Exercise 7.7

Answer 1E.

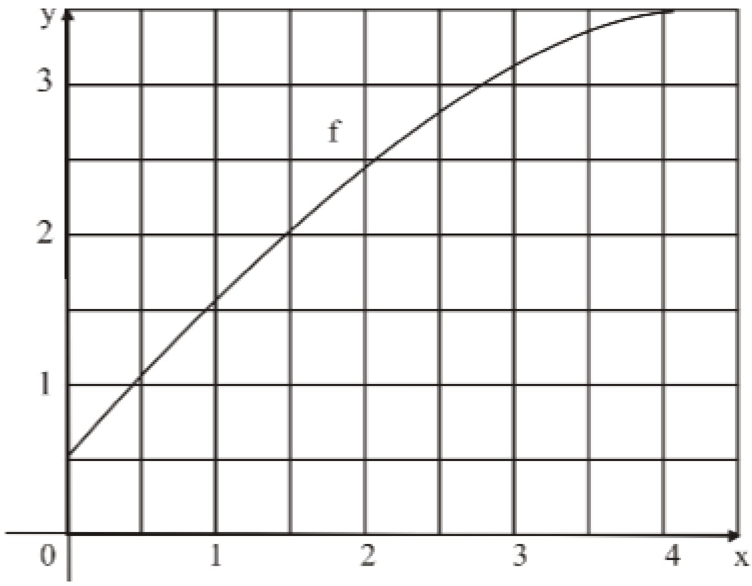


Fig. 1

(A)
$$L_2 = \frac{(4-0)}{2}\left[f(0)+f(2)\right]$$
$$= 2\left[0.5+2.5\right]$$
$$= 6 \Rightarrow \boxed{L_2 = 6}$$

$$R_2 = \frac{(4-0)}{2}\left[f(2)+f(4)\right]$$
$$= 2\left[2.5+3.5\right]$$
$$= 12 \Rightarrow \boxed{R_2 = 12}$$

$$M_2 = \frac{(4-0)}{2}\left[f(1)+f(3)\right]$$
$$\approx 2.(1.6+3.2)$$
$$\approx 4.8\times 2 \Rightarrow \boxed{M_2 \approx 9.6}$$

(B) Since this graph is concave downward
So L_2 is an underestimate, R_2 and M_2 are overestimates since the tangent lines will be over the graph

(C) $T_2 = \frac{(4-0)}{2} \times \frac{1}{2}[f(0)+2f(2)+f(4)]$

$= 1\times(0.5+2\times2.5+3.5)$

$= (0.5+5+3.5) = 9$

So $\boxed{T_2 = 9}$

Since line joining the points $(0, f(0))$ and $(2, f(2))$ and the line joining the points $(2, f(2))$ and $(4, f(4))$, both are lying under the graph

So $T_2 < I = \int_0^4 f(x)\,dx$

(D) For any value of n, $L_n < T_n < I < M_n < R_n$

Answer 2E.

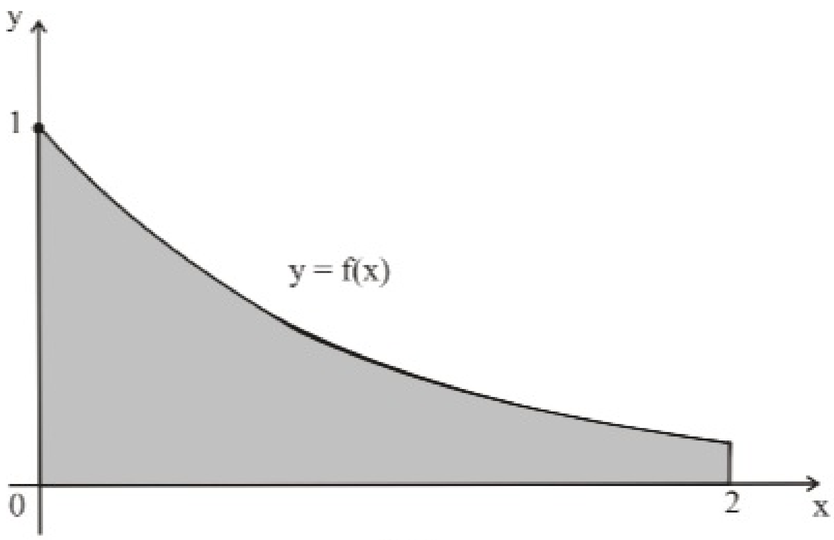


Fig. 1

(A) Since given graph of $f(x)$ is concave upward. So R_n will be underestimate and L_n will be overestimate. Since the graph lies above the tangent lines so M_n will be under estimate since M_n is the better approximation then L_n and R_n so we have $R_n < M_n < L_n$ and trapezoidal approximation $T_n = \frac{R_n + L_n}{2}$

The given estimates are 0.7811, 0.8675, 0.8632, and 0.9540

So lowest value $0.7811 = R_n$ and highest value $0.9540 = L_n$

Average of R_n and $L_n \approx 0.8675$ so $T_n = 0.8675$

And $M_n = 0.8632$

Thus $\boxed{R_n = 0.7811,\ M_n = 0.8632,\ T_n = 0.8675,\ L_n = 0.9540}$

(B) Since graph of $f(x)$ is concave upward so $M_n < \int_0^2 f(x)\,dx < T_n$

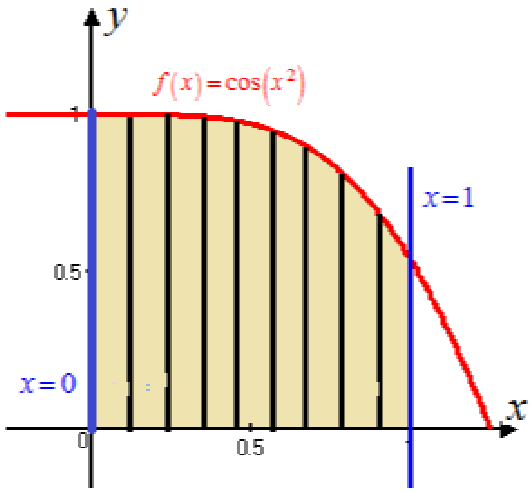
So $0.8632 < \int_0^2 f(x)\,dx < 0.8675$

Answer 3E.

Consider the integral:

$$\int_0^1 \cos\left(x^2\right) dx.$$

The graph of the region using trapezoidal rule is shown below:



(a) The given interval is $[0,1], n=4$, and $f(x)=\cos\left(x^2\right)$.

Then $\Delta x=\frac{1}{4}$ and sub-intervals are $\left[0,\frac{1}{4}\right],\left[\frac{1}{4},\frac{2}{4}\right],\left[\frac{2}{4},\frac{3}{4}\right]$, and $\left[\frac{3}{4},1\right]$.

And mid-points of the sub-intervals are $\frac{1}{8},\frac{3}{8},\frac{5}{8}$, and $\frac{7}{8}$.

The trapezoidal rule is

$$\int_a^b f(x)dx \approx T_n=\frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+....2f(x_{n-1})+f(x_n)\right].$$

Where $\Delta x=\frac{b-a}{n}$ and $x_i=a+i\Delta x$.

Then by applying the trapezoidal rule on the integral, we get

$$\int_0^1 \cos\left(x^2\right)dx \approx T_4=\frac{\Delta x}{2}\left[f(0)+2f\left(\frac{1}{4}\right)+2f\left(\frac{2}{4}\right)+2f\left(\frac{3}{4}\right)+f(1)\right]$$

$$=\frac{1}{8}\left[\cos(0)+2\cos\left(\left(\frac{1}{4}\right)^2\right)+2\cos\left(\left(\frac{1}{2}\right)^2\right)+2\cos\left(\left(\frac{3}{4}\right)^2\right)+\cos(1)\right]$$

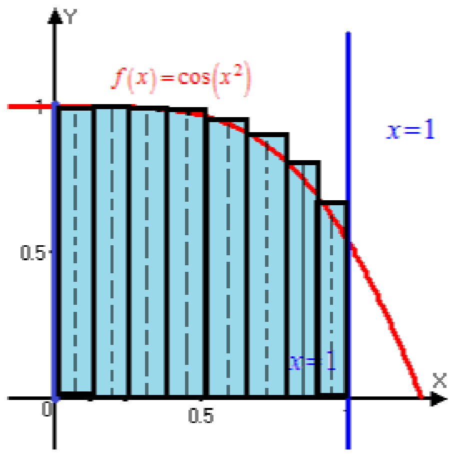
$$\approx 0.895759.$$

Hence the value of the integral is $\boxed{T_4\approx 0.895759}$.

Since the graph of $f(x)=\cos\left(x^2\right)$ is concave downward, so the lines joining the points $(0,\cos 0)$ and $\left(\frac{1}{4},\cos\frac{1}{16}\right),\left(\frac{1}{4},\cos\frac{1}{16}\right)$ and $\left(\frac{1}{2},\cos\frac{1}{4}\right),\left(\frac{1}{2},\cos\frac{1}{4}\right)$ and $\left(\frac{3}{4},\cos\frac{9}{16}\right)$, and $\left(\frac{3}{4},\cos\frac{9}{16}\right)$ and $(1,\cos 1)$ are lying under the graph,

So the value, $T_4\approx 0.895759$ is an underestimate.

(b) The graph of the region using mid-point rule is shown below:



The mid-point rule is

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\left[f(\bar{x}_1)+f(\bar{x}_2)+f(\bar{x}_3)+\ldots.f(\bar{x}_{n-1})+f(\bar{x}_n)\right].$$

Where $\Delta x = \dfrac{b-a}{n}$ and $\bar{x}_i = \dfrac{1}{2}(x_{i-1}+x_i)$

$= \text{Mid-point of } [x_{i-1}, x_i]$.

By using the mid points rule, we have

$$\int_0^1 \cos(x^2)\,dx \approx M_4 = \Delta x\left[f\left(\frac{1}{8}\right)+f\left(\frac{3}{8}\right)+f\left(\frac{5}{8}\right)+f\left(\frac{7}{8}\right)\right]$$

$$=\frac{1}{4}\left[\cos\left(\left(\frac{1}{8}\right)^2\right)+\cos\left(\left(\frac{3}{8}\right)^2\right)+\cos\left(\left(\frac{5}{8}\right)^2\right)+\cos\left(\left(\frac{7}{8}\right)^2\right)\right]$$

≈ 0.908907

Therefore, the value of the integral is $M_4 \approx 0.908907$.

Since the graph is lying under the tangent lines so $\boxed{M_4 \approx 0.908907}$ is an overestimate.

So the true value of the integral will be in between T_4 and M_4.

That is $T_4 < \int_0^1 \cos(x^2)\,dx < M_4$.

Answer 4E.

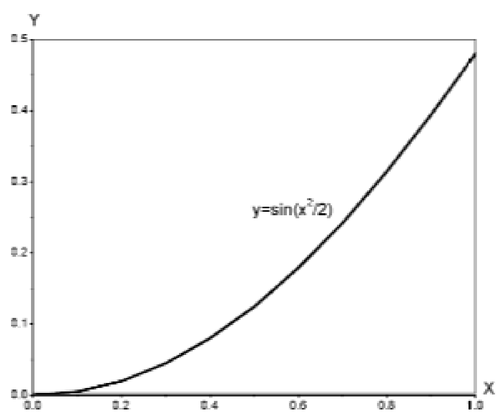


Fig.1

(A) Since this graph is concave upward. So L_2 and M_2 will be under estimates and R_2 and T_2 will be overestimate

(B) Since L_n and M_n are underestimates and M_n is better approximation then L_n

So $L_n < M_n < I$

And R_n and T_n are overestimates and T_n is the average of L_n and R_n so

$I < T_n < R_n$

Then we have $L_n < M_n < I < T_n < R_n$

(C) Here n = 5 and interval is [0, 1] then $\Delta x = \frac{1}{5} = 0.2$

Then sub intervals are [0, 0.2], [0.2, 0.4], [0.4, 0.6], [0.6, 0.8], [0.8, 1] and mid points are 0.1, 0.3, 0.5, 0.7, 0.9,

$$L_5 = 0.2\left[f(0)+f(0.2)+f(0.4)+f(0.6)+f(0.8)\right]$$

$$L_5 = 0.2\left[0+0.019999+0.79915+0.17903+0.314567\right] \approx 0.118702$$

$$\boxed{L_5 \approx 0.118702}$$

$$R_5 = 0.2\left[f(0.2)+f(0.4)+f(0.6)+f(0.8)+f(1)\right]$$

$$R_5 \approx 0.2\left[0.019999+0.079915+0.17903+0.314567+0.479426\right]$$

$$\boxed{R_5 \approx 0.214587}$$

$$M_5 = 0.2\left[f(0.1)+f(0.3)+f(0.5)+f(0.7)+f(0.9)\right]$$

$$M_5 = 0.2\left[0.005+0.044985+0.124675+0.242556+0.394019\right]$$

$$\boxed{M_5 \approx 0.162247}$$

$$T_5 = \frac{0.2}{2}\left[f(0)+2f(0.2)+2f(0.4)+2f(0.6)+2f(0.8)+f(1)\right]$$

$$\boxed{T_5 \approx 0.166644}$$

Midpoint rule gives the best estimate of I since from the graph we have $I \approx 0.16371405$

Answer 5E.

Let $f(x) = \frac{x}{1+x^2}, n = 10$

Here $a = 0, b = 2$ and $n = 10$

Therefore
$$\Delta x = \frac{b-a}{n} = \frac{2}{10} = \frac{1}{5}$$

Therefore $x_0 = 0, x_1 = \frac{1}{5}, x_2 = \frac{2}{5}, x_3 = \frac{3}{5}, x_4 = \frac{4}{5}$

$x_5 = 1, x_6 = \frac{6}{5}, x_7 = \frac{7}{5}, x_8 = \frac{8}{5}, x_9 = \frac{9}{5}, x_{10} = 2$

Therefore Midpoint Rule

(a)
$$M_{10} = \frac{1}{5}\begin{bmatrix} f\left(\frac{1}{10}\right)+f\left(\frac{3}{10}\right)+f\left(\frac{5}{10}\right)+f\left(\frac{7}{10}\right)+f\left(\frac{9}{10}\right)+ \\ f\left(\frac{11}{10}\right)+f\left(\frac{13}{10}\right)+f\left(\frac{15}{10}\right)+f\left(\frac{17}{10}\right)+f\left(\frac{19}{10}\right) \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} f(0.1)+f(0.3)+f(0.5)+f(0.7)+f(0.9) \\ +f(1.1)+f(1.3)+f(1.5)+f(1.7)+f(1.9) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0.0990+0.2752+0.4+0.4698+0.4972 \\ +0.4977+0.4833+0.4615+0.4370+0.4121 \end{bmatrix}$$

$$= \frac{1}{5}[4.0328]$$

$$\approx 0.806560$$

(b) Simpson's rule

$$s_{10} = \frac{\Delta x}{3}\left[f(0)+4\left[f\left(\frac{1}{5}\right)+f\left(\frac{3}{5}\right)+f(1)+f\left(\frac{7}{5}\right)+f\left(\frac{9}{5}\right)\right]\right.$$
$$\left.+2\left[\left[f\left(\frac{2}{5}\right)+f\left(\frac{4}{5}\right)+f\left(\frac{6}{5}\right)+f\left(\frac{8}{5}\right)\right]+f(2)\right]$$
$$= \frac{1}{15}\left[f(0)+4\left[f(0.2)+f(0.6)+f(1)+f(1.4)+f(1.8)\right]\right.$$
$$\left.+2\left[f(0.4)+f(0.8)+f(1.2)+f(1.6)+f(2)\right]\right.$$
$$= \frac{1}{15}\left[0+4\left[0.1923+0.4412+0.5+0.4729+0.4245\right]\right.$$
$$\left.+2\left[0.3448+0.4878+0.4918+0.4494\right]+0.4\right]$$
$$= \frac{1}{15}\left[8.1236+3.5476+0.4\right]$$
$$= 0.80475$$

$$\int_0^2 \frac{x}{1+x^2}dx = \frac{1}{2}\int_0^2 \frac{2x}{1+x^2}dx$$
$$= \frac{1}{2}\left[\log\left(1+x^2\right)\right]_0^2 \qquad \left[\text{because} \int \frac{f'(x)}{f(x)}dx = \log\left(f(x)\right)+c\right]$$
$$= \frac{1}{2}\left[\log 5\right]$$
$$= 0.8047189$$
$$\approx 0.804719$$

Therefore error is Midpoint rule is

$$0.804719 - 0.806560$$
$$\boxed{E_M = 0.001841}$$

Error in Simpson's rule is

$$E_S \approx 0.80479 - 0.80475$$
$$\approx 0.000040$$

Answer 6E.

Consider the following integral with specified value of n:

$$\int_0^{\pi} x\cos x\,dx, \qquad n = 4$$

The objective is to approximate the given integral using Midpoint Rule and Simpson's Rule.

First, find the exact value of the given integral.

Use integration by parts to solve the integral.

Let $u = x$ and $dv = \cos dx$ so that $du = dx$ and $v = \sin x$.

Then

$$\int_0^{\pi} x\cos x dx = x\sin x\Big]_0^{\pi} - \int_0^{\pi} \sin x dx$$
$$= \left[(\pi\sin\pi)-(0\sin 0)\right] - \int_0^{\pi} \sin x dx$$
$$= -\left[-\cos x\right]_0^{\pi}$$
$$= \left[\cos\pi - \cos 0\right]$$
$$= -2$$

Therefore, the exact value of the given definite integral is,

$$\int_0^{\pi} x\cos x dx = -2$$

For both methods:

$$\Delta x = \frac{\pi - 0}{4}$$
$$= \frac{\pi}{4}$$

(a)

Midpoint Rule:

The midpoints of the four subintervals are $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}$ and $\frac{7\pi}{8}$.

Therefore, the Midpoint Rule gives

$$\begin{aligned}\int_0^{\pi} x\cos x dx &\approx \frac{\pi}{4}\left[f\left(\frac{\pi}{8}\right)+f\left(\frac{3\pi}{8}\right)+f\left(\frac{5\pi}{8}\right)+f\left(\frac{7\pi}{8}\right)\right]\\ &= \frac{\pi}{4}\left[\frac{\pi}{8}\cos\frac{\pi}{8}+\frac{3\pi}{8}\cos\frac{3\pi}{8}+\frac{5\pi}{8}\cos\frac{5\pi}{8}+\frac{7\pi}{8}\cos\frac{7\pi}{8}\right]\\ &\approx \boxed{-1.945744}\end{aligned}$$

The Midpoint Rule error is,

$$\begin{aligned}E_M &= -2-(-1.945744)\\ &= \boxed{-0.054256}\end{aligned}$$

(b)

Simpson's Rule:

Approximate the given integral using Simpson's Rule.

Then,

$$\begin{aligned}\int_0^{\pi} x\cos x dx &\approx \frac{\frac{\pi}{4}}{3}\left[f(0.0)+4f\left(\frac{\pi}{4}\right)+2f\left(\frac{\pi}{2}\right)+4f\left(\frac{3\pi}{4}\right)+f(\pi)\right]\\ &= \frac{\pi}{12}\left[0\cos 0+4\cdot\frac{\pi}{4}\cos\frac{\pi}{4}+2\cdot\frac{\pi}{2}\cos\frac{\pi}{2}+4\cdot\frac{3\pi}{4}\cos\frac{3\pi}{4}+\pi\cos\pi\right]\\ &\approx \boxed{-1.985611}\end{aligned}$$

The Simpson's Rule error is,

$$\begin{aligned}E_S &= -2-(-1.985611)\\ &= \boxed{-0.014389}\end{aligned}$$

Answer 7E.

Let $f(x) = \sqrt{x^3 - 1}, n = 10$

Here $a = 1, b = 2$

Therefore $\Delta x = \frac{1}{10}$
$= 0.1$

Therefore $x_0 = 1, x_1 = 1.1, x_2 = 1.2, x_3 = 1.3, x_4 = 1.4$
$x_5 = 1.5, x_6 = 1.6, x_7 = 1.7, x_8 = 1.8, x_9 = 1.9, x_{10} = 2.$

(a) Trapezoidal rule.

$$\int_1^2 \sqrt{x^3-1}dx = \frac{0.1}{2}\left[f(1)+2\begin{bmatrix}f(1.1)+f(1.2)+f(1.3)+f(1.4)+f(1.5)\\ +f(1.6)+f(1.7)+f(1.8)+f(1.9)\end{bmatrix}+f(2)\right]$$
$$= 1.506361$$

(b) Midpoint rule:

$$\int_1^2 \sqrt{x^3-1}dx = 0.1\begin{bmatrix}f(1.05)+f(1.15)+f(1.25)+f(1.35)+f(1.45)\\ +f(1.55)+f(1.65)+f(1.75)+f(1.85)\end{bmatrix}$$
$$= 1.518362$$

(c) Simpson's rule:

$$\int_1^2 \sqrt{x^3-1}dx = \frac{0.1}{3}\Big[f(1)+4\big(f(1.1)\big)+f(1.3)+f(1.5)+f(1.7)+f(1.9)\Big]$$
$$+2\big[f(1.2)+f(1.4)+f(1.6)+f(1.8)\big]+f(2)\Big]$$
$$=1.511519$$

Answer 8E.

Let $f(x)=\frac{1}{1+x^6}, n=8$

Here $a=0, b=2$ and $n=8$

$$\text{Therefore } \Delta x = \frac{2}{8}$$
$$= \frac{1}{4}$$
$$=0.25$$

Therefore $x_0=0, x_1=0.25, x_2=0.5, x_3=0.75, x_4=1, x_5=1.25,$
$x_6=1.5, x_7=1.75$ and $x_8=2$

(a) Trapezoidal rule

$$\int_0^2 \frac{1}{1+x^6}dx = \frac{0.25}{2}\left[f(0)+2\begin{pmatrix} f(0.25)+f(0.5)+f(0.75)+f(1) \\ +f(1.25)+f(1.5)+f(1.75)\end{pmatrix}+f(2)\right]$$
$$=0.125\left[1+2\begin{bmatrix}0.99975+0.984615+0.8489119+0.5 \\ +0.20769+0.080706+0.033644\end{bmatrix}+0.0153846\right]$$
$$=0.125[1+7.3106338+0.0153846]$$
$$=1.0407523$$

(b) Midpoint rule

$$\int_0^2 \frac{1}{1+x^6}dx = 0.25\begin{bmatrix} f(0.125)+f(0.375)+f(0.625)+f(0.875) \\ +f(1.125)+f(1.35)+f(1.625)+f(1.875)\end{bmatrix}$$
$$=0.25\begin{bmatrix}0.999996+0.997226+0.943748+0.690228 \\ +0.330328+0.128899+0.051512+0.22496\end{bmatrix}$$
$$=1.0411082$$

(c) Simpson's rule

$$\int_0^2 \frac{1}{1+x^6}dx = \frac{0.25}{3}\begin{bmatrix} f(0)+4\big[f(0.5)+f(1)+f(1.5)\big] \\ +2\big[f(0.25)+f(0.75)+f(1.25)+f(1.75)\big]+f(2)\end{bmatrix}$$
$$=\frac{0.25}{3}\begin{bmatrix}1+4[0.984615+0.5+0.080706] \\ +2[0.99975+0.8489119+0.20769+0.033644]+0.0153846\end{bmatrix}$$
$$=1.0409925$$

Answer 9E.

Let $f(x)=\frac{e^x}{1+x^2}, n=10$

Here $a=0, b=2$ and $n=10$

$$\text{Therefore } \Delta x = \frac{2}{10}$$
$$= \frac{1}{5}$$
$$=0.2$$

Therefore $x_0=0, x_1=0.2, x_2=0.4, x_3=0.6, x_4=0.8, x_5=1.0$
$x_6=1.2, x_7=1.4, x_8=1.6,, x_9=1.8, x_{10}=2$

(a) Trapezoidal rule is

$$\int_0^2 \frac{e^x}{1+x^2}dx = \frac{0.2}{2}\left[f(0)+2\begin{pmatrix} f(0.2)+f(0.4)+f(0.6)+f(0.8) \\ +f(1.0)+f(1.2)+f(1.4)+f(1.6)+f(1.8)\end{pmatrix}+f(2)\right]$$
$$= 2.660833$$

(b) Midpoint rule is

$$\int_0^2 \frac{e^x}{1+x^2}dx = 0.2\begin{bmatrix} f(0.1)+f(0.3)+f(0.5)+f(0.7)+f(0.9) \\ +f(1.1)+f(1.3)+f(1.5)+f(1.7)+f(1.9)\end{bmatrix}$$
$$= 2.664377$$

(c) Simpson's rule is

$$\int_0^2 \frac{e^x}{1+x^2}dx = \frac{0.2}{3}\begin{bmatrix} f(0)+4\left[f(0.2)+f(0.6)+f(1)+f(1.4)+f(1.8)\right] \\ +2\left[(0.4)+f(0.8)+f(1.2)+f(1.6)\right]+f(2)\end{bmatrix}$$
$$= 2.663244$$

Answer 10E.

