Chapter 8. Binomial Theorem

Question-1

Expand each of the following expression $(1 - 2x)^5$

Solution:

By using Binomial Theorem, we have

$(1 - 2x)^5 = [1+(-2x)]^5$

$= {}^5C_0 + {}^5C_1(-2x) + {}^5C_2(-2x)^2 + {}^5C_3(-2x)^3 + {}^5C_4(-2x)^4 + {}^5C_5(-2x)^5$

$= 1 + 5(-2x) + 10(-2x)^2 + 10(-2x)^3 + 5(-2x)^4 + (-2x)^5$

$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

Question-2

Expand each of the following expression $\left(\frac{2}{x} - \frac{x}{2}\right)^5$

Solution:

By using Binomial Theorem, we have

$\left(\frac{2}{x} - \frac{x}{2}\right)^5 = \left[\frac{2}{x} + \left(-\frac{x}{2}\right)\right]^5$

$= {}^5C_0\left(\frac{2}{x}\right)^5\left(\frac{-x}{2}\right)^0 + {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{-x}{2}\right)^1 + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{-x}{2}\right)^2 + {}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{-x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)^1\left(\frac{-x}{2}\right)^4 + {}^5C_5\left(\frac{-x}{2}\right)^5$

$= \frac{32}{x^5} + 5\left(\frac{16}{x^4}\right)\left(\frac{-x}{2}\right) + 10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right) + 10\left(\frac{4}{x^2}\right)\left(\frac{-x^3}{8}\right) + 5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right) + \left(\frac{-x^3}{32}\right)$

$= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5x^2}{8} - \frac{x^5}{32}$

Question-3

Expand each of the following expression $(2x - 3)^6$

Solution:

By using Binomial Theorem, we have

$(2x - 3)^6 = [2x + (-3)]^6$

$= {}^6C_0 (2x)^6 (-3)^\circ + {}^6C_1(2x)^5 (-3)^1 + {}^6C_2 (2x)^4 (-3)^2 + {}^6C_3(2x)^3 (-3)^3 + {}^6C_4 (2x)^2 (-3)^4 + {}^6C_5(2x)^1 (-3)^5 + {}^6C_6(2x)^0 (-3)^6$

$= 1(2x)^6 + 6(2x)^5(-3) + 15(2x)^4(-3)^2 + 20(2x)^3(-3)^3 + 15(2x)^2(-3)^4 + 6(2x)(-3)^5 + (-3)^6$

$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$

Question-4

Expand each of the following expression $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

Solution:

By using Binomial Theorem, we have;

$$\left(\frac{x}{3}+\frac{1}{x}\right)^5 = {}^5C_0\left(\frac{x}{3}\right)^5\left(\frac{1}{x}\right)^0 + {}^5C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right) + {}^5C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2 + {}^5C_3\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3 + {}^5C_4\left(\frac{x}{3}\right)^1\left(\frac{1}{x}\right)^4 + {}^5C_5\left(\frac{x}{3}\right)^0\left(\frac{1}{x}\right)^5$$

$$= \left(\frac{x}{3}\right)^5 + 5\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right) + 10\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2 + 10\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3 + 5\left(\frac{x}{3}\right)\left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^5$$

$$= \frac{x}{243} + \frac{5x^3}{3} + \frac{10x}{3} + \frac{10}{3x} + \frac{5}{3x^3} + \frac{1}{x^5}$$

Question-5

Expand each of the following expression $\left(x+\frac{1}{x}\right)^6$

Solution:

By using Binomial Theorem, we have:

$$\left(x+\frac{1}{x}\right)^6 = {}^5C_0(x)^6\left(\frac{1}{x}\right)^0 + {}^6C_1(x)^5\left(\frac{1}{x}\right)^1 + {}^6C_2(x)^4\left(\frac{1}{x}\right)^2 + {}^6C_3(x)^3\left(\frac{1}{x}\right)^3 + {}^6C_4(x)^2\left(\frac{1}{x}\right)^4 + {}^6C_5(x)^1\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6$$

$$= x^6 + 6x^5\left(\frac{1}{x}\right) + 15x^4\frac{1}{x^2} + 20x^3.\frac{1}{x^3} + 15x^2.\frac{1}{x^4} + 6x.\frac{1}{x^5} + \frac{1}{x^6}$$

$$l = x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

Question-6

Using binomial theorem, evaluate each of the following $(96)^3$

Solution:

