Exercise 15.3

Chapter 15 Multiple Integrals 15.3 1E

Given Integral is $\int_0^4 \int_{x=0}^{x=\sqrt{y}} xy^2 \, dx \, dy = \int_0^4 \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} y^2 \, dy$

$$= \frac{1}{2}\int_0^4 \left[\sqrt{y}^{\,2} - 0^2\right] y^2 \, dy$$

$$= \frac{1}{2}\int_0^4 y^3 \, dy$$

$$= \frac{1}{2}\left[\frac{y^4}{4}\right]_0^4$$

$$= \frac{1}{8}\left[4^4 - 0^4\right]$$

= 32

Chapter 15 Multiple Integrals 15.3 2E

$$\int_0^1 \int_{y=2x}^{y=2} (x-y) \, dy \, dx = \int_0^1 \left[xy - \frac{y^2}{2}\right]_{2x}^2 dx$$

$$= \int_0^1 \left[x(2-2x) - \frac{1}{2}\left(4-4x^2\right)\right] dx$$

$$= \int_0^1 (2x-2) \, dx$$

$$= \left[\frac{2x^2}{2} - 2x\right]_0^1$$

$$= 1 - 2.1 = -1$$

Chapter 15 Multiple Integrals 15.3 3E

Given Integral is $\int_0^1 \int_{x^2}^{x} (1+2y)\,\mathrm{d}y\,\mathrm{d}x$

$$\int_0^1 \int_{y=x^2}^{y=x} (1+2y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \left[y+y^2\right]_{x^2}^{x}\,\mathrm{d}y$$

$$= \int_0^1 \left[\left(x-x^2\right)+\left(x^2-x^4\right)\right]\mathrm{d}x$$

$$= \int_0^1 \left(x-x^4\right)\,\mathrm{d}x$$

$$= \left[\frac{x^2}{2}-\frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{2}-\frac{1}{5}$$

$$= \frac{3}{10}$$

Chapter 15 Multiple Integrals 15.3 4E

Given Integral is $\int_0^2 \int_y^{2y} \mathrm{xy}\,\mathrm{d}x\,\mathrm{d}y$

$$\int_0^2 \int_{x=y}^{x=2y} \mathrm{xy}\,\mathrm{d}x\,\mathrm{d}y = \int_0^2 \left[\frac{x^2}{2}\right]_y^{2y}\,\mathrm{d}y$$

$$= \frac{1}{2}\int_0^2 \left[4y^2-y^2\right].y\,\mathrm{d}y$$

$$= \frac{3}{2}\int_0^2 y^3\,\mathrm{d}y$$

$$= \frac{3}{2}\left[\frac{y^4}{4}\right]_0^2$$

$$= \frac{3}{8}\left[2^4-0^4\right]$$

= 6

Chapter 15 Multiple Integrals 15.3 5E

We have the double integral $\int_0^1 \int_0^{s^2} \cos(s^3)\, dt\, ds$.

Let us start by removing the innermost integral.

$$\int_0^1 \int_0^{s^2} \cos(s^3)\, dt\, ds = \int_0^1 \left[\cos(s^3)\, t\right]_0^{s^2} ds$$
$$= \int_0^1 s^2 \cos(s^3)\, ds$$

The integral is simplified to $\int_0^1 s^2 \cos(s^3)\, ds$.

Now, let us evaluate the outer integral and apply the limits.

$$\int_0^1 s^2 \cos(s^3)\, ds = \left[\frac{\sin(s^3)}{3}\right]_0^1$$
$$= \frac{\sin 1}{3}$$

Thus, the iterated integral evaluates to $\boxed{\frac{\sin 1}{3}}$.

Chapter 15 Multiple Integrals 15.3 6E

We have the double integral $\int_0^1 \int_0^{e^v} \sqrt{1+e^v}\, dw\, dv$.

Let us start by removing the innermost integral.

$$\int_0^1 \int_0^{e^v} \sqrt{1+e^v}\, dw\, dv = \int_0^1 \left[w\sqrt{1+e^v}\right]_0^{e^v} dv$$
$$= \int_0^1 e^v \sqrt{1+e^v}\, dv$$

The integral is simplified to $\int_0^1 e^v \sqrt{1+e^v}\, dv$.

Now, let us evaluate the outer integral and apply the limits.

$$\int_0^1 e^v \sqrt{1+e^v}\, dv = \left.\frac{2\left(1+e^v\right)^{3/2}}{3}\right]_0^1$$

$$= \frac{2}{3}(1+e)^{3/2} - \frac{4\sqrt{2}}{3}$$

Thus, the iterated integral evaluates to $\boxed{\frac{2}{3}(1+e)^{3/2} - \frac{4\sqrt{2}}{3}}$.

Chapter 15 Multiple Integrals 15.3 7E

Given integral is $= \int_{-1}^{1} \int_{x=-(y+2)}^{x=y} y^2 \, dx \, dy$

$$= \int_{-1}^{1} y^2 \, [x]_{-(y+2)}^{y} \, dy$$

$$= \int_{-1}^{1} y^2.\left(2y+2\right) dy$$

$$= \int_{-1}^{1} \left(2y^3 + 2y^2\right) dy$$

$$= 0 + \left[2\frac{y^3}{3}\right]_{-1}^{1}$$ (First integral is zero as function is an odd function)

=4/3

Chapter 15 Multiple Integrals 15.3 8E

Given integral is $= \int_0^1 \int_0^{x^2} \frac{y}{x^5+1}\, dx\, dy$

$$= \int_0^1 \frac{1}{x^5+1} \left[\frac{y^2}{2}\right]_0^{x^2} dx$$

$$= \frac{1}{2} \int_0^1 \frac{x^4}{x^5+1}\, dx$$

Let $x^5 + 1 = t$

Therefore $5x^4 \mathrm{dx} = \mathrm{dt} \, \mathrm{or} \, x^4 \mathrm{dx} = \mathrm{dt}/5$

New limits for t =1 to 2.

Now integral is $= \frac{1}{2} \int_1^2 \frac{1}{t} \frac{\mathrm{dt}}{5}$

$$= \frac{1}{10} [\ln t]_1^2$$

$$= \frac{1}{10} \left[\ln 2 - \ln 1\right]$$

$$= \frac{1}{10} \ln 2$$

Chapter 15 Multiple Integrals 15.3 9E

Given integral is $= \int_0^{\pi} \int_{y=0}^{y=\sin x} x\, dy\, dx$

$$= \int_0^{\pi} x [y]_0^{\sin x}\, dx$$

$$= \int_0^{\pi} x \sin x\, dx$$

$$= [-x\cos x]_0^{\pi} - \int_0^{\pi} 1.(-\cos x)\, dx$$

$$= \left[-\pi\cos\pi + 0\right] + [\sin x]_0^{\pi}$$

$$= -\pi(-1) + \sin\pi - \sin 0 = \pi$$

Chapter 15 Multiple Integrals 15.3 10E

Consider the following double-integral:

$$\iint_D x^3 dA \quad \ldots\ldots \quad (1)$$

Limits of integration are as follows:

$$\begin{cases} 1 \le x \le e \\ 0 \le y \le \ln(x) \end{cases} \quad \ldots\ldots \quad (2)$$

Use (2), you can rewrite (1) is as follows:

$$\int_1^e \int_0^{\ln(x)} x^3 dy dx \quad \ldots\ldots \quad (3)$$

To compute a type I double integral, it is first necessary to establish the order in which you integrate the variables.

In this case, starting with the inner-most differential and moving outwards, first integrate our function $f(x) = x^3$ with respect to y from 0 to $\ln(x)$ and integrate the resulting function of x with respect to x from 1 to *e*.

Integrate with respect to *y* and taking the limits of integration, we get:

$$\iint_D x^3 dA = \int_1^e x^3 y dx \Big|_{y=0}^{y=\ln(x)}$$

$$= \int_1^e x^3 (\ln(x) - 0) dx$$

$$= \int_1^e x^3 \ln(x) dx \quad \ldots\ldots \quad (4)$$

Now, you have a single-variable integral, which we can solve with integration by parts if we let *v*=ln(*x*), and *du*=*x*3*dx*, thereby making $dv = \frac{1}{x} dx$ and $u = \frac{x^4}{4}$.

Now integrate (4) is as follows:

$$\int_1^e x^3 \ln(x) dx = \frac{x^4}{4} \ln(x) \Big|_1^e - \int_1^e \frac{x^4}{4} \frac{1}{x} dx \quad \text{Since} \quad \int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$= \frac{(e)^4}{4} \ln(e) - \frac{(1)^4}{4} \ln(1) - \int_1^e \frac{x^3}{4} dx$$

$$= \frac{(e)^4}{4} - \frac{x^4}{16} \Big|_1^e$$

$$= \frac{(e)^4}{4} - \left(\frac{e^4}{16} - \frac{1}{16} \right)$$

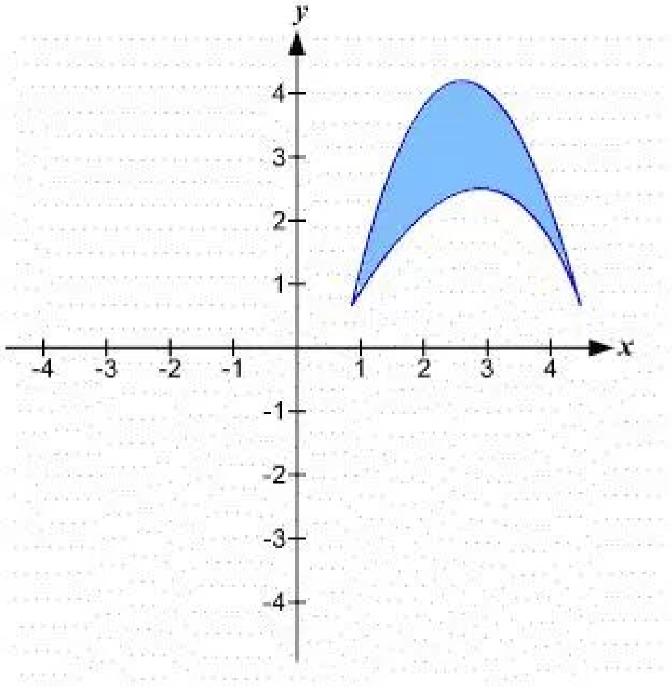
Therefore, the double integral of $\iint_D x^3 dA$ is $\boxed{\frac{3e^4}{16} + \frac{1}{16}}$.

Chapter 15 Multiple Integrals 15.3 11E

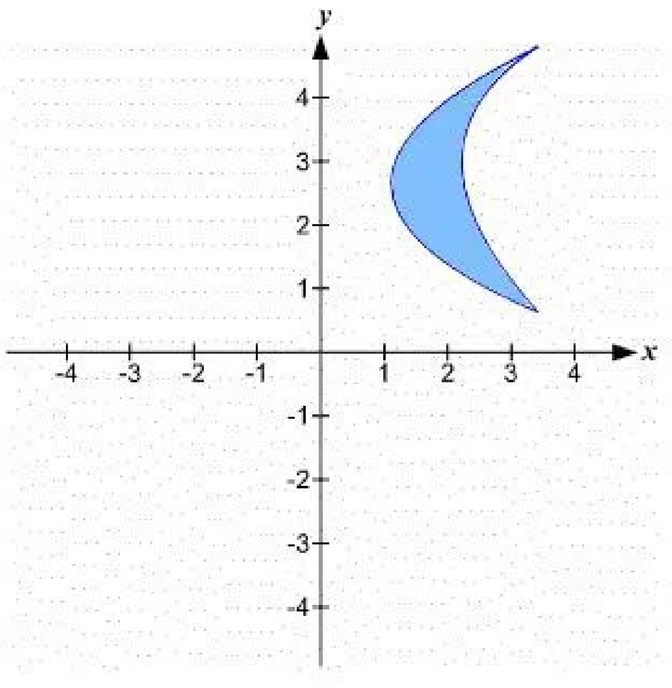
A plane region D is defined as type I if it lies between the graphs of two continuous functions of x, that is, $D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$.

Now, a plane region of type II can be expressed as
$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$.

(a) Now, let us sketch the region which is type I and not type II.



(b) Sketch the region which is type II and not type I.



Chapter 15 Multiple Integrals 15.3 12E

(a)

Draw an example of a region that is both type I and type II.

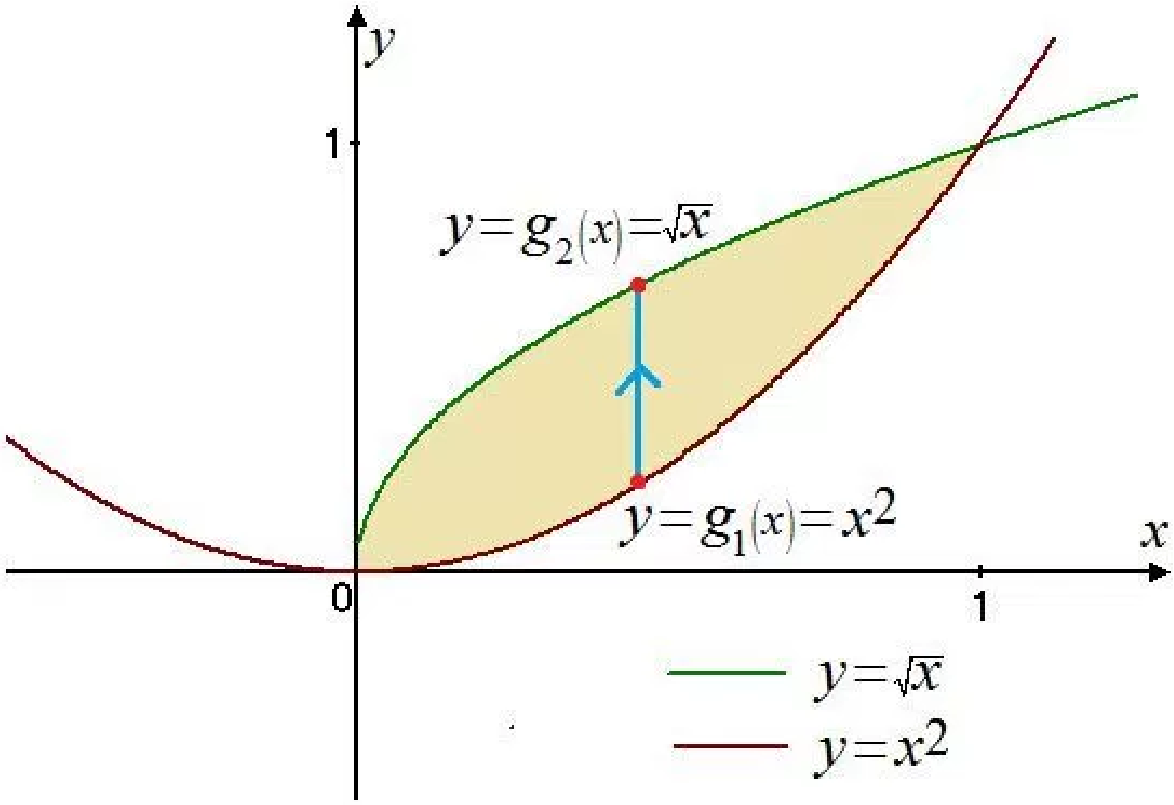
A plane region *D* is defined as type I if it lies between the graphs of two continuous functions of *x*, that is, $D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$.

A plane region of type II can be expressed as $D = \{(x, y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$.

Consider a region R bounded above by $y = \sqrt{x}$ below by $y = x^2$.

Type I region : $dydx$:

Region R is shown below:



On the region R:

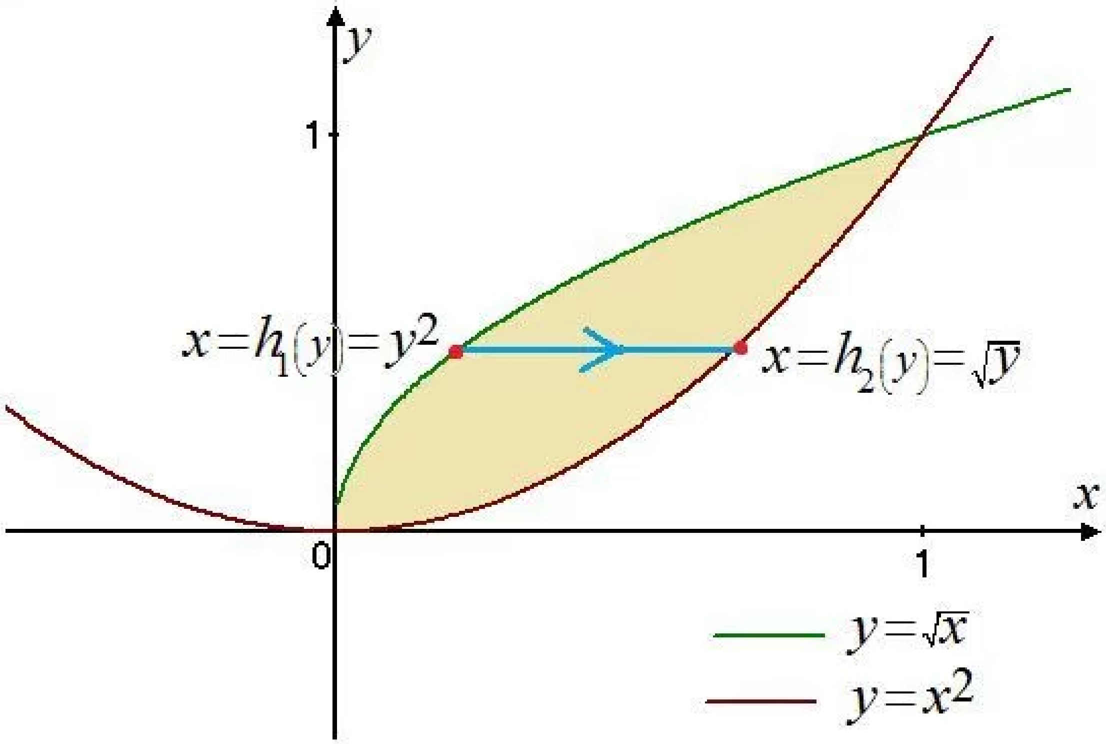
The variable y enters at $y = x^2$ and leaves at $y = \sqrt{x}$.

And the variable x varies from $x = 0$ to $x = 1$.

The region can be written as $R = \{(x, y) \mid x^2 \le y \le \sqrt{x}, 0 \le x \le 1\}$.

Type II region : $dxdy$:

Region R is shown below:



On the region R:

The variable x enters at $x=y^2$ and leaves at $x=\sqrt{y}$.

And the variable y varies from $y=0$ to $y=1$.

The region can be written as $R=\left\{(x,y)\mid 0\le y\le 1, y^2\le x\le\sqrt{y}\right\}$.

Therefore, the region R can be expressed in both the Type I and Type II regions.

(b)

Draw an example of a region that is neither type I nor type II.

Consider a region R bounded by the two curves $y=x^{\frac{1}{3}}$ and $y=\frac{x}{4}$.

Here, the region R can be expressed neither of type I, nor of type II regions.

So divide the region R into two regions R_1 and R_2 of type I or II.

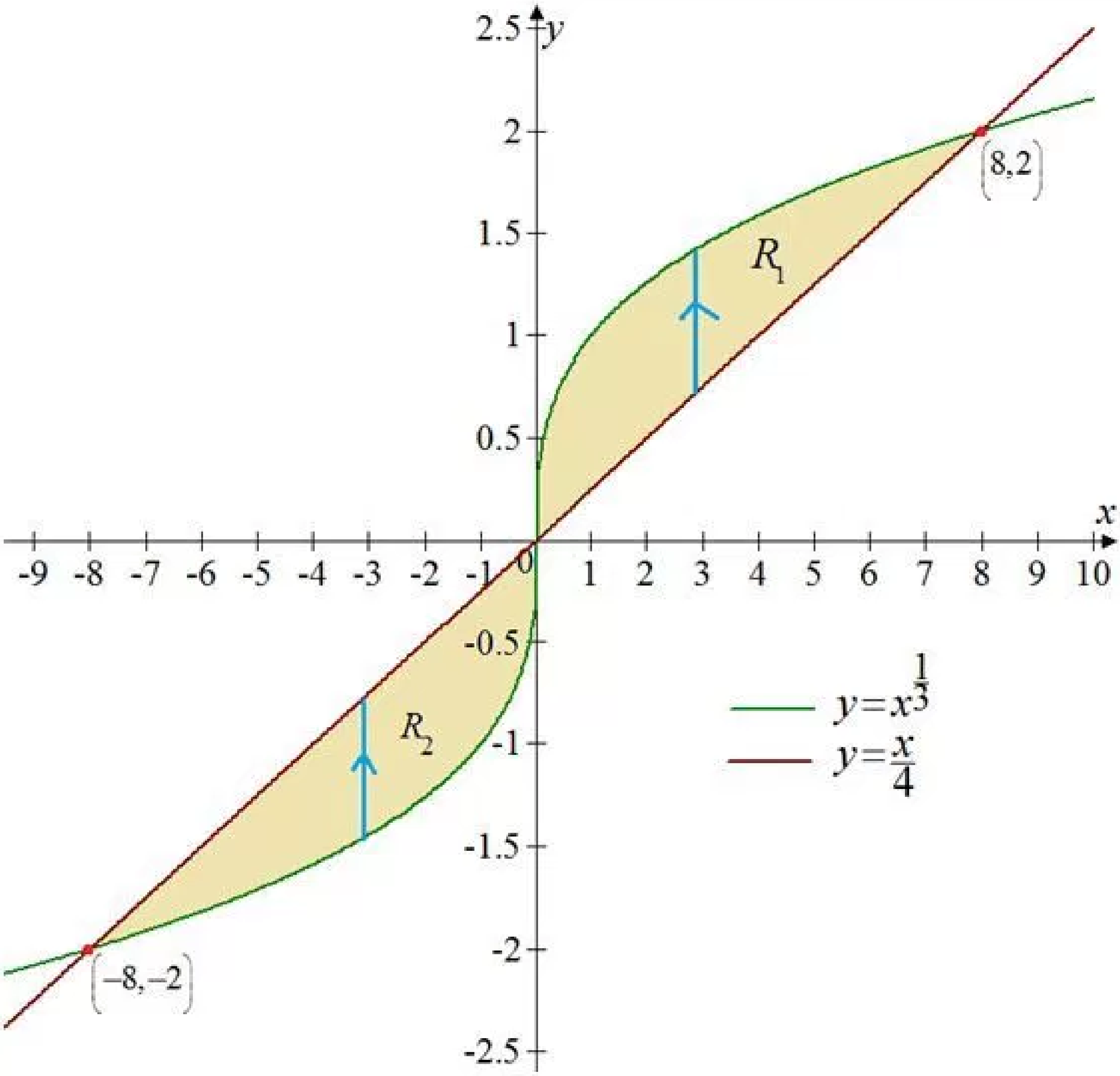
And break the integral into integrals over the sub regions and use additivity of the integral to evaluate it.

Divide the region R into two regions R_1 and R_2 of type I:

The region R_1 can be written as $R_1=\left\{(x,y)\mid 0\le x\le 8, \frac{x}{4}\le y\le x^{\frac{1}{3}}\right\}$.

The region R_2 can be written as $R_2=\left\{(x,y)\mid -8\le x\le 0, x^{\frac{1}{3}}\le y\le \frac{x}{4}\right\}$.

Observe the below figure:

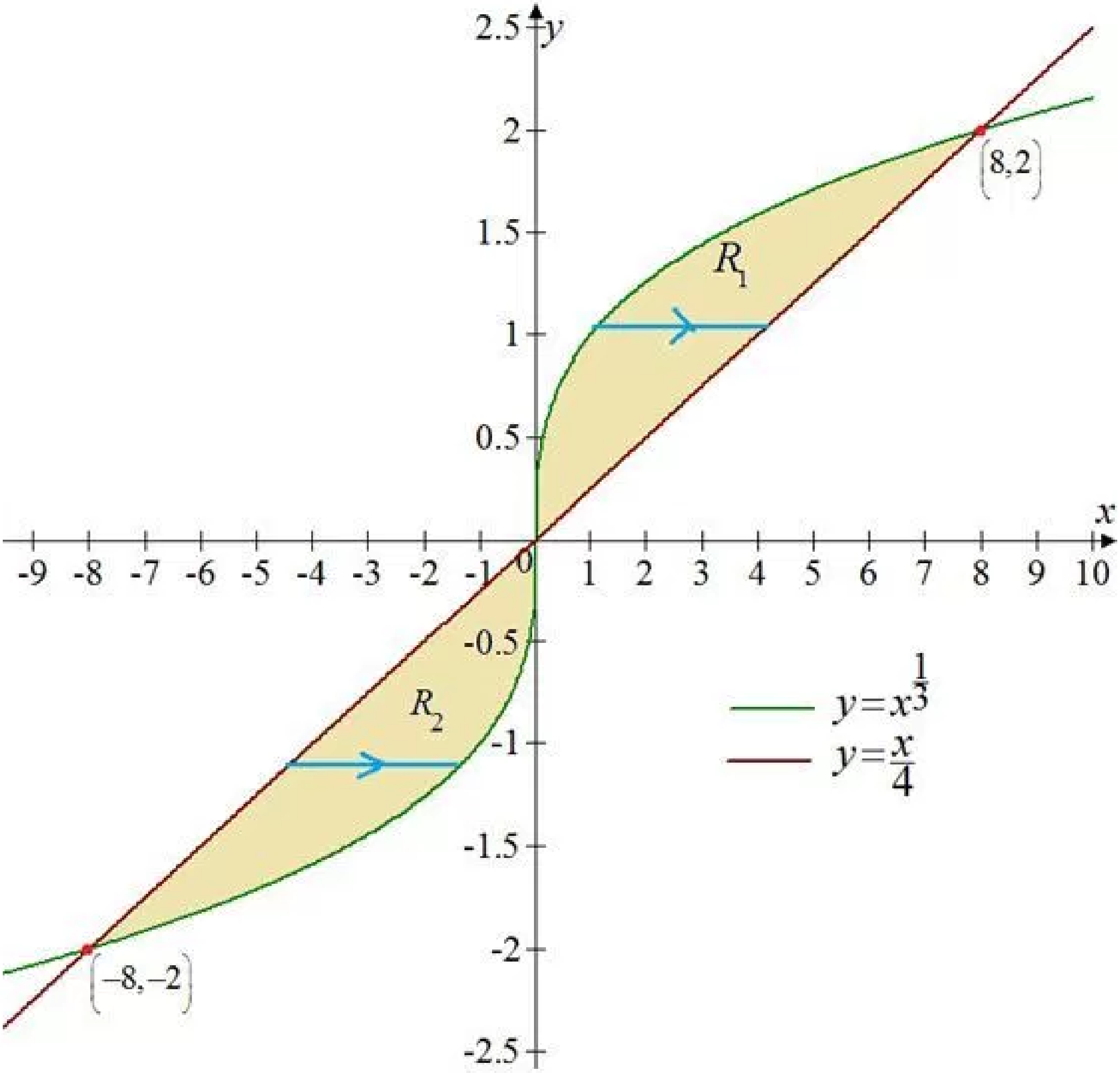


Divide the region R into two regions R_1 and R_2 of type II:

The region R_1 can be written as $R_1 = \{(x,y) \mid y^3 \le x \le 4y, 0 \le y \le 2\}$.

The region R_2 can be written as $R_2 = \{(x,y) \mid -2 \le y \le 0, 4y \le x \le y^3\}$.