Consider the following integral:

$$\int_0^{\frac{\pi}{2}} \sqrt[3]{1+\cos x}$$

a)

Write the trapezoidal rule.

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)\right]$$

Where $\Delta x = \frac{b-a}{n}$ and $x_i = a+i\Delta x$

$$\Delta x = \frac{\frac{\pi}{2}-0}{4}$$
$$= \frac{\pi}{8}$$

Divide the interval $\left[0,\frac{\pi}{2}\right]$ into points of common difference $\frac{\pi}{8}$.

The points are $x_0 = 0, x_1 = \frac{\pi}{8}, x_2 = \frac{\pi}{4}, x_3 = \frac{3\pi}{8}, x_4 = \frac{\pi}{2}$

Put these values in the Trapezoidal rule.

$$T_4 = \frac{\frac{\pi}{8}}{2}\left[f(0)+2f\left(\frac{\pi}{8}\right)+2f\left(\frac{\pi}{4}\right)+2f\left(\frac{3\pi}{8}\right)+f\left(\frac{\pi}{2}\right)\right]$$
$$= \frac{\pi}{16}\left[\sqrt[3]{1+\cos(0)}+2\sqrt[3]{1+\cos\left(\frac{\pi}{8}\right)}+2\sqrt[3]{1+\cos\left(\frac{\pi}{4}\right)}+2\sqrt[3]{1+\cos\left(\frac{3\pi}{8}\right)}+\sqrt[3]{1+\cos\left(\frac{\pi}{2}\right)}\right]$$
$$= \frac{\pi}{16}\left[\sqrt[3]{1+1}+2\sqrt[3]{1+0.9239}+2\sqrt[3]{1+0.7071}+2\sqrt[3]{1+0.3827}+\sqrt[3]{1+0}\right]$$
$$= \frac{\pi}{16}(1.2599+2.4875+2.3903+2.2281+1)$$
$$= \frac{\pi}{16}(9.3658)$$
$$\approx \boxed{1.838971}$$

Hence, the value of the given integral using Trapezoidal Rule is $\boxed{1.838971}$.

b)

The objective is to approximate the given integral using the Midpoint rule.

Write the Midpoint Rule.

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\left[f\left(\bar{x}_1\right)+f\left(\bar{x}_2\right)+\cdots+f\left(\bar{x}_n\right)\right]$$

Where $\Delta x = \dfrac{b-a}{n}$ and $\bar{x}_i = \dfrac{(x_{i-1}+x_i)}{2}$

Therefore,

$$\Delta x = \frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}$$

Thus, the interval points are, $x_0 = 0, x_1 = \dfrac{\pi}{8}, x_2 = \dfrac{\pi}{4}, x_3 = \dfrac{3\pi}{8}, x_4 = \dfrac{\pi}{2}$

Calculate the midpoints.

$$\overline{x_1} = \frac{0+\frac{\pi}{8}}{2} = \frac{\pi}{16} \qquad \overline{x_2} = \frac{\frac{\pi}{8}+\frac{\pi}{4}}{2} = \frac{3\pi}{16}$$

$$\overline{x_3} = \frac{\frac{\pi}{4}+\frac{3\pi}{8}}{2} = \frac{5\pi}{16} \qquad \overline{x_4} = \frac{\frac{3\pi}{8}+\frac{\pi}{2}}{2} = \frac{7\pi}{16}$$

Put these values in the Midpoint rule.

$$\begin{aligned}
M_4 &= \frac{\pi}{8}\left[f\left(\frac{\pi}{16}\right)+f\left(\frac{3\pi}{16}\right)+f\left(\frac{5\pi}{16}\right)+f\left(\frac{7\pi}{16}\right)\right] \\
&= \frac{\pi}{8}\left[\sqrt[3]{1+\cos\left(\frac{\pi}{16}\right)}+\sqrt[3]{1+\cos\left(\frac{3\pi}{16}\right)}+\sqrt[3]{1+\cos\left(\frac{5\pi}{16}\right)}+\sqrt[3]{1+\cos\left(\frac{7\pi}{16}\right)}\right] \\
&= \frac{\pi}{8}\left[\sqrt[3]{1+0.9808}+\sqrt[3]{1+0.8315}+\sqrt[3]{1+0.5556}+\sqrt[3]{1+0.1951}\right] \\
&= \frac{\pi}{8}(1.2559+1.2235+1.1587+1.0612)
\end{aligned}$$

$$= \frac{\pi}{8}(4.6993)$$

$$\approx \boxed{1.845411}$$

Hence, the value of the given integral using Midpoint Rule is $\boxed{1.845411}$.

c)

The objective is to approximate the given integral using Simpsons rule.

Write the Simpsons rule:

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}\left[\begin{array}{l} f(x_0)+4f(x_1)+2f(x_2)\cdots \\ \qquad +2f(x_{n-2})+4f(x_{n-1})+f(x_n)\end{array}\right]$$

Where $\Delta x = \dfrac{b-a}{n}$.

$$\Delta x = \frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}$$

Therefore, the points are $x_0 = 0, x_1 = \dfrac{\pi}{8}, x_2 = \dfrac{\pi}{4}, x_3 = \dfrac{3\pi}{8}, x_4 = \dfrac{\pi}{2}$.

Put these values in the Simpsons rule.

$$S_4 = \frac{\frac{\pi}{8}}{3}\left[f(0)+4f\left(\frac{\pi}{8}\right)+2f\left(\frac{\pi}{4}\right)+4f\left(\frac{3\pi}{8}\right)+f\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{\pi}{24}\left[\sqrt[3]{1+\cos(0)}+4\sqrt[3]{1+\cos\left(\frac{\pi}{8}\right)}+2\sqrt[3]{1+\cos\left(\frac{\pi}{4}\right)}+4\sqrt[3]{1+\cos\left(\frac{3\pi}{8}\right)}+\sqrt[3]{1+\cos\left(\frac{\pi}{2}\right)}\right]$$

$$= \frac{\pi}{24}\left[\sqrt[3]{1+1}+4\sqrt[3]{1+0.9239}+2\sqrt[3]{1+0.7071}+4\sqrt[3]{1+0.3827}+\sqrt[3]{1+0}\right]$$

$$= \frac{\pi}{24}(1.2599+4.9749+2.3903+4.4563+1)$$

$$= \frac{\pi}{24}(14.0814)$$

$$\approx \boxed{1.843251}$$

Hence, the value of the given integral using Midpoint Rule is $\boxed{1.843251}$.

Answer 11E.

Let $f(x) = \sqrt{\ln x}, n = 6$

Here $a = 1, b = 4$

Therefore $\Delta x = \dfrac{4-1}{6} = \dfrac{3}{6} = \dfrac{1}{2}$

Therefore $x_0 = 1, x_1 = 1.5, x_2 = 2, x_3 = 2.5, x_4 = 3, x_5 = 3.5, x_6 = 4.$

(a) Trapezoidal rule is

$$\int_1^4 \sqrt{\ln x}dx = \frac{1}{4}\left[f(1)+2\left[f(1.5)+f(2)+f(2.5)+f(3)+f(3.5)\right]+f(4)\right]$$
$$= 2.591334$$

(b) Midpoint rule is

$$\int_1^4 \sqrt{\ln x}dx = \frac{1}{2}\left[f(1.25)+f(1.75)+f(2.25)+f(2.75)+f(3.25)+f(3.75)\right]$$
$$= 2.681046$$

(c) Simpson's rule.

$$\int_1^4 \sqrt{\ln x}dx = \frac{1}{6}\left[f(1)+4\left(f(2)+f(3)\right)+2\left[f(1.5)+f(2.5)+f(3.5)\right]+f(4)\right]$$
$$= 2.631976$$

Answer 12E.

Let $f(x)=\sin(x^3), n=10$

Here $a=0, b=1$

Therefore $\Delta x=0.1$

Therefore $x_0=0, x_1=0.1, x_2=0.2, x_3=0.3, x_4=0.4, x_5=0.5,$

$x_6=0.6, x_7=0.7, x_8=0.8, x_9=0.9, x_{10}=1.$

(a) Trapezoidal rule

$$\int_0^1 \sin(x^3)dx=\frac{0.1}{2}\left[f(0)+2\begin{bmatrix}f(0.1)+f(0.2)+f(0.3)+f(0.4)+f(0.5)\\+f(0.6)+f(0.7)+f(0.8)+f(0.9)\end{bmatrix}+f(1)\right]$$

$$=0.05\left[0+2\begin{bmatrix}0.000999+0.0007999+0.0269967\\+0.0639563+0.124674+0.214324\\+0.336313+0.489922+0.666124\end{bmatrix}+0.841471\right]$$

$$=0.05[38482164+0.841471]$$

$$=0.23448437$$

(b) Midpoint rule

$$\int_0^1 \sin(x^3)dx=0.1\begin{bmatrix}f(0.05)+f(0.15)+f(0.25)+f(0.35)+f(0.45)\\+f(0.55)+f(0.65)+f(0.75)+f(0.85)+f(0.95)\end{bmatrix}$$

$$=0.1\begin{bmatrix}0.0001249+0.0003374+0.0156243+0.042862\\+0.090998+0.1656085+0.271186+0.409472\\+0.576243+0.756127\end{bmatrix}$$

$$=0.23285831$$

(c) Simpson's rule.

$$\int_0^1 \sin(x^3)dx=\frac{0.1}{3}\begin{bmatrix}f(0)+4[f(0.2)+f(0.4)+f(0.6)+f(0.8)]\\+2[f(0.1)+f(0.3)+f(0.5)+f(0.7)+f(0.9)]+f(1)\end{bmatrix}$$

$$\frac{0.1}{3}\begin{bmatrix}0+4[0.0007999+0.0639563+0.214324+0.489922]\\+2\begin{bmatrix}0.000999+0.026996+0.124674\\+0.336313+0.666124\end{bmatrix}+0.841471\end{bmatrix}$$

$$=\frac{0.1}{3}[3.07660088+2.3102134+0.841471]$$

$$=0.2076094$$

Answer 13E.

a) Estimate the following integral using the trapezoidal rule

$$\int_0^4 e^{\sqrt{t}}\sin t\,dt, n=8$$

Write the Trapezoidal Rule:

$$\int_a^b f(x)dx\approx T_n=\frac{\Delta x}{2}[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+2f(x_n)]$$

Where $\Delta x=\dfrac{b-a}{n}$ and $x_i=a+i\Delta x$

Evaluate Δt.

$$\Delta t = \frac{4-0}{8}$$
$$= \frac{1}{2}$$

Divide the interval $[0,4]$ into points of common difference 0.5.

These points are $x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5, x_4 = 2, x_5 = 2.5, x_6 = 3, x_7 = 3.5$ and $x_8 = 4$.

Put these values and $\Delta t = 0.5$ in the Trapezoidal formula.

$$T_8 = \frac{1}{4}\begin{bmatrix} f(0)+2f(0.5)+2f(1)+2f(1.5)+2f(2) \\ +2f(2.5)+2f(3)+2f(3.5)+f(4) \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} e^{\sqrt{0}}\sin(0)+2e^{\sqrt{0.5}}\sin(0.5)+2e^{\sqrt{1}}\sin(1)+2e^{\sqrt{1.5}}\sin(1.5)+2e^{\sqrt{2}}\sin(2) \\ +2e^{\sqrt{2.5}}\sin(2.5)+2e^{\sqrt{3}}\sin(3)+2e^{\sqrt{3.5}}\sin(3.5)+e^{\sqrt{4}}\sin(4) \end{bmatrix}$$
$$\approx \boxed{4.513618}$$

b) Estimate the following integral using the midpoint rule

$$\int_0^4 e^{\sqrt{t}} \sin t\,dt, n = 8$$

Write the Midpoint rule.

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\left[f\left(\bar{x}_1\right)+f\left(\bar{x}_2\right)+\cdots+f\left(\bar{x}_n\right)\right]$$

Where $\Delta x = \dfrac{b-a}{n}$ and $\bar{x}_i = \dfrac{x_{i-1}+x_i}{2}$

Use the points $x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5, x_4 = 2, x_5 = 2.5, x_6 = 3, x_7 = 3.5$ and $x_8 = 4$ to calculate $\bar{x}_i = \dfrac{x_{i-1}+x_i}{2}$.

The points $\bar{x}_i$ are $\overline{x_0} = 0.25, \overline{x_1} = 0.75, \overline{x_2} = 1.25, \overline{x_3} = 1.75, \overline{x_4} = 2, \overline{x_5} = 2.25,$

$\overline{x_6} = 2.75, \overline{x_7} = 3.25$ and $\overline{x_8} = 3.75$

Put these values and $\Delta t = \dfrac{1}{2}$ in the Midpoint formula.

$$M_8 = \frac{1}{2}\begin{bmatrix} f(0.25)+f(0.75)+f(1.25)+f(1.75)+f(2.25) \\ +f(2.75)+f(3.25)+f(3.75) \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} e^{\sqrt{0.25}}\sin(0)+e^{\sqrt{0.75}}\sin(0.75)+e^{\sqrt{1.25}}\sin(1.25) \\ +e^{\sqrt{1.75}}\sin(1.75)+e^{\sqrt{2.25}}\sin(2.25) \\ +e^{\sqrt{2.75}}\sin(2.75)+e^{\sqrt{3.25}}\sin(3.25)+e^{\sqrt{3.75}}\sin(3.75) \end{bmatrix}$$
$$\approx \boxed{4.748256}$$

c) Estimate the following integral using the Simpsons rule

$$\int_0^4 e^{\sqrt{t}} \sin t\,dt, n = 8$$

The Simpsons rule is

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\begin{bmatrix} f(x_0)+4f(x_1)+2f(x_2)\cdots \\ +2f(x_{n-2})+4f(x_{n-1})+f(x_n) \end{bmatrix}$$

Where $\Delta x = \dfrac{b-a}{n}$

Put the values $x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5, x_4 = 2, x_5 = 2.5, x_6 = 3, x_7 = 3.5$

$x_8 = 4$ and $\Delta t = \frac{1}{2}$ in the Simpsons rule.

$$S_8 = \frac{1}{6}\begin{bmatrix} f(0)+4f(0.5)+2f(1)+4f(1.5)+2f(2) \\ +4f(2.5)+2f(3)+4f(3.5)+f(4) \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} e^{\sqrt{0}}\sin(0)+4e^{\sqrt{0.5}}\sin(0.5)+2e^{\sqrt{1}}\sin(1)+4e^{\sqrt{1.5}}\sin(1.5)+2e^{\sqrt{2}}\sin(2) \\ +4e^{\sqrt{2.5}}\sin(2.5)+2e^{\sqrt{3}}\sin(3)+4e^{\sqrt{3.5}}\sin(3.5)+e^{\sqrt{4}}\sin(4) \end{bmatrix}$$

$$\approx \boxed{4.675111}$$

Answer 15E.

(A) We have $f(x) = \frac{\cos x}{x}$, $\quad$ n = 8

Interval is [1, 5], $\quad$ then $\quad \Delta x = \frac{5-1}{8} = \frac{1}{2}$

And then $x_0 = 1, x_1 = \frac{3}{2}, x_2 = \frac{4}{2}, x_3 = \frac{5}{2}, x_4 = \frac{6}{2}, x_5 = \frac{7}{2}, x_6 = \frac{8}{2}, x_7 = \frac{9}{2}, x_8 = 5$

We used $x_i = a + i\Delta x$, a = 1

Then by Trapezoidal rule

$$\int_1^5 \frac{\cos x}{x}dx \approx T_8 = \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+.....+2f(x_7)+f(x_8)\right]$$

$$= \frac{1}{4}\left[f(1)+2f\left(\frac{3}{2}\right)+2f\left(\frac{4}{2}\right)+....+2f\left(\frac{9}{2}\right)+f(5)\right]$$

$$\boxed{T_8 \approx -0.495333}$$

(B) Subintervals are $\left[1,\frac{3}{2}\right],\left[\frac{3}{2},2\right],\left[2,\frac{5}{2}\right],---\left[\frac{9}{2},5\right]$

And the midpoints are $\frac{5}{4},\frac{7}{4},\frac{9}{4},\frac{11}{4},\frac{13}{4},\frac{15}{4},\frac{17}{4},\frac{19}{4}$

Then by midpoint rule

$$\int_1^5 \frac{\cos x}{x}dx \approx M_8 = \Delta x\left[f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+f\left(\frac{9}{4}\right)+.....+f\left(\frac{19}{4}\right)\right]$$

$$= \frac{1}{2}\left[f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+f\left(\frac{9}{4}\right)+......+f\left(\frac{19}{4}\right)\right]$$

$$\boxed{M_8 \approx -0.543321}$$

(C) By Simpson's rule

$$S_8 = \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+.....+4f(x_7)+f(x_8)\right]$$

$$= \frac{1}{6}\left[f(1)+4f\left(\frac{3}{2}\right)+2f(2)+4f\left(\frac{5}{2}\right)+......+4f\left(\frac{9}{2}\right)+f(5)\right]$$

$$\boxed{S_8 \approx -0.526123}$$

Answer 16E.

(a)

Consider,

$f(x) = \ln(x^3+2)$, $n = 10$

Since $a = 4, b = 6$, $\Delta x = \frac{6-4}{10} = 0.2$

Let $x_0 = 4,\ x_1 = 4.2,\ x_2 = 4.4,\ x_3 = 4.6,\ x_4 = 4.8,\ x_5 = 5.0, x_6 = 5.2,\ x_7 = 5.4, x_8 = 5.6$,

$x_9 = 5.8,\ x_{10} = 6$ $\quad x_i = a + i\Delta x$ w here $a = 4$

Then by Trapezoidal rule

$$\int_4^6 \ln\left(x^3+2\right)dx = T_{10} = \frac{\Delta x}{2}\left[f\left(x_0\right)+2f\left(x_1\right)+2f\left(x_2\right)+...+2f\left(x_9\right)+f\left(x_{10}\right)\right]$$

$$=\frac{0.2}{2}\begin{bmatrix} f(4)+2f(4.2)+2f(4.4)+2f(4.6)+2f(4.8)+2f(5) \\ +2f(5.2)+2f(5.4)+2f(5.6)+2f(5.8)+f(6) \end{bmatrix}$$

$$=0.1\begin{bmatrix} \ln\left(4^3+2\right)+2\ln\left(\left(4.2\right)^3+2\right)+2\ln\left(\left(4.4\right)^3+2\right)+2\ln\left(\left(4.6\right)^3+2\right)+ \\ 2\ln\left(\left(4.8\right)^3+2\right)+2\ln\left(\left(5\right)^3+2\right)+2\ln\left(\left(5.2\right)^3+2\right)+2\ln\left(\left(5.4\right)^3+2\right)+ \\ 2\ln\left(\left(5.6\right)^3+2\right)+2\ln\left(\left(5.8\right)^3+2\right)+\ln\left(6^3+2\right) \end{bmatrix}$$

$$=0.1\begin{bmatrix} 4.189655+8.663781+8.936042+9.197016+9.447541+ \\ 9.688374+9.920199+10.143636+10.359248+10.569544+ \\ 5.384495 \end{bmatrix}$$

$$=0.1[96.499531]$$

$$\boxed{T_{10}\approx 9.649953}$$

(b)

Subintervals are [4, 4.2], [4.2, 4.4], -- [5.8, 6]

And midpoints are 4.1, 4.3, 4.5, 4.7, 4.9, 5.1, 5.3, 5.5, 5.7, 5.9

And by mid points rule,

$$\int_4^6 \ln\left(x^3+2\right)dx$$

$$\approx M_{10}$$

$$=\Delta x\begin{bmatrix} f(4.1)+f(4.3)+f(4.5)+f(4.7)+f(4.9)+ \\ f(5.1)+f(5.3)+f(5.5)+f(5.7)+f(5.9) \end{bmatrix}$$

$$=0.2\begin{bmatrix} \ln\left(\left(4.1\right)^3+2\right)+\ln\left(\left(4.3\right)^3+2\right)+\ln\left(\left(4.5\right)^3+2\right)+\ln\left(\left(4.7\right)^3+2\right) \\ +\ln\left(\left(4.9\right)^3+2\right)+\ln\left(\left(5.1\right)^3+2\right)+\ln\left(\left(5.3\right)^3+2\right)+\ln\left(\left(5.5\right)^3+2\right)+ \\ \ln\left(\left(5.7\right)^3+2\right)+\ln\left(\left(5.9\right)^3+2\right) \end{bmatrix}$$

$$=0.2\begin{bmatrix} 4.261567+4.400689+4.533943+4.661768+4.784562+ \\ 4.902686+5.016465+5.126194+5.232140+5.334548 \end{bmatrix}$$

$$=0.2[48.254561]$$

$$=9.650912$$

$$\boxed{M_{10}\approx 9.650912}$$

(c)

By Simpson's rule

$$\int_4^6 \ln\left(x^3+2\right)dx \approx S_{10} = \frac{\Delta x}{3}\left[f\left(x_0\right)+4f\left(x_1\right)+2f\left(x_2\right)+......+4f\left(x_9\right)+f\left(x_{10}\right)\right]$$

$$=\frac{1}{5}\left[f(4)+4f(4.2)+2f(4.4)+.......+4f(5.8)+f(6)\right]$$

$$=\frac{0.2}{3}\begin{bmatrix} f(4)+2f(4.2)+2f(4.4)+2f(4.6)+2f(4.8)+2f(5) \\ +2f(5.2)+2f(5.4)+2f(5.6)+2f(5.8)+f(6) \end{bmatrix}$$

$$=\frac{1}{15}\begin{bmatrix} \ln\left(4^3+2\right)+4\ln\left(\left(4.2\right)^3+2\right)+2\ln\left(\left(4.4\right)^3+2\right)+4\ln\left(\left(4.6\right)^3+2\right)+ \\ 2\ln\left(\left(4.8\right)^3+2\right)+4\ln\left(\left(5\right)^3+2\right)+2\ln\left(\left(5.2\right)^3+2\right)+4\ln\left(\left(5.4\right)^3+2\right)+ \\ 2\ln\left(\left(5.6\right)^3+2\right)+4\ln\left(\left(5.8\right)^3+2\right)+\ln\left(6^3+2\right) \end{bmatrix}$$

$$=\frac{1}{15}[144.758]$$

$$=9.650533$$

$$\boxed{S_{10}\approx 9.650533}.$$

Answer 17E.

Let $f(x)=e^{e^x}, n=10$

Here $a=-1, b=1$

Therefore $\Delta x=\frac{2}{10}$

$=\frac{1}{5}$

$=0.2$

Therefore $x_0=-1, x_1=-0.8, x_2=-0.6, x_3=-0.4, x_4=-0.2, x_5=0,$
$x_6=0.2, x_7=0.4, x_8=0.6, x_9=0.8, x_{10}=1$

(a) Trapezoidal rule is

$$\int_{-1}^{1} e^{e^x}dx=\frac{0.2}{2}\left[f(-1)+2\begin{bmatrix}f(-0.8)+f(-0.6)+f(-0.4)+f(-0.2)\\+f(0)+f(0.2)+f(0.4)+f(0.6)+f(0.8)\end{bmatrix}+f(1)\right]$$

$=8.363853$

(b) Midpoint rule is

$$\int_{-1}^{1} e^{e^x}dx=0.2\begin{bmatrix}f(-0.9)+f(-0.7)+f(-0.5)+f(-0.3)+f(-0.1)\\+f(0.1)+f(0.3)+f(0.5)+f(0.7)+f(0.9)\end{bmatrix}$$

$=8.163298$

(c) Simpson's rule is

$$\int_{-1}^{1} e^{e^x}dx=\frac{0.2}{3}\begin{bmatrix}f(-1)+4[f(-0.8)+f(-0.4)+f(0)+f(0.4)+f(0.8)]\\+2[f(-0.6)+f(-0.2)+f(0.2)+f(0.6)]+f(1)\end{bmatrix}$$

$=8.235114$

Answer 18E.

a) Estimate the following integral using the trapezoidal rule.

$$\int_0^4 \cos\sqrt{x}\,dx, n=10$$

Write the trapezoidal rule.

$$\int_a^b f(x)dx \approx T_n=\frac{\Delta x}{2}[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+2f(x_n)]$$

Where $\Delta x=\frac{b-a}{n}$ and $x_i=a+i\Delta x$

$$\Delta x=\frac{4-0}{10}$$
$$=\frac{2}{5}$$

Divide the interval $[0,4]$ into points of common difference 0.4.