We express 96 as the sum or difference of two numbers whose powers are easier to calculate, and then use Binomial Theorem

Write 96 = 100 - 4

Therefore

$(96)^3 = (100 - 4)^3$

$= {}^3C_0\,(100)^3 - {}^3C_1(100)^2(4) + {}^3C_2(100)^1(4)^2 - {}^3C_3(4)^3$

$= 1000000 - 3\,(10000)\,(4) + 3\,(100)\,(16) - (64)$

$= 1000000 - 120000 + 4800 - 64$

$= 884736$

Question-7

Using binomial theorem, evaluate each of the following $(102)^5$

Solution:

$(102)^5 = (100 + 2)^5$

$= {}^5C_0(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2 + {}^5C_3(100)^2(2)^3 + {}^5C_4(100)(2)^4 + {}^5C_5(2)^5$

$= 10000000000 + 5(100000000)(2) + 10(1000000)(4) + 10(10000)(8) + 5(100)(16) + 32$

$= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32$

$= 11040808032$

Question-8

Using binomial theorem, evaluate each of the following $(101)^4$

Solution:

$(101)^4 = (100 + 1)^4$

$= {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)^1(1)^3 + {}^4C_4(1)^4$

$= 100000000 + 4(1000000) + 6\ (10000) + 4\ (100) + 1$

$= 100000000 + 4000000 + 60000 + 400 + 1$

$= 104060401$

Question-9

Using binomial theorem, evaluate each of the following $(99)^5$

Solution:

$(99)^5 = (100 - 1)^5$

$= {}^5C_0(100)^5 - {}^5C_1(100)^4.1 + {}^5C_2(100)^3.(1)^2 - {}^5C_3(100)^2.(1)^2 + {}^5C_4(100)^1(1)^4 - {}^5C_5(1)^5$

$= (100)^5 - 5 \times (100)^4 + 10 \times (100)^3 - 10 \times (100)^2 + 5 \times 100 - 1$

$= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1$

$= 10010000500 - 500100001$

$= 9509900499$

Question-10

Using Binomial Theorem indicate which number is larger $(1.1)^{10000}$ or 1000.

Solution:

Splitting 1.1 and using Binomial Theorem to write the first few terms we have

$(1.1)^{10000} = (1+0.1)^{10000}$

$= {}^{10000}C_0 + {}^{10000}C_1(0.1) + {}^{10000}C_2(0.1)^2$ + other positive terms.

$= 1 + 10000$ ´ (0.1) + other positive terms

$= 1 + 1000$ + other positive terms

>1000

Hence, $(1.1)^{10000} > 1000$.

Question-11

Find $(a + b)^4 - (a - b)^4$. Hence, evaluate $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$.

Solution:

$(a + b)^4 - (a - b)^4$

$= [{}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4] - [{}^4C_0a^4 + {}^4C_la^3b + {}^4C_2a^2b^2 - {}^4C_3 ab^3 + {}^4C_4 b^4] = 2 \times {}^4C_1 a^3b + 2 \times {}^4C_3 ab^3$

$= 2[4a^3b + 4ab^3]$

$= 8ab [a^2 + b^2]$

Thus, $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$

$= 8 \sqrt{3} . \sqrt{2} [3 + 2] = 8\sqrt{6} (5) = 40\sqrt{6}$

Question-12

Find $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate $(\sqrt{2} + 1)^6 - (\sqrt{2} - 1)^6$

Solution:

$(x + 1)^6 + (x - 1)^6$

$= [{}^6C_0 x^6 + {}^6C_1 x^5 + {}^6C_2 x^4] + [{}^6C_3 x^3 + {}^6C_4 x^2 + {}^6C_5 x^1 - {}^6C_6] +$