Observe the below figure:



Chapter 15 Multiple Integrals 15.3 13E

Consider the double integral,

$$\iint_D x\,dA \quad \ldots\ldots (1)$$

Enclosed by the lines $y = x,\ \ y = 0,\ \text{and}\ \ x = 1$

Type I:

Therefore, the limits of integration are

$$\begin{cases} 0 \le y \le x \\ 0 \le x \le 1 \end{cases} \quad \ldots\ldots (2)$$

Rewrite the equation (1), by using the limits of integration in (2), to get:

$$\iint_D x\,dA = \int_0^1 \int_0^x x\,dy\,dx$$

To compute a type I double integral, it is first necessary to establish the order in which we integrate the variables.

In this case, starting with the inner-most differential and moving outwards, so first integrate our function, *x*, with respect to *y* from 0 to *x* and integrate the resulting function of *x* with respect to *x* from 0 to 1.

$$\begin{aligned}\iint_D x\,dA &= \int_0^1 xy\,dx\Big|_{y=0}^{y=x} \\ &= \int_0^1 \left(x^2 - 0x\right)dx \\ &= \int_0^1 x^2\,dx\end{aligned}$$

$$= \frac{x^3}{3}\Big|_0^1$$

$$= \frac{1^3}{3} - \frac{0^3}{3}$$

$$= \frac{1}{3}$$

Thus, the value of the double integral is $\iint_D x\,dA = \boxed{\frac{1}{3}}$.

Now,

Type II:

The region $D=\{(x,y):0\le y\le 1, y\le x\le 1\}$

Then, the double integral is,

$$\iint_D x\,dA=\int_0^1\int_y^1 x\,dx\,dy$$

$$=\int_0^1\left[\frac{x^2}{2}\right]_y^1 dy$$

$$=\frac{1}{2}\int_0^1\left(1-y^2\right)dy$$

$$=\frac{1}{2}\left[y-\frac{y^3}{3}\right]_0^1$$

$$=\frac{1}{2}\left(1-\frac{1}{3}\right)$$

$$=\frac{1}{2}\left(\frac{2}{3}\right)$$

$$=\frac{1}{3}$$

Thus, the value of the double integral is $\iint_D x\,dA=\boxed{\frac{1}{3}}$.

Therefore, in two types, the value of the double integral is,

$$\boxed{\iint_D x\,dA=\frac{1}{3}}.$$

Chapter 15 Multiple Integrals 15.3 14E

Consider the double integral,

$$\iint_D xy\,dA \quad \ldots\ldots (1)$$

Enclosed by the lines $y=x^2$ and $y=3x$.

To find the limits of integration, we must set the curves equal to one another, yielding:

$$x^2=3x$$

$$x^2-3x=0$$

$$x(3-x)=0$$

$$x=0,3$$

Type I:

The limits of integration are

$$\begin{cases} x^2 \le y \le 3x \\ 0 \le x \le 3 \end{cases} \quad \ldots\ldots (2)$$

Rewrite the equation (1), by using the limits of integration in (2), to get:

$$\iint_D xy\,dA = \int_0^3 \int_{x^2}^{3x} xy\,dy\,dx$$

To compute a type I double integral, it is first necessary to establish the order in which we integrate the variables.

In this case, starting with the inner-most differential and moving outwards, we see that we will first integrate our function, xy, with respect to *y* from *x*2 to 3*x* and integrate the resulting function of *x* with respect to *x* from 0 to 3.

$$\iint_D xy\,dA = \int_0^3 \frac{xy^2}{2}\,dx \Bigg|_{y=x^2}^{y=3x}$$

$$= \int_0^3 \left(\frac{x(3x)^2}{2} - \frac{x(x^2)^2}{2} \right) dx$$

$$= \int_0^3 \left(\frac{x(9x^2)}{2} - \frac{x(x^4)}{2} \right) dx$$

$$= \frac{1}{2} \int_0^3 (9x^3 - x^5)\,dx$$

$$= \frac{1}{2} \left(\frac{9x^4}{4} - \frac{x^6}{6} \right) \Bigg|_0^3$$

$$= \frac{1}{2} \left(\frac{9(3)^4}{4} - \frac{(3)^6}{6} \right) - \frac{1}{2} \left(\frac{9(0)^4}{4} - \frac{(0)^6}{6} \right)$$

$$= \frac{1}{2}(3)^6 \left[\frac{1}{4} - \frac{1}{6} \right]$$

$$= \frac{729}{2} \left(\frac{3-2}{12} \right)$$

$$= \frac{243}{8}$$

Thus, the value of the double integral is $\iint_D xy\,dA = \boxed{\frac{243}{8}}$.

Now, **Type II**:

The region is $D=\left\{(x,y):0\le y\le 9,\frac{1}{3}y\le x\le\sqrt{y}\right\}$

Then, the double integral is,

$$\iint_D xy\,dA=\int_0^9\int_{\frac{y}{3}}^{\sqrt{y}} xy\,dx\,dy$$

$$=\int_0^9 y\left[\frac{x^2}{2}\right]_{\frac{y}{3}}^{\sqrt{y}}dy$$

$$=\frac{1}{2}\int_0^9 y\left(y-\frac{1}{9}y^2\right)dy$$

$$=\frac{1}{2}\int_0^9\left(y^2-\frac{1}{9}y^3\right)dy$$

$$=\frac{1}{2}\left[\frac{y^3}{3}-\frac{1}{9}\cdot\frac{y^4}{4}\right]_0^9$$

$$=\frac{1}{2}\left[81(3)-\frac{9(81)}{4}\right]$$

$$=\frac{81}{2}\left(3-\frac{9}{4}\right)$$

$$=\frac{81}{2}\left(\frac{3}{4}\right)$$

$$=\frac{243}{8}$$

Thus, the value of the double integral is $\iint_D xy\,dA=\boxed{\frac{243}{8}}$.

Therefore, in two types, the value of the double integral is,

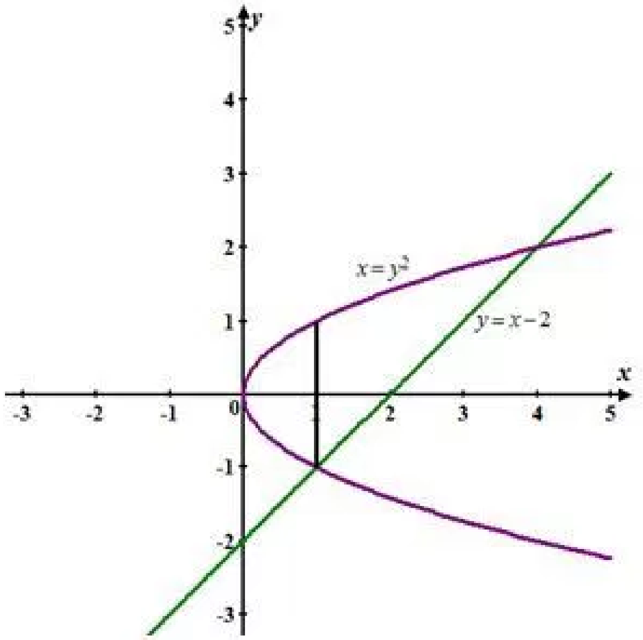
$$\boxed{\iint_D xy\,dA=\frac{243}{8}}.$$

Chapter 15 Multiple Integrals 15.3 15E

Consider the double integral $\iint_D y\,dA$, where D is bounded by the curves $y = x-2, x = y^2$.

The objective is to set up the integrals for both orders of the integration.

Let *I* be the indicated integral. In order to set up the integral integrating first on *y* it is helpful to graph the region of integration and include a vertical line in the middle.

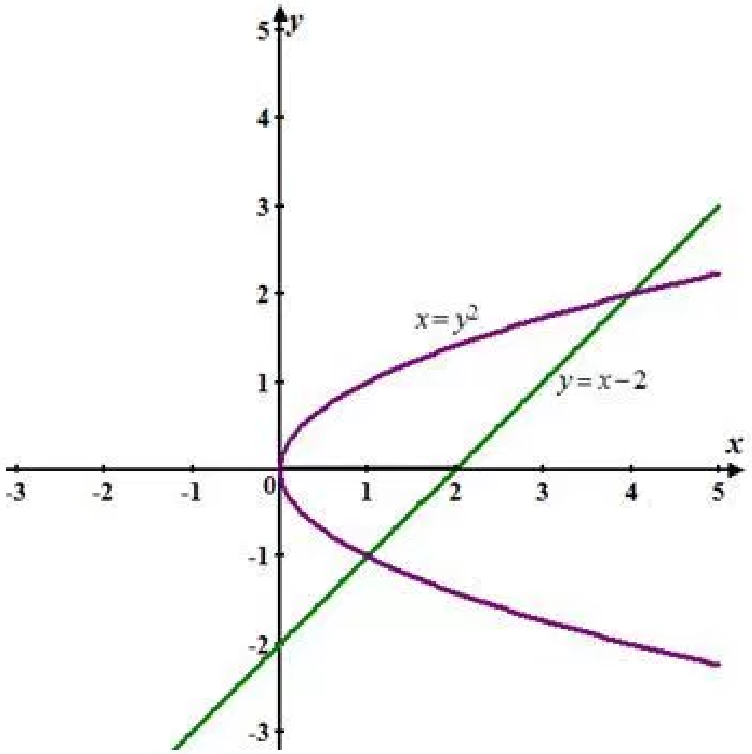


The left boundary is the curve $x = y^2$ and the right boundary is the line $y = x-2$. The vertex at the bottom is the point $(1,-1)$, the vertex at the upper right is $(4,2)$, and the left-most point is $(0,0)$. A vertical line left of $x=1$ will have both ends on the curve $x = y^2$ while a vertical line to the right of $x=1$ will have one end on the line and one end on the curve.

The region to the left can be described as $-\sqrt{x} \le y \le \sqrt{x}$ and $0 \le x \le 1$. The region to the right can be described as $x-2 \le y \le \sqrt{x}$ and $1 \le x \le 4$. Use this information to set up the integral.

$$I = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} y\,dy\,dx + \int_1^4 \int_{x-2}^{\sqrt{x}} y\,dy\,dx$$

In order to set up the integral that integrates first on *y* it is helpful to graph the region of integration and include a horizontal line in the middle.



Every horizontal line will have one end on the curve $x = y^2$ and the other end on the line $y = x - 2$. Therefore the range of the variables can be described using the inequalities $y^2 \le x \le y + 2$ and $-1 \le y \le 1$. Use this information to set up the double integral.

$$I = \int_{-1}^{2} \int_{y^2}^{y+2} y \, dx \, dy$$

Neither integrand is going to present much trouble so integrate on *x* first in order to reduce the amount of arithmetic.

$$\begin{aligned} I &= \int_{-1}^{2} \int_{y^2}^{y+2} y \, dx \, dy \\ &= \int_{-1}^{2} (xy)_{y^2}^{y+2} \, dy \\ &= \int_{-1}^{2} y\left(y + 2 - y^2\right) dy \\ &= \int_{-1}^{2} \left(-y^3 + y^2 + 2y\right) dy \end{aligned}$$

Continuation of the above step:

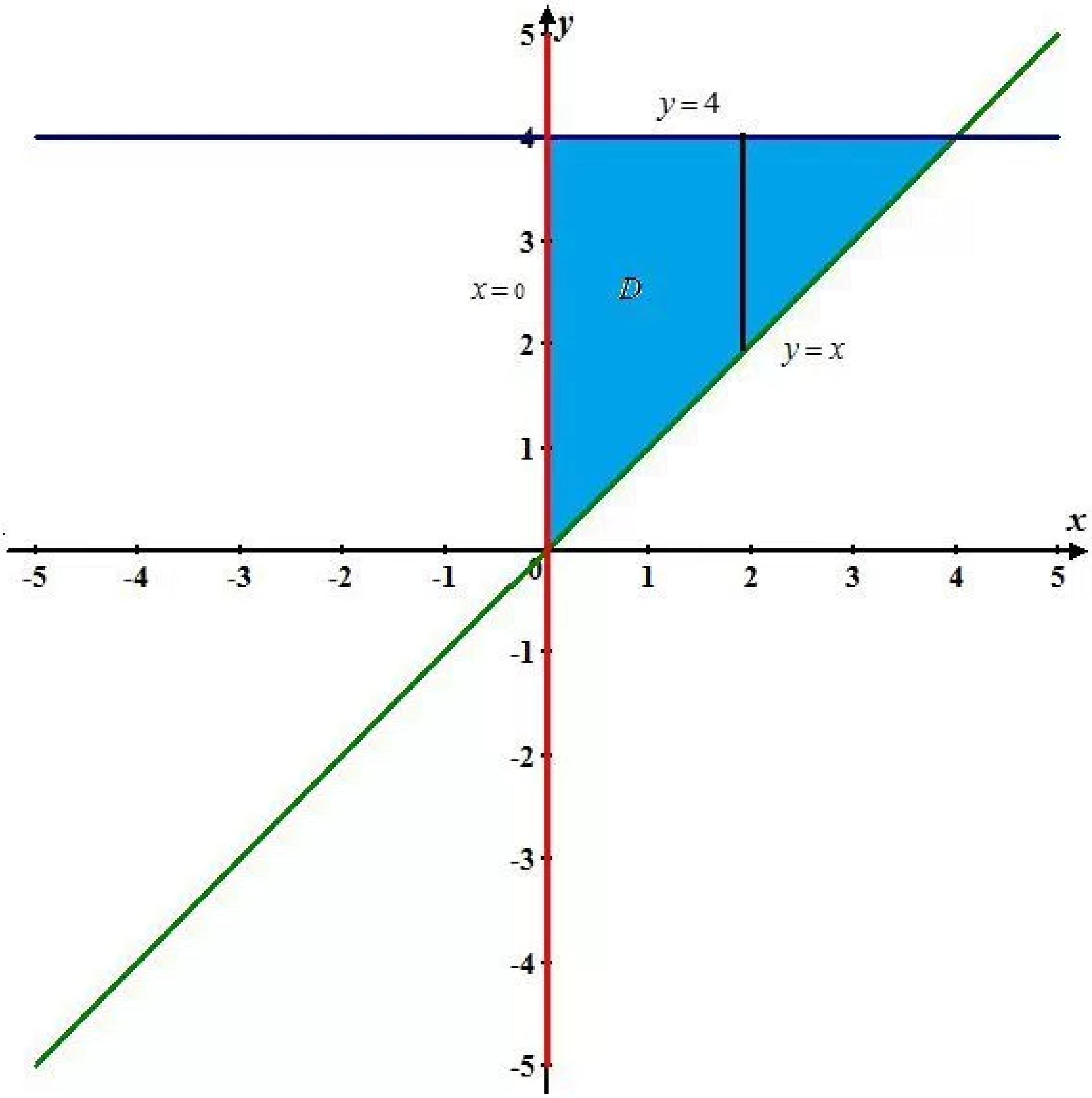
$$I = \left(-\frac{1}{4}y^4 + \frac{1}{3}y^3 + y^2\right)_{-1}^{2}$$
$$= \left(\frac{-16}{4} + \frac{8}{3} + 4\right) - \left(\frac{-1}{4} - \frac{1}{3} + 1\right)$$
$$= \frac{9}{4}$$

Thus, the value of the double integral is $\iint_D y\,dA = \boxed{\frac{9}{4}}$.

Chapter 15 Multiple Integrals 15.3 16E

Let us consider $\iint_D y^2 e^{xy}\,dA, D$ is bounded by $y = x, y = 4, x = 0$

It is easier to set up a double integral after graphing the region. To integrate on *y* first it is helpful to include a vertical line within the region. The graph is shown below.

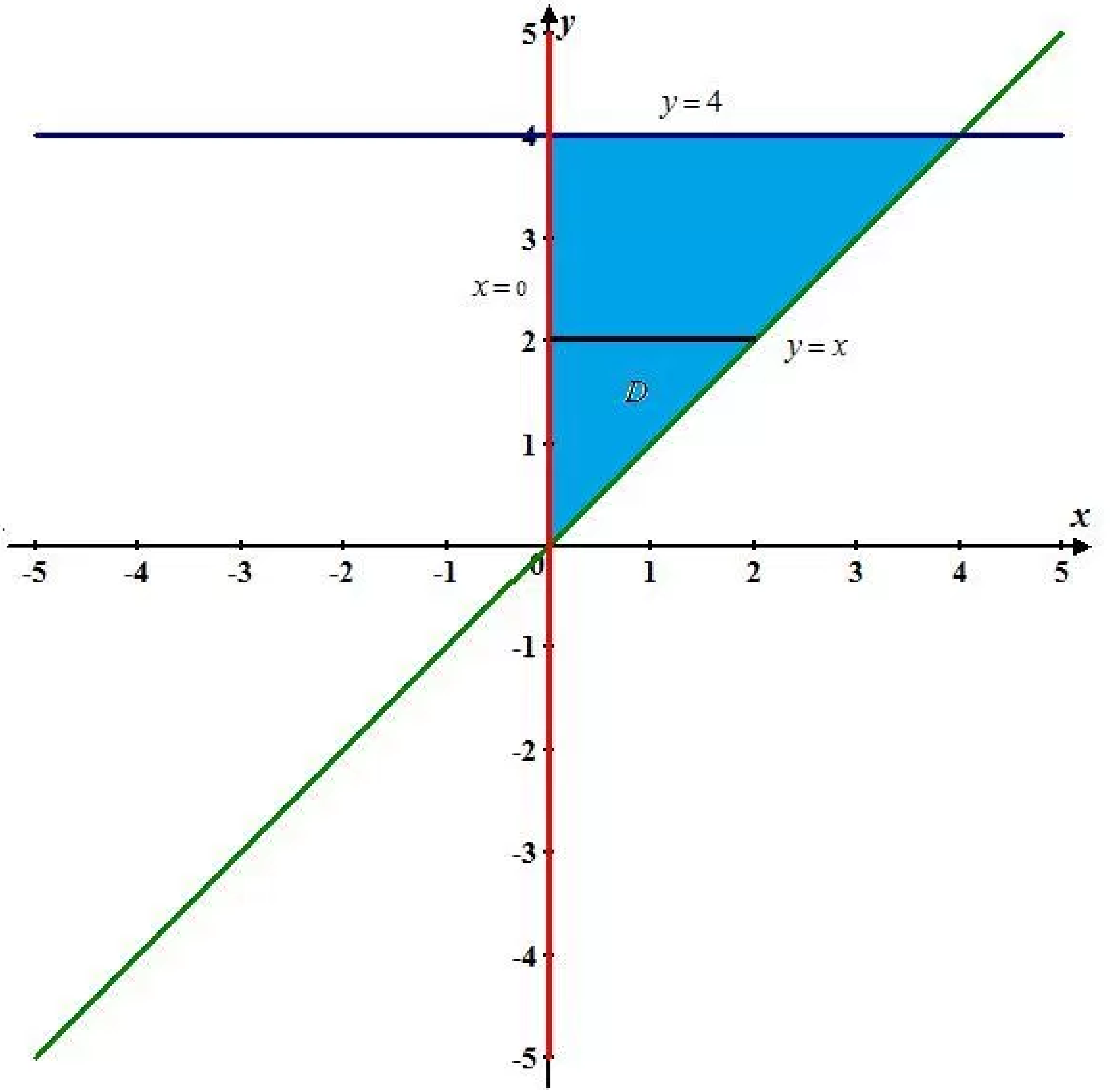


Note that *y* runs from the line $y = x$ up to the line $y = 4$. The line could slide left as far as $x = 0$ and to the right as far as $x = 4$.

Using *I* to denote the desired integral it can be set up as:

$$I = \int_0^4 \int_x^4 y^2 e^{xy}\, dy dx$$

For guidance in setting up the integral on *x* first it's helpful to have a horizontal line inside the region of integration.



This graph illustrates that *x* ranges along the horizontal line from 0 to *y*. The horizontal line can slide as low as $y = 0$ and as high as $y = 4$.

Therefore, the integral can be set up as:

$$I = \int_0^4 \int_0^y y^2 e^{xy}\, dx dy$$

The former integral requires integration by parts (applied twice) to deal with $\int y^2 e^{xy} dy$ while the latter integral can be handled easily. Integrate on *x* first. The first step is to factor out the y^2 which is acting like a constant during the inner integration.

$$I_2 = \int_0^4 y^2 \int_0^y e^{xy} dx dy$$

$$= \int_0^4 y^2 \left(\frac{1}{y} e^{xy} \Big|_0^y \right) dy$$

$$= \int_0^4 y^2 \left(\frac{1}{y} e^{y^2} - \frac{1}{y} \right) dy$$

$$= \int_0^4 y e^{y^2} - y dy$$

$$I_2 = \int_0^4 \left(y e^{y^2} - y \right) dy$$

$$= \frac{1}{2} \int_0^4 \left(2y e^{y^2} \right) dy - \int_0^4 (y) dy$$

Let $u = y^2$ then $du = 2y dy$

So,

$$I_2 = \frac{1}{2} \int_0^4 \left(2y e^{y^2} \right) dy - \int_0^4 (y) dy$$

$$= \frac{1}{2} \int e^u du - \int_0^4 (y) dy$$

$$= \frac{1}{2} \left(e^u \right) - \left(\frac{y^2}{2} \right)_0^4$$

$$= \frac{1}{2} \left(e^{y^2} \right)_0^4 - \left(\frac{16}{2} \right)$$

$$= \frac{1}{2} \left(e^{16} - 1 \right) - 8$$

$$= \frac{1}{2} e^{16} - \frac{17}{2}$$

$$= \frac{1}{2} \left(e^{16} - 17 \right)$$

Therefore, the value of the integration is $\boxed{\frac{1}{2} \left(e^{16} - 17 \right)}$.

Chapter 15 Multiple Integrals 15.3 17E

Consider the integral,

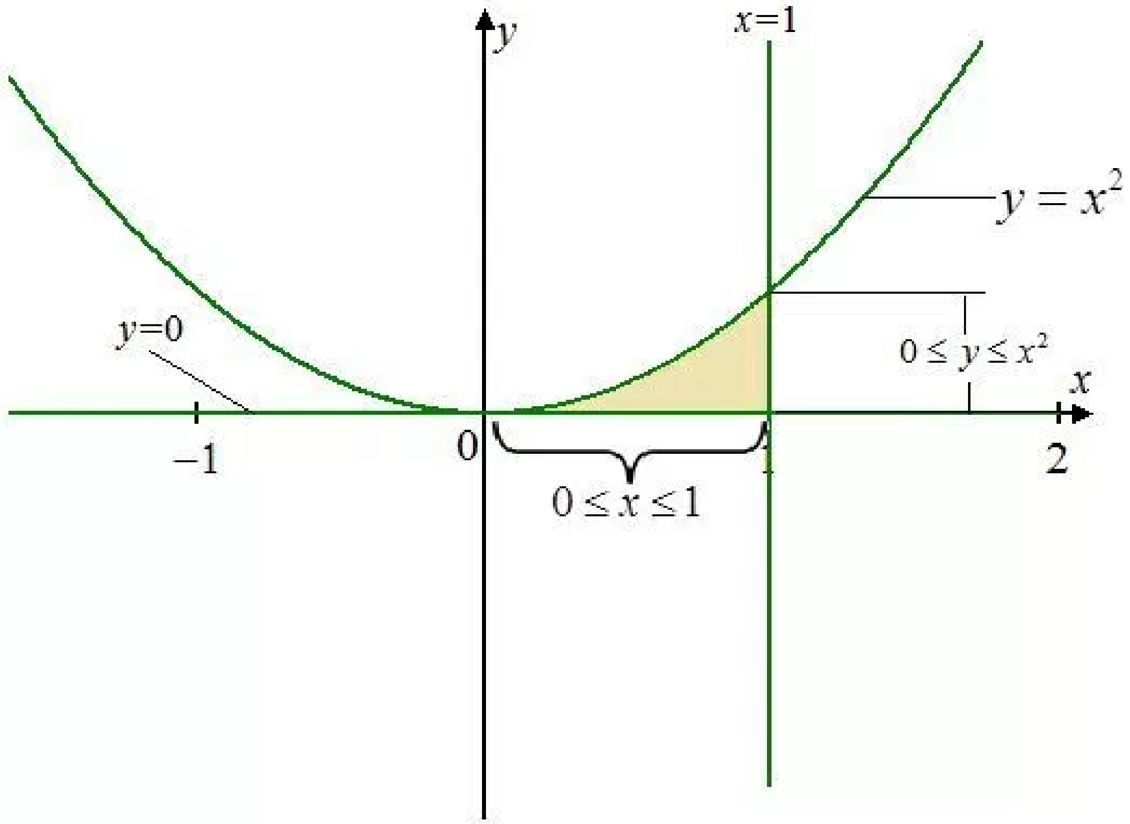
$$\iint_D x\cos y \, dA \quad \ldots\ldots (1)$$

D is bounded by $y = 0$, $y = x^2$ and $x = 1$

When $y = 0 \Rightarrow x^2 = 0$

$\Rightarrow x = 0$

Sketch the region bounded by $y = 0$, $y = x^2$ and $x = 1$



Note that the region D is a type I region of the form $D=\{(x,y)\,|\,a\le x\le b, g_1(x)\le y\le g_2(y)\}$

The region D is shown in the graph very clear. So, the region bounded by the curves is

$D=\{(x,y):0\le x\le 1, 0\le y\le x^2\}$

As this represents $xy-$ plane, $dA=dydx$

Use this limits to find the integral over D.

From (1)

$$\iint_D x\cos y\ dA=\int_0^1\int_0^{x^2} x\cos y\ dy\,dx$$

$=\int_0^1 x.\sin y\,\Big]_0^{x^2}\,dx$ Use $\int \cos x dx=\sin x+C$

$=\int_0^1 x\sin x^2 dx$ Apply the limits of y ...... (2)

To evaluate (2) use substitution.