These points are $x_0=0, x_1=0.4, x_2=0.8, x_3=1.2, x_4=1.6, x_5=2.0, x_6=2.4,$
$x_7=2.8, x_8=3.2, x_9=3.6$ and $x_{10}=4.$

Put these values in the Trapezoidal rule.

$$T_{10}=\frac{\frac{2}{5}}{2}\begin{bmatrix}f(0)+2f(0.4)+2f(0.8)+2f(1.2)+2f(1.6)\\+2f(2)+2f(2.4)+2f(2.8)+2f(3.2)\\+2f(3.6)+f(4)\end{bmatrix}$$

$$T_{10}=\frac{1}{5}\begin{bmatrix}\cos\sqrt{0}+2\cos\sqrt{0.4}+2\cos\sqrt{0.8}\\+2\cos\sqrt{1.2}+2\cos\sqrt{1.6}\\+2\cos\sqrt{2}+2\cos\sqrt{2.4}+2\cos\sqrt{2.8}\\+2\cos\sqrt{3.2}+2\cos\sqrt{3.6}+\cos\sqrt{4}\end{bmatrix}$$

$$\boxed{T_{10}\approx 0.808532}$$

b) Estimate the following integral using the Midpoint rule

$$\int_0^4 \cos\sqrt{x}\,dx, n=10$$

Write the midpoint rule.

$$\int_a^b f(x)dx \approx M_n = \Delta x\left[f\left(\bar{x}_1\right)+f\left(\bar{x}_2\right)+\cdots+f\left(\bar{x}_n\right)\right]$$

Where $\Delta x = \dfrac{b-a}{n}$ and $\bar{x}_i = 1/2\left(x_{i-1}+x_i\right)$

Use the points $x_0=0, x_1=0.4, x_2=0.8, x_3=1.2, x_4=1.6, x_5=2.0, x_6=2.4,$

$x_7=2.8, x_8=3.2, x_9=3.6$ and $x_{10}=4$ to calculate $\bar{x}_i = \dfrac{x_{i-1}+x_i}{2}.$

The points $\bar{x}_i$ are $\overline{x_0}=0.2, \overline{x_1}=0.6, \overline{x_2}=1, \overline{x_3}=1.4, \overline{x_4}=1.8, \overline{x_5}=2.2$

$\overline{x_6}=2.6, \overline{x_7}=3, \overline{x_8}=3.4$ and $\overline{x_9}=3.8.$

Put these values in the Midpoint rule.

$$M_{10} = \frac{2}{5}\begin{bmatrix} f(0.2)+f(0.6)+f(1)+f(1.4)+f(1.8) \\ +f(2.2)+f(2.6)+f(3.0)+f(3.4)+f(3.8) \end{bmatrix}$$

$$M_{10} = \frac{2}{5}\begin{bmatrix} \cos\sqrt{0.2}+\cos\sqrt{0.6}+\cos\sqrt{1} \\ +\cos\sqrt{1.4}+\cos\sqrt{1.8} \\ +\cos\sqrt{2.2}+\cos\sqrt{2.6}+\cos\sqrt{3} \\ +\cos\sqrt{3.4}+\cos\sqrt{3.8} \end{bmatrix}$$

$$\boxed{M_{10} \approx 0.80308}$$

c) Estimate the following integral using the Simpsons rule

$$\int_0^4 \cos\sqrt{x}\,dx, n=10$$

Write the Simpsons rule.

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}\begin{bmatrix} f(x_0)+4f(x_1)+2f(x_2)\cdots \\ +2f(x_{n-2})+4f(x_{n-1})+f(x_n) \end{bmatrix}$$

Where $\Delta x = \dfrac{b-a}{n}$

Divide the interval $[0,4]$ into points of common difference 0.4.

These points are $x_0=0, x_1=0.4, x_2=0.8, x_3=1.2, x_4=1.6, x_5=2.0, x_6=2.4,$

$x_7=2.8, x_8=3.2, x_9=3.6$ and $x_{10}=4.$

Put these values in the Simpsons rule.

$$S_{10} = \frac{\frac{2}{5}}{3}\begin{bmatrix} f(0)+4f(0.4)+2f(0.8)+4f(1.2)+2f(1.6) \\ +4f(2)+2f(2.4)+4f(2.8)+2f(3.2) \\ +4f(3.6)+f(4) \end{bmatrix}$$

$$S_{10} = \frac{2}{15}\begin{bmatrix} \cos\sqrt{0}+4\cos\sqrt{0.4}+2\cos\sqrt{0.8} \\ +4\cos\sqrt{1.2}+2\cos\sqrt{1.6} \\ +4\cos\sqrt{2}+2\cos\sqrt{2.4}+4\cos\sqrt{2.8} \\ +2\cos\sqrt{3.2}+4\cos\sqrt{3.6}+\cos\sqrt{4} \end{bmatrix}$$

$$\boxed{S_{10} \approx 0.804896}$$

Answer 19E.

(a)

Consider the integral as follows,

$$\int_0^1 \cos\left(x^2\right)dx$$

To find the approximations T_8 and M_8 for the integral;

First find the Trapezoidal approximation T_8 :

Trapezoidal Rule:

$$\int_a^b f(x)dx \approx T_n$$
$$= \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right]$$

Where, $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Here $f(x)=\cos\left(x^2\right), a=0, b=1$ and $n=8$.

Then,

$$\begin{aligned}\Delta x &= \frac{b-a}{n}\\ &= \frac{1-0}{8}\\ &= \frac{1}{8}\\ &= 0.125\end{aligned}$$

And,

$$\begin{aligned}x_0 &= a\\ &= 0\\ x_1 &= a+1(\Delta x)\\ &= 0+1(0.125)\\ &= 0.125\\ x_2 &= a+2(\Delta x)\\ &= 0+2(0.125)\\ &= 0.250\end{aligned}$$

Similarly,

$$\begin{aligned}x_3 &= a+3(\Delta x)\\ &= 0+3(0.125)\\ &= 0.375\\ x_4 &= a+4(\Delta x)\\ &= 0+4(0.125)\\ &= 0.500\\ x_5 &= a+5(\Delta x)\\ &= 0+5(0.125)\\ &= 0.725\\ x_6 &= a+6(\Delta x)\\ &= 0+6(0.125)\\ &= 0.75\\ x_7 &= a+7(\Delta x)\\ &= 0+7(0.125)\\ &= 0.875\\ x_8 &= a+8(\Delta x)\\ &= 0+8(0.125)\\ &= 1\end{aligned}$$

The Trapezoidal Rule,

$$\int_0^1 \cos\left(x^2\right)dx \approx T_8$$
$$=\frac{\Delta x}{2}\begin{bmatrix} f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4) \\ +2f(x_5)+2f(x_6)+2f(x_7)+f(x_8) \end{bmatrix}$$
$$=\frac{0.125}{2}\begin{bmatrix} f(0)+2f(0.125)+2f(0.250)+2f(0.375) \\ +2f(0.5)+2f(0.725)+2f(0.75)+2f(0.875)+f(1) \end{bmatrix}$$ This implies
$$=0.0625\begin{bmatrix} \cos\left((0)^2\right)+2\cos\left((0.125)^2\right)+2\cos\left((0.250)^2\right) \\ +2\cos\left((0.375)^2\right)+2\cos\left((0.5)^2\right)+2\cos\left((0.725)^2\right) \\ +2\cos\left((0.75)^2\right)+2\cos\left((0.875)^2\right)+\cos\left((1)^2\right) \end{bmatrix}$$

that,

$$\int_0^1 \cos\left(x^2\right)dx = 0.0625\begin{bmatrix} 1+1.9975586+1.9960950+1.980257176 \\ +1.937824843+1.849342522+1.691848998 \\ +1.441898761+0.540302305 \end{bmatrix}$$
$$=0.902332843$$

Therefore the required Trapezoidal approximation is $\boxed{T_8 \approx 0.902332843}$.

Find the Midpoint Rule approximation M_8 :

Midpoint Rule:

$$\int_a^b f(x)dx \approx M_n = \Delta x\left[f(\bar{x}_1)+f(\bar{x}_2)+\cdots+f(\bar{x}_n)\right]$$

Where,

$$\Delta x=\frac{b-a}{n}$$

And,

$$\bar{x}_i=\frac{1}{2}(x_{i-1}+x_i)$$
$$=\text{midpoint of }\left[x_{i-1},x_i\right]$$

Since $a=0, b=1, n=8$, then

$$\Delta x=\frac{1-0}{8}$$
$$=\frac{1}{8}$$

Now divide interval $[0,1]$ into $n=8$ subinterval of length $\Delta x=\frac{1}{8}$ with the following endpoints:

$$a=0,\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{3}{8},\frac{7}{8},1=b$$

The Midpoint Rule is,

$$\int_0^1 e^{-x^2}\,dx \approx M_8$$

$$= \frac{1}{8}\left[\begin{array}{l} f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_2+x_3}{2}\right)+f\left(\frac{x_3+x_4}{2}\right)+ \\ f\left(\frac{x_4+x_5}{2}\right)+f\left(\frac{x_5+x_6}{2}\right)+f\left(\frac{x_6+x_7}{2}\right) \end{array}\right]$$

$$= \frac{1}{8}\left[\begin{array}{l} f\left(\frac{0+\frac{1}{8}}{2}\right)+f\left(\frac{\frac{1}{8}+\frac{1}{4}}{2}\right)+f\left(\frac{\frac{1}{4}+\frac{3}{8}}{2}\right)+f\left(\frac{\frac{3}{8}+\frac{1}{2}}{2}\right)+ \\ f\left(\frac{\frac{1}{2}+\frac{5}{8}}{2}\right)+f\left(\frac{\frac{5}{8}+\frac{3}{4}}{2}\right)+f\left(\frac{\frac{3}{4}+\frac{7}{8}}{2}\right) \end{array}\right]$$

$$= \frac{1}{8}\left[\begin{array}{l} f\left(\frac{1}{16}\right)+f\left(\frac{3}{16}\right)+f\left(\frac{5}{16}\right)+f\left(\frac{7}{16}\right) \\ +f\left(\frac{9}{16}\right)+f\left(\frac{11}{16}\right)+f\left(\frac{13}{16}\right) \end{array}\right]$$

This implies that,

$$\int_0^1 \cos\left(x^2\right)dx = \frac{1}{8}\left[\begin{array}{l} \cos\left(\frac{1}{256}\right)+\cos\left(\frac{9}{256}\right)+\cos\left(\frac{25}{256}\right)+\cos\left(\frac{49}{256}\right) \\ +\cos\left(\frac{81}{256}\right)+\cos\left(\frac{121}{256}\right)+\cos\left(\frac{169}{256}\right) \end{array}\right]$$

$$= \frac{1}{8}\left[\begin{array}{l} 0.99999237+0.99938208+0.99523541+0.98173768 \\ +0.95035975+0.89036216+0.78989642 \end{array}\right]$$

$$\approx 0.90561995$$

Therefore, the required Midpoint approximation is $\boxed{M_8 \approx 0.90561995}$.

(b)

Error Bounds:

Suppose $\left|f''(x)\right| \le K$ for $a \le x \le b.$ If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$\left|E_T\right| \le \frac{K\left(b-a\right)^3}{12n^2}$$

And,

$$\left|E_M\right| \le \frac{K\left(b-a\right)^3}{24n^2}$$

To estimate the errors in the approximations of part **(a)**;

Find $f'(x)$;

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\cos\left(x^2\right)\right) \\ &= -\sin\left(x^2\right)\frac{d}{dx}\left(x^2\right) \\ &= -2x\sin\left(x^2\right) \end{aligned}$$

Find $f''(x)$ as follows:

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left[f'(x)\right] \\ &= \frac{d}{dx}\left(-2x\sin\left(x^2\right)\right) \\ &= -2x\frac{d}{dx}\left(\sin\left(x^2\right)\right) - 2\sin\left(x^2\right)\frac{d}{dx}(x) \\ &= -2x\cos\left(x^2\right)\frac{d}{dx}\left(x^2\right) - 2\sin\left(x^2\right) \end{aligned}$$

This implies that,

$$f''(x) = -4x^2\cos\left(x^2\right) - 2\sin\left(x^2\right)$$

Here, $f(x) = \cos\left(x^2\right)$ and $f''(x) = -4x^2\cos\left(x^2\right) - 2\sin\left(x^2\right)$

In the interval $0 \le x \le 1,$

$$\begin{aligned} \left|f''(x)\right| &\le \left|-4x^2\cos\left(x^2\right) - 2\sin\left(x^2\right)\right| \\ &\le \left|4x^2\cos\left(x^2\right) + 2\sin\left(x^2\right)\right| \\ &\le \left|4x^2\cos\left(x^2\right)\right| + \left|2\sin\left(x^2\right)\right| \\ &\le 4\left|x^2\cos\left(x^2\right)\right| + 2\left|\sin\left(x^2\right)\right| \end{aligned}$$

This implies that,

$$\begin{aligned} \left|f''(x)\right| &\le 4\times 1 + 2\times 1 \\ &\le 6 \end{aligned}$$

So take $K = 6$.

Error in trapezoidal rule is calculated as:

$$\begin{aligned} \left|E_T\right| &\le \frac{K(b-a)^3}{12n^2} \\ &= \frac{6(1-0)^3}{12(8)^2} \\ &= \frac{6}{768} \\ &= 0.0078 \end{aligned}$$

Therefore, the error in trapezoidal rule is $\boxed{0.0078}$.

Error in Midpoint rule is calculated as:

$$\begin{aligned} \left|E_M\right| &\le \frac{6(b-a)^3}{24n^2} \\ &= \frac{6(1-0)^3}{24(8)^2} \\ &= \frac{6}{1536} \\ &= 0.0039 \end{aligned}$$

Therefore the error in midpoint rule is $\boxed{0.0039}$.

(c)

To find n so that the approximations T_n and M_n to the integral in part **(a)** are accurate to within 0.0001.

For Trapezoidal rule:

Thus, for an error less than 0.0001, choose n so that,

$$\frac{K(b-a)^3}{12n^2} < 0.00001$$
$$\frac{6(1-0)^3}{12n^2} < 0.00001$$
$$\frac{6}{12n^2} < 0.0001$$
$$12n^2 > \frac{6}{0.0001}$$

This implies that,

$$n^2 > \frac{1}{0.0001 \times 2}$$
$$n^2 > \frac{1}{0.0002}$$
$$n > \frac{1}{\sqrt{0.0002}}$$
$$n > 70.71$$

Therefore, $n = \boxed{71}$ is the desired accuracy for T_8.

For Midpoint rule:

Thus for an error less than 0.0001, choose n so that,

$$\frac{K(b-a)^3}{24n^2} < 0.0001$$
$$\frac{6(1-0)^3}{24n^2} < 0.0001$$
$$\frac{6}{24n^2} < 0.0001$$
$$24n^2 > \frac{6}{0.0001}$$

This implies that,

$$n^2 > \frac{1}{0.0001 \times 4}$$
$$n^2 > \frac{1}{0.0004}$$
$$n > \frac{1}{\sqrt{0.0004}}$$
$$n > 50$$

Therefore, $n = \boxed{50}$ is the desired accuracy for M_8.

Answer 20E.

(a)

The integral is $\int_0^1 \cos\left(x^2\right)dx$.

The objective is to find approximations T_8, M_8 for the integral $\int_0^1 \cos\left(x^2\right)dx$.

The trapezoidal Rule:

$$\int_a^b f(x)dx \approx T_n$$
$$= \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)\right]$$

Where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Take $a=0, b=1$ and $n=8$.

$$\Delta x = \frac{1-0}{8}$$
$$= \frac{1}{8}$$

The Trapezoidal rule for $n=8$ becomes,

$$T_8 = \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+\cdots+2f(x_7)+f(x_8)\right]$$

Let $x_o = 0, x_1 = \frac{1}{8}, x_2 = \frac{2}{8}, x_3 = \frac{3}{8}, x_4 = \frac{4}{8}, x_5 = \frac{5}{8}, x_6 = \frac{6}{8}, x_7 = \frac{7}{8}, x_8 = 1$

Rewrite T_8.

$$T_8 = \frac{1}{2}\times\frac{1}{8}\begin{bmatrix} f(0)+2f(1/8)+2f(2/8)+2f(3/8)+2f(4/8) \\ +2f(5/8)+2f(6/8)+2f(7/8)+f(1) \end{bmatrix}$$
$$= \frac{1}{16}\begin{bmatrix} \cos\left(0^2\right)+2\cos\left(0.125^2\right)+2\cos\left(0.25^2\right)+2\cos\left(0.375^2\right)+2\cos\left(0.5^2\right) \\ +2\cos\left(0.625^2\right)+2\cos\left(0.75^2\right)+2\cos\left(0.875^2\right)+\cos\left(1^2\right) \end{bmatrix}$$
$$\approx 0.902333$$

Therefore, the integral value using trapezoidal Rule is $T_8 = \boxed{0.902333}$.

The Midpoint Rule:

$$\int_a^b f(x)dx \approx M_n$$
$$= \Delta x\left[f\left(\bar{x}_1\right)+f\left(\bar{x}_2\right)+\cdots+f\left(\bar{x}_n\right)\right]$$

Where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Take $a=0, b=1$ and $n=8$.

$$\Delta x = \frac{1-0}{8}$$
$$= \frac{1}{8}$$

The end points of interval are,

$$x_o = 0, x_1 = \frac{1}{8}, x_2 = \frac{2}{8}, x_3 = \frac{3}{8}, x_4 = \frac{4}{8}, x_5 = \frac{5}{8}, x_6 = \frac{6}{8}, x_7 = \frac{7}{8}, x_8 = 1$$

The mid points of intervals are,

$$\overline{x_1} = \frac{1}{16}, \overline{x_2} = \frac{3}{16}, \overline{x_3} = \frac{5}{16}, \overline{x_4} = \frac{7}{16}, \overline{x_5} = \frac{9}{16}, \overline{x_6} = \frac{11}{16}, \overline{x_7} = \frac{13}{16}, \overline{x_8} = \frac{15}{16}$$

The midpoint rule for $n=8$ becomes,

$$M_8 = \Delta x\left[f\left(\bar{x}_1\right)+f\left(\bar{x}_2\right)+\cdots+f\left(\bar{x}_8\right)\right]$$

Rewrite M_8.

$$\begin{aligned} M_8 &= \frac{1}{8}\left[\begin{array}{l} f(1/16)+f(3/16)+f(5/16)+f(7/16)+f(9/16) \\ +f(11/16)+f(13/16)+f(15/16) \end{array}\right] \\ &= \frac{1}{8}\left[\begin{array}{l} \cos(1/16)^2+\cos(3/16)^2+\cos(5/16)^2+\cos(7/16)^2 \\ +\cos(9/16)^2+\cos(11/16)^2+\cos(13/16)^2+\cos(15/16)^2 \end{array}\right] \\ &\approx 0.905620 \end{aligned}$$

Therefore, the integral can be approximated as $\boxed{0.905620}$.

b)

Estimate the errors of each approximation.

$$f(x)=\cos x^2$$
$$f'(x)=-2x\sin x^2$$
$$f''(x)=-2\sin x^2-4x^2\cos x^2$$

$|f''(x)|\le 2(1)+4(1)$ Since $\sin\theta\le 1,\cos\theta\le 1$

$$|f''(x)|\le 6$$

Recollect that if $|f''(x)|\le K$ for $a\le x\le b$ then $|E_T|\le \frac{K(b-a)^3}{12n^2}$.

So for $n=8, a=0, b=1, K=6$

$$|E_T|\le \frac{6(1^3)}{12(8^2)}$$
$$|E_T|\le 0.0078$$

Use the following formula:

$$\begin{aligned} |E_M| &\le \frac{K(b-a)^3}{24n^2} \\ &= \frac{6(1-0)^3}{24(8)^2} \end{aligned}$$

$$|E_M|\le \frac{1}{256}$$
$$|E_M|\le 0.0039$$

Therefore, the errors involved in the approximation is $\boxed{|E_T|\le 0.0078\text{and } |E_M|\le 0.0039}$.

c)

How large the value n so that the approximations are less than 0.0001.

For Trapezoidal Rule:

$$|E_T|\le 0.0001$$
$$\frac{6(1^3)}{12(n^2)}\le 0.0001$$
$$n^2\ge \frac{6}{12(0.0001)}$$
$$n^2\ge 5000$$
$$n\ge 70.7106$$

Therefore the value of n for T_n is $\boxed{71}$.

For Midpoint Rule:

$$|E_M| \le 0.0001$$

$$\frac{6(1^3)}{24(n^2)} \le 0.0001$$

$$n^2 \ge \frac{6}{24(0.0001)}$$

$$n^2 \ge 2500$$

$$n \ge 50$$

Therefore the value of n for M_n is $\boxed{50}$.

Answer 22E.

We have $f(x) = e^{x^2}$

Then $f'(x) = 2xe^{x^2}$

$$f''(x) = 2xe^{x^2}(2x) + 2e^{x^2}$$

$$= 4x^2e^{x^2} + 2e^{x^2}$$

Or $f''(x) = (4x^2 + 2)e^{x^2}$

Then $f'''(x) = (4x^2 + 2)(2x)e^{x^2} + 8xe^{x^2}$

$$= (8x^3 + 4x + 8x)e^{x^2}$$

Or $f'''(x) = (8x^3 + 12x)e^{x^2}$

And then

$$f^{(4)}(x) = (8x^3 + 12x)(2x)e^{x^2} + (24x^2 + 12)e^{x^2}$$

$$= (16x^4 + 24x^2 + 24x^2 + 12)e^{x^2}$$

Or $f^{(4)}(x) = (16x^4 + 48x^2 + 12)e^{x^2}$

Since $f^{(4)}(x)$ is an increasing function for x > 0

So $|f^4(x)| = (16x^4 + 48x^2 + 12)e^{x^2} \le (16 + 48 + 12)e$

Or $|f^4(x)| \le 76e$ for $0 \le x \le 1$

So we take k = 76e, a = 0, b = 1

For getting the error 0.00001 in Simpson's rule
We should choose n such that

$$\frac{k(b-a)^5}{180n^4} < 0.00001$$

$$\Rightarrow \frac{76e}{180n^4} < 0.00001$$

$$\Rightarrow n^4 > \frac{76e}{0.0018}$$

$$\Rightarrow n > \sqrt[4]{\frac{76e}{0.0018}} \approx 18.4$$

Next even number after 18, is 20

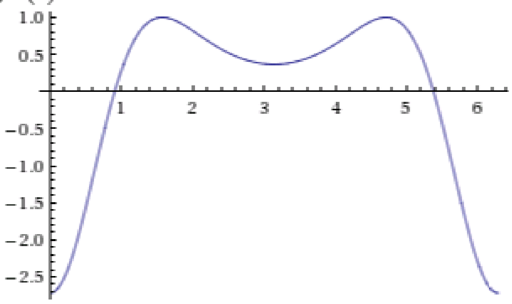
So $\boxed{n = 20}$ will give the desired result.

Answer 23E.

a) To determine $f''(x)$:

$$f(x) = e^{\cos x}$$
$$f'(x) = -\sin x e^{\cos x}$$
$$f''(x) = e^{\cos x}\left(\sin^2 x - \cos x\right)$$

Graphically, $f''(x)$ is:



So,

$$-2.8 \le f''(x) \le 1$$

Therefore, a good upper bound for $|f''(x)|$:

$$|f''(x)| = 2.8$$

---

b) Midpoint Rule:

The midpoints of the ten subintervals are $\frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}, \frac{11\pi}{10}, \frac{13\pi}{10}, \frac{15\pi}{10}, \frac{17\pi}{10}$ and $\frac{19\pi}{10}$.

So the Midpoint Rule produces

$$\int_0^{2\pi} e^{\cos x}\,dx \approx \frac{2\pi}{10}\left[f\left(\frac{\pi}{10}\right) + f\left(\frac{3\pi}{10}\right) + \ldots + f\left(\frac{17\pi}{10}\right) + f\left(\frac{19\pi}{10}\right)\right]$$
$$= \frac{2\pi}{10}\left[e^{\cos\frac{\pi}{10}} + e^{\cos\frac{3\pi}{10}} + \ldots + e^{\cos\frac{17\pi}{10}} + e^{\cos\frac{19\pi}{10}}\right]$$

Therefore

$$\boxed{M_{10} \approx 7.954926518}$$

---

c) Error Estimate For Midpoint Approximation $|E_M|$:

Use the formula $|E_M| = \frac{K(b-a)^3}{24n^2}$, where $|f''(x)| \le K$, to estimate the error for approximation M_{10}.

