CBSE Board
Class XII Mathematics
Sample Paper 4 – Solution

Part A

1. **Correct option: A**
Explanation:-

$$\begin{vmatrix} a^2 & a & 1 \\ \cos nx & \cos(n+1)x & \cos(n+2)x \\ \sin nx & \sin(n+1)x & \sin(n+2)x \end{vmatrix}$$

Let $nx = u, (n+1)x = v, (n+2)x = w$

$$\Rightarrow \begin{vmatrix} a^2 & a & 1 \\ \cos u & \cos v & \cos w \\ \sin u & \sin v & \sin w \end{vmatrix}$$

$\Rightarrow a^2 \sin(w - v) - a \sin(w - u) + \sin(v - u)$

$\Rightarrow a^2 \sin x - a \sin 2x + \sin x$

$\Rightarrow$ It is independent of n.

2. **Correct option: C**
Explanation:-

$$A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$(A + 2B) = \begin{bmatrix} -4 & 1 \\ 5 & 6 \end{bmatrix}$$

$$(A + 2B)' = \begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$$

3. **Correct option: C**
Explanation:-

$\hat{i} \cdot (\hat{j} \times k) + j \cdot (\hat{i} \times \hat{k}) + k \cdot (\hat{i} \times \hat{j})$

$= \hat{i} \cdot \hat{i} + \hat{j} \cdot (-\hat{j}) + k \cdot k$

$= 1 - 1 + 1$

$= 1$

4. **Correct option: D**

Explanation:-

$P(A) = 0.25,\ P(B) = 0.5,\ P(A \cap B) = 0.14$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = 0.25 + 0.5 - 0.14$

$P(A \cup B) = 0.61$

$P\left(\overline{A} \cap \overline{B}\right) = 1 - P(A \cup B)$

$P\left(\overline{A} \cap \overline{B}\right) = 1 - 0.61$

$P\left(\overline{A} \cap \overline{B}\right) = 0.39$

5. **Correct option: A**

Explanation:-

The given differential equation is $\frac{dy}{dx} + \frac{2y}{x} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{2y}{x}$

$\Rightarrow -\int \frac{dy}{2y} = \int \frac{dx}{x}$

$\Rightarrow -\frac{1}{2}\log y = \log x + c$

$\therefore y = x^{-2} + C$ ... (i)

When x = 1, y = 1

$\Rightarrow 1 = 1 + C$

$\Rightarrow C = 0$

$\Rightarrow y = \frac{1}{x^2}$

6. **Correct option: C**

Explanation:-

$4\cos^{-1}x + \sin^{-1}x = \pi$

$\Rightarrow 3\cos^{-1}x + \cos^{-1}x + \sin^{-1}x = \pi$

$\Rightarrow 3\cos^{-1}x + \frac{\pi}{2} = \pi$

$\Rightarrow \cos^{-1}x = \frac{\pi}{6}$

$\Rightarrow x = \cos\frac{\pi}{6}$

$\Rightarrow x = \frac{\sqrt{3}}{2}$

7. Correct option: B

Explanation:-

The parabola touches the y-axis at x = 0

i.e. y = 2 and y = −2

So, the required area is

$$A = \int_{-2}^{2} x\,dy$$

$$= \int_{-2}^{2} \left(4 - y^2\right) dy$$

$$= \left[4y - \frac{y^3}{3}\right]_{-2}^{2}$$

$$= 4(2+2) - \left(\frac{8}{3} + \frac{8}{3}\right)$$

$$= 16 - \frac{16}{3}$$

$$= \frac{32}{3} \text{ sq. units}$$

8. Correct option: B

Explanation:-

$f(x) = x - e^x + \tan(2\pi/7)$

$f'(x) = 1 - e^x$

The function f increases when $f'(x) > 0$

$\Rightarrow 1 - e^x > 0$

$\Rightarrow e^x < 1$

$\Rightarrow e^x < e^0$

$\Rightarrow x < 0$

Thus, $x \in (-\infty, 0)$

Hence, the interval of increase of the function is $(-\infty, 0)$.

9. Correct option: A

Explanation:-

$2x + y - z = 5$

Dividing both sides by 5,

$$\frac{2x}{5} + \frac{y}{5} - \frac{z}{5} = 1$$

$$\frac{x}{\frac{5}{2}} + \frac{y}{5} + \frac{z}{-5} = 1$$

It is known that the equation of a plane in intercept form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, where a, b and c are the intercepts cut off by the plane at x, y, and z-axes respectively.

Thus, the intercept cut off by the given plane on the x-axis is $\frac{5}{2}$.

10. Correct option: C

Explanation:-

$$\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}=\lambda$$

$\Rightarrow x=3+\lambda,\ y=4+2\lambda,\ z=5+2\lambda$

$\Rightarrow (3+\lambda, 4+2\lambda, 5+2\lambda)$ point is on the $x+y+z=17$ plane.