Substitute $x^2=u$

Differentiate on each side

$2x\,dx=du$

$x\,dx=\frac{du}{2}$ Divide by 2

Change the limits of integration

When $x=0$ then $u=0$

And when $x=1$ then $u=1$

So,

$$\int_0^1 x\sin x^2\ dx=\int_0^1 \sin u\ \frac{du}{2}$$

$=\frac{1}{2}\int_0^1 \sin u\ \frac{du}{2}$

$=\frac{1}{2}(-\cos u)\Big]_0^1$ Use $\int \sin x dx=-\cos x+C$

$=\frac{1}{2}(-\cos 1+\cos 0)$ Apply the limits of integration

$=\frac{1}{2}(-\cos 1+1)$ Use $\cos 0=1$

$=\frac{1}{2}(1-\cos 1)$

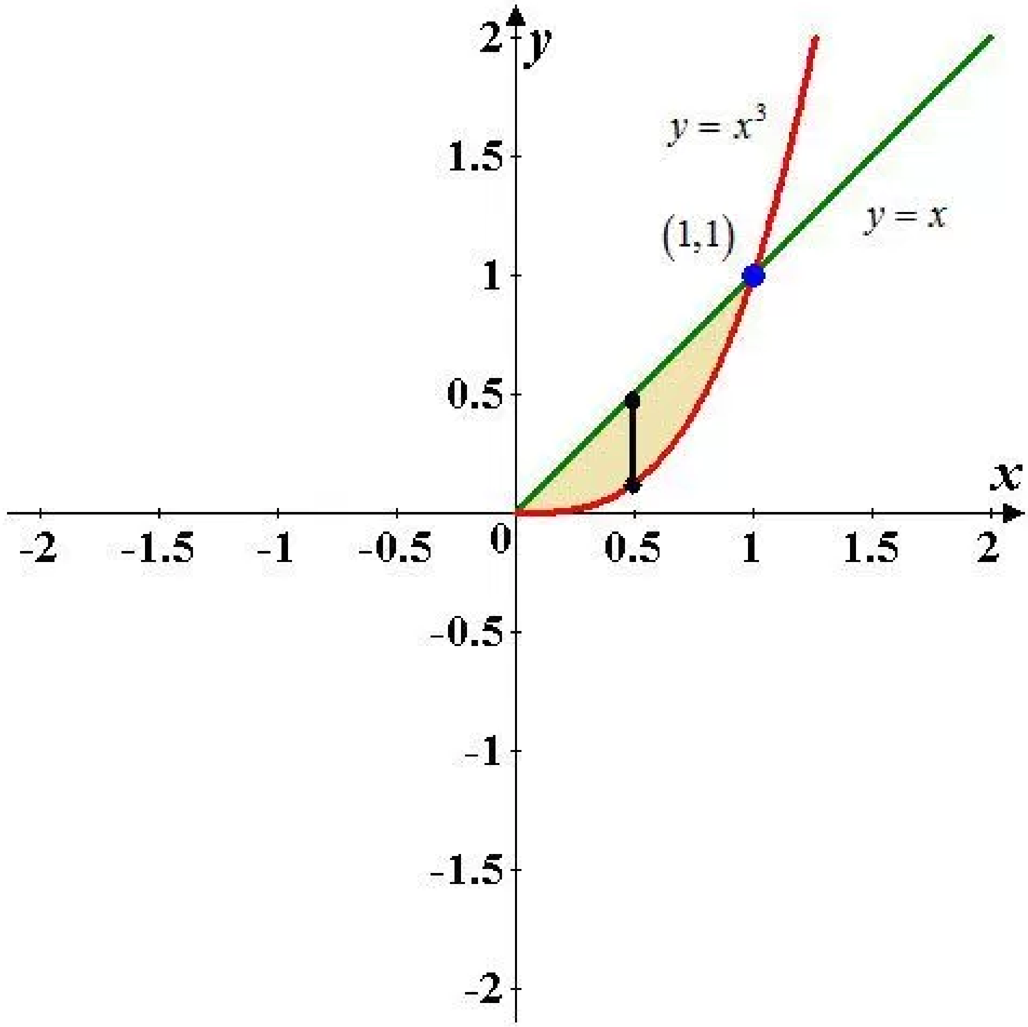
Thus, the value of the integral (1) is $\boxed{\frac{1}{2}(1-\cos 1)}$.

Chapter 15 Multiple Integrals 15.3 18E

Let us consider $\iint_D (x^2+2y)dA$, D is bounded by $y=x, y=x^3, x\geq 0$

Now evaluate the double integral.

The integrand does not favor one order of integration over the other. The region of integration is both type 1 and type 2. Therefore integrate first on *y* in order to keep whole number exponents. To set up the integral in the order *y* first then *x* it is helpful to first graph the region of integration and include a vertical line in the middle.



Find the Point of intersection of the two curves

Equating two curves $y=x^3$ in $y=x$

$$x^3=x$$
$$x^3-x=0$$
$$x(x^2-1)=0$$
$$x=0,1,-1$$

But here given $x\geq 0$

Then the points of intersection of curves are $(0,0),(1,1)$.

Region D can be expressed as $D=\{(x,y)\,|\,0\le x\le 1, x^3\le y\le x\}$ which is of type-I.

If $f(x,y)$ is continuous on type-I region D such that

$D=\{(x,y)\,|\,a\le x\le b, g_1(x)\le y\le g_2(x)\}$ then

$$\iint_D f(x,y)\,dA=\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Now

$$\iint_D (x^2+2y)\,dA$$

$$=\int_0^1\int_{x^3}^{x}(x^2+2y)\,dy\,dx$$

$$=\int_0^1\left[x^2y+y^2\right]_{y=x^3}^{x}dx \qquad \left(\text{since } \int y^n dy=\frac{y^{n+1}}{n+1}\right)$$

$$=\int_0^1\left(x^3+x^2-x^5-x^6\right)dx$$

Continue the above

$$=\left[\frac{1}{4}x^4+\frac{1}{3}x^3-\frac{x^6}{6}-\frac{x^7}{7}\right]_0^1 \qquad \left(\text{since } \int x^n dx=\frac{x^{n+1}}{n+1}\right)$$

$$=\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{7}$$

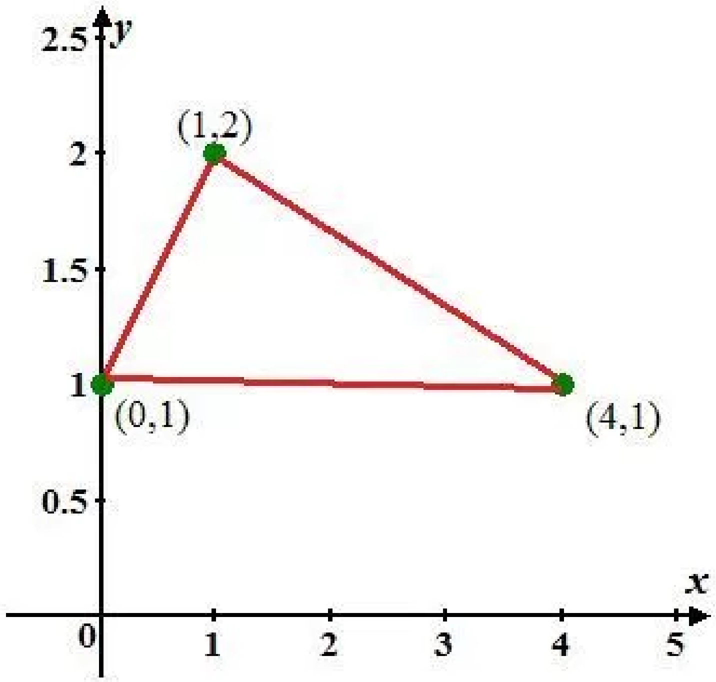
$$=\frac{23}{84}$$

Therefore, $\iint_D (x^2+2y)\,dA=\frac{23}{84}$

Chapter 15 Multiple Integrals 15.3 19E

The objective is to evaluate the double integral $\iint_D y^2\, dA$ here D is the triangular region $(0,1),(1,2),(4,1)$.

The region of the integration D is:



Determine the equations of the line passing through points $(0,1)$ and $(1,2)$.

$$y-1=\frac{2-1}{1-0}(x-0)$$
$$y-1=1\cdot x$$
$$y-1=x$$

The equations of the lines passing through points $(1,2)$ and $(4,1)$.

$$y-2=\frac{1-2}{4-1}(x-1)$$
$$y-2=-\frac{1}{3}(x-1)$$
$$-3y+6=x-1$$
$$x=7-3y$$

The range of y in the region is from 1 to 2.

Thus, the region of integration is $D=\{(x,y)\mid y-1\le x\le 7-3y, 1\le y\le 2\}$.

The integral is:

$$\iint_D y^2 dA = \int_1^2\int_{y-1}^{7-3y} y^2\,dxdy$$
$$= \int_1^2\int_{y-1}^{7-3y} y^2\,dxdy$$
$$= \int_1^2 [x]_{y-1}^{7-3y}\,y^2\,dy$$
$$= \int_1^2 [7-3y-y+1]\,y^2\,dy$$
$$= \int_1^2 y^2(8-4y)dy$$
$$= \int_1^2 (8y^2-4y^3)dy$$

Integrate with respect to y and apply limits from 1 to 2.

$$\iint_D y^2\,dA = \left(\frac{8y^3}{3}-\frac{4y^4}{4}\right)_1^2$$
$$= \left(\frac{8y^3}{3}-y^4\right)_1^2$$
$$= 1-\frac{8}{3}-16+\frac{64}{3}$$
$$= \frac{11}{3}$$

Therefore, the value of the integral $\iint_D y^2\,dA$ is $\boxed{\frac{11}{3}}$.

Chapter 15 Multiple Integrals 15.3 20E

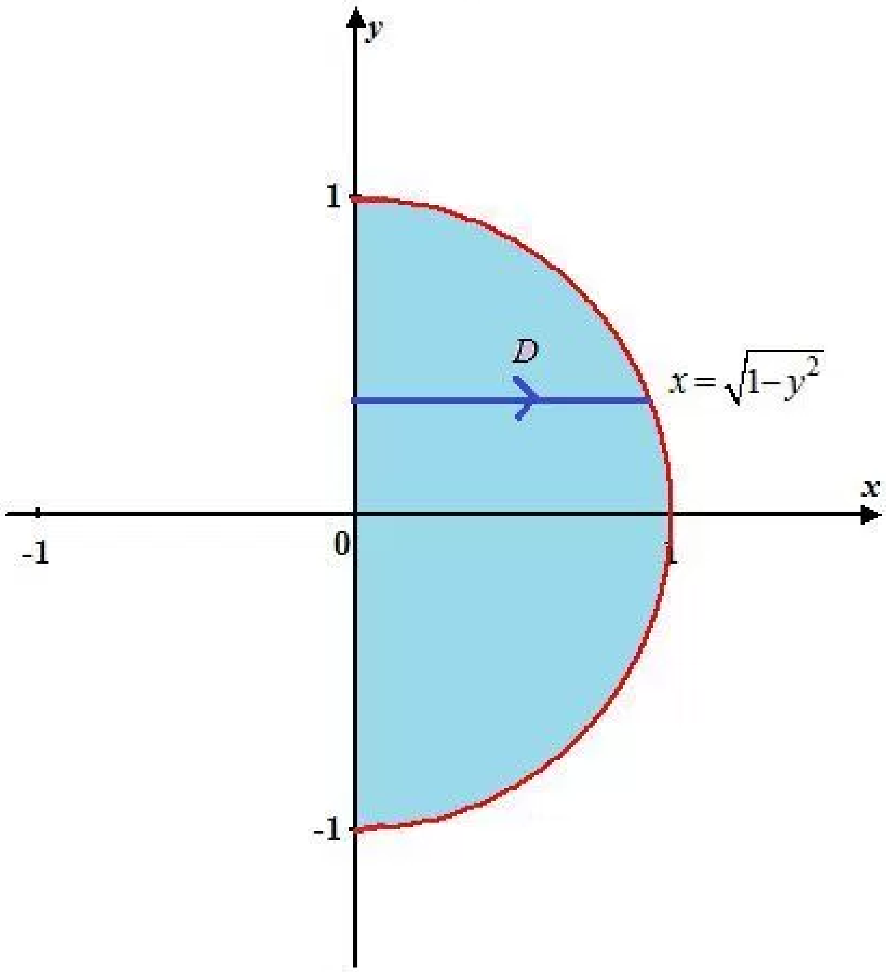
Consider the double integral,

$$\iint_D xy^2 dA$$

Where D is enclosed by $x=0$ and $x=\sqrt{1-y^2}$.

The objective is to evaluate the given double integral.

Sketch the region of integration as follows:



Find the region of the integration as follows:

From the curves $x=0$ and $x=\sqrt{1-y^2}$.

$$1-y^2=0$$
$$y^2=1$$
$$y=\pm 1$$

Thus, the region of the integration is,

$$D=\left\{(x,y)\,|\,0\le x\le \sqrt{1-y^2},-1\le y\le 1\right\}$$

Substitute these values in integral $\iint_D xy^2 dA$.

$$\iint_D xy^2 dA = \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} xy^2 dx\, dy$$
$$= \int_{-1}^{1} y^2 \left[\int_{0}^{\sqrt{1-y^2}} x\, dx\right] dy$$
$$= \int_{-1}^{1} y^2 \left[\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} dy$$
$$= \int_{0}^{1} y^2 \left(\frac{1-y^2}{2}\right) dy$$

Continue the above step,

$$\iint_D xy^2 dA = \frac{1}{2}\int_{-1}^{1}\left(y^2 - y^4\right) dy$$
$$= \frac{2}{2}\int_{0}^{1}\left(y^2 - y^4\right) dy \qquad \text{Since } f(y) = y^2 - y^4 \text{ is an even function}$$
$$= \left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1$$
$$= \left[\frac{1}{3} - \frac{1}{5}\right]$$
$$= \left[\frac{5-3}{15}\right]$$
$$= \boxed{\frac{2}{15}}$$

Thus, the required value is $\iint_D xy^2 dA = \boxed{\frac{2}{15}}$.

Chapter 15 Multiple Integrals 15.3 21E

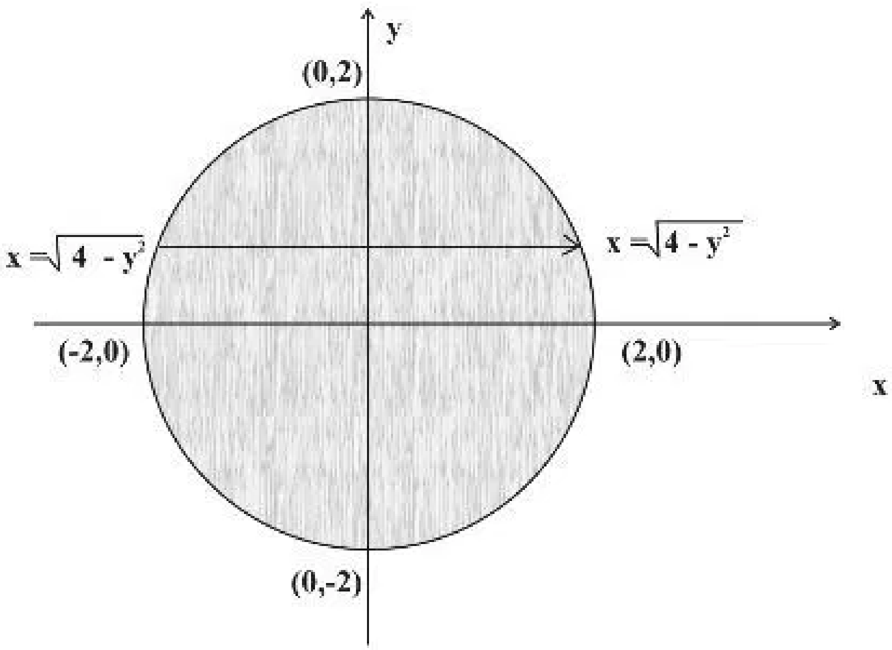
$\iint_D (2x - y) dA$, Where D is bounded by the circle centered at origin with radius 2,

i.e.

$$x^2 + y^2 = 4$$
$$\Rightarrow x^2 = 4 - y^2$$

Now $D = \left\{(x, y):\ -\sqrt{4-y^2} \le x \le \sqrt{4-y^2},\ \ -2 \le y \le 2\right\}$

Then



$$\iint_D (2x - y)\,dA = \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{+\sqrt{4-y^2}} (2x - y)\,dx\,dy$$

$$= \int_{-2}^{2} \left(x^2 - xy\right)_{x=-\sqrt{4-y^2}}^{x=+\sqrt{4-y^2}} dy$$

$$= \int_{-2}^{2} \left[\left(4 - y^2\right) - y\sqrt{4-y^2} - \left(4 - y^2\right) - y\sqrt{4-y^2}\right] dy$$

$$= \int_{-2}^{2} -2y\sqrt{4-y^2}\,dy$$

$$= \frac{2}{3}\left[\left(4 - y^2\right)^{3/2}\right]_{-2}^{2}$$

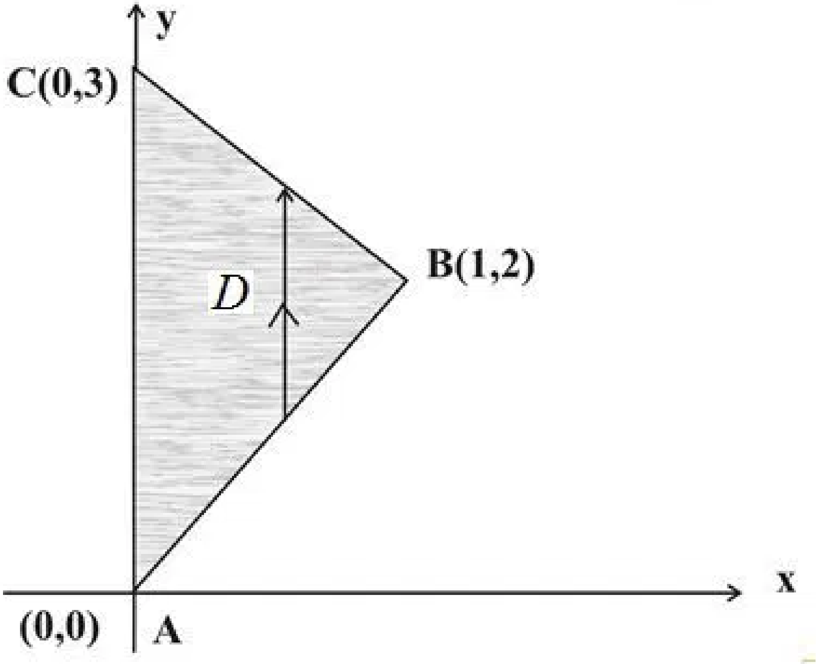
$$= \frac{2}{3}\left[\left(4-4\right)^{3/2} - \left(4-4\right)^{3/2}\right]$$

$$= \boxed{0}$$

Chapter 15 Multiple Integrals 15.3 22E

To evaluate $\iint_D 2xy\,dA$, where D is the triangular region with vertices $(0,0),(1,2),(0,3)$:

The sketch of the triangular region D is shown below:



Equation of side AB is,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
$$y - 0 = \frac{2-0}{1-0}(x - 0)$$
$$y = 2x$$

Equation of side *BC* is,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
$$y - 2 = \frac{3-2}{0-1}(x-1)$$
$$y - 2 = -x + 1$$
$$y = 3 - x$$

Then from the equations of *AB* and *BC* is $D = \{(x, y);\ 2x \le y \le 3 - x,\ 0 \le x \le 1\}$

So we can evaluate the integral as,

$$\iint_D 2xy\ dA = \int_0^1 \int_{y=2x}^{3-x} 2xy\ dydx$$
$$= \int_0^1 x\left(\int_{y=2x}^{3-x} 2y dy\right) dx$$
$$= \int_0^1 x\left(2 \cdot \frac{y^2}{2}\right)_{y=2x}^{3-x} dx \qquad \text{Use } \int y^n dy = \frac{y^{n+1}}{n+1}, n \ne -1$$
$$= \int_0^1 \left(xy^2\right)_{y=2x}^{y=3-x} dx$$
$$= \int_0^1 x\left[(3-x)^2 - (2x)^2\right] dx$$
$$= \int_0^1 x\left[9 + x^2 - 6x - 4x^2\right] dx$$

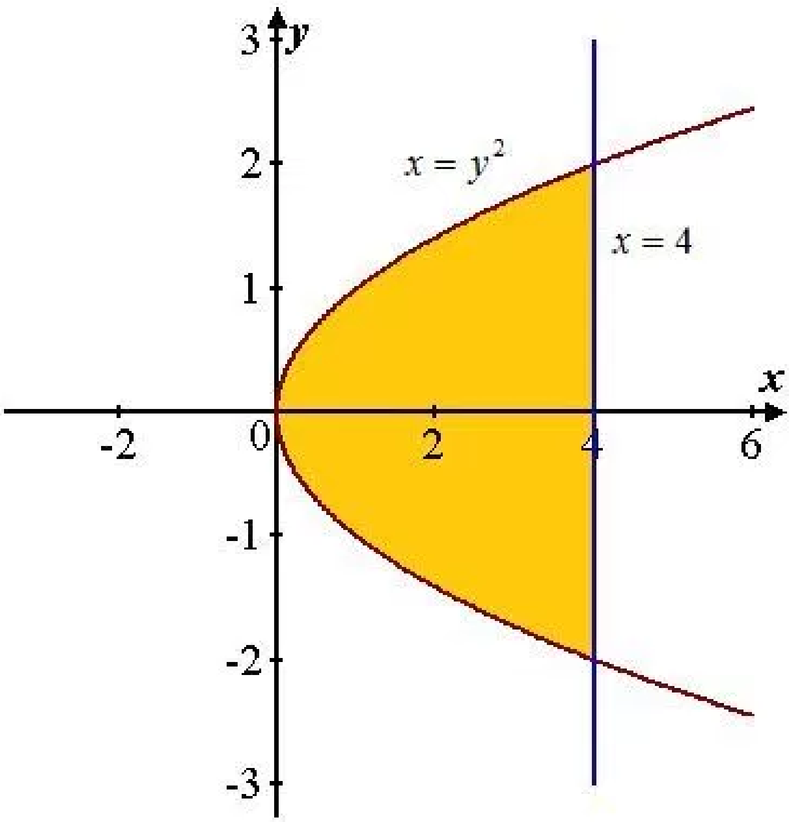
Continue this,

$$\iint_D 2xy\ dA = \int_0^1 x\left[-3x^2 - 6x + 9\right] dx$$
$$= \int_0^1 \left[-3x^3 - 6x^2 + 9x\right] dx$$
$$= \left[\frac{-3}{4}x^4 - 2x^3 + \frac{9}{2}x^2\right]_0^1$$
$$= \frac{-3}{4} - 2 + \frac{9}{2} - 0$$
$$= \boxed{\frac{7}{4}}$$

Chapter 15 Multiple Integrals 15.3 24E

Consider the plane equation $z = 1 + x^2 y^2$

The region enclosed by the parabola $x = y^2$ and $x = 4$ which is shown below.



The objective is to find the volume of the solid.

The plane is, $z = 1 + x^2 y^2$

And

$x = y^2,\ x = 4$

Thus, $y^2 \le x \le 4 \text{ and } -2 \le y \le 2$

Hence the region $D = \{(x, y) | y^2 \le x \le 4,\ -2 \le y \le 2\}$

Now the integral becomes,

$$V = \iint_D (z)\,dA$$

$$= \int_{-2}^{2}\int_{y^2}^{4}\left(1 + x^2y^2\right)dxdy$$

$$= \int_{-2}^{2}\left[x + \frac{x^3}{3}y^2\right]_{y^2}^{4} dy$$

$$= \int_{-2}^{2}\left(\left(4 - y^2\right) - \frac{y^8}{3} + \frac{64y^2}{3}\right)dy$$

$$= \left(4y + \frac{61}{9}y^3 - \frac{y^9}{27}\right)_{-2}^{2}$$

$$= \left(4(2) + \frac{61}{9}(2)^3 - \frac{(2)^9}{27}\right) - \left(4(-2) + \frac{61}{9}(-2)^3 - \frac{(-2)^9}{27}\right)$$

$$= 2\left(4(2) + \frac{61}{9}(2)^3 - \frac{(2)^9}{27}\right)$$

$$= 2\left(8 + \frac{488}{9} - \frac{512}{27}\right)$$

$$= 2\left(\frac{216 + 1464 - 512}{27}\right)$$

$$= \frac{2336}{27}$$

Therefore, the volume of the solid is $\boxed{V = \frac{2336}{27}}$.

Chapter 15 Multiple Integrals 15.3 25E

Consider the surface $z = xy$ ...... (1)

And triangular region with vertices $(1,1),(4,1)$ and $(1,2)$

Find a region *D* such that volume the surface lies under *z* and above the triangular region.

To find the region find the equation of the sides of the triangle with the vertices $(1,1),(4,1)$ and $(1,2)$.

Find the equation of the sides by finding the line joining two points.

The equation of the line joining $(1,1)$ and $(4,1)$ is

$$y - y_0 = m(x - x_0)$$

$$y - 1 = \frac{1-1}{1-4}(x-1)$$

$$y - 1 = 0(x-1)$$

$$y = 1$$

The equation of the line joining $(1,1)$ and $(1,2)$ is

$$y - 1 = \frac{2-1}{1-1}(x-1)$$

$$y - 1 = \frac{1}{0}(x-1)$$

$$x - 1 = 0$$

$$x = 1$$

The equation of the line joining $(4,1)$ and $(1,2)$ is

$$y - 1 = \frac{2-1}{1-4}(x-4)$$

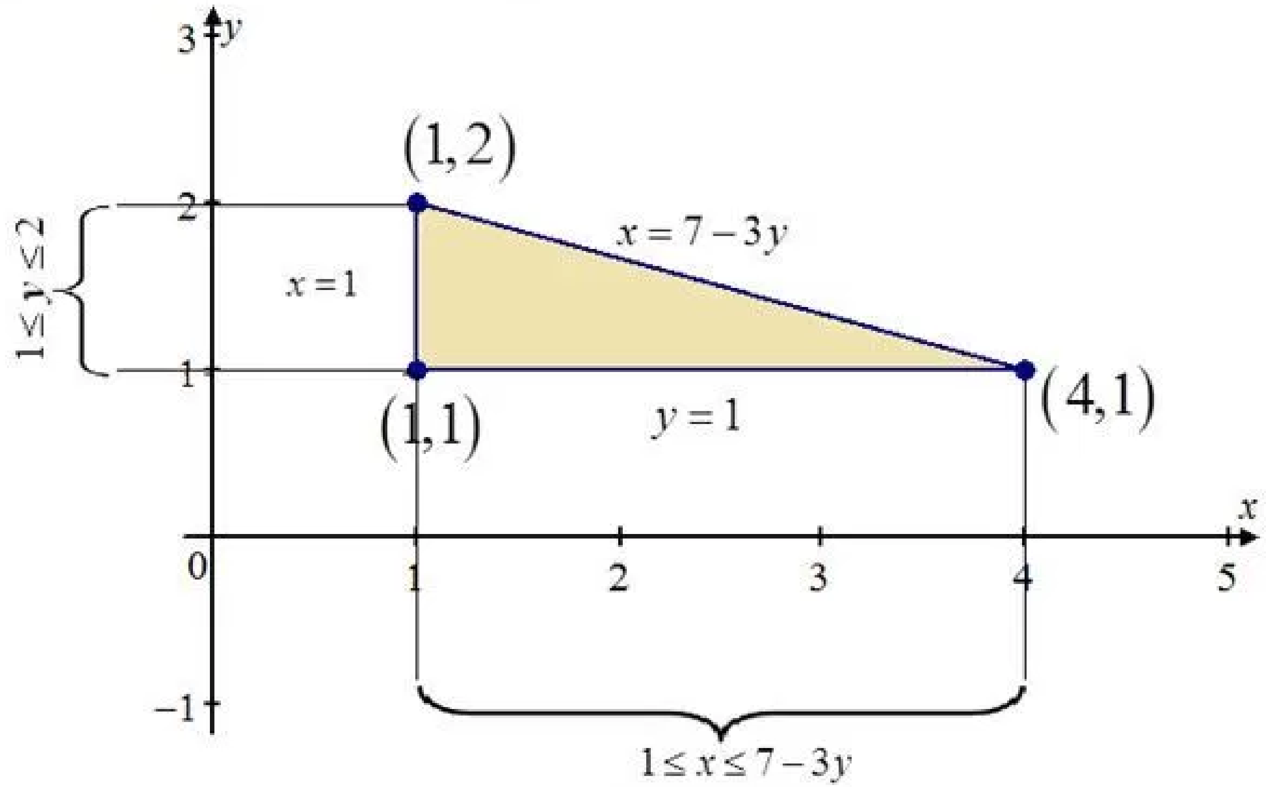
$$y - 1 = -\frac{1}{3}(x-4)$$

$-3(y-1) = x - 4$ Multiply by -3 on each side

$$-3y + 3 + 4 = x$$

$$x = 7 - 3y$$

Sketch the region bounded by this points,



From the figure, it is clear that the area of the region bounded by these vertices is of type I and is given by

$$D = \{(x, y): 1 \le y \le 2, 1 \le x \le 7 - 3y\} \quad \text{...... (2)}$$

Find the volume of the surface xy above D.

$$\iint_D xy \, dA = \int_1^2 \int_1^{7-3y} xy \, dx \, dy$$

$$= \int_1^2 \left[y \frac{x^2}{2} \right]_{x=1}^{x=7-3y} dy$$ Apply the integration with respect to x

$$= \int_1^2 \frac{y}{2} \left[(7-3y)^2 - 1^2 \right] dy$$ Apply the limits of x

$$= \frac{1}{2} \int_1^2 y \left(48 - 42y + 9y^2 \right) dy$$

$$= \frac{1}{2} \int_1^2 \left(48y - 42y^2 + 9y^3 \right) dy$$

$$= \frac{1}{2} \left[\int_1^2 48y dy - \int_1^2 42y^2 dy + \int_1^2 9y^3 dy \right]$$

$$= \frac{1}{2} \left[49 \left[\frac{y^2}{2} \right]_1^2 - 42 \left[\frac{y^3}{3} \right]_1^2 + 9 \left[\frac{y^4}{4} \right]_1^2 \right]$$ Use $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$= \frac{1}{2} \left[48 \left[\frac{2^2}{2} - \frac{1^2}{2} \right] - 42 \left[\frac{2^3}{3} - \frac{1^3}{3} \right] + 9 \left[\frac{2^4}{4} - \frac{1^4}{4} \right] \right]$$ Apply the limits

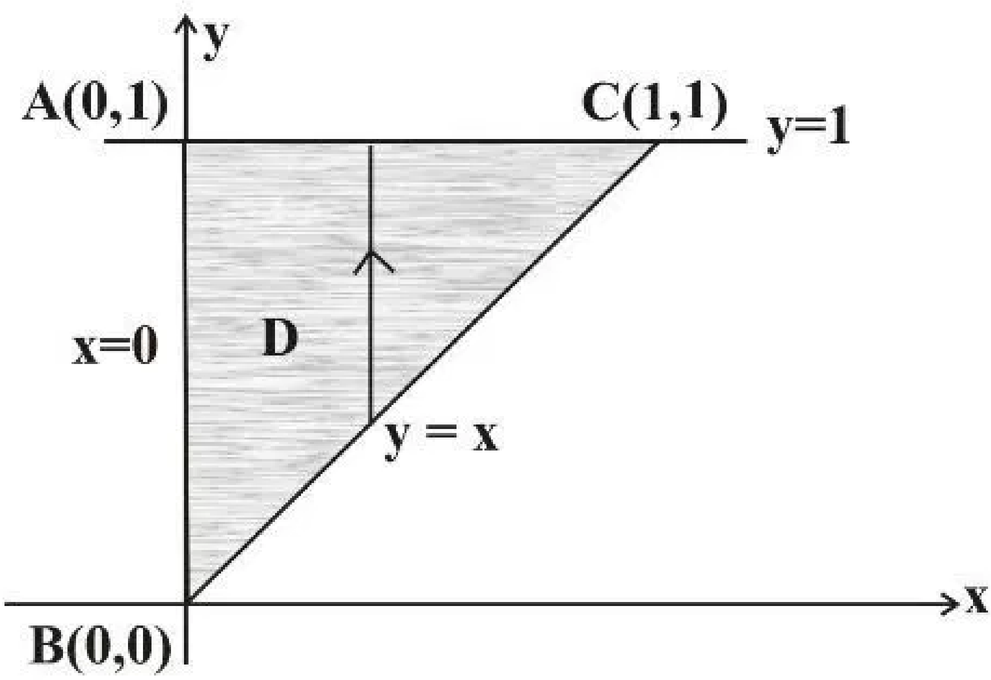
$$= \frac{1}{2} \left[48 \left(\frac{3}{2} \right) - 42 \left(\frac{7}{3} \right) + 9 \left(\frac{15}{4} \right) \right]$$

$$= \frac{31}{8}$$

Thus, the volume of the region under the surface (1) and above the region (2) is $\boxed{\frac{31}{8}}$.