Taking $K = 2.8, a = 0, b = 2\pi$ and $n = 10$ then

$$|E_M| \le \frac{2.8(2\pi - 0)^3}{24 \cdot 10^2}$$
$$\boxed{\approx 0.2894}$$

d) CAS produces:

$$\int_0^{2\pi} e^{\cos x}\,dx \approx 7.954926521$$

e) Comparing the actual error to the estimate $|E_M|$:

The actual error is:

$$E_M = \int_0^{2\pi} e^{\cos x}\,dx - M_{10}$$
$$= 7.954926521 - 7.954926518$$
$$\boxed{= 0.000000003}$$

So the actual error is much smaller than that of the estimated error of $|E_M| \approx 0.2894$.

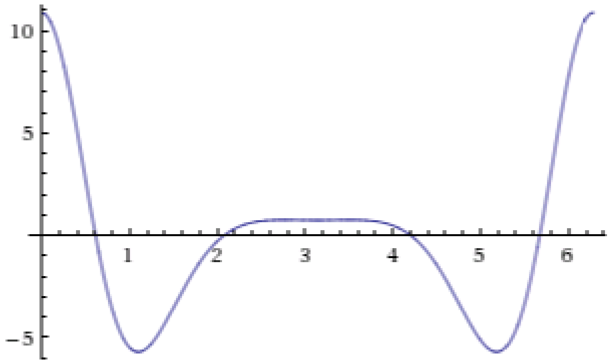
f) To determine $f^{(4)}(x)$:

$$f''(x)=e^{\cos x}\left(\sin^2 x-\cos x\right)$$

$$f'''(x)=\frac{1}{2}\sin x e^{\cos x}\left(6\cos x+\cos 2x+1\right)$$

$$f^{(4)}(x)=\frac{1}{8}e^{\cos x}\left(-4\cos x+24\cos 2x+12\cos 3x+\cos 4x-1\right)$$

Graphically, $f^{(4)}(x)$ is:



Therefore, a good upper bound for $\left|f^{(4)}(x)\right|$:

$$\left|f^{(4)}(x)\right|=10.9$$

g) Simpson's Rule:

$$\int_0^{2\pi} e^{\cos x}dx \approx \frac{\frac{2\pi}{10}}{3}\left[f(0)+4f\left(\frac{\pi}{5}\right)+2f\left(\frac{2\pi}{5}\right)...+2f\left(\frac{8\pi}{5}\right)+4f\left(\frac{9\pi}{5}\right)+f(2\pi)\right]$$

$$=\frac{2\pi}{30}\left[e^{\cos 0}+4e^{\cos\frac{\pi}{5}}+2e^{\cos\frac{2\pi}{5}}+...+2e^{\cos\frac{18\pi}{5}}+4e^{\cos\frac{9\pi}{5}}+e^{\cos 2\pi}\right]$$

Therefore

$$\boxed{S_{10}\approx 7.953789422}$$

h) Estimate Error For Simpson's Approximation $|E_S|$:

Use the formula $|E_S|=\frac{K(b-a)^5}{180n^4}$, where $\left|f^{(4)}(x)\right|\le K$, to estimate the error for approximation S_{10}.

Taking $K=10.9$, $a=0$, $b=2\pi$ and $n=10$ then

$$|E_S|\le\frac{10.9(2\pi-0)^5}{180\cdot 10^4}$$

$$\boxed{\approx 0.0593}$$

i) Comparing the actual error to the estimate $|E_S|$:

The actual error is:

$$E_S=\int_0^{2\pi}e^{\cos x}dx-S_{10}$$

$$=7.954926521-7.953789422$$

$$\boxed{=0.001137}$$

So the actual error is much smaller than that of the estimated error of $|E_S|\approx 0.0593$.

j) For Accuracy Of Simpson's Approximation:

$$\frac{K(b-a)^5}{180n^4} < |E_S|$$

Therefore

$$n^4 > \frac{K(b-a)^5}{180|E_S|}$$

So producing

$$n > \sqrt[4]{\frac{K(b-a)^5}{180|E_s|}}$$

Given that $K = 10.9$, $a = 0$, $b = 2\pi$, $E_S = 0.0001$ and $n = 10$ then

$$n > \sqrt[4]{\frac{10.9(2\pi-0)^5}{180 \cdot 0.0001}}$$

$$n \approx 49.3$$

Thus $\boxed{n \geq 50}$ will ensure accuracy to be less than 0.0001 for the Simpson's approximation.

Answer 24E.

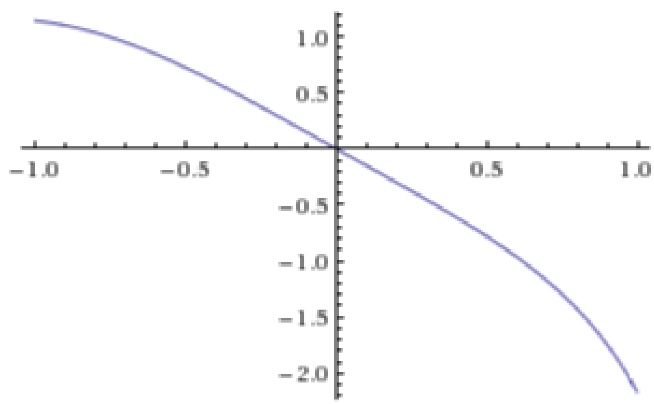
a) To determine $f''(x)$:

$$f(x) = \sqrt{4-x^3}$$

$$f'(x) = -\frac{3x^2}{2\sqrt{4-x^3}}$$

$$f''(x) = \frac{3x(x^3-16)}{4(4-x^3)^{\frac{3}{2}}}$$

Graphically, $f''(x)$ is:



So,

$$-2.17 \leq f''(x) \leq 1.14$$

Therefore, a good upper bound for $|f''(x)|$:

$$|f''(x)| = 2.17$$

b) Midpoint Rule:
The midpoints of the ten subintervals are -0.9, -0.7, -0.5, -0.3, -0.1, 0.1, 0.3, 0.5, 0.7, and 0.9.
So the Midpoint Rule produces

$$\int_{-1}^{1}\sqrt{4-x^3}\,dx \approx \frac{1}{5}\left[f(-0.9)+f(-0.7)+\ldots+f(0.7)+f(0.9)\right]$$
$$=\frac{1}{5}\left[\sqrt{4-(-0.9)^3}+\sqrt{4-(-0.7)^3}+\ldots+\sqrt{4-0.7^3}+\sqrt{4-0.9^3}\right]$$

Therefore

$$\boxed{M_{10}\approx 3.995804152}$$

---

c) Error Estimate For Midpoint Approximation $|E_M|$:

Use the formula $|E_M|=\frac{K(b-a)^3}{24n^2}$, where $|f''(x)|\le K$, to estimate the error for approximation M_{10}.
Taking $K=2.17, a=-1, b=1$ and $n=10$ then

$$|E_M|\le\frac{2.17(1-(-1))^3}{24\cdot 10^2}$$
$$\boxed{\approx -0.00723}$$

d) CAS produces:

$$\int_{-1}^{1}\sqrt{4-x^3}\,dx\approx 3.995487677$$

e) Comparing the actual error to the estimate $|E_M|$:
The actual error is:

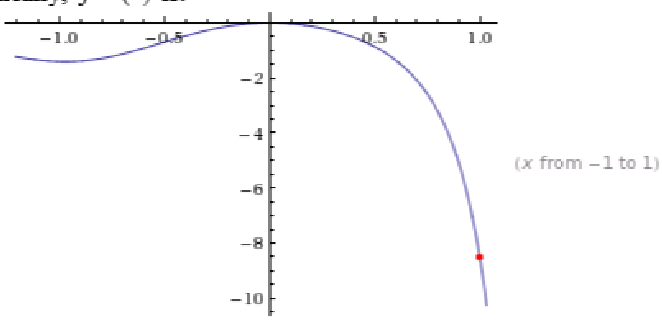
$$E_M=\int_0^{2\pi}e^{\cos x}dx-M_{10}$$
$$=3.995487677-3.995804152$$
$$\boxed{=-0.0003165}$$

So the actual error is much smaller than that of the estimated error of $|E_M|\approx -0.00723$.

f) To determine $f^{(4)}(x)$:

$$f''(x)=\frac{3x(x^3-16)}{4(4-x^3)^{\frac{3}{2}}}$$
$$f'''(x)=\frac{3(x^6-8x^3-704)}{8(4-x^3)^{\frac{5}{2}}}$$
$$f^{(4)}(x)=\frac{9x^2(x^6-8x^3-704)}{16(4-x^3)^{\frac{7}{2}}}$$

Graphically, $f^{(4)}(x)$ is:



Therefore, a good upper bound for $|f^{(4)}(x)|$:

$$|f^{(4)}(x)|=8.55$$

g) Simpson's Rule:

$$\int_{-1}^{1}\sqrt{4-x^3}\,dx \approx \frac{\frac{1}{5}}{3}\left[f(-1)+4f(-0.8)+2f(-0.6)\ldots+2f(0.6)+4f(0.8)+f(1)\right]$$

$$=\frac{1}{15}\left[\sqrt{4-(-1)^3}+4\sqrt{4-(-0.8)^3}+2\sqrt{4-(-0.6)^3}+\ldots\right.$$

$$\left.\ldots+2\sqrt{4-(0.6)^3}+4\sqrt{4-(0.8)^3}+\sqrt{4-1^3}\right]$$

Therefore

$$\boxed{S_{10}\approx 3.99544979}$$

h) Estimate Error For Simpson's Approximation $|E_S|$:

Use the formula $|E_S|=\frac{K(b-a)^5}{180n^4}$, where $\left|f^{(4)}(x)\right|\le K$, to estimate the error for approximation S_{10}.

Taking $K=8.55, a=-1, b=1$ and $n=10$ then

$$|E_S|\le \frac{8.55(1-(-1))^5}{180\cdot 10^4}$$

$$\boxed{\approx 0.000152}$$

i) Comparing the actual error to the estimate $|E_S|$:

The actual error is:

$$E_S=\int_{-1}^{1}\sqrt{4-x^3}\,dx-S_{10}$$

$$=3.995487677-3.99544979$$

$$\boxed{=0.0000379}$$

So the actual error is much smaller than that of the estimated error of $|E_S|\approx 0.000152$.

j) For Accuracy Of Simpson's Approximation:

$$\frac{K(b-a)^5}{180n^4}<|E_S|$$

Therefore

$$n^4>\frac{K(b-a)^5}{180|E_S|}$$

So producing

$$n>\sqrt[4]{\frac{K(b-a)^5}{180|E_s|}}$$

Given that $K=8.55, a=-1, b=1, E_S=0.0001$ and $n=10$ then

$$n>\sqrt[4]{\frac{8.55(1-(-1))^5}{180\cdot 0.0001}}$$

$$n\approx 11.1$$

Thus $\boxed{n\ge 12}$ will ensure accuracy to be less than 0.0001 for the Simpson's approximation.

Answer 25E.

We have $f(x)=xe^x$, Interval is [0, 1],

For $n=5$, $\Delta x=\frac{1}{5}$

And subintervals are $\left[0,\frac{1}{5}\right],\left[\frac{1}{5},\frac{2}{5}\right],\left[\frac{2}{5},\frac{3}{5}\right],\left[\frac{3}{5},\frac{4}{5}\right],\left[\frac{4}{5},1\right]$

Midpoints are $\frac{1}{10},\frac{3}{10},\frac{5}{10},\frac{7}{10},\frac{9}{10}$

Taking left end points of the subintervals

$$L_5 = \sum_{i=1}^{5} f(x_{i-1})\Delta x$$

$$= \frac{1}{5}\left[f(0) + f\left(\frac{1}{5}\right) + f\left(\frac{2}{5}\right) + f\left(\frac{3}{5}\right) + + f\left(\frac{4}{5}\right) \right]$$

$$= \frac{1}{5}\left[\left(0e^{0}\right) + \left(\frac{1}{5}e^{1/5}\right) + \left(\frac{2}{5}e^{2/5}\right) + \left(\frac{3}{5}e^{3/5}\right) + \left(\frac{4}{5}e^{4/5}\right) \right] \approx 0.742943$$

$$\boxed{L_5 \approx 0.742943}$$

Taking right end points of the intervals

$$R_5 = \sum_{i=1}^{5} f(x_i)\Delta x = \frac{1}{5}\left[f\left(\frac{1}{5}\right) + f\left(\frac{2}{5}\right) + f\left(\frac{3}{5}\right) + f\left(\frac{4}{5}\right) + f(1) \right]$$

$$= \frac{1}{5}\left[\left(\frac{1}{5}e^{1/5}\right) + \left(\frac{2}{5}e^{2/5}\right) + \left(\frac{3}{5}e^{3/5}\right) + \left(\frac{4}{5}e^{4/5}\right) + 1e^{1} \right] \approx 1.286599$$

$$\boxed{R_5 \approx 1.286599}$$

Taking midpoint of the subintervals

$$M_5 = \sum_{i=1}^{5} f(\bar{x}_i)\Delta x = \frac{1}{5}\left[f\left(\frac{1}{10}\right) + f\left(\frac{3}{10}\right) + f\left(\frac{5}{10}\right) + f\left(\frac{7}{10}\right) + f\left(\frac{9}{10}\right) \right]$$

$$= \frac{1}{5}\left[\left(\frac{1}{10}e^{1/10}\right) + \left(\frac{3}{10}e^{3/10}\right) + \left(\frac{5}{10}e^{5/10}\right) + \left(\frac{7}{10}e^{7/10}\right) + \left(\frac{9}{10}e^{9/10}\right) \right]$$

$$\boxed{M_5 \approx 0.992621}$$

By Trapezoidal rule

$$T_5 = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5) \right]$$

$$= \frac{1}{10}\left[f(0) + 2f\left(\frac{1}{5}\right) + 2f\left(\frac{2}{5}\right) + 2f\left(\frac{3}{5}\right) + 2f\left(\frac{4}{5}\right) + f(1) \right]$$

$$= \frac{1}{10}\left[0e^{0} + 2\left(\frac{1}{5}e^{1/5}\right) + 2\left(\frac{2}{5}e^{2/5}\right) + 2\left(\frac{3}{5}e^{3/5}\right) + 2\left(\frac{4}{5}e^{4/5}\right) + 1e^{1} \right] \approx 1.014771$$

$$\boxed{T_5 \approx 1.014771}$$

For $n = 10$, $\quad \Delta x = \frac{1}{10}$

Subintervals are

$$\left[0, \frac{1}{10}\right], \left[\frac{1}{10}, \frac{2}{10}\right], \left[\frac{2}{10}, \frac{3}{10}\right], \left[\frac{3}{10}, \frac{4}{10}\right], \left[\frac{4}{10}, \frac{5}{10}\right]$$

And $\left[\frac{5}{10}, \frac{6}{10}\right], \left[\frac{6}{10}, \frac{7}{10}\right], \left[\frac{7}{10}, \frac{8}{10}\right], \left[\frac{8}{10}, \frac{9}{10}\right], \left[\frac{9}{10}, 1\right]$

Midpoints are $\frac{1}{20}, \frac{3}{20}, \frac{5}{20}, \frac{7}{20}, \frac{9}{20}, \frac{11}{20}, \frac{13}{20}, \frac{15}{20}, \frac{17}{20}, \frac{19}{20}$

Taking left end points

$$L_{10} = \frac{1}{10}\left[f(0) + f\left(\frac{1}{10}\right) + f\left(\frac{2}{10}\right) + f\left(\frac{3}{10}\right) + \dots\dots + f\left(\frac{9}{10}\right) \right]$$

$$\boxed{L_{10} \approx 0.867782}$$

Taking right end point

$$R_{10} = \frac{1}{10}\left[f\left(\frac{1}{10}\right) + f\left(\frac{2}{10}\right) + f\left(\frac{3}{10}\right) + \dots\dots + f\left(\frac{9}{10}\right) + f(1) \right]$$

$$\boxed{R_{10} \approx 1.139610}$$

Taking midpoints

$$M_{10} = \frac{1}{10}\left[f\left(\frac{1}{20}\right) + f\left(\frac{3}{20}\right) + f\left(\frac{5}{20}\right) + \dots\dots + f\left(\frac{19}{20}\right) \right]$$

$$\boxed{M_{10} \approx 0.998152}$$

By Trapezoidal rule

$$T_{10} = \frac{1}{20}\left[f(0) + 2f\left(\frac{1}{10}\right) + 2f\left(\frac{2}{10}\right) + \dots\dots + 2f\left(\frac{9}{10}\right) + f(1)\right]$$

$$\boxed{T_{10} \approx 1.003696}$$

For $n = 20$ $\quad\quad \Delta x = \frac{1}{20}$

Subintervals are $\left[0, \frac{1}{20}\right], \left[\frac{1}{20}, \frac{2}{20}\right], \dots\dots\dots, \left[\frac{19}{20}, 1\right]$

Midpoints are $\frac{1}{40}, \frac{3}{40}, \frac{5}{40}, \dots\dots\dots, \frac{39}{40}$

Taking left end points

$$L_{20} = \frac{1}{20}\left[f(0) + f\left(\frac{1}{20}\right) + \dots\dots\dots + f\left(\frac{19}{20}\right)\right]$$

$$\boxed{L_{20} \approx 0.932967}$$

Taking right end points

$$R_{20} = \frac{1}{20}\left[f\left(\frac{1}{20}\right) + f\left(\frac{2}{20}\right) + \dots\dots + f(1)\right]$$

$$\boxed{R_{20} \approx 1.068881}$$

Taking midpoint

$$M_{20} = \frac{1}{20}\left[f\left(\frac{1}{40}\right) + f\left(\frac{3}{40}\right) + f\left(\frac{5}{40}\right) + \dots\dots + f\left(\frac{39}{40}\right)\right]$$

$$\boxed{M_{20} \approx 0.999538}$$

By Trapezoidal rule

$$T_{20} = \frac{1}{40}\left[f(0) + 2f\left(\frac{1}{20}\right) + 2f\left(\frac{2}{20}\right) + \dots\dots + 2f\left(\frac{19}{20}\right) + f(1)\right]$$

$$\boxed{T_{20} \approx 1.000924}$$

$$E_M = 1 - M_5 = 1 - 0.992621$$

$$\boxed{E_M = 0.007379}$$

Now we calculate actual value

$$\int_0^1 xe^x dx = \left[xe^x - e^x\right]_0^1 = \left[e^1 - e^1 - 0 + e^0\right] = 1$$

$$\boxed{\int_0^1 xe^x dx = 1}$$

For $n = 5$

Now we calculate errors

$$E_L = 1 - L_5 = 1 - 0.742943 = 0.257057$$

$$\boxed{E_L = 0.257057}$$

$$E_R = 1 - R_5 = 1 - 1.286599$$

$$\boxed{E_R = -0.286599}$$

$$E_T = 1 - T_5 = 1 - 1.014771$$

$$\boxed{E_T = -0.014771}$$

For $n = 10$, $E_L = 1 - L_{10}$ $\boxed{E_L = 0.132218}$

$E_R = 1 - R_{10}$ $\boxed{E_R = -0.139610}$

$E_T = 1 - T_{10}$ $\boxed{E_T = -0.003696}$

$E_M = 1 - M_{10}$ $\boxed{E_M = 0.001848}$

Similarly for n = 20

$E_L = 0.067033,\ E_R = -0.068881,\ E_T = -0.000924,\ E_M = 0.000462$

Now we make a table of the results got above

n	L_n	R_n	T_n	M_n
5	0.742943	1.286599	1.014771	0.992621
10	0.867782	1.139610	1.003696	0.998152
20	0.932967	1.068881	1.000924	0.999538

n	E_L	E_R	E_T	E_M
5	0.257057	-0.286599	-0.014771	0.007379
10	0.132218	-0.139610	-0.003696	0.001848
20	0.067033	-0.068881	-0.000924	0.000462

We see that from the table of errors

E_L and E_R decrease by a factor of about 2, when we double the value of n

E_T and E_M decrease by a factor of about 4, when we double the value of n

Answer 26E.

Consider the integral

$$\int_1^2 \frac{1}{x^2}\,dx$$

The objective is to find the approximations L_n, R_n, T_n, and M_n for $n = 5, 10, 20$

With $n = 5, a = 1$ and $b = 2$.

$$\Delta x = \frac{(2-1)}{5} = \frac{1}{5} = .2$$

The right approximation method gives

$$\int_1^2 \frac{1}{x^2}\,dx = R_5 = \sum_{n=1}^{5} f(x_i)\Delta x$$

$$= .2\left[f(1.2) + f(1.4) + f(1.6) + f(1.8) + f(2)\right]$$

$$= 0.2(0.6944444 + 0.510204 + 0.390625 + 0.308641 + 0.25000)$$

$$= \boxed{0.437831}$$

The left approximation method gives

$$\int_1^2 \frac{1}{x^2}\,dx = L_5 = \sum_{n=1}^{5} f(x_{i-1})\Delta x$$

$$= .2\left[f(1) + f(1.2) + f(1.4) + f(1.6) + f(1.8)\right]$$

$$= 0.2(1 + 0.6944444 + 0.510204 + 0.390625 + 0.308641)$$

$$= \boxed{0.5690549}$$

The trapezoidal rule gives

$$\int_0^1 \frac{1}{x^2}\,dx = T_5$$

$$= \frac{.2}{2}\left[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)\right]$$

$$= 0.1(1 + 1.388888 + 1.020408 + 0.78125 + 0.6172839 + 0.500000)$$

$$= \boxed{0.5307830}$$

The midpoints of the five subintervals are

$1.1, 1.3, 1.5, 1.7, 1.9;$

So the midpoints rule gives

$$\begin{aligned}\int_1^2 \frac{1}{x^2}\,dx &= M_5 \\ &= 0.2\left[f(1.1)+f(1.3)+f(1.5)+f(1.7)+f(1.9)\right] \\ &= 0.2(0.826446+0.591715+0.4444444+0.3460207+0.2770083) \\ &= \boxed{0.497127}\end{aligned}$$

The midpoints of the five subintervals are

$1.1, 1.3, 1.5, 1.7, 1.9;$

So the midpoints rule gives

$$\begin{aligned}\int_1^2 \frac{1}{x^2}\,dx &= M_5 \\ &= 0.2\left[f(1.1)+f(1.3)+f(1.5)+f(1.7)+f(1.9)\right] \\ &= 0.2(0.826446+0.591715+0.4444444+0.3460207+0.2770083) \\ &= \boxed{0.497127}\end{aligned}$$

With $n=10, a=1$ and $b=2$, we have

$$\Delta x = \frac{(2-1)}{10} = \frac{1}{10} = .1$$

The right approximation method gives

$$\begin{aligned}\int_1^2 \frac{1}{x^2}\,dx &= R_{10} = \sum_{n=1}^{10} f(x_i)\Delta x \\ &= .1\begin{bmatrix} f(1.1)+f(1.2)+f(1.3)+f(1.4)+f(1.5) \\ +f(1.6)+f(1.7)+f(1.8)+f(1.9)+f(2) \end{bmatrix} \\ &= \boxed{0.46395512}\end{aligned}$$

The midpoints of the ten subintervals are

$1.05, 1.15, 1.25, 1.35, 1.45, 1.55, 1.65, 1.75, 1.85, 1.95,$

So the midpoints rule gives

$$\begin{aligned}\int_0^1 \frac{1}{x^2}\,dx &= M_{10} \\ &= 0.1\begin{bmatrix} f(1.05)+f(1.15)+f(1.25)+f(1.35)+f(1.45)+f(1.55)+ \\ f(1.65)+f(1.75)+f(1.85)+f(1.95) \end{bmatrix} \\ &= \boxed{0.4992736}\end{aligned}$$

With $n = 20, a = 1$ and $b = 2$, we have

$$\Delta x = \frac{(1-0)}{20} = \frac{1}{20} = .05$$

The right approximation method gives

$$\int_1^2 \frac{1}{x^2}\,dx = R_{20} = \sum_{n=1}^{20} f(x_i)\Delta x$$
$$= 0.05\begin{bmatrix} f(1.05) + f(1.1) + f(1.15) + f(1.2) + \cdots \\ + f(1.95) + f(2) \end{bmatrix}$$
$$= \boxed{0.481614}$$

The left approximation method gives

$$\int_1^2 \frac{1}{x^2}\,dx = L_{20} = \sum_{n=1}^{20} f(x_{i-1})\Delta x$$
$$= 0.05\begin{bmatrix} f(1) + f(1.05) + f(1.1) + f(1.15) + f(1.2) + \\ \cdots + f(1.95) \end{bmatrix}$$
$$= \boxed{0.5191143}$$

The trapezoidal rule gives

$$\int_1^2 \frac{1}{x^2}\,dx = T_{20}$$
$$= \frac{0.05}{2}\begin{bmatrix} f(1) + 2f(1.05) + 2f(1.1) + 2f(1.15) + 2f(1.2) \\ + \cdots + 2f(1.95) + f(2) \end{bmatrix}$$
$$= \boxed{0.500364}$$

The midpoints of the ten subintervals are

$$1.025, 1.075, 1.125, 1.175, \cdots 1.975$$

So the midpoints rule gives

$$\int_1^2 \frac{1}{x^2}\,dx = M_{20}$$
$$= 0.05\begin{bmatrix} f(1.025) + f(1.075) + f(1.125) + f(1.175) \\ + \cdots + f(1.975) \end{bmatrix}$$
$$= \boxed{0.4998178}$$

From fundamental calculus

$$\int_1^2 \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^2$$
$$= \left(-\frac{1}{2}+1\right)$$
$$= 0.5$$

n	L_n	R_n	T_n	M_n
5	0.5690549	0.437831	0.5307830	0.497127
10	0.538955	0.46395512	0.501455	0.4992736
20	0.5191143	0.481614	0.500364	0.4998178

n	E_L	E_R	E_T	E_M
5	-0.0690549	0.062169	-0.030783	0.02873
10	-0.038955	0.036044	-0.001455	0.0007264
20	-0.01591143	-0.018386	-0.000364	0.0001822

Observe that we get more accurate approximation when we increase the value of n.