$[{}^6C_0 x^6 - {}^6C_1 x^5 + {}^6C_2 x^4 - {}^6C_3 x^3 + {}^6C_4 x^2 + {}^6C_5x^1 - {}^6C_6]$

$= 2[{}^6C_0x^6 + {}^6C_2 x^4 + {}^6C_4 x^2 + {}^6C_6]$

$= 2[x^6 + 15x^4 + 15x^2 + 1]$

Thus, $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$

$= 2 [(\sqrt{2})^6 + 15(\sqrt{2})^4 + 15 (\sqrt{2}) + 1]$

$= 2 [8 + 15 (4) + 15 (2) + 1]$

$= 2 [8 + 60 + 30 + 1] = 198.$

Question-13

Show that 9^{n+1} - 8n - 9 is divisible by 64, whenever n is a positive integer.

Solution:

$n = 1 \Rightarrow 9^{n+1} - 8n - 9 = 9^2 - 8 - 9$

$= 81 - 17 = 64 = 1(64)$

$n = 2 \Rightarrow 9^{n+1} - 8n - 9 = 9^3 - 8(2) - 9$

$= 729 - 16 - 9 = 704 = 11\ (64)$

From n = 3, 4, 5,.....$9^{n+1} - 8n - 9 = 9(1 + 8)n - 8n - 9$

$= 9\ [{}^nC_0 + {}^nC_1 \,.\, 8 + {}^nC_2 . 8^2 + \ldots\ {}^nC_n\ 8^n] - 8n - 9$

$= 9[1 + 8n + {}^nC_2 . 8^2 + \ldots\ {}^nC_n\ 8n] - 8n - 9$

$= 9 + 72n + 9.\ {}^nC_2 .\ 8^2 + \ldots\ 9\ {}^nC_n\ 8^n - 8n - 9$

$= 8^2\ [n + 9\ ({}^nC_2 + {}^nC_3 . 8 + \ldots\ {}^nC_n\ 8^{n-2})]$

which is divisible by 64.

Question-14

Prove that $\sum_{r=0}^{n} 3^r\ {}^nC_r = 4^n$.

Solution:

L.H.S = $3^0C(n,0) + 3^1\ C(n,1) + 3^2\ C(n,2) + \text{---}\ 3^r(n, r) + \text{---} + 3^n\ C(n, n)$

$= C(n,0) + C(n,1)\ 3^1 + C(n,2)\ 3^2 + C(n,3)\ 3^3 + \text{---} + C(n,n)3^n$

This is in the form of $(1+3)^n$

$= (1+3)^n = 4^n$ = R.H.S

Question-15

Prove that x^5 in $(x + 3)^8$

Solution:

Suppose x^5 occurs in the (r + 1)th term of the expansion $(x + 3)^8$

Now $T_{r+1} = {}^nC_r\ a^{n-r}\ b^r = {}^8C_r\ x^{8-r}\ 3^r$

Comparing the indices of x in x^5 and in T_{r+1}, we get r = 3

Thus, the coefficient of x^5 is

${}^8C_3(3)^3 = 1512$

Question-16

Prove that a b^7 in $(a-2b)^{12}$

Solution:

Let a^5b^7 occurs in the (r + 1)th term, in the expansion of $(a - 2b)^{12}$ given by $^{12}C_r.\ a^{12-r}(-2b)^r$. Then 12 – r = 5. This gives r= 7.

Thus the coefficient of a^5b^7 is

$$^{12}C_5\,(-2)^7 = \frac{12!}{5!7!}\times(-128) = (792)\,(-128) = -101376$$

Question-17

Prove that $(x^2 - y)^6$

Solution:

We have T_{r+1} in $(a + b)^n = {}^nC_r a^{n-r}.\ b^r, 0 \le r \le n$

T_{r+1} in $(x^2 - y)^6 = {}^6C_r(x^2)^{6-r}(-y)^r$

$= {}^6C_r\, x^{12-2r}(-y)^r$

Question-18

Prove that $(x^2 - yx)^{12}$, x ¹ 0

Solution:

T_{r+1} in $(x^2 - yx)^{12} = {}^{12}C_r\,(x^2)^{12-r}(-yx)^r$

$= {}^{12}C_r\, x^{24-2r}(-1)^r(y)^r(x)^r$

$= {}^{12}C_r\, x^{24-r}y^r(-1)^r$

Question-19

Find the 4th term in the expansion of $(x - 2y)^{12}$.