$\Rightarrow 3+\lambda+4+2\lambda+5+2\lambda=17$

$\Rightarrow \lambda=1$

$\Rightarrow (3+\lambda, 4+2\lambda, 5+2\lambda)=(4,6,7)$

Distance between $(4,6,7)$ and $(3,4,5)$

$=\sqrt{1+4+4}=3$

11. Correct option: B

Explanation:-

Given $a * b = a + b - ab$

Let the identity element be e, then

$a * e = a$

$\Rightarrow a + e - ae = a$

$\Rightarrow e = 0$

12. Correct option: A

Explanation:-

We have,

$$f(x)=\begin{cases} kx^2, & x \le 2 \\ 3, & x > 2 \end{cases}$$

As f is continuous at x = 2

$\Rightarrow \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

$\Rightarrow \lim_{x\to 2} kx^2 = 3$

$\Rightarrow k(2)^2 = 3$

$\Rightarrow 4k = 3$

$\Rightarrow k = \frac{3}{4}$

13. Correct option: B

Explanation:-

$y = x^3 - x + 1$

$\frac{dy}{dx} = 3x^2 - 1$

At a point where the curve cuts y-axis, x = 0

$\left.\frac{dy}{dx}\right]_{x=0} = 3x^2 - 1\Big]_{x=0} = -1$

14. Correct option: D

Explanation:-

Given: $f(x) = \frac{|x-1|}{(x-1)}$

$|x-1| = \begin{cases} (x-1), & \text{if } x > 1 \\ -(x-1), & \text{if } x < 1 \end{cases}$

(i) For x > 1, $f(x) = \frac{x-1}{x-1} = 1$

(ii) For x < 1, $f(x) = \frac{-(x-1)}{x-1} = -1$

Thus, the range of f(x) is {–1, 1}.

15. Correct option: C

Explanation:-

$\left[2\vec{a}+4\vec{b}\ \ \vec{c}\ \ \vec{d}\right] = \left[2\vec{a}\ \ \vec{c}\ \ \vec{d}\right] + \left[4\vec{b}\ \ \vec{c}\ \ \vec{d}\right]$

$\left[2\vec{a}+4\vec{b}\ \ \vec{c}\ \ \vec{d}\right] = 2\left[\vec{a}\ \ \vec{c}\ \ \vec{d}\right] + 4\left[\vec{b}\ \ \vec{c}\ \ \vec{d}\right]$

$\lambda = 2, \mu = 4$

$\Rightarrow \lambda + \mu = 6$

16. Correct option: A

Explanation:-

Given: $\sin^{-1} x = y$

We know that the range of the principal value of sin is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

So, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Hence, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

17. Correct option: B

Explanation:-

Given: $y = e^{Cx}$

$\ln y = Cx$ ... (i)

Differentiating both sides of (i) w.r.t x, we get

$$\frac{1}{y}\frac{dy}{dx} = C$$

Substituting this value in (i), we get

$$\ln y = \left(\frac{1}{y}\frac{dy}{dx}\right)x$$

$$y \ln y = x\frac{dy}{dx}$$

18. Correct option: B

Explanation:-

$$I = \int \frac{1}{5 + 3\cos x}dx$$

$$I = \int \frac{1}{5 + 3\left(\frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right)}dx$$

$$I = \int \frac{\sec^2\frac{x}{2}}{8 + 2\tan^2\frac{x}{2}}dx$$

$$\text{Put } \tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}dx = 2dt$$

$$I = \int \frac{dt}{4 + t^2}$$

$$I = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + c$$

$$I = \frac{1}{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{2}\right) + c$$

19. Correct option: A

Explanation:-

We have,

$$I = \int \frac{x^2}{1+x^3} dx$$

Put $1 + x^3 = t$

$$3x^2 dx = dt$$

$$x^2 dx = \frac{dt}{3}$$

$$I = \int \frac{dt}{3t}$$

$$I = \frac{1}{3}\int \frac{dt}{t}$$

$$I = \frac{1}{3}\log|t| + c$$

$$I = \frac{1}{3}\log\left|1 + x^3\right| + c$$

20. Correct option: C

Explanation:-

$$y = A\cos\omega t + B\sin\omega t$$

$$\frac{dy}{dx} = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$\frac{d^2y}{dx^2} = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

$$\frac{d^2y}{dx^2} = -\omega^2\left(A\cos\omega t + B\sin\omega t\right)$$

$$y'' = -\omega^2 y$$

Part B

21. We have

$$\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2 x/2}+\sqrt{2\sin^2 x/2}}{\sqrt{2\cos^2 x/2}-\sqrt{2\sin^2 x/2}}\right)$$