Answer 27E.

Evaluate T_6 for the integral $\int_0^2 x^4\,dx.$

Write the formula to find T_6.

$$\int_a^b f(x)dx \approx T_6 = \frac{\Delta x}{2}\left[\begin{array}{l} f(x_0)+2f(x_1)+2f(x_2)+2f(x_3) \\ +2f(x_4)+2f(x_5)+f(x_6) \end{array}\right]$$

Where

$$f(x) = x^4$$
$$a = 0$$
$$b = 2$$
$$\Delta x = \frac{b-a}{n}$$
$$= \frac{2-0}{6}$$
$$= \frac{1}{3}$$

Use the equation $x_i = a + i\Delta x$ to find $x_i, i = 0,1,2,3..6$

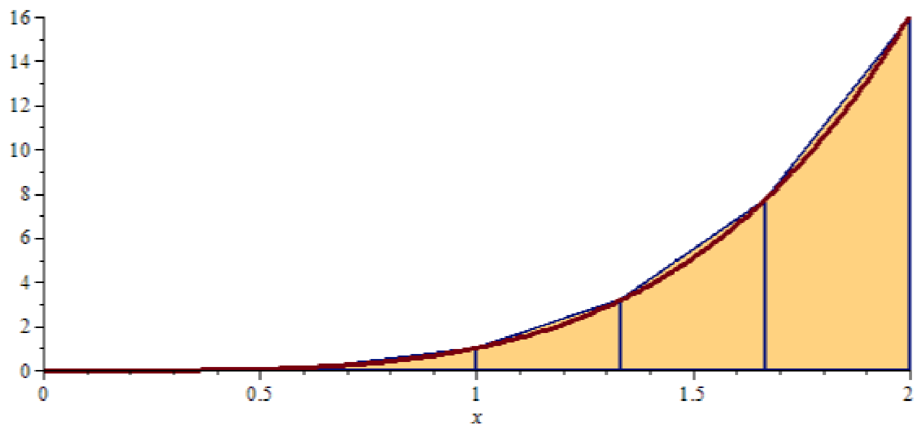
$$x_0 = 0 + (0)\frac{1}{3} \qquad x_3 = 0 + (3)\frac{1}{3}$$
$$= 0 \qquad\qquad = 1$$
$$x_1 = 0 + (1)\frac{1}{3} \qquad x_4 = 0 + (4)\frac{1}{3} \qquad x_6 = 0 + (6)\frac{1}{3}$$
$$= \frac{1}{3} \qquad\qquad = \frac{4}{3} \qquad\qquad = 2$$
$$x_2 = 0 + (2)\frac{1}{3} \qquad x_5 = 0 + (5)\frac{1}{3}$$
$$= \frac{2}{3} \qquad\qquad = \frac{5}{3}$$

Put $f(x) = x^4, a = 0, b = 2, \Delta x = \frac{1}{3}, x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1, x_4 = \frac{4}{3}, x_5 = \frac{5}{3}$ and $x_6 = 2$ in the formula.

$$\int_0^2 x^4 dx \approx T_6 = \frac{\frac{1}{3}}{2}\begin{bmatrix} f(0) + 2f\left(\frac{1}{3}\right) + 2f\left(\frac{2}{3}\right) + 2f(1) \\ +2f\left(\frac{4}{3}\right) + 2f\left(\frac{5}{3}\right) + f(2) \end{bmatrix}$$
$$= \frac{1}{6}\begin{bmatrix} (0)^4 + 2\left(\frac{1}{3}\right)^4 + 2\left(\frac{2}{3}\right)^4 + 2(1)^4 \\ +2\left(\frac{4}{3}\right)^4 + 2\left(\frac{5}{3}\right)^4 + (2)^4 \end{bmatrix}$$
$$= \frac{1627}{243} \approx 6.695473$$

Hence $T_6 \approx \boxed{6.695473}$.

The following diagram shows the area represented by T_6.



An approximation of $\int_0^2 f(x)\,dx$ using trapezoid rule, where $f(x) = x^4$ and the partition is uniform. The approximate value of the integral is 6.695473251. Number of subintervals used: 6.

Evaluate T_{12} for the integral $\int_0^2 x^4\,dx$.

Write the formula to find T_{12}.

$$\int_a^b f(x)dx \approx T_{12} = \frac{\Delta x}{2}\begin{bmatrix} f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) \\ +2f(x_4) + 2f(x_5) + 2f(x_6) + 2f(x_7) \\ +2f(x_8) + 2f(x_9) + 2f(x_{10}) + 2f(x_{11}) \\ +f(x_{12}) \end{bmatrix}$$

Where

$$f(x)=x^4$$
$$a=0$$
$$b=2$$
$$\Delta x=\frac{b-a}{n}$$
$$=\frac{2-0}{12}$$
$$=\frac{1}{6}$$

Use the equation $x_i=a+i\Delta x$ to find $x_i, i=0,1,2,3..12$

Put $f(x)=x^4, a=0, b=2, \Delta x=\frac{1}{6}, x_0=0, x_1=\frac{1}{6}, x_2=\frac{2}{6}, x_3=\frac{3}{6}, x_4=\frac{4}{6}, x_5=\frac{5}{6}, x_6=1$

$x_7=\frac{7}{6}, x_8=\frac{8}{6}, x_9=\frac{9}{6}, x_{10}=\frac{10}{6}, x_{11}=\frac{11}{6}$ and $x_{12}=12$ in the formula.

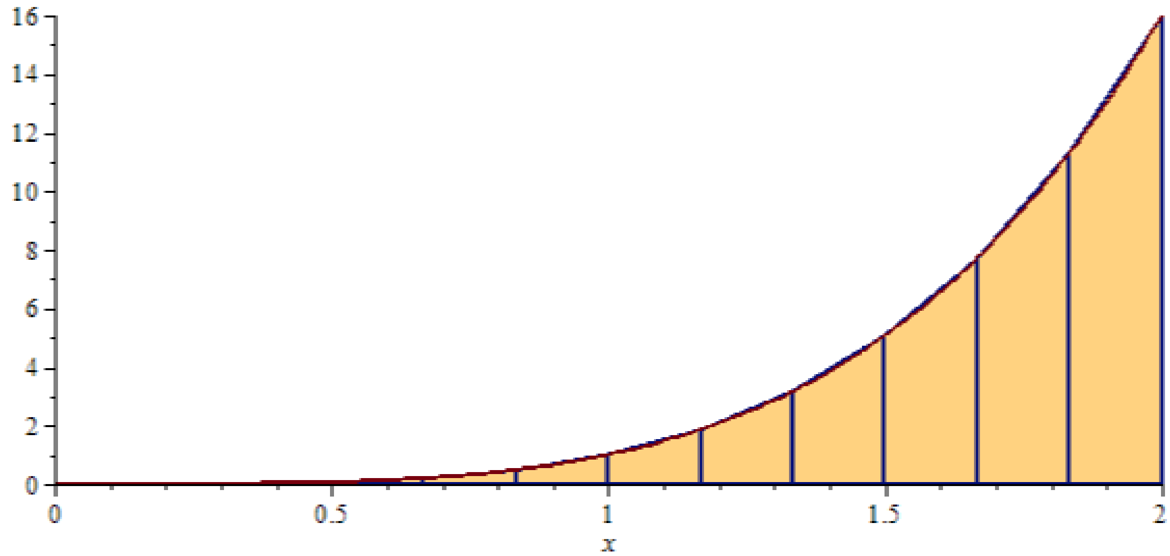
$$\int_0^2 x^4 dx \approx T_{12}=\frac{\frac{1}{6}}{2}\begin{bmatrix} f(0)+2f\left(\frac{1}{6}\right)+2f\left(\frac{2}{6}\right)+2f\left(\frac{3}{6}\right) \\ +2f\left(\frac{4}{6}\right)+2f\left(\frac{5}{6}\right)+2f(1)+2f\left(\frac{7}{6}\right) \\ +2f\left(\frac{8}{6}\right)+2f\left(\frac{9}{6}\right)+2f\left(\frac{10}{6}\right)+2f\left(\frac{11}{6}\right) \\ +f(2) \end{bmatrix}$$

$$=\frac{1}{12}\begin{bmatrix} (0)^4+2\left(\frac{1}{6}\right)^4+2\left(\frac{2}{6}\right)^4+2\left(\frac{3}{6}\right)^4 \\ +2\left(\frac{4}{6}\right)^4+2\left(\frac{5}{6}\right)^4+2(1)^4+2\left(\frac{7}{6}\right)^4 \\ +2\left(\frac{8}{6}\right)^4+2\left(\frac{9}{6}\right)^4+2\left(\frac{10}{6}\right)^4+2\left(\frac{11}{6}\right)^4 \\ +(2)^4 \end{bmatrix}$$

$$=\frac{25171}{3888}\approx 6.474022$$

Hence $T_{12}\approx \boxed{6.474022}$.

The following diagram shows the area represented by T_{12}.



An approximation of $\int_0^2 f(x)\, \mathrm{d}x$ using trapezoid rule, where $f(x) = x^4$ and the partition is uniform. The approximate value of the integral is 6.474022634.
Number of subintervals used: 12.

Evaluate M_6 for the integral $\int_0^2 x^4\, dx$.

Write the formula to find M_6.

$$\int_a^b f(x)dx \approx M_6 = \Delta x\left[f(\overline{x_1}) + f(\overline{x_2}) + f(\overline{x_3}) + f(\overline{x_4}) + f(\overline{x_5}) + f(\overline{x_6})\right]$$

Where

$$f(x) = x^4$$

$$a = 0$$

$$b = 2$$

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{2-0}{6} \\ &= \frac{1}{3}\end{aligned}$$

Use the equations $\overline{x_i} = \frac{x_i + x_{i-1}}{2}$, $x_0 = 0,\ x_1 = \frac{1}{3},\ x_2 = \frac{2}{3},\ x_3 = 1,\ x_4 = \frac{4}{3},\ x_5 = \frac{5}{3}$ and $x_6 = 2$ to find $\overline{x_i}, i = 0,1,2,3..6$

$$\overline{x_1} = \frac{x_1 + x_0}{2} = \frac{\frac{1}{3}+0}{2} = \frac{1}{6} \qquad \overline{x_3} = \frac{x_3 + x_2}{2} = \frac{5}{6} \qquad \overline{x_5} = \frac{x_5 + x_4}{2} = \frac{3}{2}$$

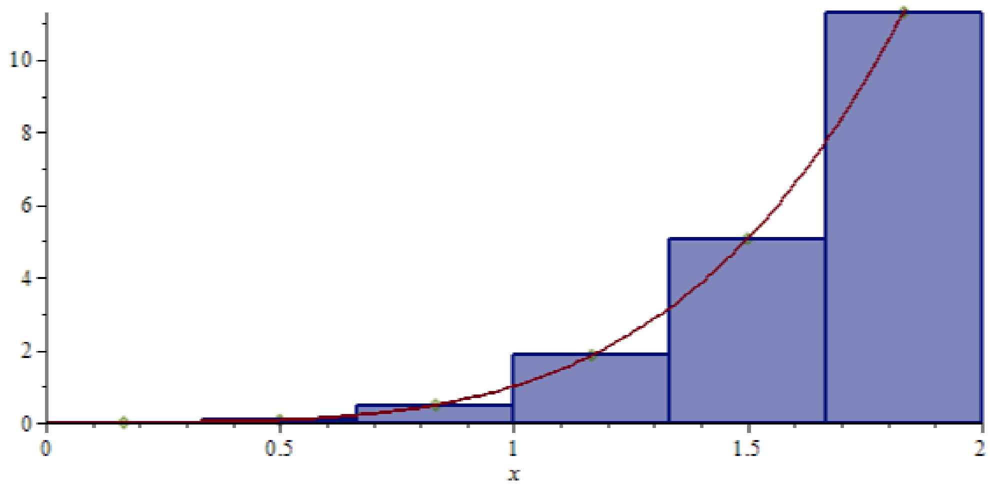
$$\overline{x_2} = \frac{x_2 + x_1}{2} = \frac{\frac{2}{3}+\frac{1}{3}}{2} = \frac{1}{2} \qquad \overline{x_4} = \frac{x_4 + x_3}{2} = \frac{7}{6} \qquad \overline{x_6} = \frac{x_6 + x_5}{2} = \frac{11}{6}$$

Put $f(x) = x^4, a = 0, b = 2, \Delta x = \frac{1}{3}, \overline{x_1} = \frac{1}{6}, \overline{x_2} = \frac{1}{2}, \overline{x_3} = \frac{5}{6}, \overline{x_4} = \frac{7}{6}, \overline{x_5} = \frac{3}{2}$ and $\overline{x_6} = \frac{11}{6}$ in the formula.

$$\begin{aligned}\int_0^2 x^4 dx \approx M_6 &= \frac{1}{3}\left[f\left(\frac{1}{6}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{5}{6}\right) + f\left(\frac{7}{6}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{11}{6}\right)\right] \\ &= \frac{1}{3}\left[\left(\frac{1}{6}\right)^4 + \left(\frac{1}{2}\right)^4 + \left(\frac{5}{6}\right)^4 + \left(\frac{7}{6}\right)^4 + \left(\frac{3}{2}\right)^4 + \left(\frac{11}{6}\right)^4\right] \\ &= \frac{12155}{1944} \approx 6.252572\end{aligned}$$

Hence $M_6 \approx \boxed{6.252572}$.

The following diagram shows the area represented by M_6.



A midpoint Riemann sum approximation of $\int_0^2 f(x)\ dx$, where $f(x) = x^4$ and the partition is uniform. The approximate value of the integral is 6.252572016. Number of subintervals used: 6.

Evaluate M_{12} for the integral $\int_0^2 x^4\, dx$.

Write the formula to find M_{12}.

$$\int_a^b f(x)dx \approx M_{12} = \Delta x\left[\begin{array}{l} f(\overline{x_1}) + f(\overline{x_2}) + f(\overline{x_3}) + f(\overline{x_4}) + f(\overline{x_5}) + f(\overline{x_6}) \\ + f(\overline{x_7}) + f(\overline{x_8}) + f(\overline{x_9}) + f(\overline{x_{10}}) + f(\overline{x_{11}}) + f(\overline{x_{12}}) \end{array}\right]$$

Where

$$f(x) = x^4$$
$$a = 0$$
$$b = 2$$
$$\Delta x = \frac{b-a}{n} = \frac{2-0}{12} = \frac{1}{6}$$

Put $f(x) = x^4, a = 0, b = 2, \Delta x = \frac{1}{6}, \overline{x_1} = \frac{1}{12}, \overline{x_2} = \frac{3}{12}, \overline{x_3} = \frac{5}{12}, \overline{x_4} = \frac{7}{12}, \overline{x_5} = \frac{9}{12}$ $, \overline{x_6} = \frac{11}{12}, \overline{x_7} = \frac{13}{12}, \overline{x_8} = \frac{15}{12}, \overline{x_9} = \frac{17}{12}, \overline{x_{10}} = \frac{19}{12}, \overline{x_{11}} = \frac{21}{12}$ and $\overline{x_{12}} = \frac{23}{12}$ in the formula.

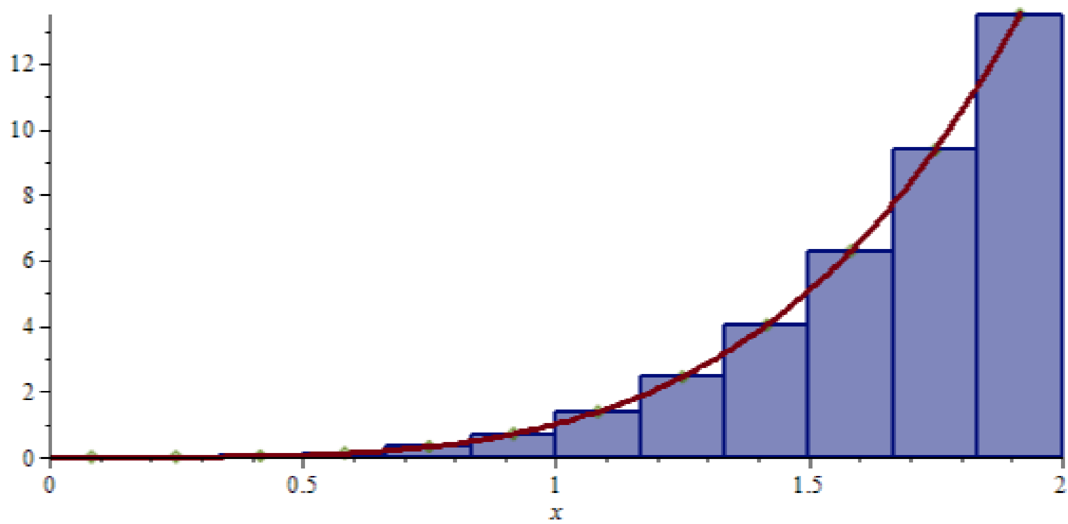
$$\int_0^2 x^4 dx \approx M_{12} = \frac{1}{6}\left[\begin{array}{l} f\left(\frac{1}{12}\right) + f\left(\frac{3}{12}\right) + f\left(\frac{5}{12}\right) + f\left(\frac{7}{12}\right) + f\left(\frac{9}{12}\right) + f\left(\frac{11}{12}\right) \\ + f\left(\frac{13}{12}\right) + f\left(\frac{15}{12}\right) + f\left(\frac{17}{12}\right) + f\left(\frac{19}{12}\right) + f\left(\frac{21}{12}\right) + f\left(\frac{23}{12}\right) \end{array}\right]$$

$$= \frac{1}{6}\left[\begin{array}{l} \left(\frac{1}{12}\right)^4 + \left(\frac{3}{12}\right)^4 + \left(\frac{5}{12}\right)^4 + \left(\frac{7}{12}\right)^4 + \left(\frac{9}{12}\right)^4 + \left(\frac{11}{12}\right)^4 \\ + \left(\frac{13}{12}\right)^4 + \left(\frac{15}{12}\right)^4 + \left(\frac{17}{12}\right)^4 + \left(\frac{19}{12}\right)^4 + \left(\frac{21}{12}\right)^4 + \left(\frac{23}{12}\right)^4 \end{array}\right]$$

$$= \frac{197915}{31104} \approx 6.363008$$

Hence $M_{12} \approx \boxed{6.363008}$.

The following diagram shows the area represented by M_{12}.



A midpoint Riemann sum approximation of $\int_0^2 f(x)\, dx$, where $f(x) = x^4$ and the partition is uniform. The approximate value of the integral is 6.363007974. Number of subintervals used: 12.

Evaluate S_6 for the integral $\int_0^2 x^4\,dx.$

Write the formula to find S_6.

$$\int_a^b f(x)\,dx \approx S_6 = \frac{\Delta x}{3}\begin{bmatrix} f(x_0)+4f(x_1)+2f(x_2)+4f(x_3) \\ +2f(x_4)+4f(x_5)+f(x_6) \end{bmatrix}$$

Where

$$\begin{aligned} f(x) &= x^4 \\ a &= 0 \\ b &= 2 \\ \Delta x &= \frac{b-a}{n} \\ &= \frac{2-0}{6} \\ &= \frac{1}{3} \end{aligned}$$

Use the equation $x_i = a + i\Delta x$ to find $x_i, i = 0,1,2,3..6$

$$\begin{aligned} x_0 &= 0+(0)\frac{1}{3} & x_3 &= 0+(3)\frac{1}{3} \\ &= 0 & &= 1 \\ x_1 &= 0+(1)\frac{1}{3} & x_4 &= 0+(4)\frac{1}{3} & x_6 &= 0+(6)\frac{1}{3} \\ &= \frac{1}{3} & &= \frac{4}{3} & &= 2 \\ x_2 &= 0+(2)\frac{1}{3} & x_5 &= 0+(5)\frac{1}{3} \\ &= \frac{2}{3} & &= \frac{5}{3} \end{aligned}$$

Put $f(x)=x^4, a=0, b=2, \Delta x=\frac{1}{3}, x_0=0, x_1=\frac{1}{3}, x_2=\frac{2}{3}, x_3=1, x_4=\frac{4}{3}, x_5=\frac{5}{3}$ and $x_6=2$ in the formula.

$$\begin{aligned} \int_0^2 x^4 dx \approx S_6 &= \frac{\frac{1}{3}}{3}\begin{bmatrix} f(0)+4f\left(\frac{1}{3}\right)+2f\left(\frac{2}{3}\right)+4f(1) \\ +2f\left(\frac{4}{3}\right)+4f\left(\frac{5}{3}\right)+f(2) \end{bmatrix} \\ &= \frac{1}{9}\begin{bmatrix} (0)^4+4\left(\frac{1}{3}\right)^4+2\left(\frac{2}{3}\right)^4+4(1)^4 \\ +2\left(\frac{4}{3}\right)^4+4\left(\frac{5}{3}\right)^4+(2)^4 \end{bmatrix} \\ &= \frac{1556}{243} \approx 6.403292 \end{aligned}$$

Hence $S_6 \approx \boxed{6.403292}$.

Put $f(x)=x^4, a=0, b=2, \Delta x=\frac{1}{3}, x_0=0, x_1=\frac{1}{3}, x_2=\frac{2}{3}, x_3=1, x_4=\frac{4}{3}, x_5=\frac{5}{3}$ and $x_6=2$ in the formula.

$$\int_0^2 x^4 dx \approx S_6 = \frac{\frac{1}{3}}{3}\begin{bmatrix} f(0)+4f\left(\frac{1}{3}\right)+2f\left(\frac{2}{3}\right)+4f(1) \\ +2f\left(\frac{4}{3}\right)+4f\left(\frac{5}{3}\right)+f(2) \end{bmatrix}$$
$$=\frac{1}{9}\begin{bmatrix} (0)^4+4\left(\frac{1}{3}\right)^4+2\left(\frac{2}{3}\right)^4+4(1)^4 \\ +2\left(\frac{4}{3}\right)^4+4\left(\frac{5}{3}\right)^4+(2)^4 \end{bmatrix}$$
$$=\frac{1556}{243}\approx 6.403292$$

Hence $S_6 \approx \boxed{6.403292}$.

n	E_T	E_M	E_S
6	$6.695473-6.4 = \boxed{0.295473}$	$6.252572-6.4 = \boxed{-0.147428}$	$6.403292-6.4 = \boxed{0.003292}$
12	$6.474023-6.4 = \boxed{0.074023}$	$6.363008-6.4 = \boxed{-0.036992}$	$6.400206-6.4 = \boxed{0.000206}$

From this table it is observed that error is almost zero for the approximation S_{12} it means two things can be said, **error becomes zero as *n* increases** and the best approximation can be obtained by using the **Simpsons Rule.**

Evaluate S_{12} for the integral $\int_0^2 x^4\, dx.$

Write the formula to find S_{12}.

$$\int_a^b f(x)dx \approx S_{12} = \frac{\Delta x}{3}\begin{bmatrix} f(x_0)+4f(x_1)+2f(x_2)+4f(x_3) \\ +2f(x_4)+4f(x_5)+2f(x_6)+4f(x_7) \\ +2f(x_8)+4f(x_9)+2f(x_{10})+4f(x_{11}) \\ +f(x_{12}) \end{bmatrix}$$

Where

$$f(x)=x^4$$
$$a=0$$
$$b=2$$
$$\Delta x=\frac{b-a}{n}$$
$$=\frac{2-0}{12}$$
$$=\frac{1}{6}$$

Use the equation $x_i=a+i\Delta x$ to find $x_i, i=0,1,2,3..12$

Put $f(x)=x^4, a=0, b=2, \Delta x=\frac{1}{6}, x_0=0, x_1=\frac{1}{6}, x_2=\frac{2}{6}, x_3=\frac{3}{6}, x_4=\frac{4}{6}, x_5=\frac{5}{6}, x_6=1$

$x_7=\frac{7}{6}, x_8=\frac{8}{6}, x_9=\frac{9}{6}, x_{10}=\frac{10}{6}, x_{11}=\frac{11}{6}$ and $x_{12}=12$ in the formula.

$$\int_0^2 x^4 dx \approx T_{12} = \frac{\frac{1}{6}}{3}\begin{bmatrix} f(0)+4f\left(\frac{1}{6}\right)+2f\left(\frac{2}{6}\right)+4f\left(\frac{3}{6}\right) \\ +2f\left(\frac{4}{6}\right)+4f\left(\frac{5}{6}\right)+2f(1)+4f\left(\frac{7}{6}\right) \\ +2f\left(\frac{8}{6}\right)+4f\left(\frac{9}{6}\right)+2f\left(\frac{10}{6}\right)+4f\left(\frac{11}{6}\right) \\ +f(2) \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} (0)^4+4\left(\frac{1}{6}\right)^4+2\left(\frac{2}{6}\right)^4+4\left(\frac{3}{6}\right)^4 \\ +2\left(\frac{4}{6}\right)^4+4\left(\frac{5}{6}\right)^4+2(1)^4+4\left(\frac{7}{6}\right)^4 \\ +2\left(\frac{8}{6}\right)^4+4\left(\frac{9}{6}\right)^4+2\left(\frac{10}{6}\right)^4+4\left(\frac{11}{6}\right)^4 \\ +(2)^4 \end{bmatrix}$$

$$= \frac{6221}{972} \approx 6.400206$$

Hence $S_{12} \approx \boxed{6.400206}$.