Solution:

4th term in $(x - 2y)^{12} = T_4 = T_{3+1}$

$= {}^{12}C_3(x)^{12-3}(-2y)^3$

$[T_{r+1}$ in $(a + b)^n = {}^nC_r\, a^{n-r}\, b^r]$

$= {}^{12}C_3(x)^9\,(-2)^3\,(y)^3$

$= \frac{12\times11\times10}{3\times2}\times -8\times x^9y^3$

$= -1760\, x^9\, y^3$

Question-20

Find the 13th term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$

Solution:

13th term in $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$ $= T_{13} = T_{12+1}$

$$= {}^{18}C_{12}(9x)^{18-12}\left(-\frac{1}{3\sqrt{x}}\right)^{12}$$

$$= {}^{18}C_{12}(9x)^{6}\left(-\frac{1}{3}\right)^{12}\left(-\frac{1}{\sqrt{x}}\right)^{12}$$

$$= {}^{18}C_{12}(9x)^{6}\left(\frac{-1}{3}\right)^{12}\left(x^{\frac{-1}{2}}\right)^{12}$$

$$= {}^{18}C_{12}(9x)^{6}\left(\frac{-1}{3}\right)^{12}(x)^{-6}$$

$$= {}^{18}C_{12}(3^2)^6(x)^6(-1)^{12}(3)^{-12}(x)^{-6}$$

$$= {}^{18}C_{12}(3^{12})(x)^6(3)^{-12}(x)^{-6}$$

$$= {}^{18}C_{12} = 18564$$

Question-21

Find the 13th term in the expansion of $\left(3 - \frac{x^3}{6}\right)^{7}$

Solution:

The index of $\left(3 - \frac{x^3}{6}\right)^{7}$, is 7, which is an odd natural number.

So, Middle terms are $T_{\frac{7+1}{2}}$ and $T_{\frac{7+3}{2}}$

$$T_{\frac{7+1}{2}} = T_4 = T_{3+1} = {}^7C_3(3)^{7-3}\left(-\frac{x^3}{6}\right)^3$$

$$= {}^7C_3(3)^4(x^3)^3(-1)(6^{-1})^3$$

$$= {}^7C_3(81)(x)^9\,(6)^{-3}(-1)^3$$

$$= \frac{-35 \times 81}{216}x^9 = \frac{-105}{8}x^9$$

$$T_{\frac{7+3}{2}} = T_5 = T_4 + 1 = {}^7C_4(3)^{7-4}\left(-\frac{x^3}{6}\right)^4$$

$$= {}^7C_4(3)^3\,(-1)^4(x^3)(6)^{-4}$$

$$= {}^7C_4(27)\,(x)^{12}(6)^{-4}$$

$$= (35)(27)(6)^{-4}(x)^{12}$$

$$= \frac{35 \times 27}{1296}x^{12} = \frac{35}{48}x^{12}$$

Question-22

$\left(\frac{x}{3} + 9y\right)^{10}$

Solution:

The index $\left(\frac{x}{3} + 9y\right)^{10}$ is 10, which is an even natural number.

Hence, Middle term = $T_{\frac{10+2}{2}} = T_6 = T_{5+1}$

$$= {}^{10}C_5\left(\frac{x}{3}\right)^{10-5}(9y)^5$$

$$= {}^{10}C_5\left(\frac{x}{3}\right)^5(9y)^5$$

$$= {}^{10}C_5(x)^5\left(\frac{1}{3}\right)^5(9)^5(y)^5$$

$$= {}^{10}C_5(3)^{-5}\left(3^2\right)^5(x)^5(y)^5$$

$$= {}^{10}C_5(3)^{-5}(3)^{10}x^5y^5$$

$$= {}^{10}C_5 3^5 x^5 y^5 = (252)(243)x^5 y^5$$

$$= 61236\, x^5y^5$$

Question-23

In the expansions of $(1 + a)^{m+n}$, using Binomial Theorem, prove that coefficients of a^m and a^n are equal.