$$= \tan^{-1}\left(\frac{\cos x/2 + \sin x/2}{\cos x/2 - \sin x/2}\right)$$

$$= \tan^{-1}\left(\frac{-\cos x/2 + \sin x/2}{-\cos x/2 - \sin x/2}\right) \qquad \left(\because \pi < x < \frac{3\pi}{2} \Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}\right)$$

$$= \tan^{-1}\left(\frac{\cos x/2 - \sin x/2}{\cos x/2 + \sin x/2}\right)$$

$$= \tan^{-1}\left(\frac{1 - \tan x/2}{1 + \tan x/2}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right)$$

$$= \frac{\pi}{4} - \frac{x}{2}$$

22. $2ye^{x/y}dx + \left(y - 2x\,e^{x/y}\right)dy = 0$

$$\frac{dx}{dy} = \frac{2xe^{\frac{x}{y}} - y}{2ye^{\frac{x}{y}}} \qquad \ldots (1)$$

Let

$$F(x, y) = \frac{2xe^{\frac{x}{y}} - y}{2ye^{\frac{x}{y}}}$$

Then,

$$F(\lambda x, \lambda y) = \frac{\lambda\left(2xe^{\frac{x}{y}} - y\right)}{\lambda\left(2ye^{\frac{x}{y}}\right)} = \lambda^{o}\left[F(x, y)\right]$$

Thus, F(x, y) is a homogeneous function of degree zero.

Therefore, the given differential equation is a homogeneous differential equation.

OR

The equation of the family of circles touching x-axis at the origin is

$(x - 0)^2 + (y - a)^2 = a^2$, where a is a parameter.

$x^2 + y^2 - 2ay = 0$ ... (i)

This equation contains only one arbitrary constant, thus we differentiate it only once, we get

$$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0$$

$$\Rightarrow a\frac{dy}{dx} = x + y\frac{dy}{dx}$$

$$\Rightarrow a = \frac{\left(x + y\frac{dy}{dx}\right)}{\frac{dy}{dx}}$$

Substituting the value of 'a' in equation (i), we get

$$\Rightarrow x^2 + y^2 = 2y\left\{\frac{\left(x + y\frac{dy}{dx}\right)}{\frac{dy}{dx}}\right\}$$

$$\Rightarrow \left(x^2 + y^2\right)\frac{dy}{dx} = 2y\left(x + y\frac{dy}{dx}\right)$$

This is the required differential equation of all the circles touching the x-axis at the origin.

23. Given that

$$\vec{b} = 2\hat{i} + 4\hat{j} - 5k$$

$$\vec{c} = \lambda\hat{i} + 2\hat{j} + 3k$$

Now consider the sum of the vectors $\vec{b} + \vec{c}$:

$$\vec{b} + \vec{c} = \left(2\hat{i} + 4\hat{j} - 5k\right) + \left(\lambda\hat{i} + 2\hat{j} + 3k\right)$$

$$\Rightarrow \vec{b} + \vec{c} = (2 + \lambda)\hat{i} + 6\hat{j} - 2k$$

Let $\hat{n}$ be the unit vector along the sum of vectors $\vec{b} + \vec{c}$:

$$\hat{n} = \frac{(2 + \lambda)\hat{i} + 6\hat{j} - 2k}{\sqrt{(2 + \lambda)^2 + 6^2 + 2^2}}$$

The scalar product of $\vec{a}$ and n is 1. Thus,

$$\vec{a} \cdot \hat{n} = \left(\hat{i} + \hat{j} + k\right) \cdot \left(\frac{(2 + \lambda)\hat{i} + 6\hat{j} - 2k}{\sqrt{(2 + \lambda)^2 + 6^2 + 2^2}}\right)$$

$$\Rightarrow 1 = \frac{1(2 + \lambda) + 1 \cdot 6 - 1 \cdot 2}{\sqrt{(2 + \lambda)^2 + 6^2 + 2^2}}$$

$$\Rightarrow \sqrt{(2 + \lambda)^2 + 6^2 + 2^2} = 2 + \lambda + 6 - 2$$

$$\Rightarrow \sqrt{(2 + \lambda)^2 + 6^2 + 2^2} = \lambda + 6$$

$$\Rightarrow (2 + \lambda)^2 + 40 = (\lambda + 6)^2$$

$$\Rightarrow \lambda^2 + 4\lambda + 4 + 40 = \lambda^2 + 12\lambda + 36$$

$$\Rightarrow 4\lambda + 44 = 12\lambda + 36$$

$$\Rightarrow 8\lambda = 8$$

$$\Rightarrow \lambda = 1$$

24. * is a binary operation on A defined by (a, b) * (c, d) = (ac, b + ad) for (a, b), (c, d) ∊ A

(i) Let (x, y) ∊ A is an identity element of *

Therefore, (x, y) * (a, b) = (a, b) = (a, b) * (x, y) for any (a, b) ∊ A

i.e. (xa, y + xb) = (a, b) = (ax, b + ay)

Equating the corresponding elements, we get

$xa = a \Rightarrow x = 1$

Also, $y + xb = b \Rightarrow y = 0$

So, we have (x, y) = (1, 0)

Thus, (1, 0) is the identity element of *.