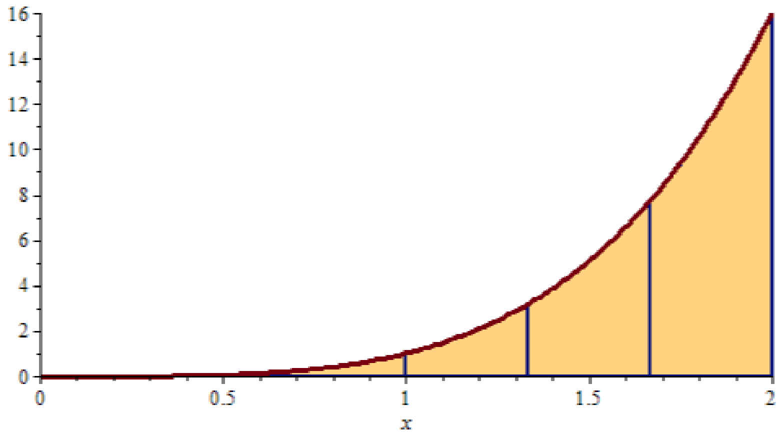
$$\int_0^2 x^4 dx \approx T_{12} = \frac{\frac{1}{6}}{3}\begin{bmatrix} f(0)+4f\left(\frac{1}{6}\right)+2f\left(\frac{2}{6}\right)+4f\left(\frac{3}{6}\right) \\ +2f\left(\frac{4}{6}\right)+4f\left(\frac{5}{6}\right)+2f(1)+4f\left(\frac{7}{6}\right) \\ +2f\left(\frac{8}{6}\right)+4f\left(\frac{9}{6}\right)+2f\left(\frac{10}{6}\right)+4f\left(\frac{11}{6}\right) \\ +f(2) \end{bmatrix}$$

$$= \frac{1}{18}\begin{bmatrix} (0)^4+4\left(\frac{1}{6}\right)^4+2\left(\frac{2}{6}\right)^4+4\left(\frac{3}{6}\right)^4 \\ +2\left(\frac{4}{6}\right)^4+4\left(\frac{5}{6}\right)^4+2(1)^4+4\left(\frac{7}{6}\right)^4 \\ +2\left(\frac{8}{6}\right)^4+4\left(\frac{9}{6}\right)^4+2\left(\frac{10}{6}\right)^4+4\left(\frac{11}{6}\right)^4 \\ +(2)^4 \end{bmatrix}$$

$$= \frac{6221}{972} \approx 6.400206$$

Hence $S_{12} \approx \boxed{6.400206}$.

The following diagram shows the area represented by S_{12}.



An approximation of $\int_0^2 f(x)\,dx$ using Simpson's rule, where $f(x) = x^4$ and the partition is uniform. The approximate value of the integral is 6.400205761. Number of subintervals used: 6.

Answer 28E.

Consider the integral

$$\int_1^4 \frac{1}{\sqrt{x}}dx$$

The objective is to find the approximations T_n, M_n and S_n for $n = 5,12$

With $n = 6, a = 1$ and $b = 4$, we have

$$\Delta x = (4-1)/6 = \frac{3}{6} = 0.5$$

and so the trapezoidal rule gives

$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{x}}dx &= T_6 \\ &= \frac{0.5}{2}\begin{bmatrix} f(1)+2f(1.5)+2f(2)+2f(2.5)+ \\ 2f(3)+2f(3.5)+f(4) \end{bmatrix} \\ &= \boxed{2.008965}\end{aligned}$$

The midpoints of the eight subintervals are

$$1.25, 1.75, 2.25, 2.75, 3.25, 3.75$$

So the midpoints rule gives

$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{x}}dx &= M_6 \\ &= 0.5\left[f(1.25)+f(1.75)+f(2.25)+f(2.75)+f(3.25)+f(3.75)\right] \\ &= \boxed{1.995572}\end{aligned}$$

In Simpson's rule, we obtain

$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{x}}dx &= S_6 \\ &= \frac{0.5}{3}\begin{bmatrix} f(1)+4f(1.5)+2f(2)+4f(2.5)+ \\ 2f(3)+4f(3.5)+f(4) \end{bmatrix} \\ &= \boxed{2.000468}\end{aligned}$$

With $n=12, a=1$ and $b=4$, we have

$$\Delta x=(4-1)/12=\frac{3}{12}=0.25$$

and so the trapezoidal rule gives

$$\int_1^4 \frac{1}{\sqrt{x}}dx=T_{12}$$

$$=\frac{0.25}{2}\begin{bmatrix} f(1)+2f(1.25)+2f(1.5)+2f(1.75)+2f(2)+ \\ 2f(2.25)+2f(2.5)+2f(2.75)+2f(3)+ \\ 2f(3.25)+2f(3.5)+2f(3.75)+f(4) \end{bmatrix}$$

$$=\boxed{2.002268}$$

The midpoints of the eight subintervals are

$$1.125, 1.375, 1.625, 1.875, 2.215, 2.375, 2.625, 2.875,$$
$$3.125, 3.375, 3.625, 3.875.$$

So the midpoints rule give

$$\int_1^4 \frac{1}{\sqrt{x}}dx=M_{12}$$

$$=0.25\begin{bmatrix} f(1.125)+f(1.375)+f(1.625)+f(1.875)+ \\ f(2.125)+f(2.375)+f(2.625)+f(2.875)+ \\ f(3.125)+f(3.375)+f(3.625)+f(3.875) \end{bmatrix}$$

$$=\boxed{1.998869}$$

In Simpson's rule, we have

$$\int_1^4 \frac{1}{\sqrt{x}}dx=S_{12}$$

$$=\frac{0.25}{3}\begin{bmatrix} f(1)+4f(1.25)+2f(1.5)+4f(1.75)+2f(2) \\ +4f(2.25)+2f(2.5)+4f(2.75)+2f(3)+ \\ 4f(3.25)+2f(3.5)+4f(3.75)+f(4) \end{bmatrix}$$

$$=\boxed{2.000036}$$

By using the fundamental theorem of calculus

$$\int_1^4 \frac{1}{\sqrt{x}}dx = \left[2\sqrt{x}\right]_1^4$$
$$= [4-2]$$
$$= 2$$

n	T_n	M_n	S_n
6	2.008965	1.995572	2.000468
12	2.002268	1.998869	2.000036

Corresponding errors are

n	E_T	E_M	E_S
6	−0.008965	0.004428	−0.000468
12	−0.002268	0.001131	−0.000036

Observe that we get more accurate approximation when we increase the value of n

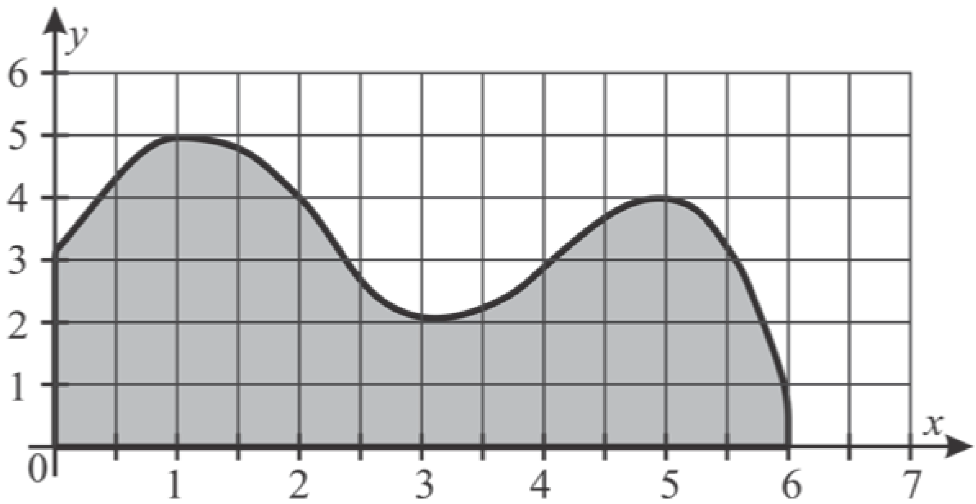
Answer 29E.

The area under the curve is determined by evaluating the definite integral over the limits of integration provided.

The definite integral was evaluated to by Reimann Sums, by making use of the left-hand or right-end approximation.

Further there are different rules developed such as Simpsons rule, mid-point rule and trapezoidal rule and eventually the rule developed was the Fundamental theorem of calculus.

Consider the graph of a function as shown below:



Represent the graph by the equation $y = f(x)$.

Observe the above diagram to determine that the area is in the interval $[0,6]$.

It follows that the area under the curve is the integral:

$$\int_0^6 f(x)dx$$

Use the trapezoidal rule to estimate the value of the integral:

$$\int_0^6 f(x)\,dx$$

Consider the trapezoidal rule for $n = 6.$

$$\int_a^b f(x)\,dx = \frac{\Delta x}{2}\begin{bmatrix} f(x_0)+2f(x_1)+2f(x_2)+2f(x_3) \\ +2f(x_4)+2f(x_5)+f(x_6) \end{bmatrix}$$

Substitute the value as shown below:

$$a = 0 \text{ and } b = 6$$

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{6-0}{6} \\ &= 1\end{aligned}$$

Use the equation $x_i = a + i\Delta x$ to find $x_i,\ \ i = 0,1,2,3..6$:

$$\begin{aligned}x_0 &= 0 \\ x_1 &= 1 \\ x_2 &= 2\end{aligned}$$

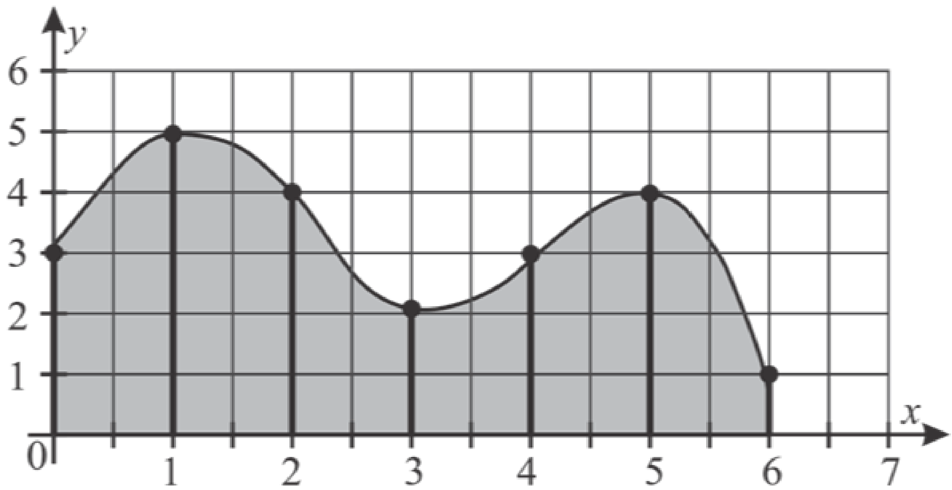
Also, consider the values shown below:

$$\begin{aligned}x_3 &= 3 \\ x_4 &= 4 \\ x_5 &= 5\end{aligned}$$

Put the above values in the formula shown below:

$$\int_0^6 f(x)\,dx = \frac{1}{2}\begin{bmatrix} f(1)+2f(2)+2f(3)+2f(4) \\ +2f(5)+2f(6)+f(7) \end{bmatrix}$$

Observe the diagram to approximate the values $f(0), f(1)...f(6)$:



So, the values of the function at the above points are:

$$\begin{aligned}f(0) &= 3 \\ f(1) &= 4.9 \\ f(2) &= 4\end{aligned}$$

Also, consider the values shown below:

$$\begin{aligned}f(3) &= 2.9 \\ f(4) &= 3 \\ f(5) &= 4 \\ f(6) &= 1\end{aligned}$$

So, the area under the curve is:

$$\int_0^6 f(x)\,dx = \frac{1}{2}\begin{bmatrix} 3+2\cdot 4.9+2\cdot 4+2\cdot 2.9 \\ +2\cdot 3+2\cdot 4+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 3+9.8+8+3.8 \\ +6+8+1 \end{bmatrix}$$

$$= 19.8$$

Hence, the area under the curve by the use of trapezoidal rule is approximately $\boxed{19.8 \text{ unit}^2}$.

Use the midpoint rule to estimate the value of the integral:

$$\int_0^6 f(x)\,dx$$

Consider the midpoint rule for $n = 6.$

$$\int_a^b f(x)\,dx \approx M_6 = \Delta x\begin{bmatrix} f(\overline{x_1})+f(\overline{x_2})+f(\overline{x_3}) \\ +f(\overline{x_4})+f(\overline{x_5})+f(\overline{x_6}) \end{bmatrix}$$

Substitute the value as shown below:

$a = 0$ and $b = 6$

$$\Delta x = \frac{b-a}{n}$$

$$= \frac{6-0}{6}$$

$$= 1$$

Use the equation $x_i = a + i\Delta x$ to find $x_i,\ i = 0,1,2,3..6$:

$x_0 = 0$

$x_1 = 1$

$x_2 = 2$

Also, consider the values shown below:

$x_3 = 3$

$x_4 = 4$

$x_5 = 5$

Put the above values in the formula shown below:

$$\int_0^6 f(x)\,dx = \frac{1}{2}\begin{bmatrix} f(1)+2f(2)+2f(3)+2f(4) \\ +2f(5)+2f(6)+f(7) \end{bmatrix}$$

Use the equations $\overline{x_i} = \frac{x_i + x_{i-1}}{2}$ to find $\overline{x_i}$, for $i = 0,1,2,3..6$.

Consider the value of $\overline{x_1}$:

$$\begin{aligned}\overline{x_1} &= \frac{x_1 + x_0}{2} \\ &= \frac{1+0}{2} \\ &= \frac{1}{2}\end{aligned}$$

Consider the value of $\overline{x_2}$:

$$\begin{aligned}\overline{x_2} &= \frac{x_2 + x_1}{2} \\ &= \frac{2+1}{2} \\ &= \frac{3}{2}\end{aligned}$$

Consider the value of $\overline{x_3}$:

$$\begin{aligned}\overline{x_3} &= \frac{x_3 + x_2}{2} \\ &= \frac{3+2}{2} \\ &= \frac{5}{2}\end{aligned}$$

Consider the value of $\overline{x_4}$:

$$\begin{aligned}\overline{x_4} &= \frac{x_4 + x_3}{2} \\ &= \frac{4+3}{2} \\ &= \frac{7}{2}\end{aligned}$$

Consider the value of $\overline{x_5}$:

$$\begin{aligned}\overline{x_5} &= \frac{x_5 + x_4}{2} \\ &= \frac{5+4}{2} \\ &= \frac{9}{2}\end{aligned}$$

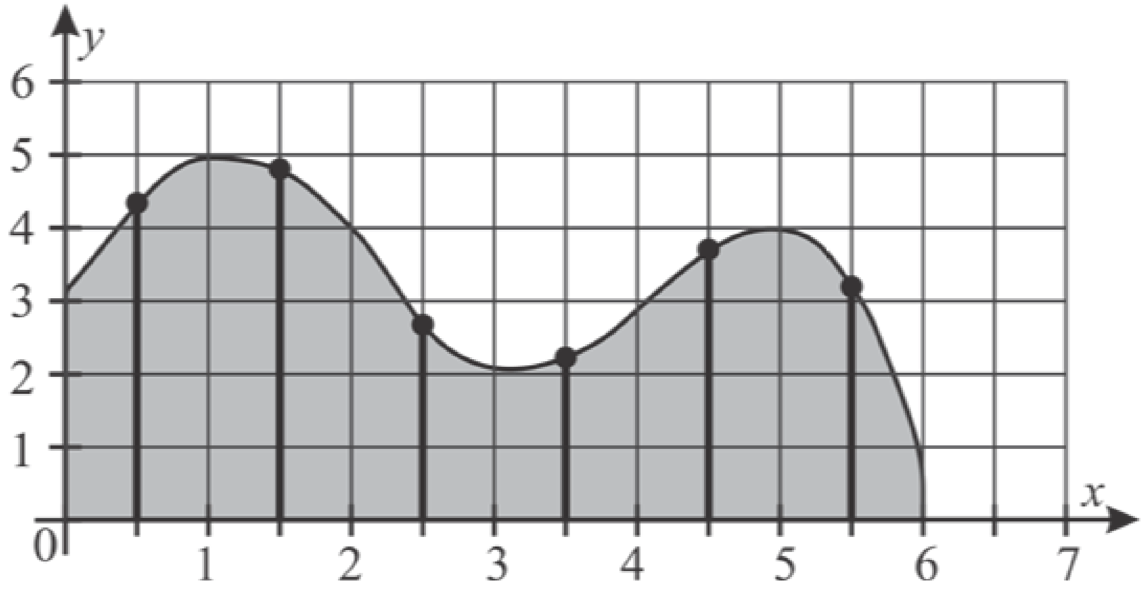
Consider the value of $\overline{x_6}$:

$$\begin{aligned}\overline{x_6} &= \frac{x_6 + x_5}{2} \\ &= \frac{11}{2}\end{aligned}$$

Put the above values in the midpoint rule formula as shown below:

$$\int_0^6 f(x)\,dx = 1\begin{bmatrix} f\left(\frac{1}{2}\right)+f\left(\frac{3}{2}\right)+f\left(\frac{5}{2}\right) \\ +f\left(\frac{7}{2}\right)+f\left(\frac{9}{2}\right)+f\left(\frac{11}{2}\right) \end{bmatrix}$$

Observe the diagram to approximate the values $f\left(\frac{1}{2}\right), f\left(\frac{3}{2}\right) \ldots f\left(\frac{11}{2}\right)$:



The approximate values are as shown below:

$f\left(\frac{1}{2}\right) = 4.5$

$f\left(\frac{3}{2}\right) = 4.8$ Also, consider the values shown below:

$f\left(\frac{5}{2}\right) = 2.8$

$f\left(\frac{7}{2}\right) = 2.2$

$f\left(\frac{9}{2}\right) = 3.6$

$f\left(\frac{11}{2}\right) = 3.3$

So, the area under the curve is:

$$\begin{aligned}\int_0^6 f(x)\,dx &= 1\begin{bmatrix} f\left(\frac{1}{2}\right)+f\left(\frac{3}{2}\right)+f\left(\frac{5}{2}\right) \\ +f\left(\frac{7}{2}\right)+f\left(\frac{9}{2}\right)+f\left(\frac{11}{2}\right) \end{bmatrix} \\ &= 1\left[4.4+4.7+2.6+2.2+3.4+3.3\right] \\ &= 20.6\end{aligned}$$

Hence, the area under the curve by the use of midpoint rule is approximately $\boxed{20.6 \text{ unit}^2}$.

Use the Simpsons rule to estimate the value of the integral:

$$\int_0^6 f(x)\,dx$$

Consider the Simpsons rule for $n = 6.$

$$\int_a^b f(x)\,dx \approx S_6 = \frac{\Delta x}{3}\left[\begin{array}{l} f(x_0)+4f(x_1)+2f(x_2)+4f(x_3) \\ +2f(x_4)+4f(x_5)+f(x_6) \end{array}\right]$$

Substitute the value as shown below:

$a = 0$ and $b = 6$

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{6-0}{6} \\ &= 1\end{aligned}$$

Use the equation $x_i = a + i\Delta x$ to find x_i, for $i = 0,1,2,\ldots,6$.

Consider the values shown below:

$$x_0 = 0$$
$$x_1 = 1$$
$$x_2 = 2$$

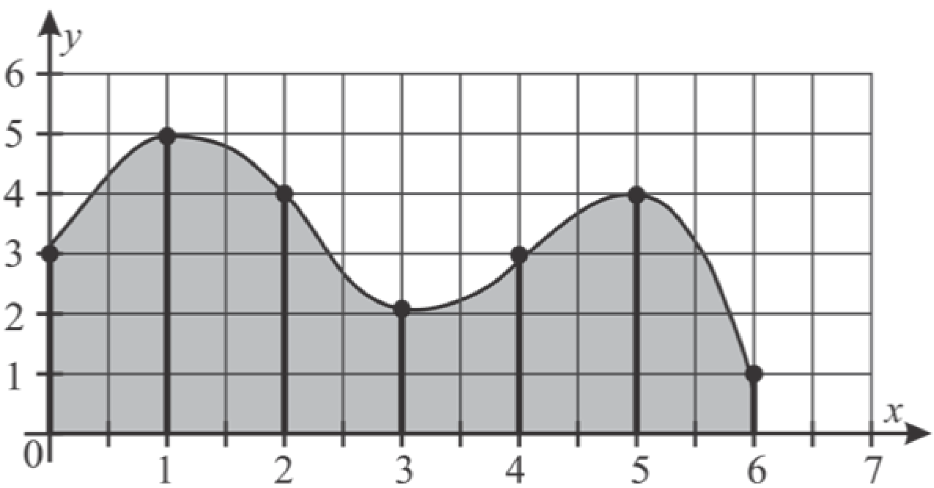
Also, consider the values shown below:

$$x_3 = 3$$
$$x_4 = 4$$
$$x_5 = 5$$
$$x_6 = 6$$

Put the above values in the formula as shown below:

$$\int_0^6 f(x)\,dx = \frac{1}{3}\left[\begin{array}{l} f(1)+4f(2)+2f(3)+4f(4) \\ +2f(5)+4f(6)+f(7) \end{array}\right]$$

Observe the diagram to approximate the values $f(0), f(1)\ldots f(6)$.



So, the values of the function at the above points are:

$$f(0) = 3$$
$$f(1) = 4.9$$
$$f(2) = 4$$

Also, consider the values shown below:

$$f(3) = 2.9$$
$$f(4) = 3$$
$$f(5) = 4$$
$$f(6) = 1$$

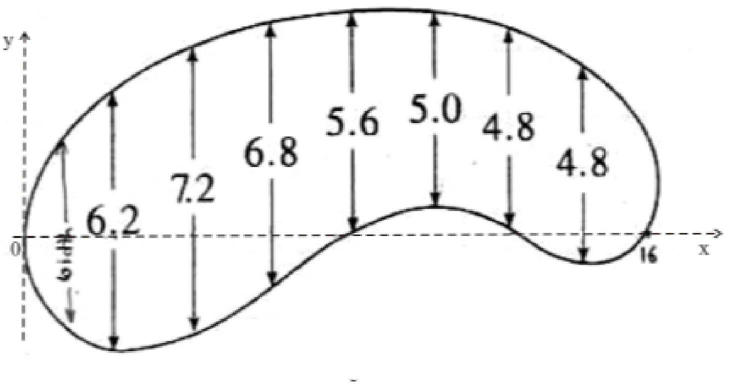
Use the above obtained values to calculate the area as shown below:

$$\int_0^6 f(x)\,dx = \frac{1}{3}\begin{bmatrix} 3+4\times4.9+2\times4+4\times2.9 \\ +2\times3+4\times4+1 \end{bmatrix}$$
$$= \frac{1}{3}\begin{bmatrix} 3+19.6+8+11.6 \\ +6+16+1 \end{bmatrix}$$
$$= 21.73$$

Hence, the area under the curve by the use of Simpsons rule is approximately $\boxed{21.73 \text{ unit}^2}$.

Answer 30E.