Chapter 15 Multiple Integrals 15.3 26E

The solid is bounded by planes $x=0, y=1, y=x, z=0$ and parabolic $z=x^2+3y^2$



The region of integration is given by

$D=\{(x, y): 0 \le x \le 1,\ x \le y \le 1\}$

Then the required volume of the solid is

$$\begin{aligned}
v &= \iint_D \left(x^2+3y^2\right) dA \\
&= \int_0^1 \int_x^1 \left(x^2+3y^2\right) dy\ dx \\
&= \int_0^1 \left[x^2y+y^3\right]_{y=x}^{y=1} dx \\
&= \int_0^1 \left(x^2+1-2x^3\right) dx \\
&= \left[\frac{1}{3}x^3+x-\frac{1}{2}x^4\right]_0^1 \\
&= \frac{1}{3}+1-\frac{1}{2}-0 \\
&= \boxed{\frac{5}{6}}
\end{aligned}$$

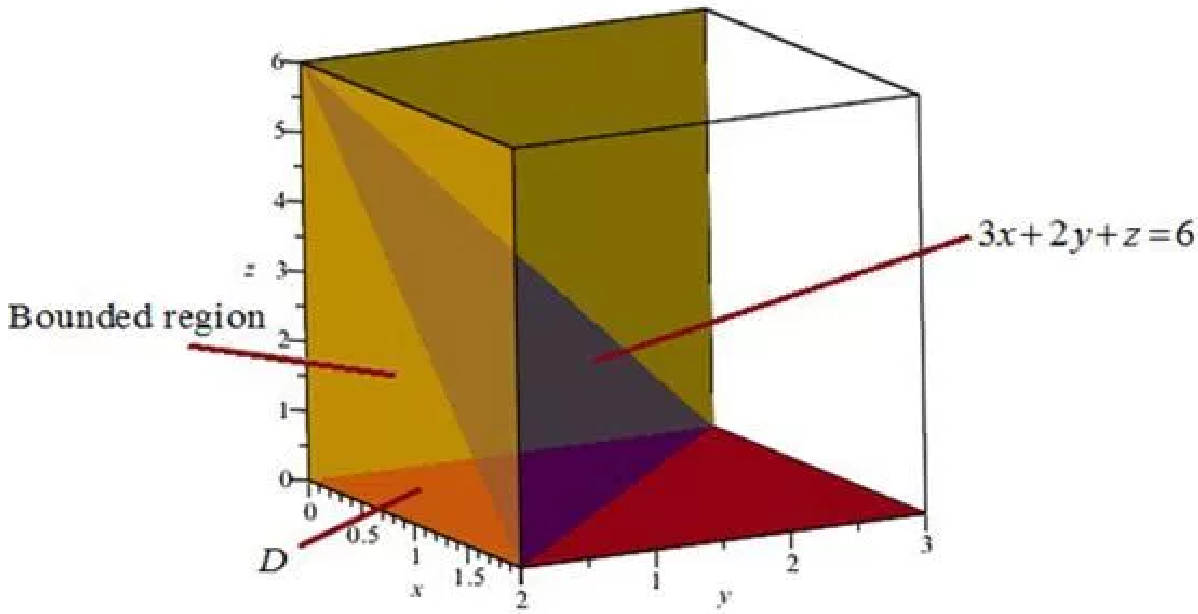
Chapter 15 Multiple Integrals 15.3 27E

Consider the equation of the plane,

$$3x+2y+z=6.$$

The objective to find the volume of the solid bounded by the coordinate planes $x=0, y=0, z=0$ and the given plane $3x+2y+z=6.$

The graph of the solid is shown in the below figure:



Find the region of the integration as follows:

The required solid is lies under the graph of the function $z=6-3x-2y$ and above the $xy-$ plane.

When $z=0$:

$$z=6-3x-2y$$
$$0=6-3x-2y$$
$$y=\frac{6-3x}{2}$$

So, y is varies from 0 to $\frac{6-3x}{2}$.

When $z=0, y=0$:

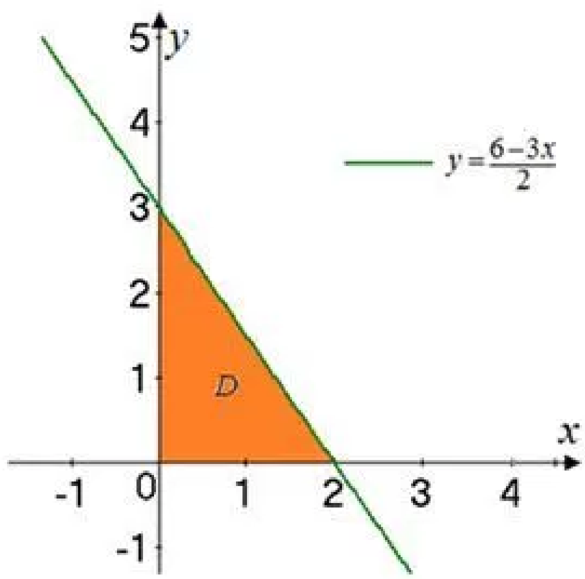
$$z=6-3x-2y$$
$$0=6-3x-2(0)$$
$$x=2$$

So, x varies from 0 to 2.

Thus, the required solid is lies under the graph of the function $z=6-3x-2y$ and above

$$D=\left\{(x,y)\,|\,0\le y\le\frac{6-3x}{2},0\le x\le 2\right\}.$$

The region D in $xy-\text{plane}$ as shown below:



The volume of the solid bounded by the coordinate plane and the plane $3x+2y+z=6$ is,

$$V=\iint_D z\,dA$$

$$=\int_0^2\int_0^{3-\frac{3x}{2}}(6-3x-2y)\,dy\,dx$$

$$=\int_0^2\left(6y-3xy-2\left(\frac{y^2}{2}\right)\right)\Bigg|_0^{3-\frac{3x}{2}}dx$$

$$=\int_0^2\left(6\left(3-\frac{3x}{2}\right)-3x\left(3-\frac{3x}{2}\right)-\left(3-\frac{3x}{2}\right)^2\right)dx$$

Continue the above step,

$$V=\int_0^2\left(18-9x-9x+\frac{9x^2}{2}-\left(9+\frac{9x^2}{4}-9x\right)\right)dx$$

$$=\int_0^2\left(\frac{9x^2}{4}-9x+9\right)dx$$

$$=\left(\frac{3x^3}{4}-\frac{9x^2}{2}+9x\right)\Bigg|_0^2$$

$$=\left(\frac{3(2)^3}{4}-\frac{9(2)^2}{2}+9(2)\right)$$

$$=6$$

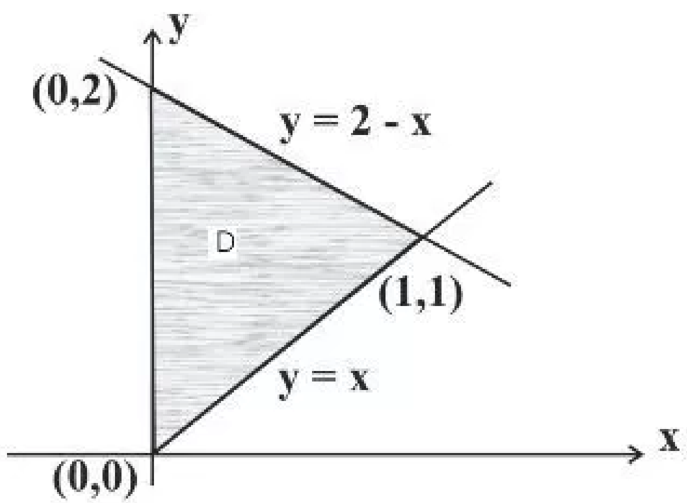
Therefore, the volume of the solid is $\boxed{6}$.

Chapter 15 Multiple Integrals 15.3 28E

The plane $y=x$ meets plane $x+y=2$ when $x+x=2$ that is $2x=2$ that is $x=1$

Therefore the region of integration is given by

$$D=\{(x,y):\ 0\le x\le 1,\ x\le y\le 2-x\}$$



The required volumes is

$$v = \iint_D z\, dA$$

$$= \int_0^1 \int_x^{2-x} x\, dy\, dx$$

$$= \int_0^1 x[y]_{y=x}^{y=2} dx$$

$$= \int_0^1 \left(2x - 2x^2\right) dx$$

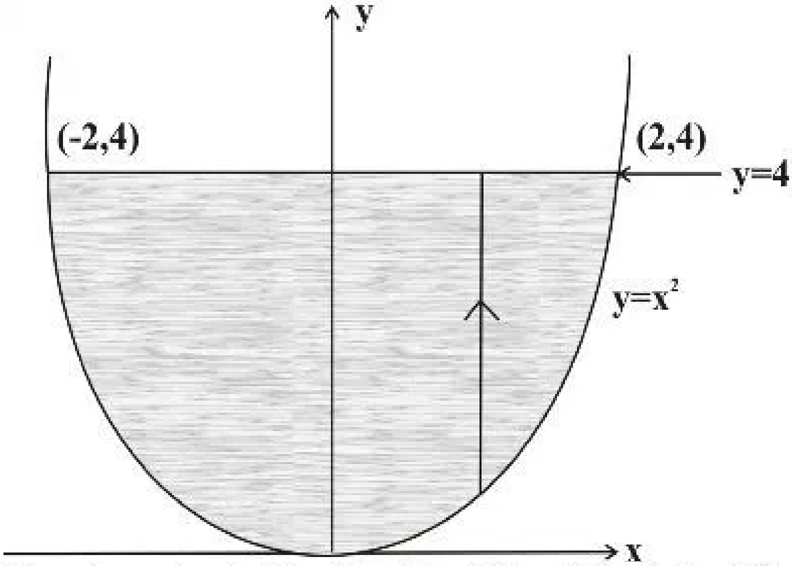
$$= \left[x^2 - \frac{2}{3}x^3\right]_0^1$$

$$= 1 - \frac{2}{3}$$

$$= \boxed{\frac{1}{3}}$$

Chapter 15 Multiple Integrals 15.3 29E

The solid is enclosed by surfaces $z = x^2$, $y = x^2$ and the planes $z = 0$ and $y = 4$



Then the region is $D = \{(x, y): -2 \le x \le 2,\ \ x^2 \le y \le 4\}$

The required volume is

$-2 \le x \le 2,\ \ x^2 \le y \le 4\}$

$$v = \iint_D x^2\, dA$$

$$= 2\int_0^2 \int_{x^2}^4 x^2\ dy\ dx$$

$$= 2\int_0^2 \left(x^2 y\right)_{y=x^2}^{y=4} dx$$

$$= 2\int_0^2 x^2\left(4 - x^2\right) dx$$

i.e. v $= 2\int_0^2 \left(4x^2 - x^4\right)dx$

$$= 2\left[\frac{4}{3}x^3 - \frac{x^5}{5}\right]_0^2$$

$$= 2\left(\frac{64}{15}\right)$$

$$= \boxed{\frac{128}{15}}$$

Chapter 15 Multiple Integrals 15.3 30E

Consider the equations of the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y, x = 0$ and $z = 0$ in first octant.

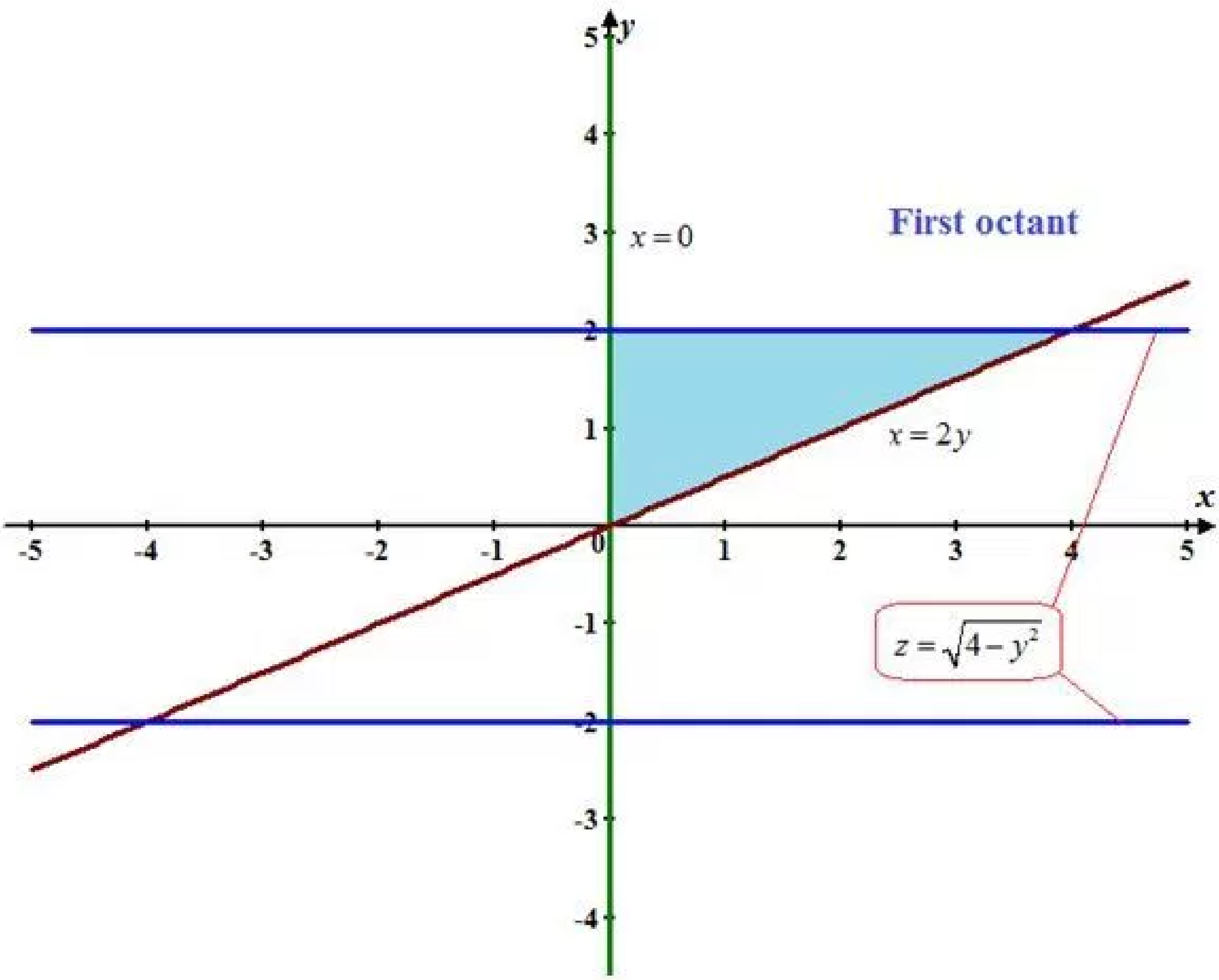
The objective is to find the volume of the solid bounded by the given curves.

Rewrite the equation of the cylinder $y^2 + z^2 = 4$ as follows:

$$y^2 + z^2 = 4$$
$$z^2 = 4 - y^2$$
$$z = \sqrt{4 - y^2}$$

The region enclosed by $x = 0$ and $x = 2y$ where $0 \le y \le 2$.

The sketch of the region bounded by the curves $z = \sqrt{4 - y^2}$, $x = 0$ and $x = 2y$ is shown as below:



From the above graph, the region is $R = \{(x, y) \mid 0 \le x \le 2y, 0 \le y \le 2\}$.

Therefore, the volume of the solid is given by,

$$V = \iint_R \sqrt{4-y^2}\, dA$$
$$= \int_0^2 \int_0^{2y} \sqrt{4-y^2}\, dx dy$$
$$= \int_0^2 [x]_0^{2y} \sqrt{4-y^2}\, dy \quad \text{Here integrate first with respect to } x$$
$$= \int_0^2 [2y-0]\sqrt{4-y^2}\, dy$$
$$= \int_0^2 2y\sqrt{4-y^2}\, dy$$
$$= 2\int_0^2 \sqrt{4-y^2}\,(ydy)$$

Take $u = \sqrt{4-y^2}$, and then

$$du = \frac{1}{2\sqrt{4-y^2}} \cdot (-2ydy)$$
$$du = \frac{-1}{\sqrt{4-y^2}} \cdot ydy$$
$$du = \frac{-1}{u} \cdot ydy \quad \text{Since } u = \sqrt{4-y^2}$$
$$-udu = ydy$$

Find the limits of u as follows:

For $y = 0$, then

$$u = \sqrt{4-y^2}$$
$$= \sqrt{4}$$
$$= 2$$

For $y = 2$, then

$$u = \sqrt{4-2^2}$$
$$= \sqrt{4-4}$$
$$= 0$$

Substitute $u=\sqrt{4-y^2}$ and $-udu = ydy$ in $2\int_0^2 \sqrt{4-y^2}\,(ydy)$.

$$V = 2\int_0^2 \sqrt{4-y^2}\,(ydy)$$
$$= 2\int_2^0 u(-udu)$$
$$= -2\int_2^0 u^2 du$$
$$= 2\int_0^2 u^2 du$$

Continuous to the above step,

$$= 2\left[\frac{u^{2+1}}{2+1}\right]_0^2$$

Here use the formula $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$$= 2\left[\frac{u^3}{3}\right]_0^2$$
$$= \frac{2}{3}\left[u^3\right]_0^2$$
$$= \frac{2}{3}\left[2^3 - 0^3\right]$$
$$= \frac{2}{3}[8-0]$$
$$= \frac{16}{3}$$

Hence, the volume of the given solid is $\boxed{V = \frac{16}{3}}$

Chapter 15 Multiple Integrals 15.3 31E

Given cylinder is $x^2 + y^2 = 1$ and planes $y = z, x = 0, z = 0$

$$x = 0 \Rightarrow y^2 = 1 \Rightarrow y = 1$$
$$z = 0 \Rightarrow y = 0$$

So $0 \le y \le 1$ and $0 \le x \le \sqrt{1-y^2}$

Then volume of the solid $z = y$ is

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} y \, dx \, dy$$

$$= \int_0^1 \left. y \, x \right]_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 y . \sqrt{1-y^2} \, dy$$

Now substitute $1 - y^2 = t \Rightarrow -2y \, dy = dt$

And when $y = 0$ then $t = 1$ and $y = 1$ then $t = 0$

So volume of the solid is

$$= \int_1^0 \sqrt{t} \left(\frac{-dt}{2} \right)$$

$$= \frac{1}{2} \int_0^1 \sqrt{t} \, dt$$

$$= \frac{1}{2} \left. \frac{t^{3/2}}{3/2} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{2}{3} (1 - 0) \right]$$

$$= \frac{1}{2} \left(\frac{2}{3} \right)$$

$$= \boxed{\frac{1}{3}}$$

Chapter 15 Multiple Integrals 15.3 32E

Consider the following cylindrical equations

$x^2 + y^2 = r^2$, and $y^2 + z^2 = r^2$

Its need to find the volume bounded by the cylinders

The equation $y^2 + z^2 = r^2$ can be written as,

$y^2 + z^2 = r^2$

$z^2 = r^2 - y^2$ Subtract y^2 on both sides

$z = \sqrt{r^2 - y^2}$ Take square root on both sides

Enclosed by the region $x = -\sqrt{r^2 - y^2}$ and $x = \sqrt{r^2 - y^2}$ where $-r < y < r$

The volume of the cylinder is

$$V = \int\int \sqrt{r^2 - y^2}\, dA$$

$$= \int_{-r}^{r} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \sqrt{r^2 - y^2}\, dxdy$$

Integrating with respect to *x* and *y* and taking the limits of integration:

$$\int_{-r}^{r} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \sqrt{r^2 - y^2}\, dxdy = \int_{-r}^{r} \left(\sqrt{r^2 - y^2}\, [x]_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \right) dy$$

$$= \int_{-r}^{r} \sqrt{r^2 - y^2} \left[\sqrt{r^2 - y^2} - \left(-\sqrt{r^2 - y^2} \right) \right] dy$$

$$= \sqrt{r^2 - y^2} \left[2\sqrt{r^2 - y^2} \right]$$

$$= \int_{-r}^{r} 2\left(r^2 - y^2 \right) dy$$

$$= 2\left(r^2 \cdot y - \frac{y^3}{3} \right)_{-r}^{r}$$

On continuation,

$$\int_{-r}^{r}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \sqrt{r^2-y^2}\,dxdy = 2\left(r^2\cdot y-\frac{y^3}{3}\right)_{-r}^{r}$$

$$=2\left(r^2(r+r)-\frac{1}{3}\left(r^3-(-r)^3\right)\right)$$

$$=2\left(2r^3-\frac{2}{3}r^3\right)$$

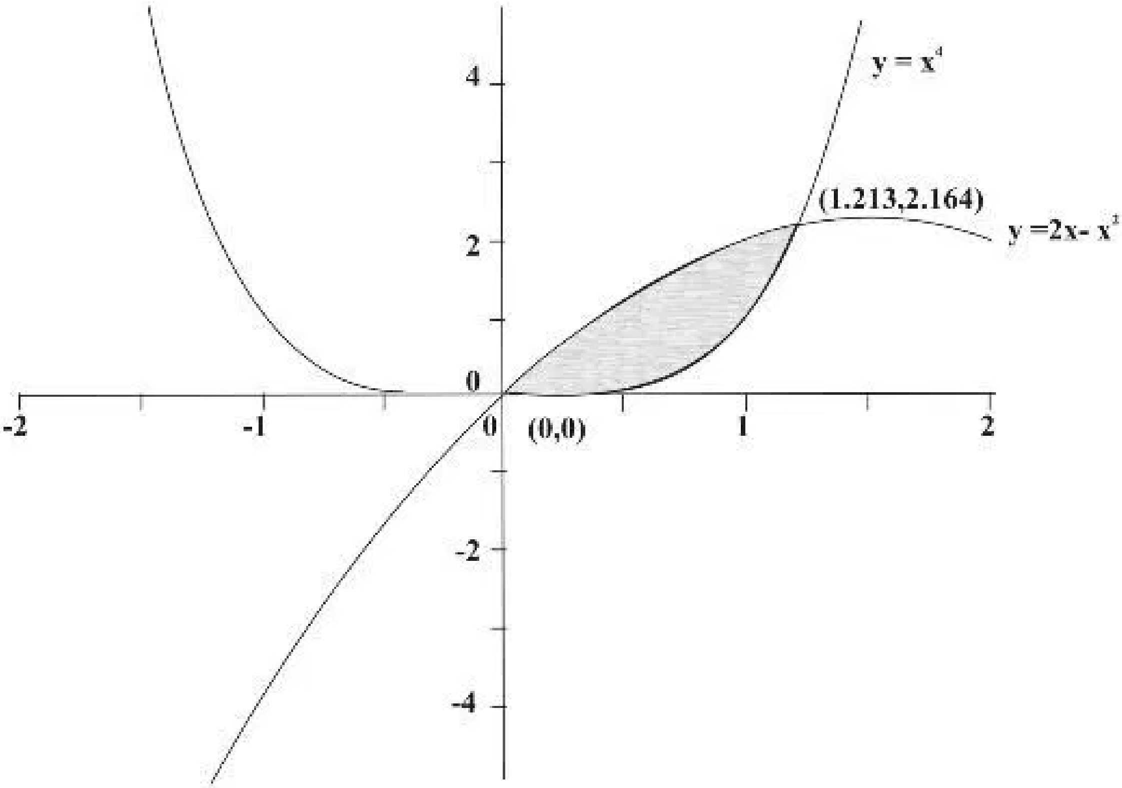
$$=2\left(\frac{6r^3-2r^3}{3}\right)$$

$$=2\left(\frac{4r^3}{3}\right)$$

$$=\frac{8r^3}{3}$$

Therefore, the volume of the cylinder is $\boxed{V=\frac{8r^3}{3}}$.

