High Order Thinking Skills (HOTS)

Q.1. Show that the points (x, y), given by $x = \frac{2at}{1+t^2}$ and $y = \frac{a(1-t^2)}{1+t^2}$ lies on a circle for all real values of t such that $-1 \le t \le 1$, where $'a'$ is any given real number.

Sol. Given, $x = \frac{2at}{(1+t^2)}, y = \frac{a(1-t^2)}{(1+t^2)}$

We have

$$x^2 + y^2 = \left(\frac{2at}{1+t^2}\right)^2 + \left(\frac{a(1-t^2)}{1+t^2}\right)^2$$

$$= \frac{4a^2t^2}{(1+t^2)^2} + \frac{a^2(1-t^2)^2}{(1+t^2)^2}$$

$$= \frac{4a^2t^2 + a^2(1-t^2)^2}{(1+t^2)^2}$$

$$= \frac{a^2[4t^2 + 1 - 2t^2 + t^4]}{(1+t^2)^2}$$

$$= \frac{a^2(1+2t^2+t^4)}{(1+t^2)^2} = \frac{a^2(1+t^2)^2}{(1+t^2)^2} = a^2$$

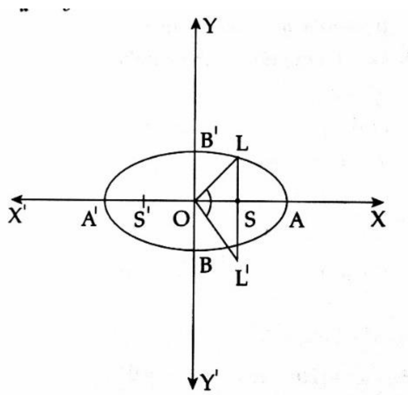
$$\Rightarrow x^2 + y^2 = a^2.$$

Thus, all points (x, y) lies on a circle with centre $(0, 0)$ and radius 'a' units.

Q.2. If a latus rectum of an ellipse subtends a right angle at he centre of the ellipse, then find the eccentricity of the ellipse.

Sol. Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$$



Given, Latus rectum LL' subtends right angle at the centre.

$\therefore\ \angle LOL' = 90^o$

Since OF is the median of $\Delta OLL'$

$\therefore\ \angle LOF = \angle L'OF = 45^o$

In the right angles triangles OFL, we have

$$tan45^o = \frac{OF}{LF}$$

$$\Rightarrow 1 = \frac{c}{\frac{b^2}{a}}$$

$$\Rightarrow \frac{b^2}{a} = c$$

$$\Rightarrow b^2 = a^2e\ \ [\therefore c = ae]$$

$$e^2 = 1 - \frac{b^2}{a^2}$$

$$= 1 - \frac{a^2e}{a^2}$$

$$= 1 - e$$

$$\Rightarrow e^2 + e - 1 = 0$$

$$\Rightarrow e = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1}$$

$$= \frac{-1 \pm \sqrt{5}}{2}$$

Since $0 < e < 1, so$

$$e = \frac{-1 + \sqrt{5}}{2}$$

Q.3. If e and e' are the eccentricities of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and is conjugate hyperbola, then prove that $\frac{1}{e^2} + \frac{1}{e^2} = 1$.

Sol. Given hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (1)$$

Here, eccentricity, $e = \sqrt{1 + \frac{b^2}{a^2}}$

$$\Rightarrow e^2 = 1 + \frac{b^2}{a^2}$$

$$\Rightarrow e^2 = \frac{a^2 + b^2}{a^2}$$

$$\Rightarrow \frac{1}{e^2} = \frac{a^2}{a^2 + b^2} \qquad (2)$$

The equation of the conjugate hyperbola of (1) is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

Here, eccentricity, $e' = \sqrt{1 + \frac{a^2}{b^2}}$

$$\Rightarrow e'^2 = 1 + \frac{a^2}{b^2}$$

$$\Rightarrow e'^2 = \frac{b^2 + a^2}{b^2}$$

$$\Rightarrow \frac{1}{e'^2} = \frac{b^2}{a^2 + b^2} \qquad (3)$$

$$(2) + (3) \Rightarrow \frac{1}{e^2} + \frac{1}{e'^2} = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 + b^2}$$

$$\Rightarrow \frac{1}{e^2} + \frac{1}{e'^2} = 1$$

Q.4. If two diameters of a circle lie along the lines $x - y = 9$ and $x - 2y = 7$ and the area of the circle is 38.5 cm^2, then find the equation of the circle.

Sol. Let the equation of the circle with centre (h, k) and radius $'r'$ be

$$(x - h)^2 + (y - k)^2 = r^2 \qquad (1)$$

The two diameters of the circle are

$$x - y = 9 \qquad (2)$$

$$x - 2y = 7 \qquad (3)$$

Since the centre of a circle always lie on the diameter, so

$$(2) \Rightarrow h - k = 9 \qquad (4)$$

$$(3) \Rightarrow h - 2k = 7 \qquad (7)$$

$(4)-(5) \Rightarrow k=2$

$(4) \Rightarrow h=11$

$\therefore$ centre $=(11,2)$

Also, $\pi r^2=38.5$

$\Rightarrow r^2=38.5\times\frac{7}{22}$

$\Rightarrow r^2=\frac{49}{4} \Rightarrow r=\frac{7}{2}$

$\therefore (1) \Rightarrow (x-11)^2+(y-2)^2=\frac{49}{4}$, is the required equation of the circle.

Q.5. The sides of a rectangle are given by the equations $x=-2, x=4, y=-2$ and $y=5$. Find the equation of the circle drawn on the diagonal of this rectangle its diameter.

Sol. The given equation of the sides are $x=-2, x=4, y=-2, y=5$

$\therefore$ The ends of any one of the diameter are $A(-2,-2)$ and $B(4,5)$.

We know equation of a circle with $(x_1, y_1), (x_2, y_2)$ as the two ends of any of its diameter is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$ (1)

So, let $(x_1, y_1)=(-2,-2)$ and $(x_2, y_2)=(4,5)$.

$(1) \Rightarrow (x+2)(x-4)+(y+2)(y-5)=0$

$\Rightarrow x^2-2x-8+y^2-3y-10=0$

$\Rightarrow x^2+y^2-2x-3y-18=0$, is the required equation of the circle.