Linear Programming

Linear Programming

- The process of taking various linear inequalities relating to a given situation and then finding the best obtainable value satisfying the required conditions is formally known as linear programming.

- The problem which seeks to maximize or minimize a linear function (e.g. profit or cost expressed in some variables) subject to some certain constraints as determined by a set of linear inequalities is called an optimization problem.

- Linear programming problems are a special case of optimization problems.

- A linear programming problem (LPP) is one that is concerned with finding the optimum value (maximum or minimum value) of a linear function (called objective function) of several variables (say x and y), subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear constraints).

- For example, consider the following problem.

  Two tailors A and B are working in a tailoring shop. Tailor A and B earn Rs 150 and Rs 200 per day respectively. Tailor A can stitch 6 shirts and 4 pants per day while tailor B can stitch 10 shirts and 4 pants per day. It is desired to produce at least 60 shirts and 32 pants at a minimum labour cost. Express the given situation in the form of a linear programming problem to find the number of days for which A and B should work.

  Solution:

  Let the tailors A and B work for x days and y days respectively.

  Then, our problem is to minimise $C = 150x + 200y$

  Subject to the constraints,

  $6x + 10y \geq 60$

  Or, $3x + 5y \geq 30$

  and $4x + 4y \geq 32$

  Or, $x + y \geq 8$, where $x \geq 0$, $y \geq 0$

  Thus, the linear programming problem is obtained as

  Minimise $C = 150x + 200y$

Subject to,

$3x + 5y \geq 30$

$x + y \geq 8$

$x, y \geq 0$

Term	Definition	Expression in above problem
Objective function	Linear function $Z = ax + by$, which has to be minimized or maximised	$C = 150x + 200y$
Decision variables	Deciding variables in the LPP	x and y
Constraints	Linear inequalities or equations or restrictions on the variables of a linear programming problem	$3x + 5y \geq 30$ $x + y \geq 8$ $x, y \geq 0$
Optimisation Problem	Problem which seeks to minimise or maximise a linear function subject to certain constraints as determined by a set of linear inequalities	Min $C = 150x + 200y$ $3x + 5y \geq 30$ $x + y \geq 8$ $x, y \geq 0$

Solved Examples

Example 1:

Two positive numbers x and y are such that their sum is at least 15 and their difference is at most 7. Express the given situation mathematically in the form of a linear programming problem, which aims at maximising the product of the two numbers.

Solution:

Let the two numbers be x and y.

It is given that $x \geq 0, y \geq 0$

The sum of x and y should be at least 15.

$\Rightarrow x + y \geq 15$

The difference of x and y should be at most 7.

$\Rightarrow x - y \leq 7$

We have to maximise the product xy.

Thus, the linear programming problem can be formulated as:

Max $P = xy$

Subject to,

$x + y \geq 15$

$x - y \leq 7$

$x, y \geq 0$

Example 2:

A decorative item dealer deals in only two items – wall hangings and artificial plants. He has Rs 15000 to invest and a space to store at the most 80 pieces. A wall hanging costs him Rs 300 and an artificial plant costs him Rs 150. He can sell a wall hanging at the profit of Rs 50 and an artificial plant at the profit of Rs 18. Assume that he can sell all the items that he bought. Make a mathematical model to maximise his profit with the given conditions.

Solution:

Let the person sell x wall hangings and y artificial plants.

	Wall-hangings	Artificial Plants	Available
No. of items he sells	x	y	80
Unit cost	300	150	15000
Unit profit	50	18	

The objective is to maximise the profit.