Chapter 15 Multiple Integrals 15.3 33E



From the graph we find that the two curves $y = x^4$ and $y = 3x - x^2$ intersect when $x = 0$ and $x = 1.213$

Then $D = \{(x, y) : 0 \le x \le 1.213,\ x^4 \le y \le 3x - x^2\}$

Therefore $\iint_D x\, dA = \int_0^{1.213} \int_{x^4}^{3x-x^2} x\, dy\, dx$

$$= \int_0^{1.213} (xy)_{y=x^4}^{y=3x-x^2} dx$$

$$= \int_0^{1.213} x(3x - x^2 - x^4) dx$$

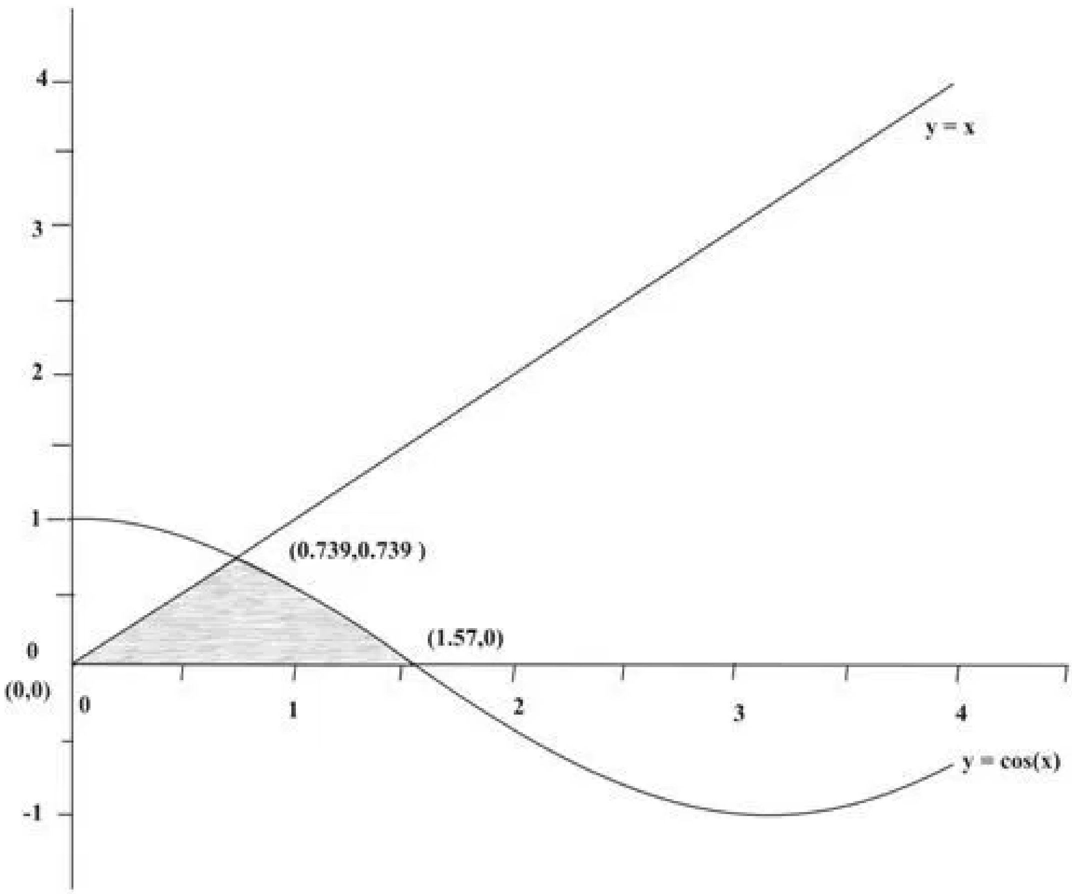
$$= \int_0^{1.213} (3x^2 - x^3 - x^5) dx$$

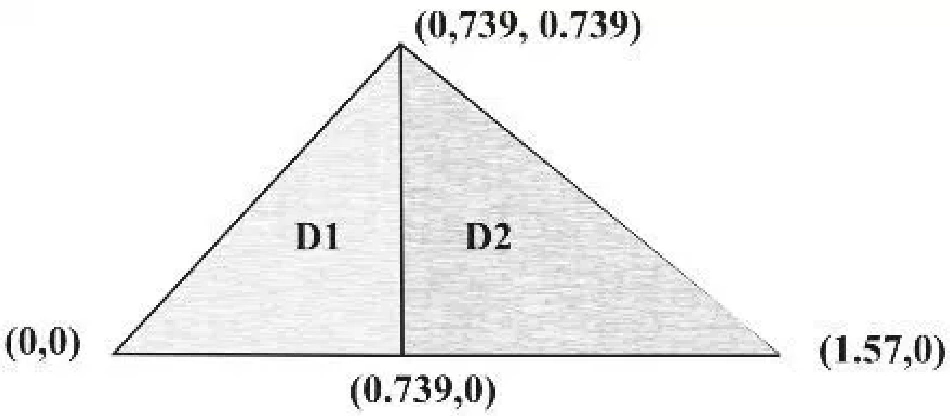
$$= \left(x^3 - \frac{x^4}{4} - \frac{x^6}{6} \right)_0^{1.213}$$

$$= (1.213)^3 - \frac{(1.213)^4}{4} - \frac{(1.213)^6}{6} - 0$$

$$= \boxed{0.7126}$$

Chapter 15 Multiple Integrals 15.3 34E





$$D = D_1 + D_2$$

Where region $D_1 = \{(x, y): 0 \le x \le 0.739,\ 0 \le y \le x\}$

And $D_2 = \{(x, y): 0.739 \le x \le 1.57,\ 0 \le y \le \cos x\}$

Therefore the volume of the solid in first octant is

$$\iint_D x\, dA = \iint_{D_1} x\, dA + \iint_{D_2} x\, dA$$

$$= \int_0^{0.739} \int_0^{x} x\, dy\, dx + \int_{0.739}^{1.57} \int_0^{\cos x} x\, dy\, dx$$

$$= \int_0^{0.739} (xy)_{y=0}^{y=x} + \int_{0.739}^{1.57} (xy)_{y=0}^{y=45x} dx$$

$$= \int_0^{0.739} x^2\, dx + \int_{0.7}^{1.57} x \cos x\, dx$$

i.e. $\iint_D x dA = \frac{1}{3}\left(x^3\right)_0^{0.739} + \left[\cos x + x \sin x\right]_{0.739}^{1.57}$

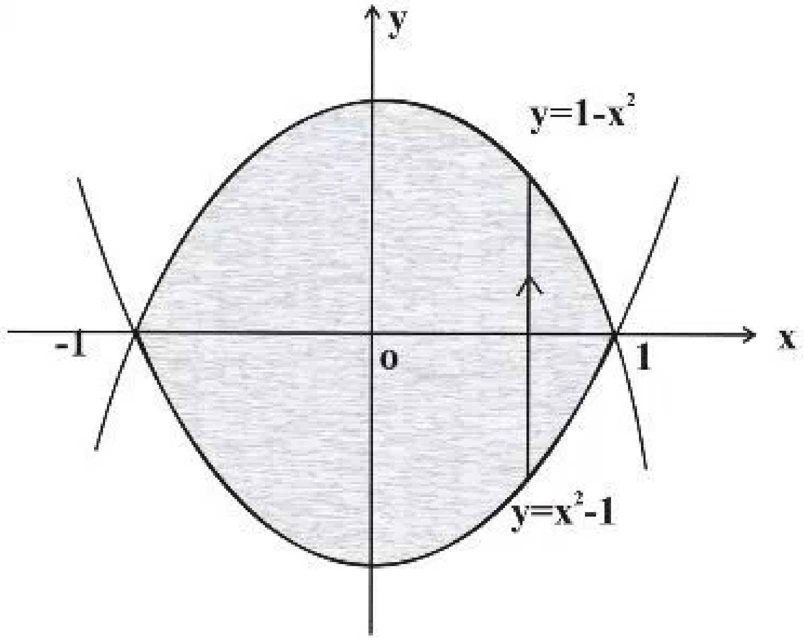
$$= \frac{1}{3}(0.739)^3 - 0 + \left[\cos 1.57 + (1.57)\sin 1.57 \cos 0.739 - 0.739 \sin 0.739\right]$$

$$= \frac{1}{3}(0.739)^3 + \left[\cos\frac{\pi}{2} + \frac{\pi}{2}\sin\frac{\pi}{2} - \cos 0.739 - 0.739\sin(0.739)\right]$$

$$= 0.1345 + \frac{\pi}{2} - 0.739 - 0.4977$$

$$= \boxed{0.4685}$$

Chapter 15 Multiple Integrals 15.3 35E



The region is $D=\{(x,y): -1\le x\le 1,\ x^2-1\le y\le 1-x^2\}$

Then volume of the solid enclosed by plane $x+y+z=2$
And the region D is

$$v_1=\iint_D (2-x-y)\,dA$$

$$=\int_{-1}^{1}\int_{x^2-1}^{1-x^2}(2-x-y)\,dy\,dx$$

$$=\int_{-1}^{1}\left[2y-xy-\frac{y^2}{2}\right]_{y=x^2-1}^{y=1-x^2}dx$$

$$=\int_{-1}^{1}\left[2(1-x^2)-x(1-x^2)-\frac{(1-x^2)^2}{2}-2(x^2-1)+x(x^2-1)+\frac{(x^2-1)^2}{2}\right]dx$$

i.e. $v_1 =\int_{-1}^{1}\left[4(1-x^2)+2x(x^2-1)\right]dx$

$$=2\int_{-1}^{1}(x^3-2x^2-x+2)\,dx$$

$$-2\left[\frac{x^4}{4}-\frac{2}{3}x^3-\frac{x^2}{2}+2x\right]_{-1}^{1}$$

$$=2\left[4-\frac{4}{3}\right]$$

$$=\frac{16}{3}$$

And the volume of solid enclosed by plane $2x + 2y - z + 10 = 0$ and the region D is

$$v_2 = \iint_D (2x+2y+10)\,dA$$

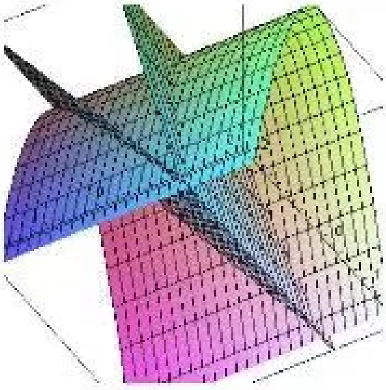
$$= \int_{-1}^{1}\int_{x^2-1}^{1-x^2} (2x+2y+10)\,dy\,dx$$

$$= \int_{-1}^{1}\left[2xy+y^2+10y\right]_{y=x^2-1}^{y=1-x^2} dx$$

$$= \int_{-1}^{1}\left[4x\left(1-x^2\right)+20\left(1-x^2\right)\right]dx$$

Chapter 15 Multiple Integrals 15.3 36E

The region enclosed by the surfaces $y = x^2, z = 3y, z = 2+y$ is



From the figure, we can see that $z = 3y$ is the lower surface and $z = 2+y$ is the upper surface while these surfaces intersect at $y = 1$

Further, at $y = 1$ we see that $x^2 = 1$ and so, $x = -1, 1$

Also, we see that the parabola $y = x^2$ can take the lowest value $x = 0$

At this point, we get $y = 0$

Putting these observations, we get

x varies from -1 through 1

y varies from 0 through x^2 and

z varies from $3y$ to $2+y$

Thus, the volume of the enclosed region is $\int_{x=-1}^{1}\left(\int_{y=0}^{x^2}\left(\int_{z=3y}^{2+y} 1dz\right)dy\right)dx$

$$= \int_{x=-1}^{1}\int_{y=0}^{x^2}(2+y-3y)dydx$$

$$= \int_{x=-1}^{1} 2y-y^2\}_0^{x^2}\,dx$$

$$= \int_{-1}^{1}\left(2x^2-x^4\right)dx$$

$$= \frac{2x^3}{3}-\frac{x^5}{5}\}_{-1}^{1}$$

$$= \frac{2}{3}(1+1)-\frac{1}{5}(1+1)$$

$$= \boxed{\frac{14}{15}}$$

Chapter 15 Multiple Integrals 15.3 **37E**

Problem: Sketch the solid

Integral:

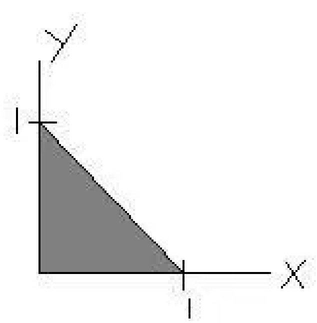
$$\int_0^1\int_0^{1-x} 1-x-y\,\mathrm{d}y\,\mathrm{d}x$$

Take the limits of integration and write them as:

$$0 \le x \le 1$$

$$0 \le y \le (1-x)$$

Plot this on an xy graph



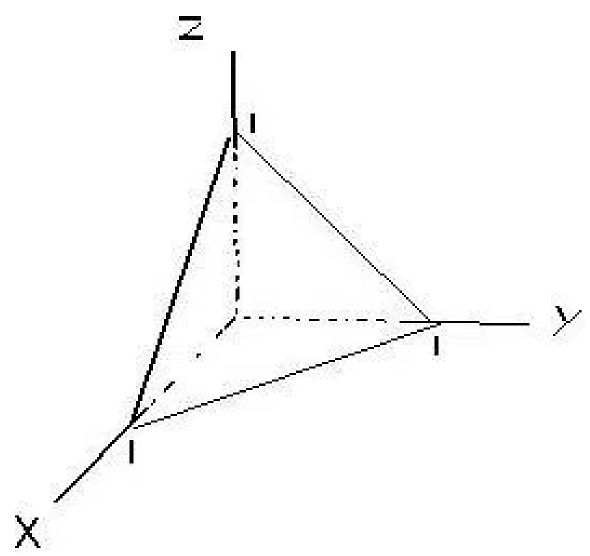
This is the D of this graph

Plug in x=0 and y=0 to get the z-axis value

$f(x,y) = z = 1-x-y$

$f(0,0) = z = 1$

So, the graph of the solid is:



$z=1-x-y$

Chapter 15 Multiple Integrals 15.3 38E

Consider the following iterated integral:

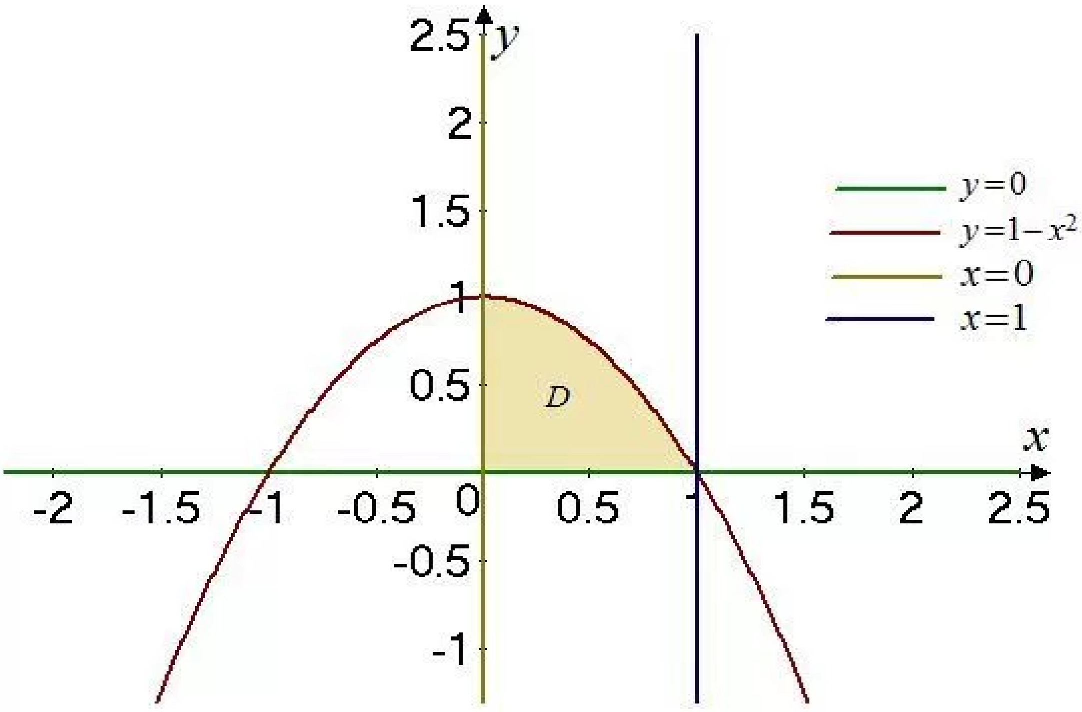
$$\int_0^1 \int_0^{1-x^2} (1-x)\, dy\, dx.$$

The objective is to sketch the solid whose volume is given by the iterated integral.

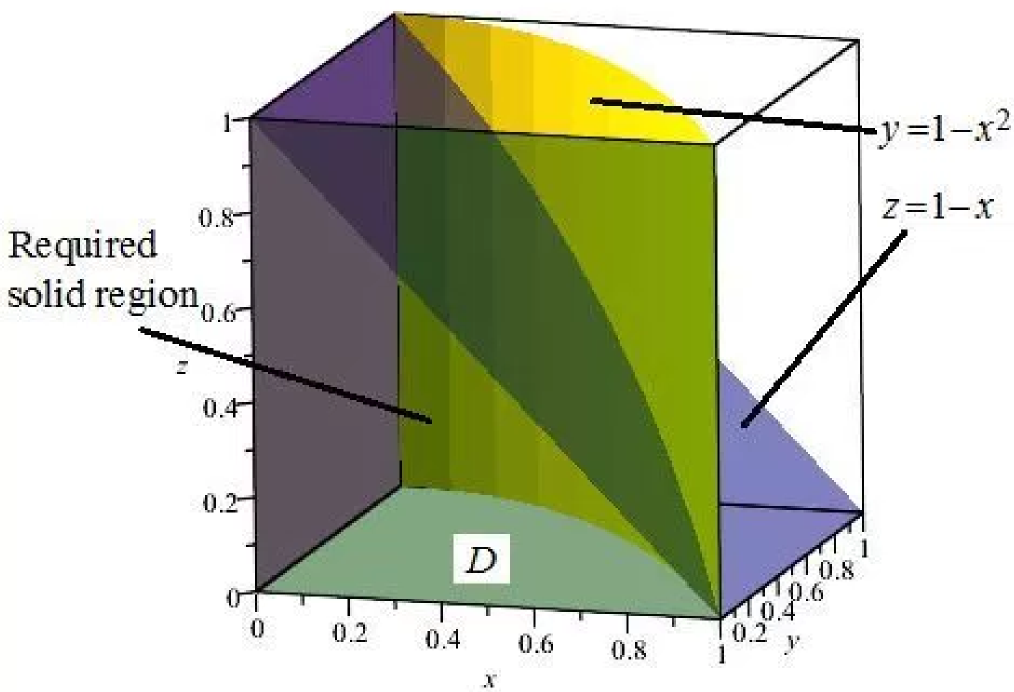
From the given integral observe the following:

$$D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1 - x^2\}.$$

The sketch of the region D in xy-plane is as follows:

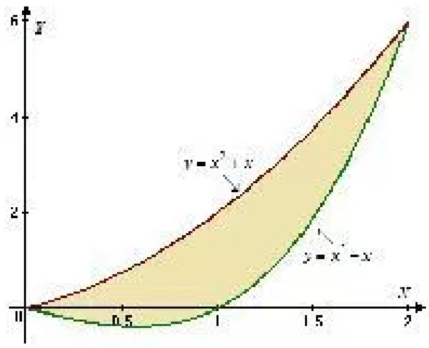


The solid region is shown as follows:



Chapter 15 Multiple Integrals 15.3 39E

The region enclosed by the curves $y = x^3 - x, y = x^2 + x$ is



From the figure, we follow that the upper curve is $y = g_2(x) = x^2 + x$ and the lower curve is $y = g_1(x) = x^3 - x$ enclosed between $x = 0, 6$ and the function is

$f(x, y) = z = x^3y^4 + xy^2$

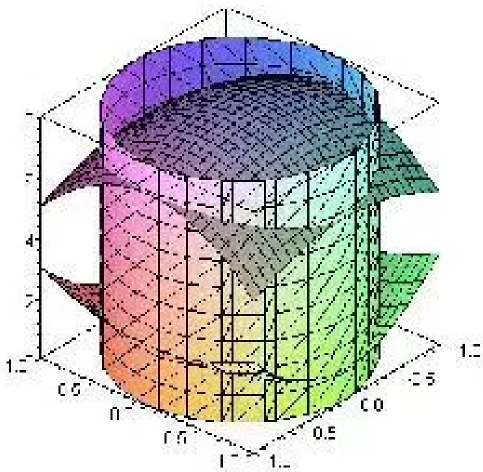
So, we use the volume $\iint_D f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} zdxdy$

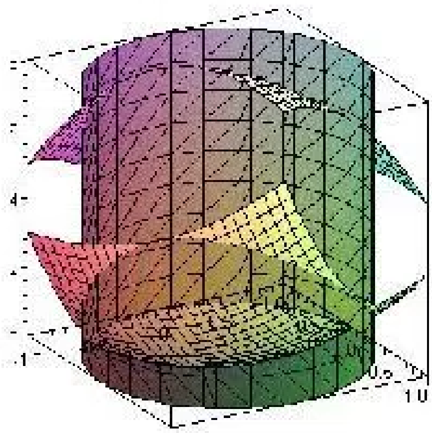
$$= \int_{x=0}^{6} \int_{y=x^3-x}^{x^2+x} (x^3y^4 + xy^2)dydx$$

Using the Maple software to evaluate this integral, we get $\boxed{\dfrac{13,984,735,616}{14,549,535}}$

Chapter 15 Multiple Integrals 15.3 40E

The region enclosed between the paraboloids $z = 2x^2 + y^2 = g_2(x, y)$ as the lower surface and $z = 8 - x^2 - 2y^2 = g_1(x, y)$ as the upper surface while the cylinder portion $x^2 + y^2 = 1$ is enclosed by these paraboloids to give the required volume.





If the area between the paraboloids within the cylindrical limits is integrated, then we get the volume.

i.e., $z = f(x,y) = g_1(x,y) - g_2(x,y)$

Volume is $\iint_D f(x,y)\,dA = \int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)\,dy\,dx$ ...... (1)

We observe that x, y varies on $x^2 + y^2 = 1$ between $g_1(x,y)$ and $g_2(x,y)$

So, $x = -1, 1$ and $y = -\sqrt{1-x^2}, \sqrt{1-x^2}$

Substituting these things in (1), we get $\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(8 - x^2 - 2y^2 - 2x^2 - y^2\right) dy\,dx$

$$= \int_{-1}^{1} \left(8 - 3x^2\right) y - y^3 \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= 2\int_{-1}^{1} 8\sqrt{1-x^2}\,dx - 2\int_{-1}^{1} \left(1-x^2\right)\sqrt{1-x^2}\,dx$$

$$= 14\int_{-1}^{1} \sqrt{1-x^2}\,dx + 2\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx$$

Using the Maple software, we get $14\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{8}\right) = \boxed{\frac{29\pi}{4}}$

Chapter 15 Multiple Integrals 15.3 41E

Use a computer algebra system to find the exact volume of the solid, which is enclosed by $z = 1 - x^2 - y^2$ and $z = 0$.

Recall, that the volume of a solid over a region can be calculated by the appropriate integral $\iint_R f(x,y)\,dA$.

Note that, $z = 1 - x^2 - y^2$ and $z = 0$ intersect in the circle, $x^2 + y^2 = 1$, or $R = \left\{(x,y) \mid -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\right\}$

Find the volume of the solid and obtain the limits of integration and appropriate function to integrate. On R, we have $0 \le z \le 1 - x^2 - y^2$, so our function is $f(x,y) = 1 - x^2 - y^2$.

From R, calculate $-1 \le x \le 1$ and $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$.

This allows us to write the volume of the solid as $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(1 - x^2 - y^2\right) dy\,dx$.

Use maple software to evaluate the integral $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(1-x^2-y^2\right)\,dy\,dx$.

Use maple software to use the symbolic command, as shown below:

Int(int(1-x^2-y^2,y=-sqrt(1-x^2)..sqrt(1-x^2)),x=-1..1);

> $int(int(1-x^2-y^2, y=-\text{sqrt}(1-x^2)..\text{sqrt}(1-x^2)), x=-1..1);$

$\frac{1}{2}\pi$

This outputs the volume as $\frac{\pi}{2}$.

Hence, the result is $\boxed{\frac{\pi}{2}}$.

Chapter 15 Multiple Integrals 15.3 42E

Consider the solid $z=x^2+y^2$ and $z=2y$.

Need to find the volume of the solid by using computer algebra system.

Recall that the volume of a solid over a region can be calculated by the appropriate integral $\iint_R f(x,y)\,dA$

Note that $z=x^2+y^2$ and $z=2y$ intersect in the circle $x^2+(y-1)^2=1$, or $R=\left\{(x,y)\,|\,-1\le x\le 1, 1-\sqrt{1-x^2}\le y\le 1+\sqrt{1-x^2}\right\}$

To find the volume of the solid, we need to obtain the limits of integration and appropriate function to integrate. On R we have that $x^2+y^2\le z\le 2y$, so our function is $f(x,y)=2y-\left(x^2+y^2\right)$.

From R we have $-1\le x\le 1$ and $1-\sqrt{1-x^2}\le y\le 1+\sqrt{1-x^2}$. This allows us to write the volume of the solid as

$$\int_{-1}^{1}\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} 2y-\left(x^2+y^2\right)\,dy\,dx$$

Next we use maple software to evaluate the integral $\int_{-1}^{1}\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} 2y-\left(x^2+y^2\right)\,dy\,dx$ using the symbolic command, which is shown below:

Int(int(2y-x^2-y^2,y=1-sqrt(1-x^2)..1+sqrt(1-x^2)),x=-1..1);

```
> int(int(2y - x^2 - y^2, y = 1 - sqrt(1 - x^2) ..1 + sqrt(1 - x^2)), x = -1
    ..1);
```

$$\frac{1}{2}\pi$$

This outputs the volume as $\frac{\pi}{2}$.

Hence the volume of a solid is $\boxed{\frac{\pi}{2}}$.

Chapter 15 Multiple Integrals 15.3 43E

Consider the double integral,

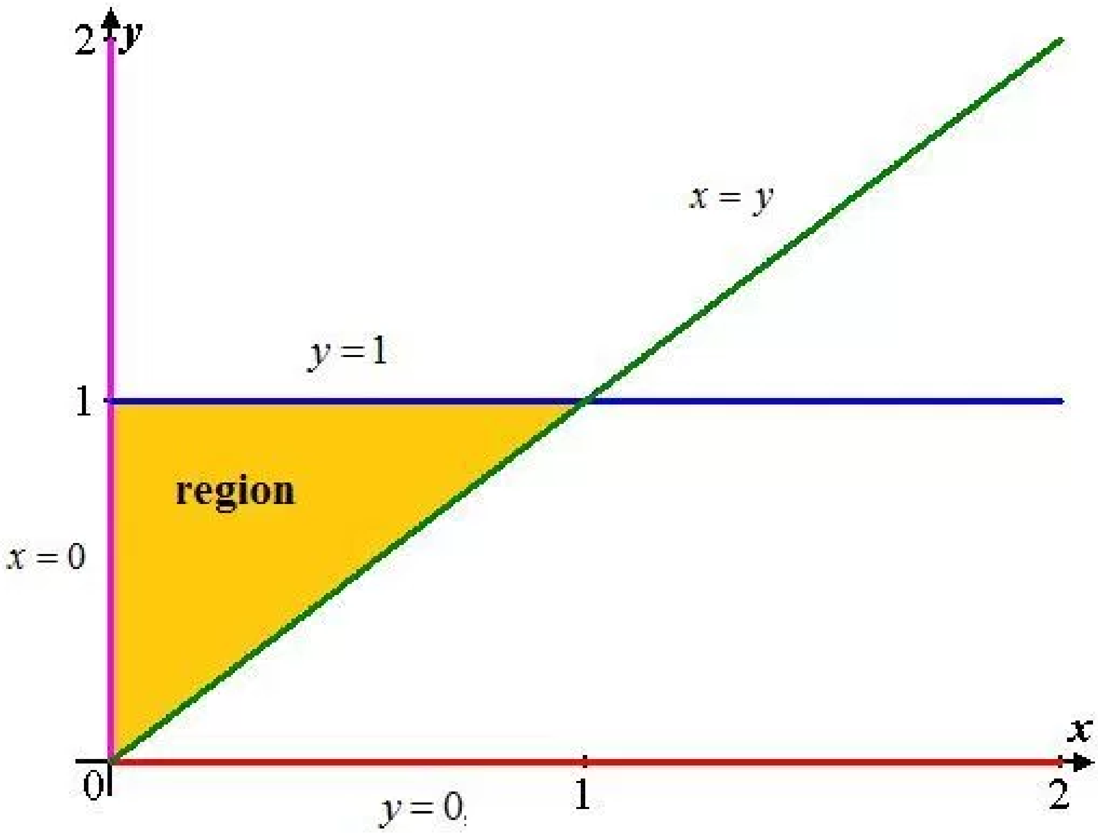
$$\int_0^1\int_0^y f(x,y)\,dx\,dy$$

The objective is to sketch the region of integration, and then change the order of integration.