Solution:

We have,

$(1 + a)^{m+n} = [{}^{m+n}C_0 + {}^{m+n}C_1 a^1 + {}^{m+n}C_2 a^2 + \ldots\ldots {}^{m+n}C_r$
$a^r + \ldots\ldots + {}^{m+n}C_{m+n}\, a^{m+n}$

Coefficient of $a^m = {}^{m+n}C_m = \frac{(m+n)!}{m!n!}$

Also the coefficient of a^n

$= {}^{m+n}C_n = \frac{(m+n)!}{n!m!}$

Clearly, ${}^{m+n}C_m = {}^{m+n}C_n$

Question-24

The coefficients of the (r-1)th, rth and (r+1)th terms in the expansion of (x+1)^{n} are in the ratio 1:3:5. Find both n and r.

Solution:

Coefficient of (r-1)th term = C(n, r-2)

Coefficient of rth term = C(n, r-1)

Coefficient of (r+1)th term = C(n, r)

Considering 1st and ^{2nd}

$$\frac{C(n,r-2)}{C(n,r-1)} = \frac{n!}{(r-2)!(n-r+2)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{(r-1)!(n-r+1)!}{(r-2)!(n-r+2)(n-r+1)!} = \frac{(r-1)(r-2)!}{(r-2)!(n-r+2)} = \frac{(r-1)}{n-r+2} = \frac{1}{3}$$

3r - 3 = n - r + 2

n - 4r = -5 ----------(1)

Considering 2nd and 3rd

$$\frac{C(n,r-1)}{C(n,r)} = \frac{3}{5} = \frac{n!}{(r-1)!(n-r+1)!} \times \frac{r!(n-r)!}{n!} = \frac{(n-r)!r(r-1)!}{(n-r+1)(n-r)!(r-1)!} = \frac{r}{n-r+1} = \frac{3}{5}$$

5r = 3n - 3r +3

3n - 8r = -3 ---------(2)

2(n - 4r = -5)

2n - 8r = -10 ---------(3)

Subtract (3) from (2)

n = 7

Substitute n = 7 in (2)

We get r = 3

n = 7, r = 3

Question-25

Prove that the coefficient of x^n in the expansion of $(1+x)^{2n}$ is twice the coefficient of x^n in the expansion $(1 + x)^{2n-1}$.

Solution:

$(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 x^1 + {}^{2n}C_2 x^2 + ... + {}^{2n}C_n x^n$

$(1+x)^{2n-1} = {}^{2n-1}C_0 + {}^{2n-1}C_1 x^1 + {}^{2n-1}C_2 x^2 + ... + {}^{2n-1}C_n x^n$

Coefficient of x^n in $(1+x)^{2n-1}$ is $({}^{2n-1}C_n)$

$${}^{2n}C_n = \frac{(2n)!}{n!n!}$$

$$= \frac{(2n)(2n-1)(2n-2)!}{n(n-1)!n!}$$

$$= \frac{(2n)(2n-1)2(n-1)!}{n(n-1)!n!}$$

$$= 2\left[\frac{2(2n-1)}{n!}\right] ....(i)$$

Now $${}^{2n-1}C_n = \frac{(2n-1)!}{n!(2n-1-n)!}$$

$$= \frac{(2n-1)!}{n!(n-1)!}$$

$$= \frac{(2n-1)(2n-2)!}{n!(n-1)!}$$

$$= \frac{(2n-1)2(n-1)!}{n!(n-1)!}$$

$$= \frac{2(2n-1)}{n!} ....(ii)$$

From (i) and (ii) we have

${}^{2n}C_n = 2. {}^{2n-1}C_n$