$\therefore Z = 50x + 18y$

It is given that,

$x + y \leq 80$

$300x + 150y \leq 15000$

$\Rightarrow 2x + y \leq 100$

Also, $x, y \geq 0$

Thus, the required linear programming problem is

Max $Z = 50x + 18y$

Subject to,

$x + y \leq 80$

$2x + y \leq 100$

$x, y \geq 0$

Graphical Solution to Linear Programming Problems

- Consider the following linear programming problem:

  Maximise $Z = x + y$

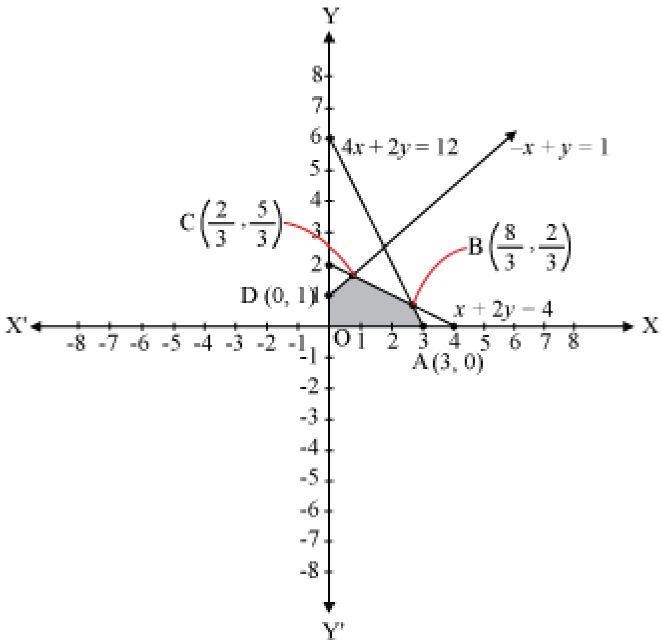
  Subject to $4x + 2y \leq 12$

  $-x + y \leq 1$

  $x + 2y \leq 4$

  $x, y \geq 0$

  The above problem can be represented on a graph as:



- The graph of this system (shown by shaded region) constitutes all the common points to all the inequalities.
  Basic constituents of this graph:

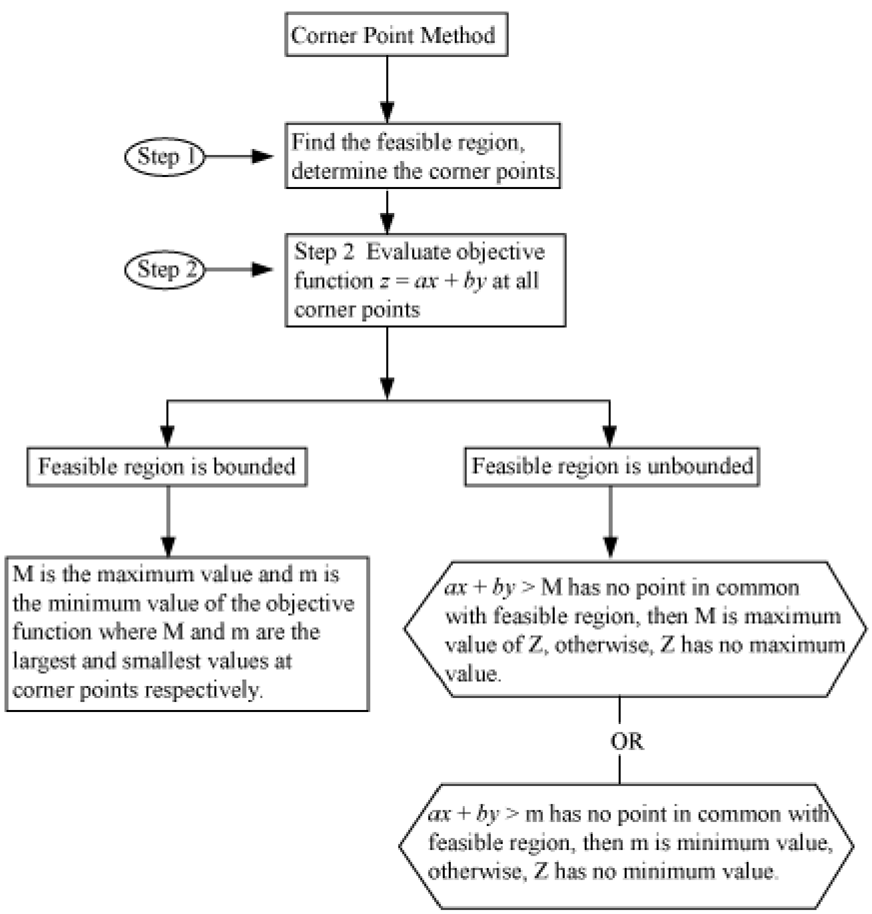
- Each point in the shaded region represents a feasible choice.