Here the x limits are $x=0$ and $x=y$

And the y limits are $y=0,\ \ y=1$

The region of integration is shown below.



Changing the order of integration:

The variable y enters at $y = x$ and leaves at $y = 1$.

And the variable x varies from $x = 0$ to $x = 1$.

The region can be written as $R = \{(x, y) \mid x \le y \le 1, 0 \le x \le 1\}$

Therefore, Change of order of integration is,

$$\int_0^1 \int_0^y f(x, y)\, dx\, dy = \boxed{\int_0^1 \int_x^1 f(x, y)\, dy dx}$$

Chapter 15 Multiple Integrals 15.3 44E

Consider the double integral,

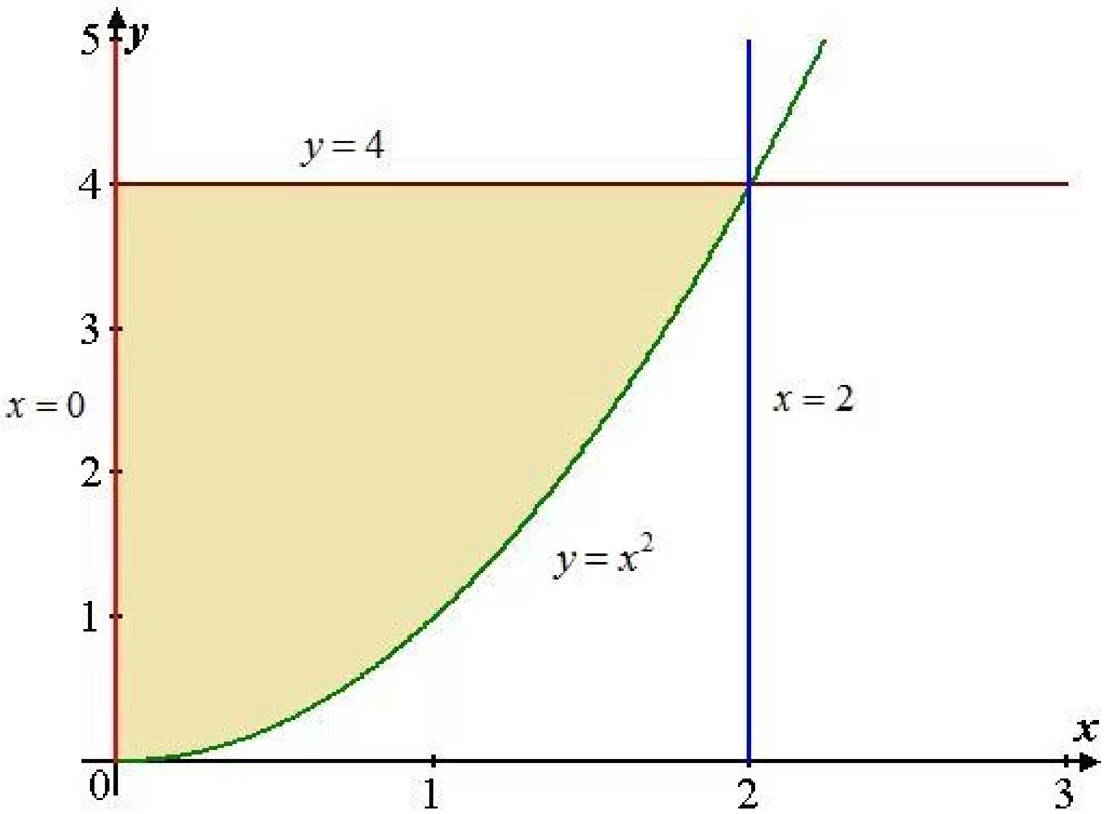
$$\int_0^2 \int_{x^2}^4 f(x, y)\, dy dx$$

The objective is to sketch the region of integration, and then change the order of integration.

Here the *x* limits are $x = 0$ and $x = 2$

And the *y* limits are $y = x^2$, $y = 4$

The region of integration is shown below.



Changing the order of integration:

The variable y enters at $y=0$ and leaves at $y=4$.

And the variable x varies from $x=0$ to $x=\sqrt{y}$.

The region can be written as $R=\left\{(x,y)\,|\,0\le y\le 4, 0\le x\le \sqrt{y}\right\}$

Therefore, Change of order of integration is,

$$\int_0^2\int_{x^2}^4 f(x,y)\,dydx = \boxed{\int_0^4\int_0^{\sqrt{y}} f(x,y)\,dxdy}$$

Chapter 15 Multiple Integrals 15.3 45E

Consider the double integral,

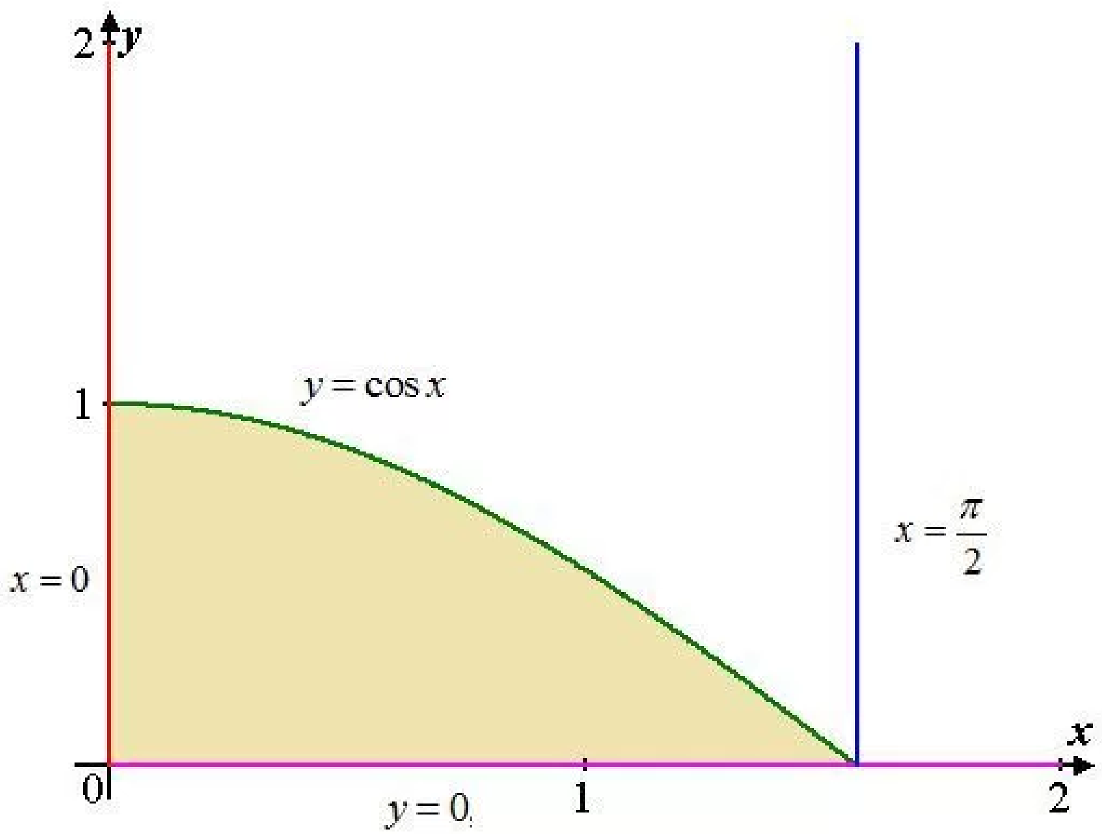
$$\int_0^{\frac{\pi}{2}}\int_0^{\cos x} f(x,y)\,dydx$$

The objective is to sketch the region of integration, and then change the order of integration.

Here the x limits are $x=0$ and $x=\frac{\pi}{2}$

And the y limits are $y=0,\ \ y=\cos x$

The region of integration is shown below.



Changing the order of integration:

The variable y enters at $y=0$ and leaves at $y=1$

And the variable x varies from $x=0$ to $x=\cos^{-1} y.$

The region can be written as $D=\{(x,y)\mid 0\le x\le \cos^{-1} y,\quad 0\le y\le 1\}$

Therefore, Change of order of integration is,

$$\int_0^{\frac{\pi}{2}}\int_0^{\cos x} f(x,y)\,dydx = \boxed{\int_0^1\int_0^{\cos^{-1} y} f(x,y)\,dxdy}$$

Chapter 15 Multiple Integrals 15.3 46E

Consider the following double integral:

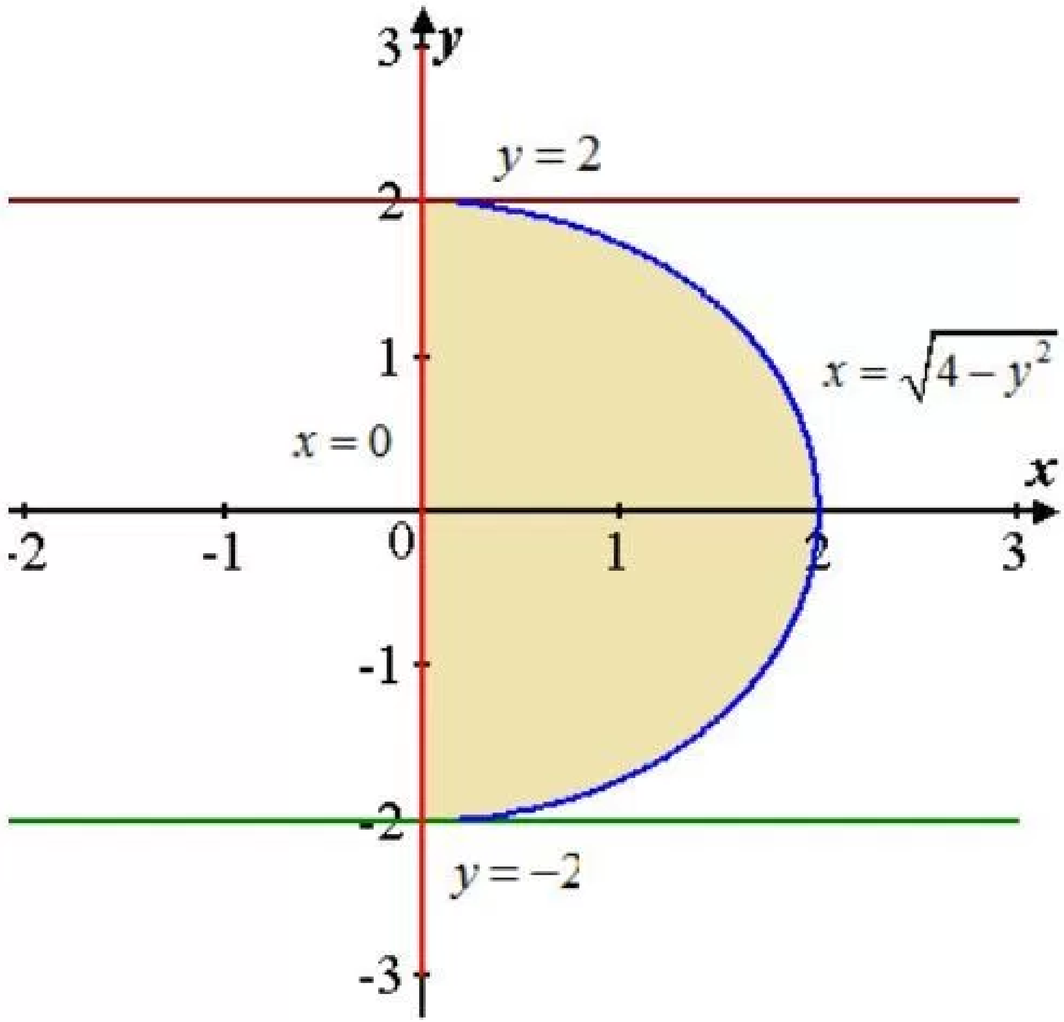
$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$$

The objective is to sketch the region of integration, and then change the order of integration.

The inner limits of integration indicate that $0\le x\le \sqrt{4-y^2}$ and the outer limits of integration indicate that $-2\le y\le 2$.

Note that the limits on *y* are the intersection points for the limits on *x*. Typically in cases like this a closed region is determined by the *x* boundary curves.

Sketch the region of integration as shown below:



Solve the equation $x=\sqrt{4-y^2}$ for *y* to get $y=\pm\sqrt{4-x^2}$. Use a vertical line intersecting the region of integration to help determine the *y* boundaries:

$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$.

The extreme values of *x* are 0 and 2.

So, the region can be written as,

$$D=\left\{(x,y)\,|\,0\le x\le 2, -\sqrt{4-x^2}\le y\le \sqrt{4-x^2}\right\}$$

Thus, the change of order of integration is,

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} f(x,y)\,dx\,dy = \boxed{\int_{0}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx}.$$

Chapter 15 Multiple Integrals 15.3 47E

Consider the double integral.

$$\int_1^2 \int_0^{\ln x} f(x,y)\,dy dx.$$

The objective is to sketch the region of integration.

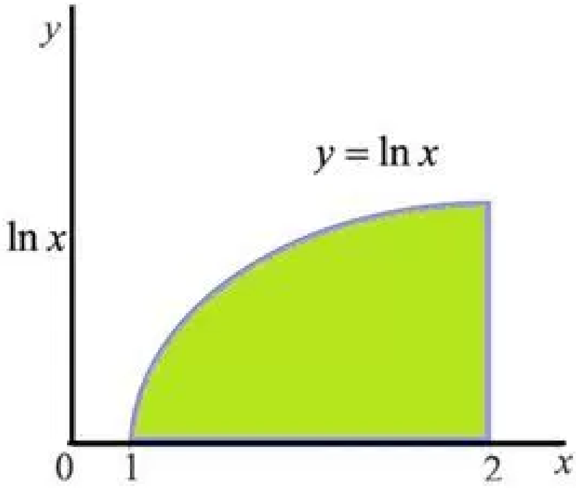
To sketch the region of integration, use the limits of integration in regards to *x* and *y* to determine the enclosed region we wish to sketch.

As the innermost integration is performed with respect to *y*, this implies that $0 \le y \le \ln x$, or that any value of *x* must lie in-between the graph of $y = \ln x$ and $y = 0$.

Since the outermost integration is performed with respect to *x*, implies that $1 \le x \le 2$, or that any value of *x* must lie in-between the lines $x = 1$ and $x = 2$.

From these limits of integration, the region of integration is the area bounded by the *x*-coordinate axis, the line $x = 2$, and the graph $y = \ln x$.

The diagram below shows the region:



The order of integration of $\int_1^2 \int_0^{\ln x} f(x,y)dydx$ written as,

From the existing integral, it is clear that the function is integrated with respect to *y* first and then integrated with respect to *x*, which means we will write the integral with respect to *x* first and then with respect to *y*.

From the sketch of the region of integration we found in part (a), see that any value of *x* is at most 2 and at least e^y, or $e^y \le x \le 2$, and that any value of *y* is at least 1 and at most ln 2, or $1 \le y \le \ln 2$. This means that $y = 1$ is the lower limit of integration with respect to *y* and $y = \ln 2$ is the upper limit of integration with respect to *y*, and that $x = 2$ is the upper limit of integration with respect to *x* and $x = e^y$ is the lower limit of integration with respect to *x*.

Hence, the order of integration of $\int_1^2 \int_0^{\ln x} f(x,y)\,dy\,dx$ changed as,

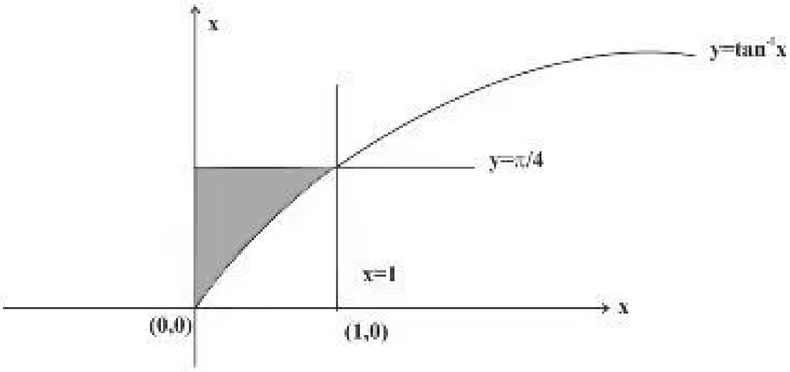
$$\boxed{\int_1^{\ln 2} \int_{e^y}^2 f(x,y)\,dx\,dy}.$$

Chapter 15 Multiple Integrals 15.3 48E

$$\int_0^1 \int_{\arctan x}^{\pi/4} f(x, y)\, dy\, dx$$

The region of integration is

$$D = \left\{(x, y) : 0 \le x \le 1,\ \arctan x \le y \le \frac{\pi}{4}\right\}$$



Now $\quad 0 \le x \le 1,\ \tan^{-1} x \le y \le \frac{\pi}{4}$

Or $\quad 0 \le x \le 1,\ x \le \tan y \le \tan\frac{\pi}{4}$

Or $\quad 0 \le x \le 1,\ x \le \tan y \le 1$

$\quad \Rightarrow 0 \le x \le \tan y$

Also $0 \le x \le 1,\ \tan^{-1} x \le y \le \frac{\pi}{4}$ gives $0 \le y \le \frac{\pi}{4}$

Then the order of integration can be changed and we can write

$$\boxed{\int_0^{\pi/4} \int_0^{\tan y} f(x, y)\, dx\, dy}$$

Chapter 15 Multiple Integrals 15.3 49E

Consider the iterated integral is $\int_0^1 \int_{3y}^3 e^{x^2}\, dx dy$

Using the Fubini's theorem $\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y) dy dx = \int_c^d \int_a^b f(x,y) dx dy$, get

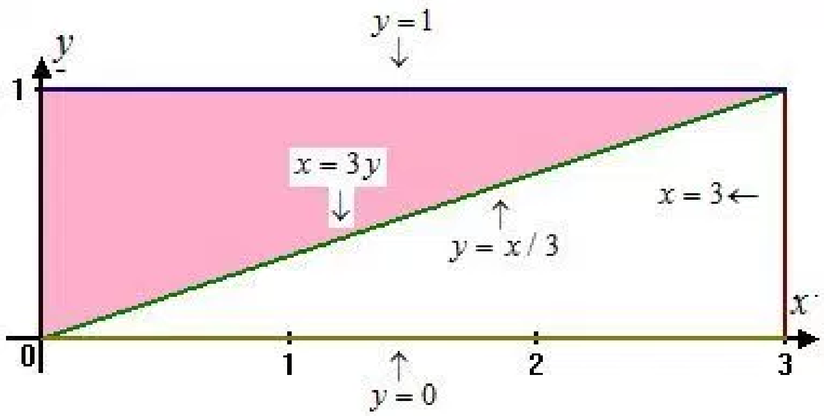
$f(x,y) = e^{x^2}$

Use the existing limits of integration in regards to *x* and *y* to help reverse the order of integration. Since the innermost integration is performed with respect to *x*, know that $3y \le x \le 3$, and since the outermost integration is performed with respect to *y*, know that $0 \le y \le 1$.

Since y is at least 0, $y = 0$ is the lower limit of integration with respect to y, and since y is bounded by $3y \le x$, $y = \frac{1}{3}x$ is the upper limit of integration with respect to y. Also since y is at least 0, $x = 0$ is the lower limit of integration with respect to x, and $x = 3$ is the upper limit of integration with respect to x

Now, by changing the order of integration, get $\int_{x=0}^{3}\int_{y=0}^{\frac{x}{3}} e^{x^2}\,dy\,dx$

Graph of the integration



First, evaluate the integral with respect to y:

$$\int_0^{\frac{1}{3}x} e^{x^2}\,dy = \left[ye^{x^2}\right]_{y=0}^{y=\frac{1}{3}x}$$
$$= \frac{1}{3}xe^{x^2} - 0$$
$$= \frac{1}{3}xe^{x^2}$$

Second, evaluate the integral with respect to x. Use the substitution $u = x^2, du = 2x$ to transform the integral and evaluate

$$\int_0^3 \frac{1}{3}xe^{x^2}\,dx = \frac{1}{3}\int_0^9 \frac{1}{2}e^{u}\,du$$
$$= \frac{1}{6}\left[e^{u}\right]_{u=0}^{u=9}$$
$$= \boxed{\frac{1}{6}\left(e^9 - 1\right)}$$

Chapter 15 Multiple Integrals 15.3 50E

Consider the double integral:

$$I = \int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2) dxdy$$

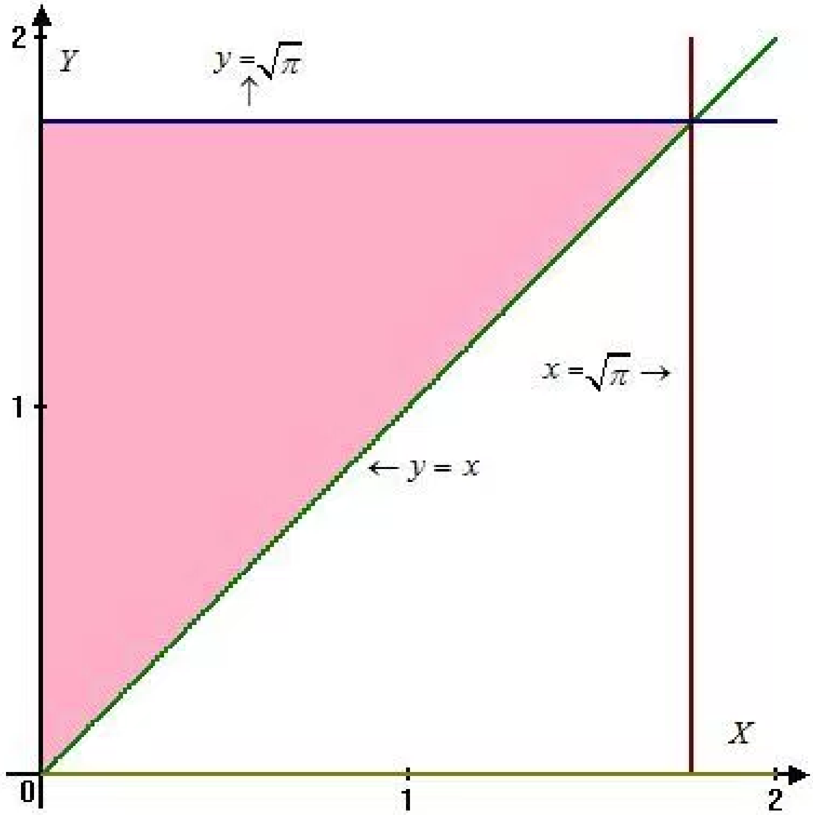
Here the region is $0 \le y \le \sqrt{\pi}$ and $y \le x \le \sqrt{\pi}.$

Objective is to evaluate the integral by reversing the order of integration.

Using the Fubini's theorem $\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$, on

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2) dxdy.$$

From this iterated integral, we follow that x varies from y through $\sqrt{\pi}$ and y varies from 0 through $\sqrt{\pi}$.



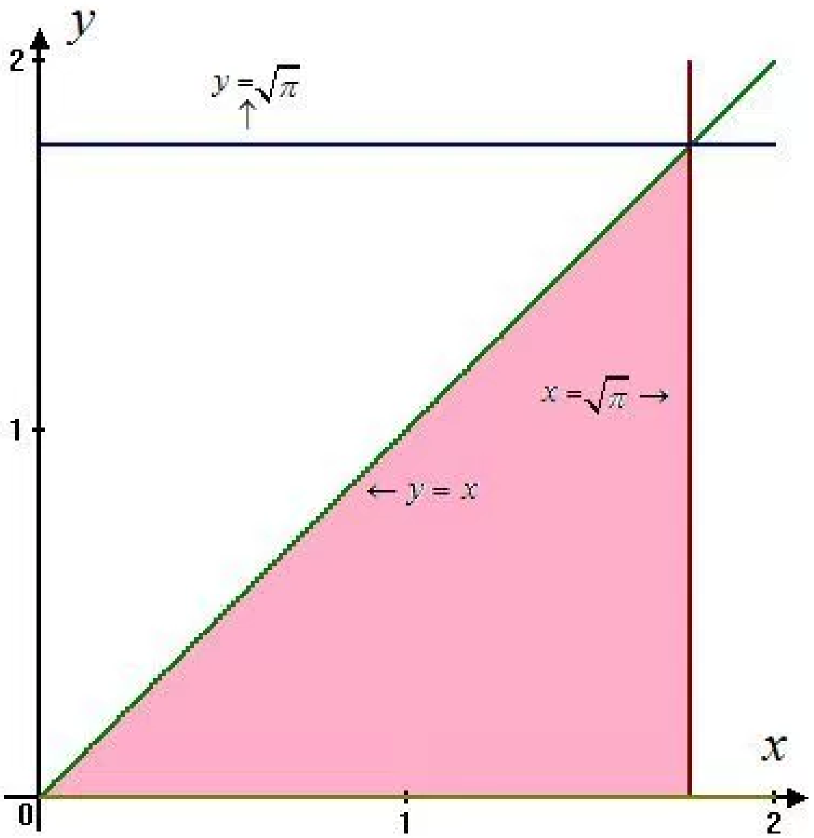
So, we follow that along the line $y = x$, y varies from 0 through x.

Also, when $y = x$, x becomes 0.

So, x varies from 0 through $\sqrt{\pi}$.

The region becomes $0 \le y \le x$ and $0 \le x \le \sqrt{\pi}$.

