrt{\frac{2g}{k+l}}\right)=1+\sinh^2\left(t\sqrt{\frac{2g}{k+l}}\right)=1+\frac{2l}{k}+\frac{l^2}{k^2}=\left(1+\frac{l}{k}\right)^2, \text{ and}$$

hence as stated.

23. Let x on one side and $a+x_1$ on the other side be in motion at time t, so that $5a=x+(x_1+a)+x_1$, *i.e.* $x+2x_1=4a$.

Then $$\frac{d}{dt}[(x+a+x_1)\dot{x}]=\{xg-(a+x_1)g\}.$$

$$\therefore\ \frac{d}{dt}[(x+6a)\dot{x}]=3g(x-2a).$$

$$\therefore\ (x+6a)^2\dot{x}^2=6g\int(x-2a)(x+6a)\,dx$$
$$=2g(x^3+6ax^2-36a^2x+40a^3), \text{ (since } \dot{x}=0 \text{ when } x=2a)$$
$$=2g(x-2a)^2(x+10a).$$

When $x=4a$, then $\dot{x}=\frac{2}{5}\sqrt{7ga}$, and the required impulsive tension

$$=5ma\dot{x}=2ma\sqrt{7ga}.$$

24. $$\frac{d}{dt}[(M+M'-mt)\dot{x}]-mu=-\mu(M+M'-mt)g.$$

$$\therefore\ (M+M'-mt)\dot{x}=\frac{\mu g}{2m}(M+M'-mt)^2+mut-\frac{\mu g}{2m}(M+M')^2.$$

The mass M' has been fired when $mt=M'$, and then

$$M\dot{x}=\frac{\mu g}{2m}[M^2-(M+M')^2]+M'u, \text{ etc.}$$