- The shaded region is called the feasible solution to the given problem.

- The common region determined by all the constraints including non-negative constraints $x, y \geq 0$ of a linear programming problem is called the feasible region for the problem.
  In the above example, OABCD is the feasible region.

- The region other than the feasible region is called infeasible region.

- Points within and on the boundary of the feasible region are called feasible solutions.
- Any point in the feasible region that gives an optical value of the objective function is called optical solution.
- Let R be the feasible region for a linear programming problem and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function Z has both a maximum and minimum value on R and each of these occurs at corner (vertex) point of R.
- If R is unbounded, then maximum or minimum value of the objective function may or may not exist. If it exists, then it occurs at a corner point of R.
- Corner point method for solving linear programming method consists of the following steps.



- Some important different types of linear programming problems:
- Manufacturing Problem
- Transportation Problem
- Diet Problem

Solved Examples

Example 1:

Solve the following linear programming problem graphically.

Minimise $Z = x + y$

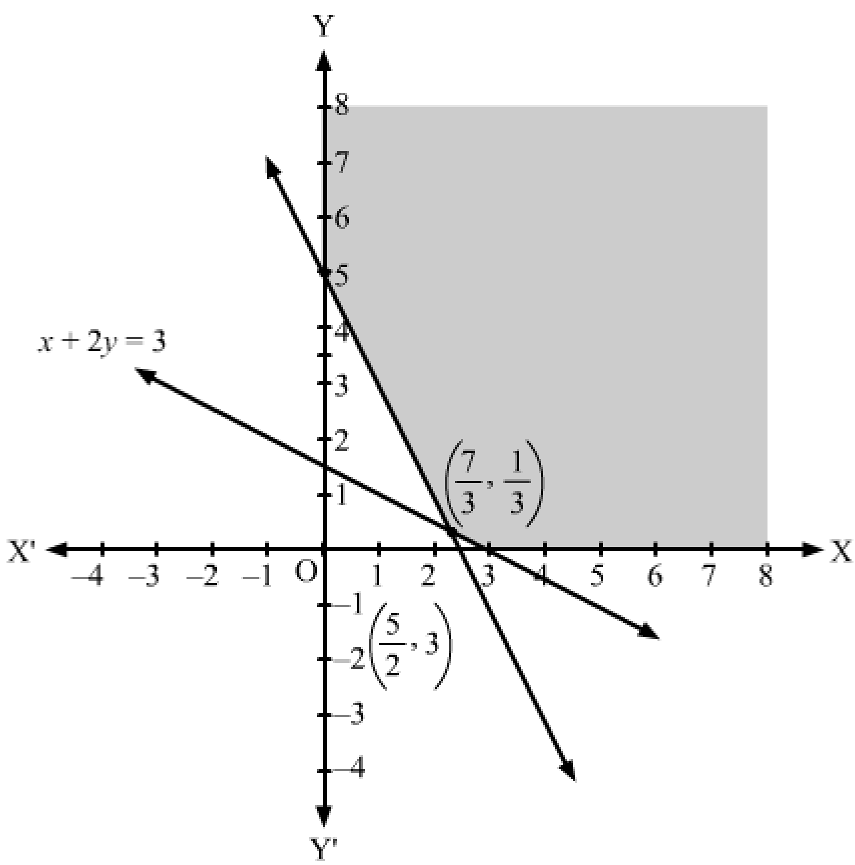
Subject to $x + 2y \geq 3$

$2x + y \geq 5$

$x, y \geq 0$

Solution:

First we draw the lines $x + 2y = 3$ and $2x + y = 5$ on graph as



Minimum value of objective function $Z = x + y$ is obtained at $\left(\frac{7}{3}, \frac{1}{3}\right)$.