Then the region becomes as follows:



Thus, by applying the change of order of integration, the given integral becomes:

$$\int_0^{\sqrt{\pi}}\int_y^{\sqrt{\pi}}\cos\left(x^2\right)dxdy=\int_0^{\sqrt{\pi}}\int_0^{x}\cos\left(x^2\right)dydx$$

$$=\int_0^{\sqrt{\pi}}\cos\left(x^2\right)[y]_{y=0}^{y=x}dx$$

$$=\int_0^{\sqrt{\pi}}x\cos\left(x^2\right)dx$$

Let $x^2=t$

Then $2xdx=dt$

$$xdx=\frac{1}{2}dt$$

If $x=0$ then $t=0.$

If $x=\sqrt{\pi}$ then $t=\pi.$

Double integral becomes;

$$\int_0^{\sqrt{\pi}}\int_y^{\sqrt{\pi}}\cos\left(x^2\right)dxdy=\frac{1}{2}\int_0^{\pi}\cos(t)dt$$

$$=\frac{1}{2}\left[\sin(t)\right]_0^{\pi}$$

$$=\frac{1}{2}\left[\sin(\pi)-\sin(0)\right]$$

$$=\frac{1}{2}(0-0)$$

$$=\boxed{0}.$$

Chapter 15 Multiple Integrals 15.3 51E

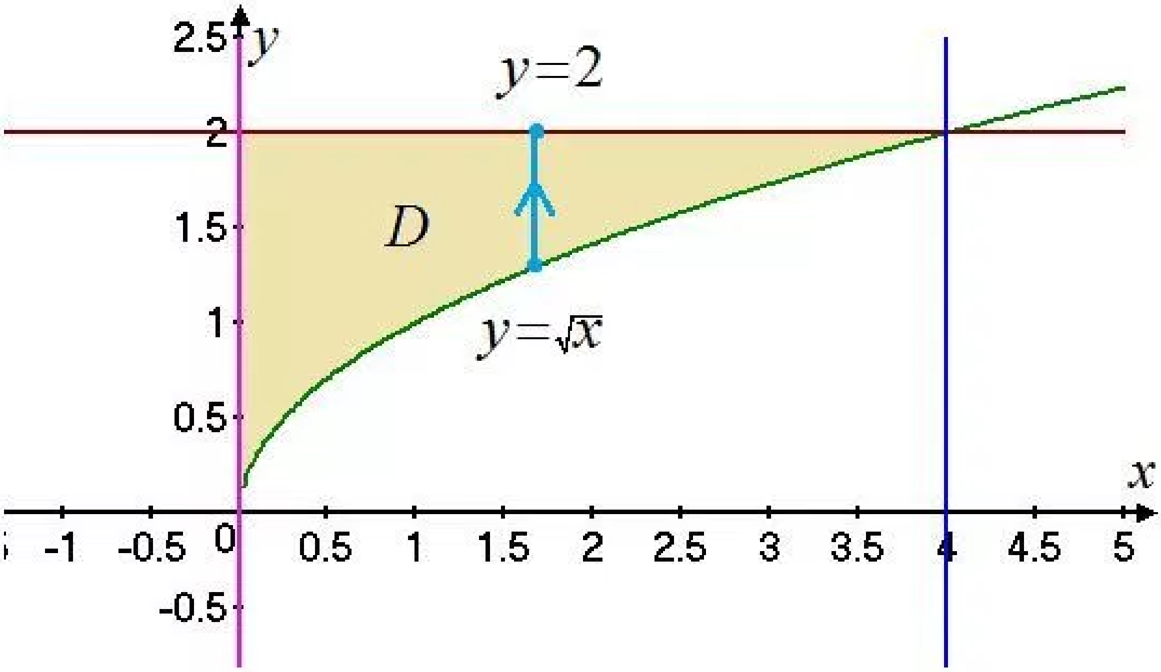
Consider the integral $\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\, dydx.$

From the given double integral, notice that the variable y varies from $\sqrt{x}$ to 2 and the variable x varies from 0 to 4.

So the region D can be written as follows:

$D = \left\{(x, y) \mid \sqrt{x} \le y \le 2, 0 \le x \le 4\right\}.$

The sketch of the region D (Type I) is shown below:



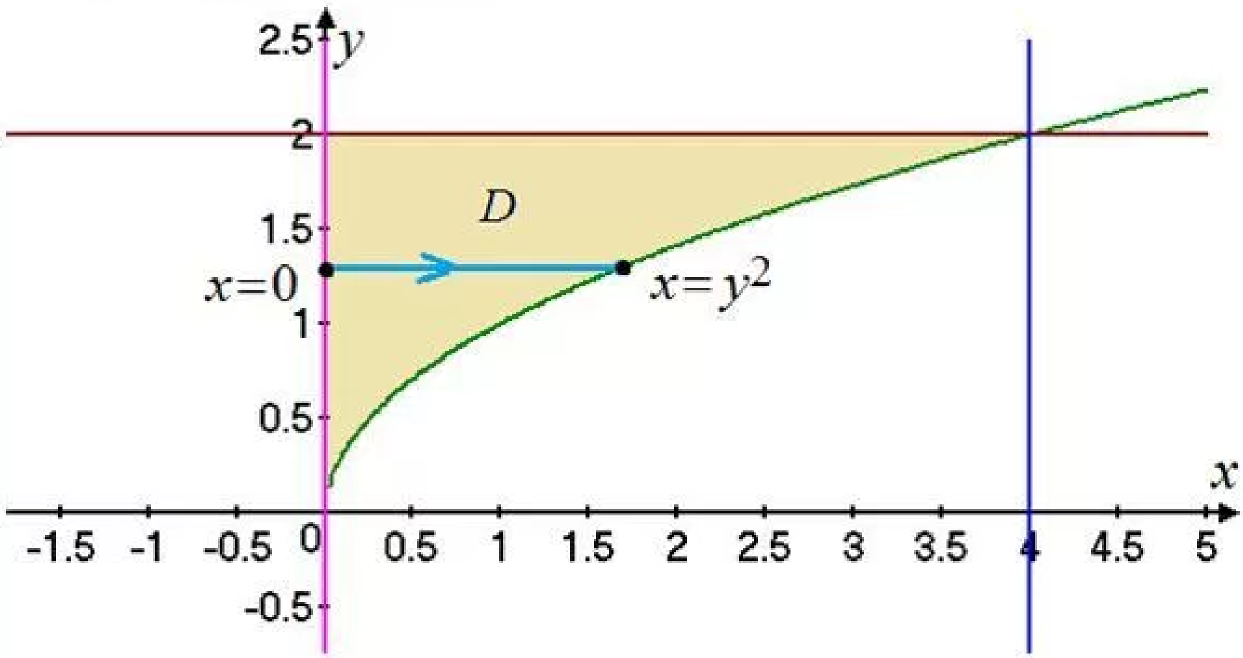
Reverse the order of integration:

Rewrite the equation $y=\sqrt{x}$ as follows:

$$y=\sqrt{x}$$
$$y^2=\left(\sqrt{x}\right)^2$$
$$y^2=x$$
$$x=y^2$$

The alternative description of D is $D=\{(x,y)\mid 0\le y\le 2, 0\le x\le y^2\}$.

The region D (Type II) is shown below:



Evaluate the integral:

$$\int_0^4\int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dydx = \int_0^2\int_0^{y^2} \frac{1}{y^3+1}\,dxdy$$

$$= \int_0^2 \frac{1}{y^3+1}[x]_0^{y^2}\,dy$$

$$= \int_0^2 \frac{y^2}{y^3+1}dy$$

$$= \frac{1}{3}\int_1^9 \frac{1}{t}dt \qquad \left[\begin{array}{l}\text{Put } y^3+1=t \Rightarrow 3y^2dy=dt \\ \text{When } y\to 0 \text{ then } t\to 1 \\ \text{When } y\to 2 \text{ then } t\to 9\end{array}\right]$$

$$= \frac{1}{3}[\ln t]_1^9$$

$$= \frac{1}{3}[\ln 9 - \ln 1]$$

$$= \frac{1}{3}\left[\ln 3^2 - 0\right]$$

$$= \frac{2}{3}\ln 3$$

Therefore, $\int_0^4\int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dydx = \boxed{\frac{2}{3}\ln 3}$.

Chapter 15 Multiple Integrals 15.3 52E

Consider the following integral

$$\int_0^1\int_x^1 e^{x/y}\,dy\,dx$$

Use the existing limits of integration in regards to x and y to help reverse the order of integration. Since the outermost integration is performed with respect to x, we know that $0 \le x \le 1$, and since the innermost integration is performed with respect to y, we know that $x \le y \le 1$.

Since x is at least 0, $x=0$ is the lower limit of integration with respect to x, and since x is bounded by $x \le y$, $x=y$ is the upper limit of integration with respect to x. Also since y is at most 1, $y=1$ is the upper limit of integration with respect to y, and since x is at least 0, $y=0$ is the lower limit of integration with respect to y.

From these limits of integration, the integral is as follows:

$$\int_0^1 \int_0^y e^{x/y}\, dx\, dy$$

First evaluate the integral with respect to x:

$$\begin{aligned}\int_0^y e^{x/y}\, dx &= \left[y\left(e^{x/y}\right)\right]_{x=0}^{x=y} \\ &= ye^1 - ye^0 \\ &= y(e-1)\end{aligned}$$

Second evaluate the integral we respect to y:

$$\begin{aligned}\int_0^1 y(e-1)\, dy &= (e-1)\left[\frac{1}{2}y^2\right]_{y=0}^{y=1} \\ &= (e-1)\left(\frac{1}{2}(1) - \frac{1}{2}(0)\right) \\ &= \frac{1}{2}(e-1)\end{aligned}$$

Therefore, the integral of $\int_0^1 y(e-1)\, dy$ is $\boxed{\frac{1}{2}(e-1)}$.

Chapter 15 Multiple Integrals 15.3 53E

Consider the following integral:

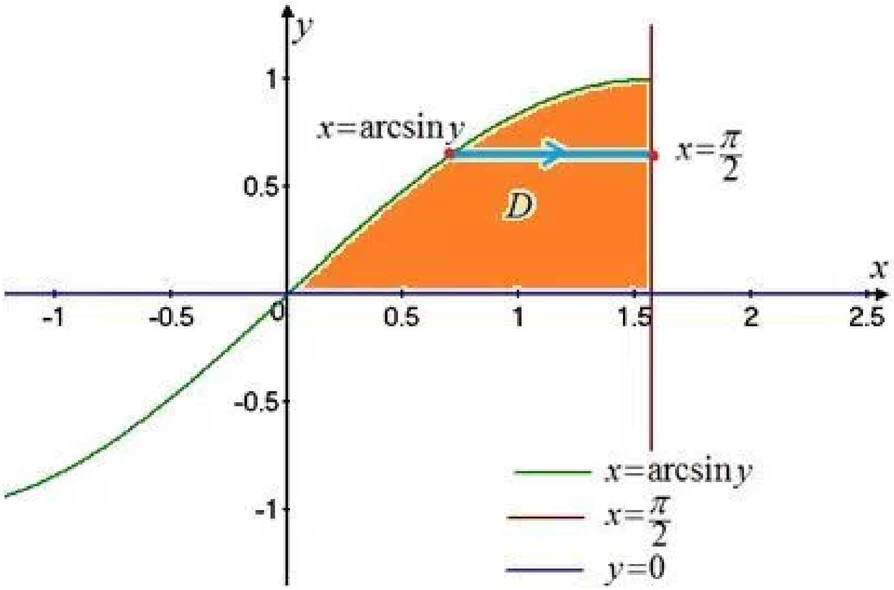
$$\int_0^1 \int_{\arcsin y}^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}\, dxdy.$$

From the given double integral, notice that the variable y varies from 0 to 1 and the variable x varies from $\arcsin y$ to $\frac{\pi}{2}$.

So the region D can be written as follows:

$$D = \left\{(x, y) \mid 0 \le y \le 1, \arcsin y \le x \le \frac{\pi}{2}\right\}.$$

The sketch of the region D (Type I) is shown below:



Reverse the order of integration:

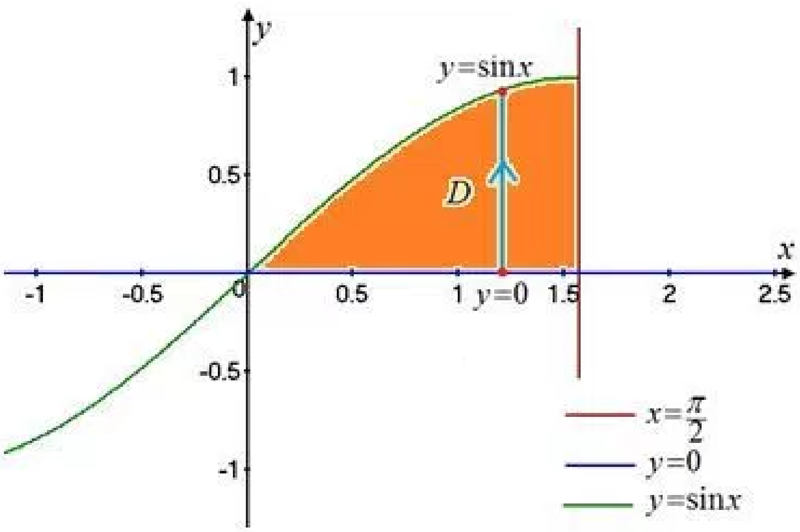
$dydx$:

Rewrite the equation $x = \arcsin y$ as follows:

$$x = \arcsin y$$
$$x = \sin^{-1} y$$
$$\sin x = y$$

The alternative description of D is $D = \left\{(x, y) \mid 0 \le y \le \sin x, 0 \le x \le \frac{\pi}{2}\right\}$.

The region D (Type II) is shown below:



Evaluate the integral:

$$\int_0^1 \int_{\arcsin y}^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}\ dxdy = \int_0^{\frac{\pi}{2}} \int_0^{\sin x} \cos x\sqrt{1+\cos^2 x}\ dydx$$

$$= \int_0^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}\,[y]_0^{\sin x}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x \cos x\sqrt{1+\cos^2 x}\, dx$$

$$= -\frac{1}{2}\int_2^1 \sqrt{t}\,dt \qquad \left[\begin{array}{l} \text{Put } 1+\cos^2 x = t \\ \Rightarrow -2\cos x \sin x dx = dt \\ \text{When } x \to 0 \text{ then } t \to 2 \\ \text{When } x \to \frac{\pi}{2} \text{ then } t \to 1 \end{array}\right]$$

$$= -\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\right]_2^1$$

$$= -\frac{1}{2}\cdot\frac{2}{3}\left[(1)^{\frac{3}{2}} - (2)^{\frac{3}{2}}\right]$$

$$= -\frac{1}{3}\left[1-(2)^{\frac{3}{2}}\right]$$

$$= -\frac{1}{3}\left[1-2\sqrt{2}\right]$$

$$= \frac{1}{3}\left(2\sqrt{2}-1\right)$$

Therefore, the integral of $\int_0^1 \int_{\arcsin y}^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}\ dxdy$ is $\boxed{\frac{1}{3}\left(2\sqrt{2}-1\right)}$.

Chapter 15 Multiple Integrals 15.3 54E

Given that $0 < y < 8$ and $y^{1/3} < x < 2$

By reversing the order

$$0 < x < 2,\ 0 < y < x^3$$

So $$\int_0^8 \int_{y^{1/3}}^2 e^{x^4} dx\, dy$$

$$= \int_0^2 \int_0^{x^3} e^{x^4} dy\, dx$$

$$= \int_0^2 e^{x^4} . y \Big]_0^{x^3} dx$$

$$= \int_0^2 e^{x^4} . x^3 dx$$

Substitute $x^4 = t \Rightarrow 4x^3\, dx = dt$

When $x = 0$ then $t = 0$ and when $x = 2$ then $t = 16$

So $$\int_0^8 \int_{y^{1/3}}^2 e^{x^4} dx dy$$

$$= \int_0^{16} e^t \frac{dt}{4}$$

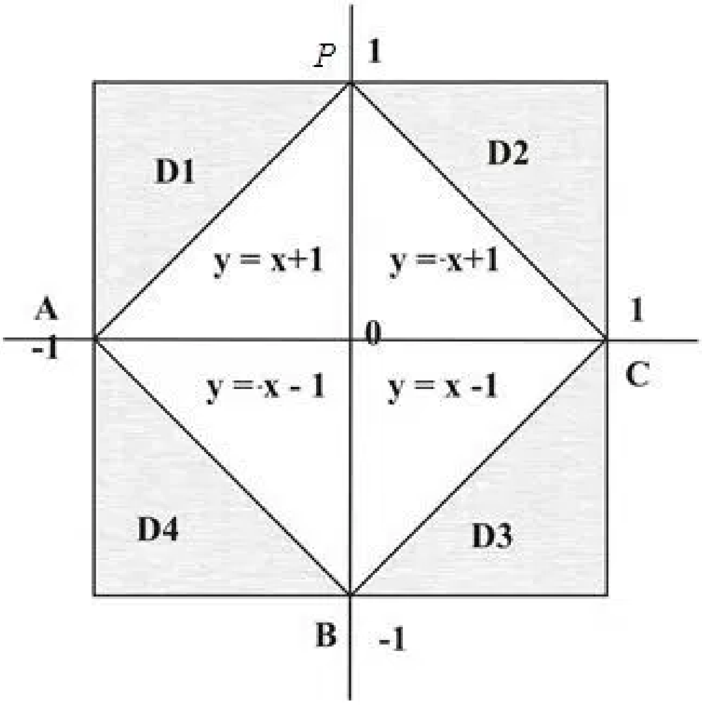
$$= \frac{1}{4} e^t \Big]_0^{16}$$

$$= \frac{1}{4}\left(e^{16} - e^0\right)$$

$$= \boxed{\frac{1}{4}\left(e^{16} - 1\right)}$$

Chapter 15 Multiple Integrals 15.3 55E

Consider the integral $\iint_D x^2 dA$



Equation of line AB is $y = -x - 1$

Equation of line BC is $y = x-1$

Equation of line CP is $y = -x + 1$

Equation of line AP is $y = x + 1$

The region is the union of four regions $D1$, $D2$, $D3$ and $D4$ as shown in the figure

Now $D_1 = \{(x, y): -1 \le x \le D,\ x+1 \le y \le 1\}$

Then $\iint_{D_1} x^2\, dA = \int_{-1}^{0}\int_{x+1}^{1} x^2\, dy\, dx$

$$= \int_{-1}^{0} \left(x^2 y\right)_{y=x+1}^{y=1} dx$$

$$= \int_{-1}^{0} \left[x^2 - x^2 (x+1) \right] dx$$

$$= -\int_{-1}^{0} x^3 \ dx$$

$$= \frac{-1}{4} \left[x^4 \right]_{-1}^{0} \text{ Since } \int x^3 dx = \frac{x^4}{4} + C$$

$$= \frac{-1}{4} (0-1)$$

$$= \frac{1}{4}$$

Now $D_2 = \{ (x, y) : 0 \le x \le 1, -x+1 \le y \le 1 \}$

Then $\iint_{D_2} x^2 \, dA = \int_0^1 \int_{-x+1}^{1} x^2 \ dy \, dx$

$$= \int_0^1 \left(x^2 \ y \right)_{y=-x+1}^{y=1} dx$$

$$= \int_0^1 \left(x^2 (1) - x^2 (-x+1) \right) dx$$

$$= \int_0^1 \left(x^2 + x^3 - x^2 \right) dx$$

$$= \int_0^1 x^3 dx$$

$$= \left[\frac{x^4}{4} \right]_0^1 \text{ Since } \int x^3 dx = \frac{x^4}{4} + C$$

$$= \frac{1}{4}$$

Also $D_3 = \{(x, y): 0 \le x \le 1, -1 \le y \le x-1\}$

Then $\iint\limits_{D_3} x^2 \; dA = \int_0^1 \int_{-1}^{x-1} x^2 \, dy \; dx$

$= \int_0^1 \left(x^2 y\right)_{y=-1}^{y=x-1} dx$

$= \int_0^1 \left(x^2(x-1) - x^2(-1)\right) dx$

$= \int_0^1 \left(x^3 - x^2 + x^2\right) dx$

$= \int_0^1 x^3 \; dx$

$= \left[\frac{x^4}{4}\right]_0^1$ Since $\int x^3 dx = \frac{x^4}{4} + C$

$= \frac{1}{4}$

And $D_4 = \{(x, y): -1 \le x \le 0, \; -1 \le y \le -x-1\}$

Then $\iint\limits_{D_4} x^2 \, dA = \int_{-1}^{0} \int_{-1}^{-x-1} x^2 \, dy \; dx$

$= \int_{-1}^{0} \left(x^2 \; y\right)_{y=-1}^{y=-x-1} dx$

$= \int_{-1}^{0} \left(x^2(-x-1) - x^2(-1)\right) dx$

$= \int_{-1}^{0} \left(-x^3 - x^2 + x^2\right) dx$

$= \int_{-1}^{0} \left(-x^3\right) dx$

$= -\left[\frac{x^4}{4}\right]_{-1}^{0}$ Since $\int x^3 dx = \frac{x^4}{4} + C$

$= \frac{1}{4}$

Hence $\iint\limits_{D} x^2 \, dA = \iint\limits_{D_1} x^2 \; dA + \iint\limits_{D_2} x^2 \; dA + \iint\limits_{D_3} x^2 dA + \iint\limits_{D_4} x^2 dA$

$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$

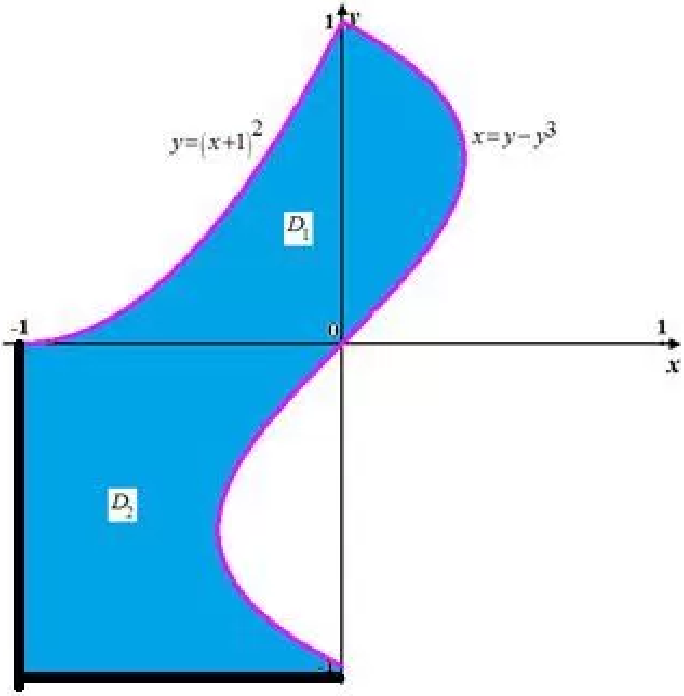
$= \boxed{1}$

Chapter 15 Multiple Integrals 15.3 56E

Consider the following regions:

$$x = y - y^3, y = (x+1)^2$$

The region of integration D can be partitioned into the union of two regions as shown in the following graph:



From the above graph, with this partition, the following can be determined:

$$I = \iint_D y\,dA$$
$$= \iint_{D_1} y\,dA + \iint_{D_2} y\,dA$$

Both regions should be integrated first on x.

For D_1, the equation for the left boundary needs to be inverted giving $\sqrt{y} - 1 \le x \le y - y^3$ and for D_2, the boundaries are $-1 \le x \le y - y^3$. Set up the limits of the double integral.

$$I = \int_0^1 y \int_{\sqrt{y}-1}^{y-y^3} dx dy + \int_{-1}^0 y \int_{-1}^{y-y^3} dy$$

For both terms, the inner integral is x; so move straight to evaluating the limits of integration.

$$I=\int_0^1 y\int_{\sqrt{y}-1}^{y-y^3} dxdy+\int_{-1}^0 y\int_{-1}^{y-y^3} dxdy$$
$$=\int_0^1 y\left(y-y^3-\sqrt{y}+1\right)dy+\int_{-1}^0 y\left(y-y^3+1\right)dy$$
$$=\int_0^1\left(-y^4+y^2-y^{\frac{3}{2}}+y\right)dy+\int_{-1}^0\left(-y^4+y^2+y\right)dy$$

Integrate on y as shown below:

$$I=\left(-\frac{1}{5}y^5+\frac{1}{3}y^3-\frac{2}{5}y^{\frac{5}{2}}+\frac{1}{2}y^2\right|_0^1+\left(-\frac{1}{5}y^5+\frac{1}{3}y^3+\frac{1}{2}y^2\right|_{-1}^0$$
$$=\left(\left(-\frac{1}{5}(1)^5+\frac{1}{3}(1)^3-\frac{2}{5}(1)^{\frac{5}{2}}+\frac{1}{2}(1)^2\right)-0+0-0+0\right)$$
$$+\left(-0+0+0-\left(-\frac{1}{5}(-1)^5+\frac{1}{3}(-1)^3+\frac{1}{2}(-1)^2\right)\right)$$
$$=-\frac{1}{5}+\frac{1}{3}-\frac{2}{5}+\frac{1}{2}-\frac{1}{5}+\frac{1}{3}-\frac{1}{2}$$
$$=-\frac{4}{5}+\frac{2}{3}$$
$$=\boxed{-\frac{2}{15}}$$

Chapter 15 Multiple Integrals 15.3 57E

Consider the following integral:

$$\iint_Q e^{-\left(x^2+y^2\right)^2}dA$$

Here, Q is the quarter circle with center at the origin and radius $\frac{1}{2}$ in the first quadrant.

Consider the following property:

If $m\le f\left(x,y\right)\le M$ for all $\left(x,y\right)$ in D, then

$$mA\left(D\right)\le\iint_D f\left(x,y\right)dA\le MA\left(D\right)$$

As the square of a quantity cannot be negative,

$$0 \le \left(x^2 + y^2\right)^2$$

$$0 \ge -\left(x^2 + y^2\right)^2 \text{ ...... (1)}$$

And, as Q is the quarter circle with center at the origin and radius $\frac{1}{2}$,

$$x^2 + y^2 \le \left(\frac{1}{2}\right)^2$$

$$x^2 + y^2 \le \frac{1}{4}$$

$$\left(x^2 + y^2\right)^2 \le \frac{1}{16}$$

$$-\left(x^2 + y^2\right)^2 \ge -\frac{1}{16} \text{ ...... (2)}$$

From inequality (1) and (2),

$$-\frac{1}{16} \le -\left(x^2 + y^2\right)^2 \le 0$$

$$e^{-\frac{1}{16}} \le e^{-\left(x^2+y^2\right)^2} \le e^0$$

$$e^{-\frac{1}{16}} \le f(x,y) \le 1$$

Therefore, the minimum and maximum value of the function $f(x,y) = e^{-\left(x^2+y^2\right)^2}$ is:

$$m = e^{-\frac{1}{16}} \text{ and } M = 1$$

The area of quarter circle Q will be,

$$A(Q) = \frac{1}{4} \times \left(\pi r^2\right)$$

$$= \frac{1}{4} \times \pi \left(\frac{1}{2}\right)^2$$

$$= \frac{\pi}{16}$$

Substitute $A(Q)=\frac{\pi}{16}, m=e^{-\frac{1}{16}}$ and $M=1$, the value of the integral will be,

$$mA(D)\le \iint_D f(x,y)\,dA \le MA(D)$$

$$e^{-\frac{1}{16}}\times\frac{\pi}{16}\le \iint_Q e^{-(x^2+y^2)^2}\,dA \le 1\times\frac{\pi}{16}$$

$$\frac{\pi}{16}e^{-\frac{1}{16}}\le \iint_Q e^{-(x^2+y^2)^2}\,dA \le \frac{\pi}{16}$$

Therefore, value of the integral is $\boxed{\frac{\pi}{16}e^{-\frac{1}{16}}\le \iint_Q e^{-(x^2+y^2)^2}\,dA \le \frac{\pi}{16}}$.

Chapter 15 Multiple Integrals 15.3 58E

Consider the integral:

$$\iint_T \sin^4(x+y)\,dA$$

Here, *T* is the triangle enclosed by the lines $y=0, y=2x$ and $x=1$.