202-08.07-30E



We have the interval [0, 16] if we divide this interval in to n = 8 subintervals then

$$\Delta w = \frac{16}{8} = 2$$

And subintervals are [0, 2], [2, 4], [4, 6], [6, 8], [8, 10], [10, 12], [12, 14], [14, 16]

Then area of swimming pool by Simpson's rule

$$A \approx \frac{\Delta w}{3}\left[w(0)+4w(2)+2w(4)+4w(6)+2w(8)+4w(10)+2w(12)+4w(14)+w(16)\right]$$
$$A \approx \frac{2}{3}\left[0+4(6.2)+2(7.2)+4(6.8)+2(5.6)+4(5.0)+2(4.8)+4(4.8)+0\right]$$
$$= \frac{2}{3}\left[24.8+14.4+27.2+11.2+20+9.6+19.2\right]$$
$$= 84.2\overline{6}$$
$$\boxed{A \approx 84.2\overline{6}}\ \text{m}^2$$

Answer 31E.

Apply the Midpoint rule on the given data, we have

$$\int_1^5 f(x)\,dx = 0.5\begin{bmatrix} f(1.25)+f(1.75)+f(2.25)+f(2.75)+f(3.25) \\ +f(3.75)+f(4.25)+f(4.75) \end{bmatrix}$$
$$= 0.5\left[2.65+3.1+3.45+3.7+39+4.05+4+3.7\right]$$
$$= 14.275$$

Suppose $|f''(x)| \le k$ for $a \le x \le b$. If E_M are the errors in Midpoint Rules then

$$|E_M| \le \frac{k(b-a)^3}{24n^2}$$
$$= \frac{3(4)^3}{248^2} = 0.125$$

Answer 32E.

(a)

Consider the following table which represents the values of the function *g*.

x	$g(x)$		x	$g(x)$
0.0	12.1		1.0	12.2
0.2	11.6		1.2	12.6
0.4	11.3		1.4	13.0
0.6	11.1		1.6	13.2
0.8	11.7			

The objective is to calculate $\int_0^{1.6} g(x)dx$ by using Simpson's Rule.

The Simpson's Rule is defined as:

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right)$$

Where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Given that there are nine values for the function g then the number of subintervals is $n = 8$.

The values of x are from 0 to 1.6 so $a = 0$ from $b = 1.6$

Therefore

$$\Delta x = \frac{1.6-0}{8}$$

$$= 0.2$$

Therefore

$$\int_0^{1.6} g(x)dx \approx S_8$$

$$= \frac{0.2}{3}\left(f(0)+4f(0.2)+2f(0.4)+4f(0.6)+2f(0.8)+\right.$$
$$\left.+4f(1)+2f(1.2)+4f(1.4)+f(1.6)\right)$$

$$= \frac{0.2}{3}\left(12.1+4\cdot(11.6)+2\cdot(11.3)+4\cdot(11.1)+2\cdot(11.7)+\right.$$
$$\left.+4\cdot(12.2)+2\cdot(12.6)+4\cdot(13.0)+13.2\right)$$

$$= \frac{0.2}{3}(12.1+46.4+22.6+44.4+23.4+48.8+25.2+52+13.2)$$

$$= \frac{0.2}{3}(288.1)$$

$$= 19.206$$

$$\approx \boxed{19.21}$$

(b)

Consider $-5 \le g^{(4)}(x) \le 2$ for $0 \le x \le 1.6$.

Need to estimate the error involved in the approximation for part (a).

The error involved in Simpson's Rule is:

$$|E_S| \le \frac{K(b-a)^5}{180n^4} \text{ where } |g^{(4)}(x)| \le K \text{ for } a \le x \le b$$

Since $-5 \le g^{(4)}(x) \le 2$, therefore for Simpson's Rule error approximation $|g^{(4)}(x)| \le 5$ for $0 \le x \le 1.6$

Since the Simpson's Rule error bound is

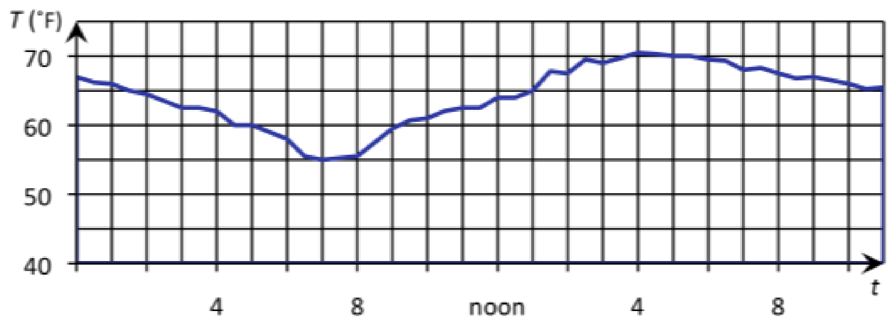
$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

Now substitute $K=5, a=0, b=1.6$ and $n=8$ in $|E_S| \le \frac{K(b-a)^5}{180n^4}$, obtain the error.

$$\begin{aligned}|E_S| &\le \frac{5(1.6-0)^5}{180 \cdot 8^4} \\ &= \frac{5(1.6)^5}{737280} \\ &= \frac{52.4288}{737280} \\ &\approx \boxed{0.0000711}\end{aligned}$$

Answer 33E.

Consider the following graph which represents the temperature in a New York City on September 19, 2009.



The objective is to find the average temperature on that day using Simpson's Rule with $n=12$.

Since the average value of a function on $[a,b]$ is defined as:

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

The Simpson's Rule is defined as:

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right)$$

Where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

So the formula for an average value using Simpson's Rule is:

$$\frac{1}{b-a}\int_a^b f(x)dx \approx \frac{1}{b-a}S_n$$

$$= \frac{1}{b-a}\cdot\frac{\Delta x}{3}\left(f(x_0)+4f(x_1)+2f(x_3)+\cdots\right.$$

$$\left.\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right)$$

Since the temperature readings are throughout the day, so $a=0$ and $b=24$. And also $n=12$.

Therefore

$$\Delta x=\frac{24-0}{12}$$
$$=2$$

We are now in a position to calculate the average temperature using Simpson's Rule:

$$f_{ave}=\frac{1}{b-a}\int_a^b f(x)dx$$
$$=\frac{1}{24-0}\int_0^{24} f(x)dx$$
$$\approx\frac{1}{24}S_{12}$$
$$=\frac{1}{24}\cdot\frac{2}{3}(f(0)+4f(2)+2f(4)+4f(6)+2f(8)+4f(10)+2f(12)+$$
$$+4f(14)+2f(16)+4f(18)+2f(20)+4f(22)+f(24))$$
$$=\frac{1}{24}\cdot\frac{2}{3}[67+4(64.5)+2(62)+4(58)+2(55.5)+4(61)+2(64)+4(67.5)$$
$$+2(70.5)+4(69.5)+2(67.5)+4(66)+64]$$
$$=\frac{1}{24}\cdot\frac{2}{3}[67+258+124+232+111+244+128+270+141+278+135+264+64]$$
$$=\frac{1}{24}\cdot\frac{2}{3}[2316]$$
$$=\frac{1}{24}\cdot 1544$$
$$=\boxed{64.3}$$

Answer 34E.

202-08.07-32E

The given data is

Time t (s)	0	0.5	1.0	1.5	2	2.5	3	3.5	4	4.5	5
V(m/s)	0	4.67	7.34	8.86	9.73	10.22	10.51	10.67	10.76	10.81	10.81

Here width of the interval is $\Delta t=0.5$

And

$t_0=0,\ t_1=0.5,\ t_2=1,\ t_3=1.5,\ t_4=2,\ t_5=2.5,\ t_6=3,\ t_7=3.5,\ t_8=4,\ t_9=4.5,\ t_{10}=5$

Then we can estimate the distance traveled during 5 seconds as follows

$$\text{Distance}\approx S_{10}=\frac{\Delta t}{3}[v(t_0)+4v(t_1)+2v(t_2)+4v(t_3)+......+4v(t_9)+v(t_{10})]$$
$$=\frac{0.5}{3}[0+(4\times4.67)+(2\times7.34)+(4\times8.86)+(2\times9.73)+(4\times10.22)+$$
$$(2\times10.51)+(4\times10.67)+(2\times10.76)+(4\times10.81)+10.81]$$
$$=\frac{0.5}{3}[18.68+14.68+35.44+19.46+40.88+21.02+42.68+21.52+43.24+10.81]$$
$$=\frac{0.5\times268.41}{3}\approx44.735\text{ m}$$

So the distance covered in 5 seconds $\approx\boxed{44.735\text{ m}}$

Answer 35E.

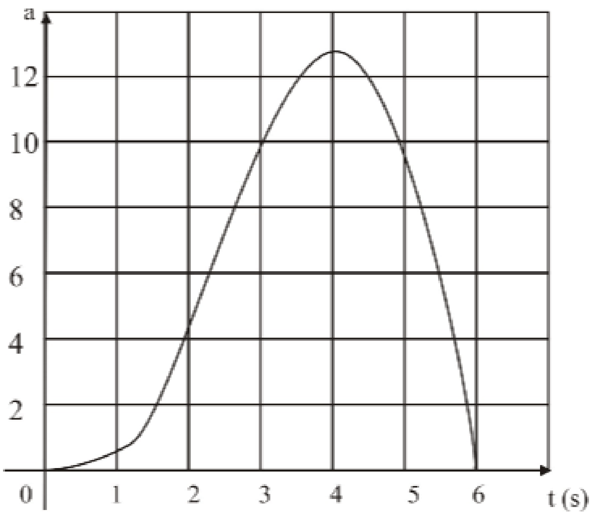


Fig. 1

This is given graph of acceleration a (t) of a car measured in ft / s^2
Now we can estimate the increase in velocity during 6 seconds by Simpson's rule as follows
Here we divide the time interval [0, 6] in to 6 subintervals, then $\Delta t = 1$ Seconds

$$V = \int_0^6 a(t)dt \approx S_6 = \frac{\Delta t}{3}\left[a(0)+4a(1)+2a(2)+4a(3)+2a(4)+4a(5)+a(6)\right]$$

$$= \frac{113.2}{3} \approx 37.73333......$$

So $\boxed{V \approx 37.7\overline{3} \text{ ft/s}}$

Answer 36E.

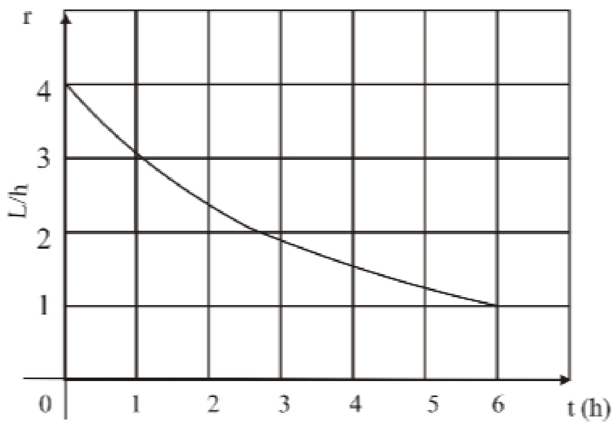


Fig. 1

This is the given graph of rate of leaking water *r (t)* in liters per hour. Time interval is [0, 6]
If we divide the interval in to 6 subintervals then $\Delta t = 1$ and subintervals are [0, 1], [1, 2], [2, 3], [3, 4], [4, 5], [5, 6]

Then we can estimate the total amount of water by Simpson's rule as follows

$$W = \int_0^6 r(t)dt \approx S_6 = \frac{\Delta t}{3}\left[r(0)+4r(1)+2r(2)+4r(3)+2r(4)+4r(5)+r(6)\right]$$

$$= \frac{1}{3}[4+4\times3+2\times2.3+4\times1.8+2\times1.5+4\times1.2+1]$$

$$= \frac{1}{3}[4+12+4.6+7.2+3+4.8+1]$$

$\boxed{W \approx 12.2 \text{ Liters}}$

Answer 37E.

The following table shows the power consumption *P* in megawatts in County S from midnight to 6:00 AM on a day in December.

t	*P*	*t*	*P*
0:00	1814	3:30	1611
0:30	1735	4:00	1621
1:00	1686	4:30	1666
1:30	1646	5:00	1745
2:00	1637	5:30	1886
2:30	1609	6:00	2052
3:00	1604		

Since power is the derivative of energy, the energy used from midnight to 6:00 AM is given by the integral, $\int_{0:00}^{6:00} P(t)dt$. To calculate this integral, use Simpson's Rule.

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots\right.$$
$$\left.+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$$

Here, *n* is even and $\Delta x = \frac{b-a}{n}$.

Since the power is given in a 30-second interval for six hours, there are 12 sub-intervals. Use Simpson's Rule to calculate *S*12.

$$S_{12} = \frac{6-0}{3(12)}\left[P(0:00)+4f(0:30)+2(1:00)\ldots+4f(5:30)+f(6:00)\right]$$
$$= \frac{6}{36}\left[1814+4(1735)+2(1686)+4(1646)+\ldots+4(1886)+2052\right]$$
$$\approx 10{,}177$$

Therefore, $\boxed{10{,}177 \text{ megawatt-hours}}$ were used from midnight to 6:00 AM.

Answer 38E.

202-08.07-36E

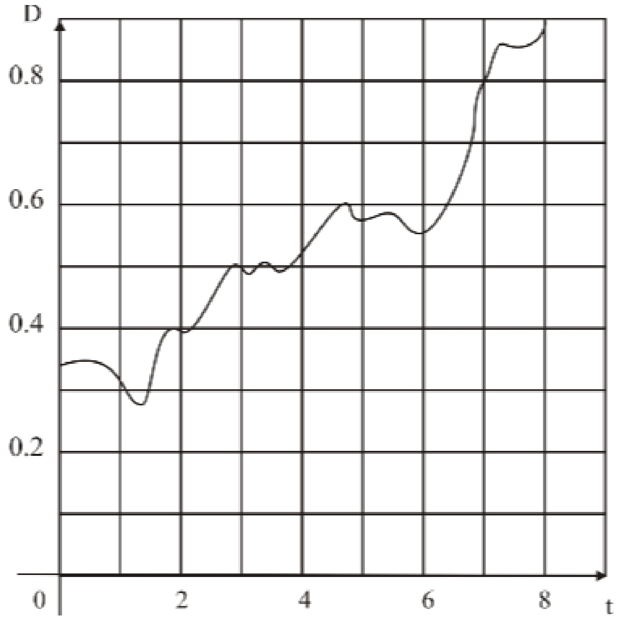


Fig. 1

This is the given graph of D measured in megabits per second and time is in seconds. Time interval is [0, 8]. If we divide this interval in to 8 subintervals then $\Delta t = 1$, then we can estimate the total amount of data transmitted during the time interval [0, 8], by Simpson's rule as follows

$$\text{Amount} = \int_0^8 D(t)\,dt \times 3600 \quad [\text{Since time is in hours and 1 hour} = 3600 \text{ seconds}]$$

$$\int_0^8 D(t)dt \approx S_8 = \frac{\Delta t}{3}\left[D(0)+4D(1)+2D(2)+4D(3)+2D(4)+4D(5)+2D(6)+4D(7)+D(8)\right]$$

$$= \frac{1}{3}\left[(0.35)+4(0.32)+2(0.41)+4(0.5)+2(0.51)+4(0.56)+2(0.56)+4(0.83)+0.88\right]$$

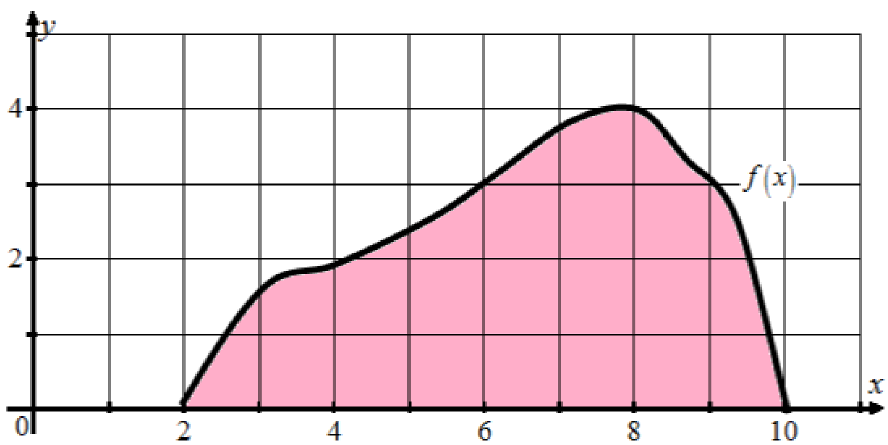
$$= \frac{13.03}{3} = \boxed{4.34\overline{3}}$$

So total amount of data transmitted $= 4.34\overline{3} \times 3600$ megabits

$\approx \boxed{15636}$ megabits

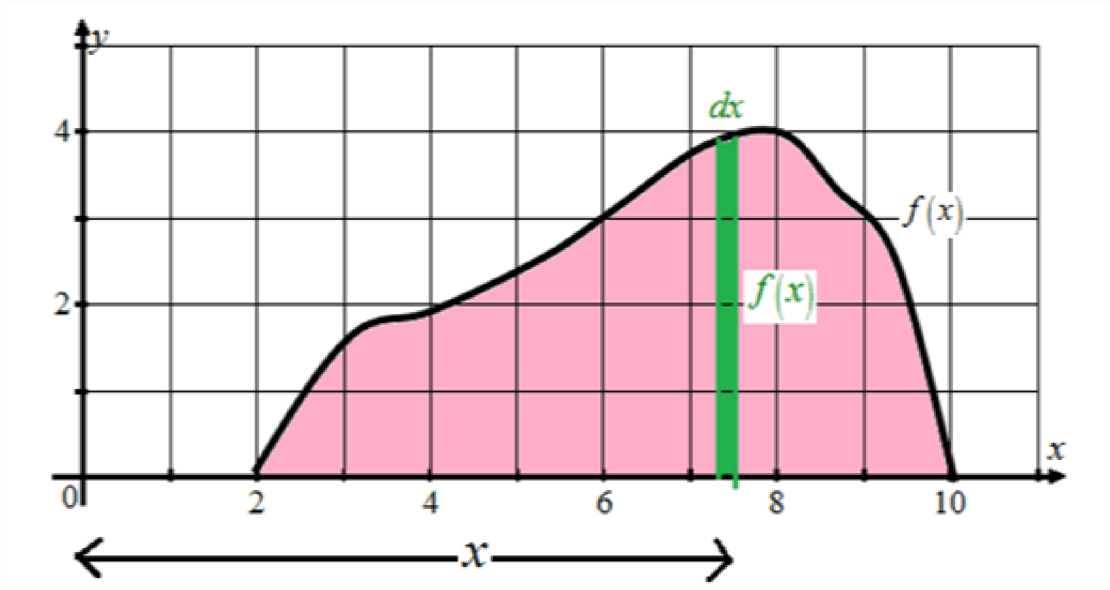
Answer 39E.

Consider the region

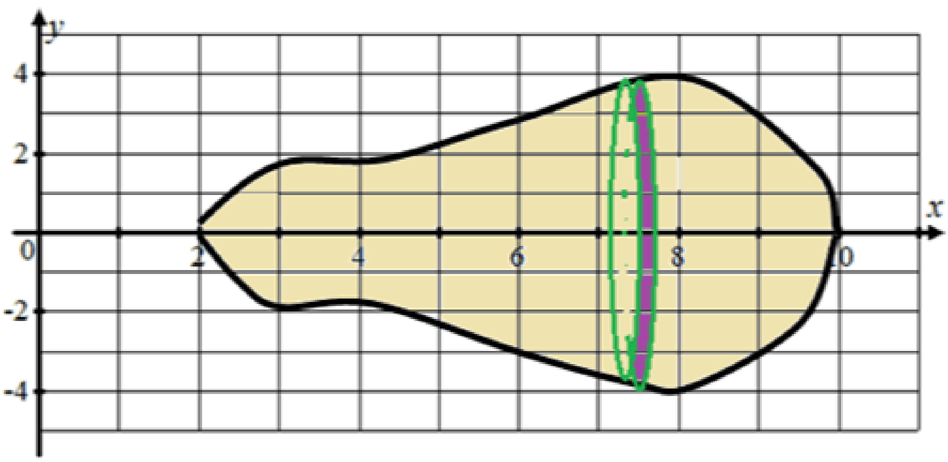


a)

Rotating the region about the *x*-axis, the region divide into vertical stripes is shown below.



The obtained solid region is shown below.



From the disk method the volume of the solid is

$$V = \pi \int_a^b f^2(x)\,dx$$

In this case $a = 2, b = 10$

$$V = \pi \int_2^{10} f^2(x)\,dx$$

Using Simpson's rule finds the above integral with $n = 8$

$$h = \frac{b-a}{n} = \frac{10-2}{8} = 1$$

x	2	3	4	5	6	7	8	9	10
$f(x)$	0	1.5	1.9	2.1	3	3.9	4	3	0
$f^2(x)$	0	2.25	3.61	4.41	9	15.21	16	9	0

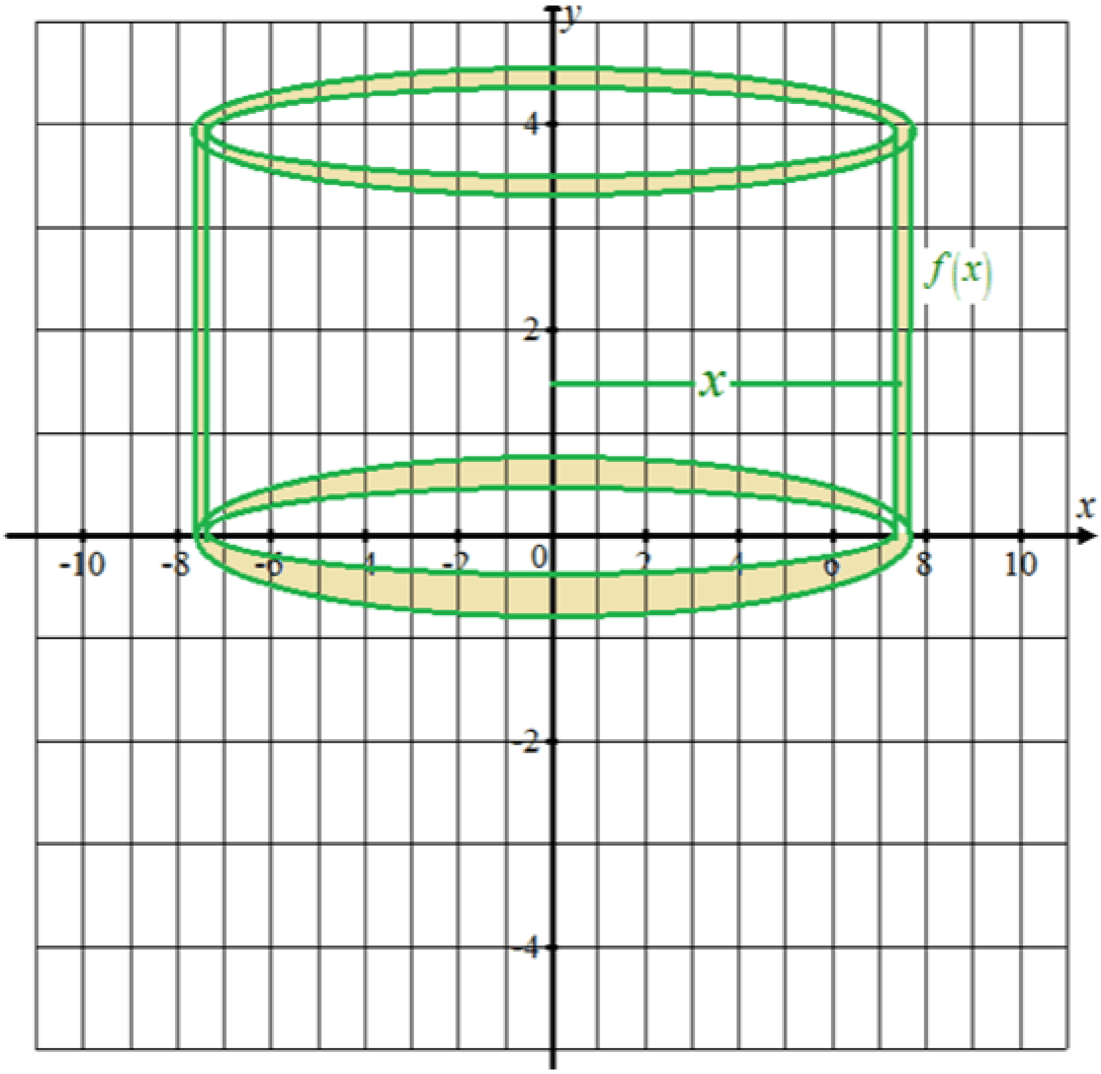
$$\int_2^{10} f^2(x)\,dx \approx \frac{1}{3}\left(0 + 4(2.25) + 2(3.61) + 4(4.41) + 2(9) + 4(15.21) + 2(16) + 4(9) + 0\right)$$

$$\approx \frac{1}{3}(150.84)$$

$$V \approx \pi(60.23)$$

$$\approx \boxed{189.28}$$

The vertical slice forms a shell shown below.