Thus, optimal solution is $x = \frac{7}{3}, y = \frac{1}{3}$

Optimal value = $\frac{7}{3} + \frac{1}{3} = \frac{8}{3}$

Example 2:

A pharmaceutical company manufactures two types of drugs A and B. The combined production of the drugs should not exceed 900 units per week. The demand for drug B is at most half of drug A. Also, the production level of drug A is less than the sum of 3 times the production level of drug B and 500 units. If the company makes a profit of Rs 10 and Rs 15 respectively on selling of one unit of drug A and B each, then how many of each drug should be produced in a week in order to maximise profit?

[Assume that all drugs manufactured are able to sell]

Solution:

Let x units of drug A are produced and y units of drug B are produced in a week.

Objective function is

Maximise $Z = 10x + 15y$

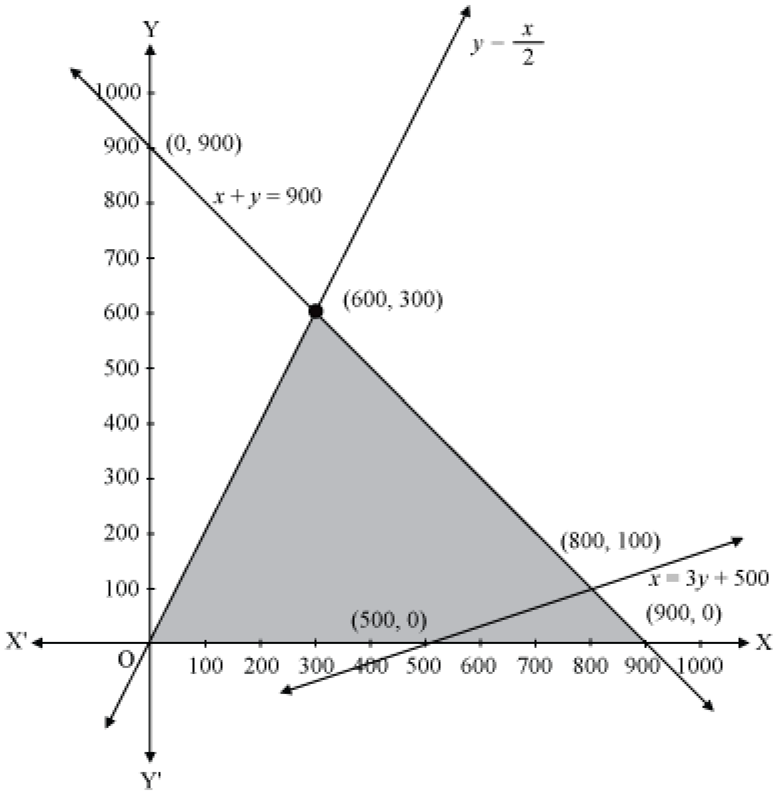
Subject to constraints $x + y \leq 900$

$y \leq \frac{x}{2}$

$x \leq 3y + 500$

$x, y \geq 0$

The inequalities can now be represented on graph as



It is observed that the corner points are (0, 0), (500, 0), (800, 100), and (600, 300).

The value of $Z = 10x + 15y$ is maximum at the corner point (600, 300).

Therefore, feasible solution is $x = 600$, $y = 300$

Optimal value $= 10x + 15y = 10\,(600) + 15(300) = 6000 + 4500 = 10{,}500$

Thus, in order to maximise the profit, 600 units of drug A and 300 units of drug B are to be produced.