As the minimum value of sine function is -1 and its maximum value is 1,

$$-1\le \sin(x+y)\le 1$$

$$0\le \sin^2(x+y)\le 1$$

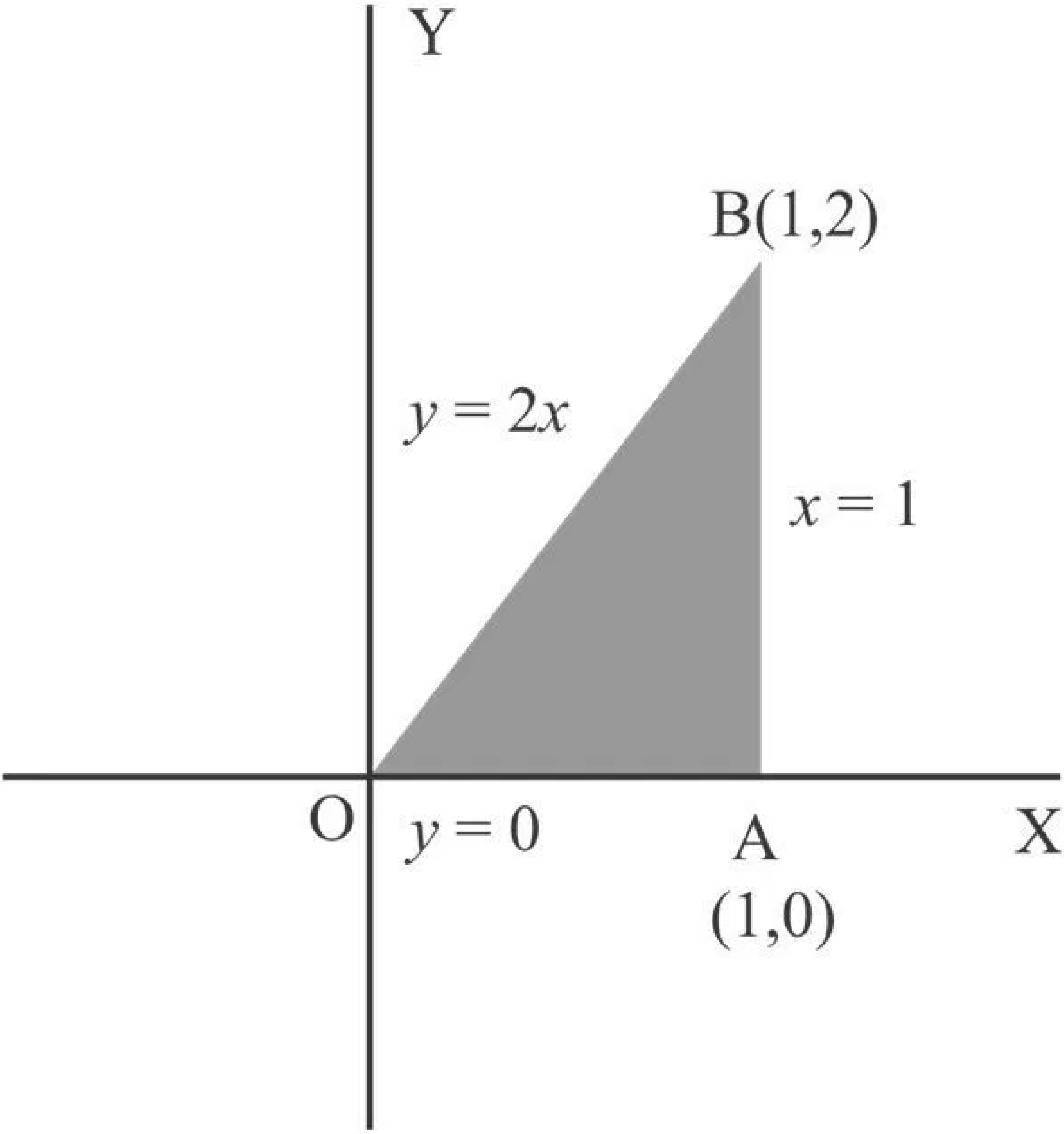
$$0\le \sin^4(x+y)\le 1$$

$$0\le f(x,y)\le 1$$

Therefore, the minimum and maximum value of given function $f(x,y)=\sin^4(x+y)$ is,

$m=0$ and $M=1$

Consider the triangle *T*:



In right-angled triangle OAB,

$$\text{OA} = 1 \text{ units}$$

$$\text{AB} = 2 \text{ units}$$

So, area of triangle *T* will be,

$$\begin{aligned} A(T) &= \frac{1}{2} \times \text{OA} \times \text{AB} \\ &= \frac{1}{2} \times 1 \times 2 \\ &= 1 \text{ square units} \end{aligned}$$

Substitute $A(T)=1, m=0$ and $M=1$, the value of the integral will be,

$$mA(T) \le \iint_T f(x,y)\,dA \le MA(T)$$

$$0\times 1 \le \iint_T \sin^4(x+y)\,dA \le 1\times 1$$

$$0 \le \iint_T \sin^4(x+y)\,dA \le 1$$

Therefore, value of the integral is $\boxed{0 \le \iint_T \sin^4(x+y)\,dA \le 1}$.

Chapter 15 Multiple Integrals 15.3 59E

Problem: We need to find the average value of the following function over the region D

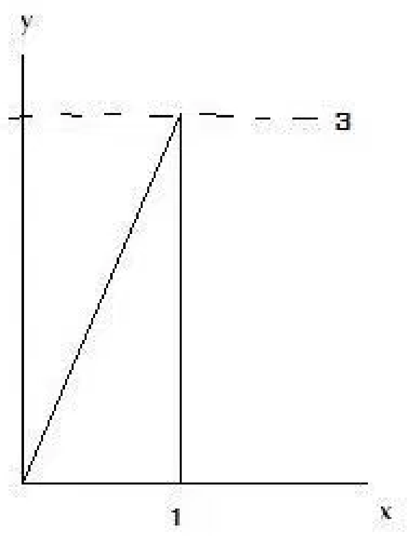
Function:

f(x,y)=xy

Region:

D is a triangle with vertices (0,0), (1,0), and (1,3)

So the graph of D looks like:



Notice the slope of the hypotenus is y=3x

Now we need to set up the integral:

$$f_{\text{ave}} = \frac{1}{A(D)} \iint f\left(x, y\right) \mathrm{DA}$$

From D we get:

$$0 \leq x \leq 1$$

and

$$0 \leq y \leq 3x$$

Area of D:

A(D) = 3/2

So, the integral can now be solved! :

$$f_{ave} = \frac{1}{\frac{3}{2}} \int_0^1 \int_0^{3x} xy \, dy \, dx$$

$$= \frac{2}{3} \int_0^1 \left[\frac{1}{2} xy^2 \right]_0^{3x} dx$$

$$= \frac{1}{3} \int_0^1 X(3X)^2 \, dX$$

$$= \frac{1}{3} \int_0^1 9x^3 \, dx$$

$$= \frac{3}{12} \left[x^4 \right]_0^1$$

$$= \frac{3}{4}(1) = \frac{3}{4}$$

ANSWER : $\frac{3}{4}$

Chapter 15 Multiple Integrals 15.3 60E

Consider the function:

$$f(x, y) = x \sin y$$

The region D can be defined as:

$$0 \le y \le x^2, \; 0 \le x \le 1$$

Use the formula of the average value of a function:

$$f_{avg} = \frac{1}{A}\iint_D f(x,y)\,dA$$

Where, area *A* of the region *D*.

Therefore,

$$\begin{aligned} A &= \iint_D dA \\ &= \int_0^1 \int_0^{x^2} dy\,dx \\ &= \int_0^1 [y]_0^{x^2} dx \\ &= \int_0^1 x^2 dx \\ &= \frac{[x^3]_0^1}{3} \\ &= \frac{1}{3} \end{aligned}$$

Substitute the value of *A* in the formula.

$$\begin{aligned} f_{avg} &= 3\int_0^1 \int_0^{x^2} x \sin y\,dy\,dx \\ &= 3\int_0^1 [-x\cos y]_0^{x^2} dx \\ &= 3\int_0^1 \left(-x\cos(x^2) + x\right) dx \\ &= 3\int_0^1 x\left(1-\cos(x^2)\right) dx \\ &= \frac{3}{2}\int_0^1 2x\left(1-\cos(x^2)\right) dx \end{aligned}$$

Substitute $x^2 = u$ and $2x = du$.

$$\begin{aligned} \frac{3}{2}\int_0^1 2x\left(1-\cos(x^2)\right) dx &= \frac{3}{2}\int_0^1 \left(1-\cos(u)\right) du \\ &= \frac{3}{2}\left[u - \sin(u)\right]_0^1 \\ &= \frac{3(1-\sin 1)}{2} \end{aligned}$$

Hence, the average value of *f* over the region *D* is $\boxed{\frac{3(1-\sin 1)}{2}}$.

Chapter 15 Multiple Integrals 15.3 61E

Given m and M are real numbers such that

$$m \le f(x,y) \le M \text{ , for all } (x,y) \text{ in D}$$

We have $m \le f(x,y) \le M$

Then taking double integral over region D

$$\iint_D m\,dA \le \iint_D f(x,y)\,dA \le \iint_D M\,dA$$

(Using property 8, if $f(x,y) \ge y(x,y) \forall (x,y)$ in D,

Then $\iint_D f(x,y)\,dA \ge \iint_D g(x,y)\,dA$)

From property 7, we have $\iint_D c\,f(x,y)\,dA = c\iint_D f(x,y)\,dA$

Then we get

$$m\iint_D 1.dA \le \iint_D f(x,y)\,dA \le M\iint_D 1.dA$$

But from property 10, $\iint_D 1.dA = A(D)$

Therefore $m\,A(D) \le \iint_D f(x,y)\,dA \le M\,A(D)$

Hence proved

Chapter 15 Multiple Integrals 15.3 62E

Consider the following double integral which is written as a sum of iterated integrals:

$$\iint_D f(x,y)\,dA = \int_0^1\int_0^{2y} f(x,y)\,dx\,dy + \int_1^3\int_0^{3-y} f(x,y)\,dx\,dy$$

The integral limit of the first integral are:

$x = 0$ to $x = 2y$

And limit of y is:

$y = 0$ to $y = 1$

Hence, the region of the first integral is;

$$D_1 = \{(x,y): 0 \le y \le 1,\ 0 \le x \le 2y\}$$

Now, the integral limit of the second integral is:

$x = 0$ to $x = 3 - y$

And

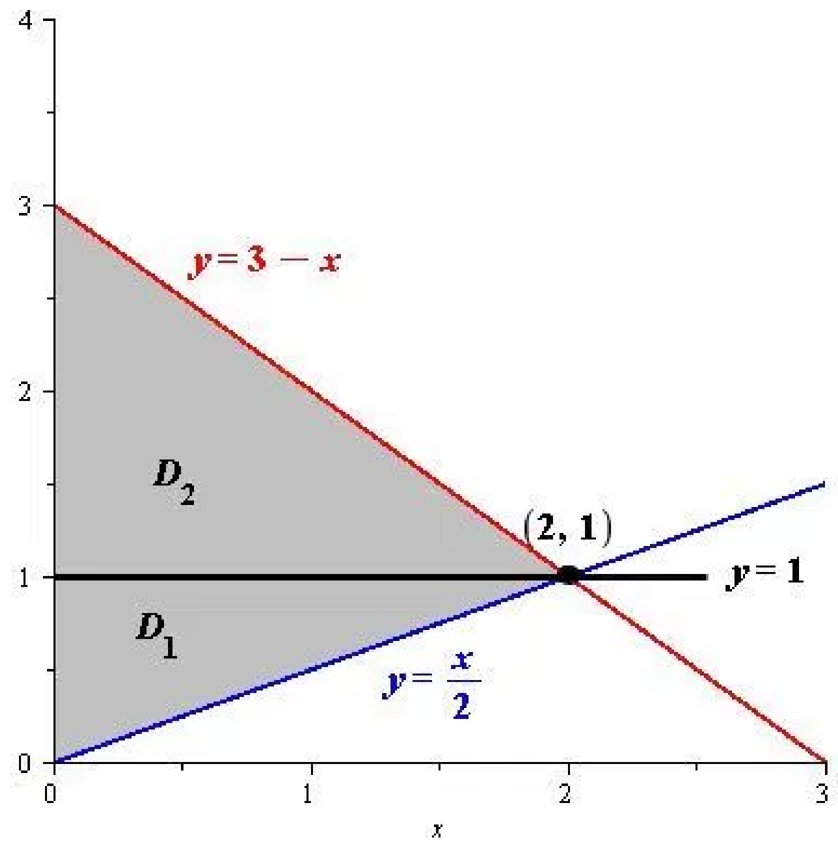
$y = 1$ to $y = 3$

Hence the region of the second integral is given by:

$$D_2 = \{(x,y): 1 \le y \le 3,\ 0 \le x \le 3 - y\}$$

Now, since it is known that the region D is the union of two regions D1 and D2, hence the sketch of the region D is given by the union of the graphs of these two regions.

The sketch of the region is:



From the graph of the domain the integral limits of the iterated integrals can be easily determined.

Therefore, the required iterated integral with reversed order of integration is:

$$\iint_D f(x,y)\,dA = \int_0^2 \int_{x/2}^{1} f(x,y)\,dy\,dx + \int_0^2 \int_1^{3-x} f(x,y)\,dy\,dx$$

Chapter 15 Multiple Integrals 15.3 63E

Consider the integral,

$$\iint_D (x+2)\,dA$$

The region is $D=\left\{(x,y)\middle|0\le y\le\sqrt{9-x^2}\right\}$

The objective is to evaluate the integral.

Here $y=0,\ y=\sqrt{9-x^2}$ then

$$y^2=9-x^2$$

$$x^2+y^2=9$$

This is a circle with radius 3.

And the *x* limits are,

$$x^2=9$$

$$x=\pm 3$$

Thus, $D=\left\{(x,y)\middle|-3\le x\le 3,\ 0\le y\le\sqrt{9-x^2}\right\}$

The integral can be evaluated as follows:

$$\iint_D (x+2)\,dA = \int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} (x+2)\,dy\,dx$$

$$= \int_{-3}^{3} (x+2)[y]_0^{\sqrt{9-x^2}}\,dx$$

$$= \int_{-3}^{3} (x+2)\sqrt{9-x^2}\,dx$$

$$= \int_{-3}^{3} x\sqrt{9-x^2}\,dx + 2\int_{-3}^{3} \sqrt{9-x^2}\,dx$$

Since $\int_{-a}^{a} f(x) = 0$ When $f(x)$ is an odd function.

Hence, by symmetry, $\int_{-3}^{3} x\sqrt{9-x^2}\,dx = 0$

Thus, the integral becomes,

$$\iint_D (x+2)\,dA = 0 + 2\int_{-3}^{3} \sqrt{9-x^2}\,dx$$

$$= 2\left[\frac{1}{2}x\sqrt{-x^2+9} + \frac{9}{2}\arcsin\left(\frac{1}{3}x\right)\right]_{-3}^{3}$$

$$= 2\left[\left(0+\frac{9}{4}\pi\right) - \left(-\frac{9}{4}\pi\right)\right]$$

$$= 2\left[\frac{9}{2}\pi\right]$$

$$= \boxed{9\pi}$$

Therefore, the answer is $\iint_D (x+2)\,dA = \boxed{9\pi}$

Chapter 15 Multiple Integrals 15.3 64E

Consider the integral,

$$\iint_D \sqrt{R^2 - x^2 - y^2}\, dA$$

Here D is a disk with center at the origin and radius R.

So,

$$x^2 + y^2 = R^2$$
$$y = \pm\sqrt{R^2 - x^2}$$

If $y = 0$, then

$$x^2 = R^2 \Rightarrow x = \pm R$$

Therefore the region is $D = \left\{(x, y) \middle| -R \le x \le R, -\sqrt{R^2 - x^2} \le y \le \sqrt{R^2 - x^2}\right\}$

Thus,

Then, we have

$$\iint_D \sqrt{R^2 - x^2 - y^2}\, dA = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2 - x^2 - y^2}\, dy dx$$
$$= \int_{-R}^{R} \left[\frac{1}{2}\left(R^2 - x^2\right)\pi\right]$$
$$= \left[\frac{1}{2}\pi\left(-\frac{1}{3}x^3 + R^2 x\right)\right]_{-R}^{R}$$
$$= \frac{1}{3}\pi R^3 - \left(-\frac{1}{3}\pi R^3\right)$$
$$= \boxed{\frac{2}{3}\pi R^3}$$

Therefore; $\iint_D \sqrt{R^2 - x^2 - y^2}\, dA = \boxed{\frac{2\pi}{3} R^3}$

Chapter 15 Multiple Integrals 15.3 65E

Consider the integral,

$$\iint_D (2x+3y)\,dA$$

Here D is the rectangle $D=\{(x,y)|0\le x\le a,\quad 0\le y\le b\}$

The objective is to evaluate the integral.

Since the limits for *x* and *y* are $0\le x\le a,\quad 0\le y\le b$

Therefore the integral can be evaluated as follows:

$$\iint_D (2x+3y)\,dA=\int_0^a\int_0^b (2x+3y)\,dy\,dx$$

$$=\int_0^a\left[2xy+\frac{3y^2}{2}\right]_0^b dx$$

$$=\int_0^a\left(2xb+\frac{3}{2}b^2\right)dx$$

$$=\left[2b\frac{x^2}{2}+\frac{3}{2}b^2x\right]_0^a$$

$$=ba^2+\frac{3}{2}ab^2$$

$$=\boxed{a^2b+\frac{3}{2}ab^2}$$

Therefore; $\iint_D (2x+3y)\,dA=\boxed{a^2b+\frac{3}{2}ab^2}$

Chapter 15 Multiple Integrals 15.3 66E

Find the value of the given integral.

$$\iint_D \left(2+x^2y^3-y^2\sin x\right)\,dA$$
$$=\int_{-1}^{1}\int_{0}^{1-x}\left(2+x^2y^3-y^2\sin x\right)dydx$$
$$=\int_{-1}^{1}\left[2y+\frac{x^2y^4}{4}-\frac{y^3\sin x}{3}\right]_0^{1-x}dx$$
$$=\int_{-1}^{1}\left(2(1-x)+\frac{x^2(1-x)^4}{4}-\frac{(1-x)^3\sin x}{3}\right)dx$$
$$=\begin{bmatrix}2x-x^2+\frac{x^7}{28}-\frac{x^6}{6}+\frac{3x^5}{10}-\frac{x^4}{4}+\frac{x^3}{12}+\left(x^2-2x-1\right)\sin x\\ -\frac{1}{3}\left(x^3-3x^2-3x+5\right)\cos x\end{bmatrix}_{-1}^{1}$$ Therefore,
$$=\frac{508}{105}+\frac{4\cos 1}{3}$$
$$\approx 5.5585$$
$$\iint_D \left(2+x^2y^3-y^2\sin x\right)\,dA=\boxed{5.5585}.$$

Chapter 15 Multiple Integrals 15.3 67E

Since this problem specifies the use of geometry and symmetry rather than the regular procedure for solving double integrals, we try to be clever about the shape of the integral rather than solving with the usual procedure.

Split the integral up across the addition signs:

$$\iint_D ax^3dA+\iint_D by^3dA+\iint_D \sqrt{a^2-x^2}\,dA$$

We deal with each of these integrals separately.

First calculate $\iint_D ax^3dA$. We notice that this is an odd function in x and is symmetric across the origin. We therefore choose to integrate in terms of x first:

$$\int_{-b}^{b}\int_{-a}^{a}ax^3\,dxdy$$

Since this function is symmetric across the origin in x, its integral in x (the inside integral) equals 0. This leaves us with

$$\int_{-b}^{b}0dy$$

This equals 0.

Now calculate $\iint_D by^3 dA$. We notice that this is an odd function in y and is symmetric across the origin. We therefore choose to integrate in terms of y first:

$$\int_{-a}^{a}\int_{-b}^{b} by^3 dydx$$

Since this function is symmetric across the origin in y, its integral in y (the inside integral) equals 0. This leaves us with

$$\int_{-a}^{a} 0dx$$

This equals 0.

Finally, calculate $\iint_D \sqrt{a^2 - x^2}\,dA$. We examine what this function is geometrically. Let

$$z = \sqrt{a^2 - x^2}$$
$$z^2 = a^2 - x^2$$
$$x^2 + z^2 = a^2$$

Parallel to the xz-plane, this is a circle of radius a, so the shape is a cylinder with axis the y-axis such that every cross-section at any value of y is a circle of radius a. Since we are integrating $\sqrt{a^2 - x^2}$ and not $-\sqrt{a^2 - x^2}$, we are looking at the volume of the positive half of the cylinder above the xy-plane, with each cross-section a semicircle.

The formula for the volume of a cylinder is the area of the circular base times the height, or $V = \pi r^2 h$. The volume of a half-cylinder is half that, or $V = \dfrac{\pi r^2 h}{2}$.

Since the region of integration goes from $-a$ to a in the x, it encompasses the whole diameter of the half-cylinder of radius a, and the limits in the y, $-b$ and b, give the half-cylinder a geometric height of $2b$. Therefore, plugging into the formula for volume, we get

$$\begin{aligned} V &= \frac{\pi r^2 h}{2} \\ &= \frac{\pi a^2 (2b)}{2} \\ &= \pi a^2 b \end{aligned}$$

This is the volume of the half-cylinder. Since the volume under the surface equals the integral, we have $\iint_D \sqrt{a^2 - x^2}\,dA = \pi a^2 b$.

Since the first two integrals evaluated to 0, the sum of all three integrals, and therefore the total solution, is $\boxed{\pi a^2 b}$.

Chapter 15 Multiple Integrals 15.3 68E

Graph the solid bounded by the plane $x+y+z=1$ and the paraboloid $z=4-x^2-y^2$ and find its exact volume (Use your CAS to do the graphing, to find the equations of the boundary curves of the region of integration, and to evaluate the double integral).

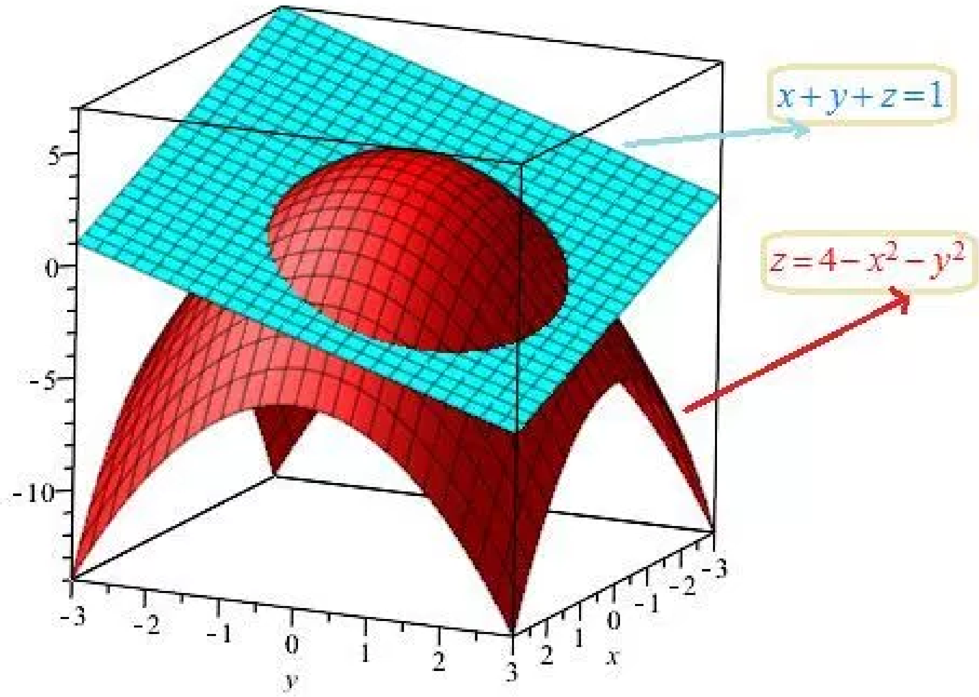
In this problem, use Maple, (input commands and operations will vary among calculators and computer algebra systems).

First obtain the graph of the solid figure using the Maple command

Plot3d({1-x-y,4-x^2-y^2},x=-3..3,y=-3..3,axes=boxed);

The diagram shown below :

$plot3d(\{1-x-y, 4-x^2-y^2\}, x=-3..3, y=-3..3, axes=boxed);$



From this see that $z=4-x^2-y^2$ lies below $x+y+z=1$, $1-x-y\le z\le 4-x^2-y^2$ or and the solid figure is the "cap" of the paraboloid sliced by the plane.

The boundary region of integration is as shown below: $\frac{1}{2}\left(1-\sqrt{14}\right)<x<\frac{1}{2}\left(1+\sqrt{14}\right)$ and

$\frac{1}{2}-\frac{1}{2}\sqrt{13+4x-4x^2}<y<\frac{1}{2}+\frac{1}{2}\sqrt{13+4x-4x^2}$

and allows us to conclude that our boundary of integration is

$$R=\left\{(x,y)\,|\,\frac{1}{2}\left(1-\sqrt{14}\right)<x<\frac{1}{2}\left(1+\sqrt{14}\right),\frac{1}{2}-\frac{1}{2}\sqrt{13+4x-4x^2}<y<\frac{1}{2}+\frac{1}{2}\sqrt{13+4x-4x^2}\right\}$$

Recall that the volume of a solid over a rectangular region can be calculated by the appropriate integral $\int_a^b \int_c^d f(x,y)\,dy\,dx$

To find the volume of the solid, need to obtain the limits of integration and appropriate function to integrate. Since the solid lies in between

$1-x-y \le z \le 4-x^2-y^2$, our function is $f(x,y)=\left(4-x^2-y^2\right)-\left(1-x-y\right)$.

From the boundary region R, to obtained

$\frac{1}{2}\left(1-\sqrt{14}\right)<x<\frac{1}{2}\left(1+\sqrt{14}\right)$ and $\frac{1}{2}-\frac{1}{2}\sqrt{13+4x-4x^2}<y<\frac{1}{2}+\frac{1}{2}\sqrt{13+4x-4x^2}$

This allows us to write the volume of the solid as

$$\int_{\frac{1}{2}(1-\sqrt{14})}^{\frac{1}{2}(1+\sqrt{14})} \int_{\frac{1}{2}-\frac{1}{2}\sqrt{13+4x-4x^2}}^{\frac{1}{2}+\frac{1}{2}\sqrt{13+4x-4x^2}} \left(4-x^2-y^2\right)-\left(1-x-y\right)\,dy\,dx$$

Next, use Maple to evaluate the integral

$$\int_{\frac{1}{2}(1-\sqrt{14})}^{\frac{1}{2}(1+\sqrt{14})} \int_{\frac{1}{2}-\frac{1}{2}\sqrt{13+4x-4x^2}}^{\frac{1}{2}+\frac{1}{2}\sqrt{13+4x-4x^2}} \left(4-x^2-y^2\right)-\left(1-x-y\right)\,dy\,dx$$

Int(int(4-x^2-y^2-1+x+y,y=((1/2)-(1/2)

using symbolic commands

- $int\left(int\left(4-x^2-y^2-1+x+y, y=\left(\frac{1}{2}-\frac{1}{2}\sqrt{13+4x-4x^2}\right) .. \left(\frac{1}{2}+\frac{1}{2}\sqrt{13+4x-4x^2}\right)\right), x=\left(\frac{1}{2}-\frac{1}{2}\sqrt{14}\right) .. \left(\frac{1}{2}+\frac{1}{2}\sqrt{14}\right)\right);$

$\frac{49}{8}\pi$

Which outputs the volume as $\boxed{\frac{49\pi}{8}}$.