We consider a vertical strip in the shaded region at the distance x from the origin. If we rotate this reign about y – axis, then we get a cylindrical shell with radius x and height $f(x)$ [Let the given graph be the graph of the function $f(x)$]

Then volume of the resulting solid obtained after rotation is

$$V = \int_2^{10} (\text{Circumference of shell}) \times (\text{height})\, dx$$

$$V = \int_2^{10} (2\pi x) \times f(x)\, dx$$

$$V = 2\pi \int_2^{10} x\, f(x)\, dx$$

Let $F(x) = x\,f(x)$, interval is [2, 10]

If we divide this interval in to n = 8 subintervals then $\Delta x = \frac{10-2}{8} = 1$

And subintervals are [2, 3], [3, 4], [4, 5], [5, 6], [6, 7], [7, 8], [8, 9], [9, 10] Then we can estimate the volume of the solid by Simpson's rule

$$V \approx 2\pi.\frac{\Delta x}{3}\left[F(2)+4F(3)+2F(4)+4F(5)+2F(6)\right.$$
$$\left.+4F(7)+2F(8)+4F(9)+F(10)\right]$$

$$V \approx 2\pi.\frac{\Delta x}{3}\left[2.(0)+4.3.(1.5)+2.4(1.9)+4.5.(2.2)+2.6.(3)\right.$$
$$\left.+4.7(3.8)+2.8.(4)+4.9.(3.1)+10.(0)\right]$$

$$V \approx 2\pi.\frac{1}{3}[0+18+15.2+44+36+106.4+64+111.6+0]$$

$$V \approx \frac{2\pi}{3}[395.2] \approx 828$$

$\boxed{V \approx 828}$ Cubic units

Answer 40E.

This table shows the values of a force function $f(x)$ where x is measured in meters and $f(x)$ in newtons

x	0	3	6	9	12	15	18
f (x)	9.8	9.1	8.5	8.0	7.7	7.5	7.4

Here $\Delta x = 3$
Then we can estimate work done by the force in moving an object a distance of 18 m by Simpson's rule as follows

$$W \approx \frac{\Delta x}{3}\left[f(0)+4f(3)+2f(6)+4f(9)+2f(12)+4f(15)+f(18)\right]$$
$$= 1.[9.8+4\times 9.1+2\times 8.5+4\times 8+2\times 7.7+4\times 7.5+7.4]$$
$$= 9.8+36.4+17+32+15.4+30+7.4$$
$$= 148 \text{ J}$$

So work done $\boxed{W = 148\,J}$

Answer 41E.

To show that $\frac{1}{2}(T_n + M_n) = T_{2n}$ first use the definitions of T_n and M_n to rewrite the left side of the equation

$$\frac{1}{2}(T_n+M_n)=\frac{1}{2}\left\{\frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)\right]\right.$$
$$\left.+\frac{b-a}{n}\left[f(\bar{x}_1)+f(\bar{x}_2)+\ldots+f(\bar{x}_n)\right]\right\}$$

In the formula for the Midpoint Rule, $\bar{x}_n$ is the midpoint of [xn-1, x_n]. Therefore, $\bar{x}_1$ is halfway between x0 and x1, $\bar{x}_2$ is halfway between x1 and x2, and so on. As a result,

$$x_0 < \bar{x}_1 < x_1 < \bar{x}_2 < x_2 \ldots < \bar{x}_n < x_n$$

Define a new variable t_n as $t_0 = x_0$, $t_1 = \bar{x}_1$, $t_2 = x_1$, $t_3 = \bar{x}_2$, $t_4 = x_2$ and so on. Rewrite $\frac{1}{2}(T_n + M_n)$ in terms of t_n .

$$\frac{1}{2}(T_n+M_n)=\frac{1}{2}\left\{\frac{b-a}{2n}\left[f(t_0)+2f(t_2)+2f(t_4)+\ldots+2f(t_{2n-2})+f(t_{2n})\right]\right.$$
$$\left.+\frac{b-a}{n}\left[f(t_1)+f(t_3)+\ldots+f(t_{2n-1})\right]\right\}$$

Then combine the two halves of the result by rewriting the second half.

$$\frac{1}{2}(T_n+M_n)=\frac{1}{2}\left\{\frac{b-a}{2n}\left[f(t_0)+2f(t_2)+2f(t_4)+\ldots+2f(t_{2n-2})+f(t_{2n})\right]\right.$$
$$\left.+\frac{b-a}{2n}\left[2f(t_1)+2f(t_3)+\ldots+2f(t_{2n-1})\right]\right\}$$
$$=\frac{b-a}{2(2n)}\left[f(t_0)+2f(t_1)+2f(t_2)+\ldots+2f(t_{2n-1})+f(t_{2n})\right]$$

Notice that because of the way we defined t_n, the result is equal to *T*2n.

$$\frac{1}{2}(T_n+M_n)=T_{2n}$$

Answer 42E.

Calculate the period *T* using the formula below, where $k=\sin\left(\frac{1}{2}\theta_0\right)$, *g* is the acceleration due to gravity (9.8 m/ s^s), $L=1\text{ m}$, and $\theta_0=42°$.

$$T=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}}$$

Substitute the values of *k*, *g*, *L*, and *0 into the formula.*

$$T=4\sqrt{\frac{1\text{ m}}{9.8\text{ m/s}^2}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-\sin^2 21°\sin^2 x}}$$

Then calculate the integral $\int_0^{\pi/2}\frac{dx}{\sqrt{1-\sin^2 21°\sin^2 x}}$ *using Simpson's Rule shown below.*

$$\int_a^b f(x)\,dx\approx S_n=\frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots\right.$$
$$\left.+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$$

Where n is even and $\Delta x=\frac{b-a}{n}$.

Use Simpson's Rule to calculate *S*10. In the equation below, $g(x)=\frac{1}{\sqrt{1-\sin^2 21°\sin^2 x}}$ to make the equation shorter

$$S_{10}=\frac{\pi/2-0}{3(10)}\left[g(0)+4g\left(\frac{\pi}{20}\right)+2g\left(\frac{\pi}{10}\right)+4g\left(\frac{3\pi}{20}\right)+\ldots\right.$$
$$\left.+4g\left(\frac{9\pi}{20}\right)+g\left(\frac{\pi}{2}\right)\right]$$

Then calculate $g(0)$, $g\left(\frac{\pi}{20}\right)$, ..., $g\left(\frac{\pi}{2}\right)$ with the equation $g(x)=\frac{1}{\sqrt{1-\sin^2 21°\sin^2 x}}$. For instance,

$$g(0)=\frac{1}{\sqrt{1-\sin^2 21°\sin^2 0}}=1$$

$$g\left(\frac{\pi}{20}\right)=\frac{1}{\sqrt{1-\sin^2 21°\sin^2\frac{\pi}{20}}}\approx 1.003152755$$

Simplify *S*10.

$$S_{10}\approx\frac{\pi}{60}\left[1+4(1.003152755)+\ldots+1.147351597\right]$$
$$\approx\frac{\pi}{60}\left[32.134349809\right]$$
$$\approx 1.682550621$$

Therefore, $\int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}} \approx 1.682550621.$

Finally, calculate T by using the original equation $T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}}$.

$$\begin{aligned} T &= 4\sqrt{\frac{1\text{ m}}{9.8\text{ m/s}^2}} \cdot 1.682550621 \\ &\approx 2.15 \end{aligned}$$

Thus, the period of the pendulum is about $\boxed{2.15 \text{ seconds}}$.

Answer 43E.

$$\text{Given: } I(\theta) = \frac{N^2 \sin^2 k}{k^2} \qquad \text{--- (1)}$$

$$k = \frac{(\pi N d \sin\theta)}{\lambda} \qquad \text{--- (2)}$$

$N = 10000$

$d = 10^{-4}$ m

$\lambda = 632.8\times10^{-9}$ m

Then from (2) we have

$$k = \frac{(\pi\times10000\times10^{-4}.\sin\theta)}{632.8\times10^{-9}}$$

$$\Rightarrow \quad k = \frac{\pi\sin\theta}{632.8\times10^{-9}}$$

Then from (1) we have

$$I(\theta) = 10^8.\frac{\sin^2\left(\frac{\pi\sin\theta}{632.8\times10^{-9}}\right)}{\frac{\pi^2\sin^2\theta}{(632.8\times10^{-9})^2}}$$

$$\Rightarrow \quad I(\theta) = \frac{(632.8\times10^{-9})^2\times10^8\sin^2\left(\frac{\pi\sin\theta}{632.8\times10^{-9}}\right)}{\pi^2\sin^2\theta} \qquad \text{--- (3)}$$

We have $-10^{-6} < \theta < 10^{-6}$

We divide this interval in n = 10 subintervals then

$$\Delta\theta = \frac{10^{-6}+10^{-6}}{10} = \frac{2\times10^{-6}}{10}$$

$$\Rightarrow \quad \Delta\theta = 2\times10^{-7}$$

Subintervals are $[-10^{-6}, -0.8\times10^{-6}]$, $[-0.8\times10^{-6}, -0.6\times10^{-6}]$, $[-0.6\times10^{-6}, -0.4\times10^{-6}]$, $[-0.4\times10^{-6}, -0.2\times10^{-6}]$, $[-0.2\times10^{-6}, 0]$, $[0, 0.2\times10^{-6}]$, $[0.2\times10^{-6}, 0.4\times10^{-6}]$, $[0.4\times10^{-6}, 0.6\times10^{-6}]$, $[0.6\times10^{-6}, 0.8\times10^{-6}]$, $[0.8\times10^{-6}, 10^{-6}]$

Then midpoints are -0.9×10^{-6}, -0.7×10^{-6}, -0.5×10^{-6}, -0.3×10^{-6}, -0.1×10^{-6}, 0.1×10^{-6}, 0.3×10^{-6}, 0.5×10^{-6}, 0.7×10^{-6}, 0.9×10^{-6}

By midpoints rule

$$\begin{aligned} \int_{-10^{-6}}^{10^{-6}} I(\theta)d\theta \approx M_{10} &= \Delta\theta\left[I(-0.9\times10^{-6}) + I(-0.7\times10^{-6}) + \ldots\ldots + I(0.9\times10^{-6})\right] \\ &= 0.2\times10^{-6}\left[I(-0.9\times10^{-6}) + I(-0.7\times10^{-6}) + \ldots\ldots + I(0.9\times10^{-6})\right] \\ &\approx 59.41 \end{aligned}$$

$$\Rightarrow \boxed{\int_{-10^{-6}}^{10^{-6}} I(\theta)d\theta \approx 59.41}$$

Answer 44E.

We have $f(x)=\cos(\pi x)$
Interval is $=[0, 20]$
And $\quad n=10$

Then $\quad \Delta x=\dfrac{20}{10}=2$

Then subintervals are $[0, 2], [2, 4], [4, 6], \ldots\ldots [18, 20]$

Then by Trapezoidal rule

$$\int_0^{20}\cos(\pi x)\,dx \approx T_{10}=\frac{\Delta x}{2}\left[f(0)+2\left(f(2)+f(4)+f(6)+\ldots+f(18)\right)+f(20)\right]$$

$$=1.\left[\cos 0+2\left\{\cos(2\pi)+\cos(4\pi)+--+\cos(18\pi)\right\}+\cos(20\pi)\right]$$

Since $\quad \cos 2n\pi=1$

Then $\quad T_{10}=\left[1+2.(1+1+1+1+1+1+1+1+1)+1\right]$

$\quad T_{10}=1+18+1=20$

So $\quad \boxed{\int_0^{20}\cos(\pi x)\,dx \approx 20}$

Now we have calculate the actual value

$$\int_0^{20}\cos(\pi x)\,dx=\left[\frac{\sin(\pi x)}{\pi}\right]_0^{20}=0$$

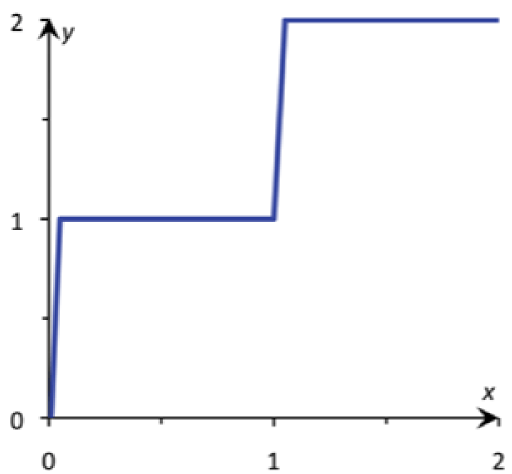
Comparison:-

The actual value of the integral is 0, because the integral $\int_0^{20}\cos(\pi x)\,dx$ represents the net area under the graph of $\cos(\pi x)$. Since the graph covers same area below the x – axis as above the x – axis so net area = 0
But for n = 10, in trapezoidal rule, the portion of the graph lying above x – axis is considered. So this is representing the area lying only above x – axis that is why it is not correct
If we take n = 20 then we can get the same answer

Answer 45E.

Though Simpson's Rule will be more accurate than the right endpoint approximation for most continuous functions (for any value of *n*), there are many different functions that this will not be true for.

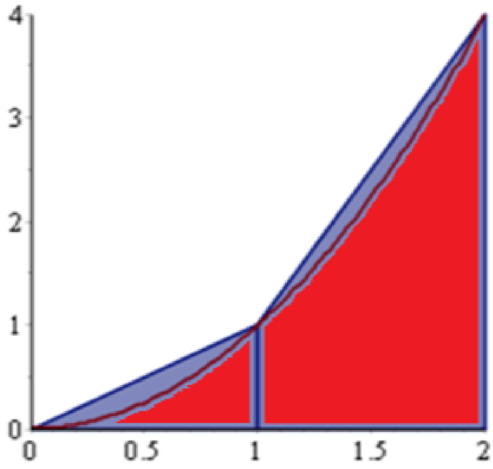
The function *f* graphed below is a continuous function because there are no jumps or no holes. In this function, the right endpoint approximation with $n=2$ is more accurate than Simpson's Rule because the graph is very close to the two rectangles that the right endpoint approximation is based on anyway. Since Simpson's Rule is based on using parabolas to approximate the function, it will underestimate the area under the function.



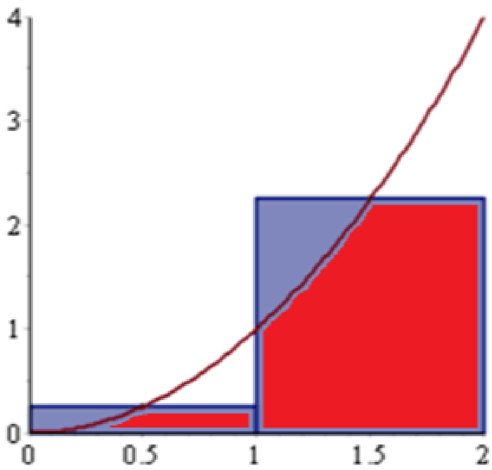
Answer 46E.

Consider the continuous function $f(x) = x^2$ on the interval $[0,2]$

Area under the curve $f(x) = \sin x$ using Trapezoidal Rule is shown below:



Area under the curve $f(x) = \sin x$ using Midpoint Rule is shown below:



From the above diagrams, loss of area by Midpoint Rule is more than that of Trapezoidal rule.
That is area by Trapezoidal is accurate than that of Midpoint rule.

Answer 47E.

Since f is a positive function and $f''(x) < 0$ for $a \le x \le b$

So $f(x)$ is concave downward in the interval $[a, b]$

So tangent line will be above the graph so mid point approximation will be overestimation of $\int_a^b f(x)dx$

So $\int_a^b f(x)dx < M_n$

Since $f(x)$ is concave downward so the line joining the points $(a, f(a))$ and $(b, f(b))$ will be under the graph

Here T_n will be underestimation of $\int_a^b f(x)dx$

Then $T_n < \int_a^b f(x)dx$

Then we have $T_n < \int_a^b f(x)dx < M_n$

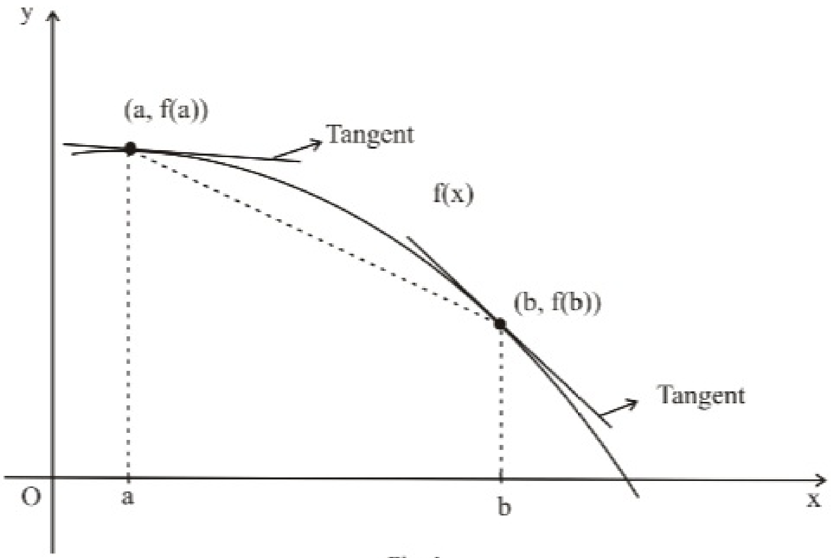


Fig. 1

Answer 48E.

Show that if f is a polynomial of degree 3 or lower, then Simpson's Rule gives the exact value of $\int_a^b f(x)dx$.

To prove this, it is sufficient to show that Simpson's Rule produces and exact value on just two subintervals, therefore let the x values be x_{i-1}, x_i and x_{i+1}. These are the same as x_{i-h}, x_i and x_{i+h} where $\Delta x = h = \dfrac{b-a}{h}$.

Now, let $f(x) = Ax^3 + Bx^2 + Cx + D$ and integrate it over $[x_{i-h}, x_{i+h}]$

Therefore

$$\int_{x_{i-h}}^{x_{i+h}} Ax^3 + Bx^2 + Cx + Ddx$$

Using substitution, let $u = x - x_i$, so that $x = u + x_i$ and $du = dx$.

When $x = x_i + h$ then $u = x_i + h - x_i = h$

When $x = x_i - h$ then $u = x_i - h - x_i = -h$

Therefore

$$\int_{x_{i-h}}^{x_{i+h}} Ax^3 + Bx^2 + Cx + Ddx = \int_{-h}^{h} A(u+x_i)^3 + B(u+x_i)^2 + C(u+x_i) + Ddu$$
$$= \int_{-h}^{h} au^3 + bu^2 + du + ddu$$

Where $a = A$

$b = 3Ax_i + B$

$c = 3Ax_i^2 + 2Bx_i + C$

$d = Ax_i^3 + Bx_i^2 + Cx_i + D$

Now let $g(x) = ax^3 + bx^2 + cx + d$, this will be used later in the proof.

$\int_{-h}^{h} au^3 + bu^2 + du + ddu$ can be split into odd and even integrals:

$$\int_{-h}^{h} au^3 + bu^2 + du + ddu = \int_{-h}^{h} au^3 + cudu + \int_{-h}^{h} bu^2 + ddu$$

The first integral is odd over $[-h, h]$ so it evaluates to zero, whereas the second is even, so:

$$\int_{-h}^{h} bu^2 + ddu = 2\int_{-h}^{h} bu^2 + ddu$$

Using the integral formula:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

Then

$$2\int_{-h}^{h} bu^2 + d\,du = 2\left[\frac{bu^3}{3} + du\right]_0^h$$

$$= 2\left(\frac{bh^3}{3} + dh\right)$$

$$= \frac{h}{3}\left(2bh^2 + bh\right)$$

$$= \frac{h}{3}\left(g(-h) + 4g(0) + g(h)\right)$$

This is exactly the same as $\frac{h}{3}\left(f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\right)$ which is Simpson's Rule with two subintervals.

Therefore, if a polynomial is of degree 3 or lower, then Simpson's Rule gives the exact value of $\int_a^b f(x)dx$.

Answer 49E.

To show that $\frac{1}{2}(T_n + M_n) = T_{2n}$ first use the definitions of T_n and M_n to rewrite the left side of the equation

$$\frac{1}{2}(T_n + M_n) = \frac{1}{2}\left\{\frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right] + \frac{b-a}{n}\left[f(\bar{x}_1) + f(\bar{x}_2) + \ldots + f(\bar{x}_n)\right]\right\}$$

In the formula for the Midpoint Rule, $\bar{x}_n$ is the midpoint of [xn-1, x_n]. Therefore, $\bar{x}_1$ is halfway between x0 and x1, $\bar{x}_2$ is halfway between x1 and x2, and so on. As a result,

$$x_0 < \bar{x}_1 < x_1 < \bar{x}_2 < x_2 \ldots < \bar{x}_n < x_n$$

Define a new variable t_n as $t_0 = x_0$, $t_1 = \bar{x}_1$, $t_2 = x_1$, $t_3 = \bar{x}_2$, $t_4 = x_2$ and so on. Rewrite $\frac{1}{2}(T_n + M_n)$ in terms of t_n .

$$\frac{1}{2}(T_n + M_n) = \frac{1}{2}\left\{\frac{b-a}{2n}\left[f(t_0) + 2f(t_2) + 2f(t_4) + \ldots + 2f(t_{2n-2}) + f(t_{2n})\right] + \frac{b-a}{n}\left[f(t_1) + f(t_3) + \ldots + f(t_{2n-1})\right]\right\}$$

Then combine the two halves of the result by rewriting the second half.

$$\frac{1}{2}(T_n + M_n) = \frac{1}{2}\left\{\frac{b-a}{2n}\left[f(t_0) + 2f(t_2) + 2f(t_4) + \ldots + 2f(t_{2n-2}) + f(t_{2n})\right] + \frac{b-a}{2n}\left[2f(t_1) + 2f(t_3) + \ldots + 2f(t_{2n-1})\right]\right\}$$

$$= \frac{b-a}{2(2n)}\left[f(t_0) + 2f(t_1) + 2f(t_2) + \ldots + 2f(t_{2n-1}) + f(t_{2n})\right]$$

Notice that because of the way we defined t_n, the result is equal to *T*2n.

$$\frac{1}{2}(T_n + M_n) = T_{2n}$$

Answer 50E.

202-08.07-48E

We have Simpson's approximation for the interval $[x_0, x_n]$

$$S_n = \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)4f(x_3)+\ldots\ldots+4f(x_{n-1})+f(x_n)\right]$$

If we double the number of n then subintervals will $[x_0, \bar{x}_1]$, $[\bar{x}_1, x_1]$, $[x_1, \bar{x}_2]$, $[\bar{x}_2, x_2]$, ------ $[\bar{x}_n, x_n]$

Where $\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots\ldots$ are the mid pints of the intervals $[x_0, x_1], [x_1, x_2], \ldots\ldots\ldots, [x_{n-1}, x_n]$

Then Simpson's approximation for 2n

$$S_{2n} = \frac{1}{2}\cdot\frac{\Delta x}{3}\left[f(x_0)+4f(\bar{x}_1)+2f(x_1)+4f(\bar{x}_2)+2f(x_2)+\ldots\ldots+4f(\bar{x}_n)+f(x_n)\right]$$

$$=\frac{1}{6}\Delta x\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots\ldots+f(x_n)+4f(\bar{x}_1)+4f(\bar{x}_2)+--+4f(\bar{x}_n)\right]$$

$$=\frac{1}{2}\left[\frac{\Delta x}{3}\left(f(x_0)+2f(x_1)+2f(x_2)+\ldots+f(x_n)\right)\right]+\frac{4\Delta x}{6}\left[4f(\bar{x}_1)+f(\bar{x}_2)+\ldots\ldots+f(\bar{x}_n)\right]$$

$$=\frac{1}{2}T_n+\frac{2}{3}M_n$$

Since $\quad T_n = \frac{\Delta x}{3}\left(f(x_0)+2f(x_1)+2f(x_2)+--+2f(x_{n-1})+f(x_n)\right)$

And $\qquad M_n = \Delta x\left(f(\bar{x}_1)+f(\bar{x}_2)+--+f(\bar{x}_n)\right)$

Thus we have $\boxed{S_{2n} = \frac{1}{2}T_n+\frac{2}{3}M_n}$