(ii) Let (p, q) ∊ A is an inverse element of *

⇒ (p, q) * (a, b) = (1, 0) = (a, b) * (p, q) for any (a, b) ∊ A

⇒ (pa, q + pb) = (1, 0) = (ap, b + aq)

Equating the corresponding elements, we get

$pa = 1 \Rightarrow p = \frac{1}{a}$

$q + pb = 0 \Rightarrow q = -\frac{b}{a}$

Therefore, $(p, q) = \left(\frac{1}{a}, -\frac{b}{a}\right)$

Thus, the inverse of (a, b) is $\left(\frac{1}{a}, -\frac{b}{a}\right)$ where $a \neq 0$

Inverse of (5, 3) is $\left(\frac{1}{5}, -\frac{3}{5}\right)$

Inverse of $\left(\frac{1}{2}, 4\right)$ is $(2, -8)$

25. Using $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ in the given integral we get,

$$I = \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}} dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\, dx - \int dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\, dx - x + C$$

Let $I_1 = \int \sin^{-1}\sqrt{x}\ dx$

Put $\sin^{-1}\sqrt{x} = \theta$

$\Rightarrow \sqrt{x} = \sin\theta$

$\Rightarrow x = \sin^2\theta$

$\Rightarrow dx = 2\sin\theta \cos\theta \, d\theta$

$I_1 = \int \theta . \, 2\sin\theta\cos\theta \, d\theta$

$= \int \theta . \sin 2\theta \, d\theta$

$$= \theta\left(-\frac{\cos 2\theta}{2}\right) - \int 1 . \left(-\frac{\cos 2\theta}{2}\right) d\theta$$

$$= \frac{-\theta \cos 2\theta}{2} + \frac{\sin 2\theta}{4}$$

$$= \frac{-\theta\left(1 - 2\sin^2\theta\right)}{2} + \frac{2\sin\theta\sqrt{1-\sin^2\theta}}{4}$$

$$= \frac{-\sin^{-1}\sqrt{x}\,(1-2x)}{2} + \frac{\sqrt{x}\sqrt{1-x}}{2}$$

Hence given integral is

$$= \frac{4}{\pi}\left[\frac{-\sin^{-1}\sqrt{x}\,(1-2x)}{2} + \frac{\sqrt{x-x^2}}{2}\right] - x + C$$

$$= \frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x} + \frac{2\sqrt{x-x^2}}{\pi} - x + C$$

26. $$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we have:

$$\Delta = \begin{vmatrix} 1-1 & 1-1 & 1 \\ a-c & b-c & c \\ a^3-c^3 & b^3-c^3 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ a-c \quad a^2+ac+c^2 & b-c \quad b^2+bc+c^2 & c^3 \end{vmatrix}$$

$$= c-a \quad b-c \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & c \\ - \quad a^2+ac+c^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$, we have:

$$\Delta = c-a \quad b-c \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ b^2-a^2+bc-ac & b^2+bc+c^2 & c^3 \end{vmatrix}$$

$$= b-c \quad c-a \quad a-b \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ -\ a+b+c & b^2+bc+c^2 & c^3 \end{vmatrix}$$

$$= a-b \quad b-c \quad c-a \quad a+b+c \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ -1 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta = a-b \quad b-c \quad c-a \quad a+b+c \quad -1 \begin{vmatrix} 0 & 1 \\ 1 & c \end{vmatrix}$$

$$= a-b \quad b-c \quad c-a \quad a+b+c$$

Hence proved.

27. Given: $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Let $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

Now y = u + v

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \ldots (i)$$

Consider u = $(\sin x)^x$

Taking logarithms on both the sides, we have,

$$\log u = x \log(\sin x)$$

Differentiating with respect to x, we have,

$$\frac{1}{u}.\frac{du}{dx} = \log(\sin x) + \frac{x}{\sin x}.\cos x$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x \left(\log(\sin x) + x\cot x\right) \ \ldots (ii)$$

Consider $v = \sin^{-1}\sqrt{x}$

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}} \ldots (iii)$$

From (i), (ii) and (iii), we get

$$\frac{dy}{dx} = (\sin x)^x \left(\log(\sin x) + x \cot x\right) + \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

OR

We have

$\left(x^2+y^2\right)^2 = xy$ ____(i)

Differentiating with respect to x, we have,

$$2\left(x^2+y^2\right)\left(2x+2y.\frac{dy}{dx}\right) = y + \frac{xdy}{dx}$$

$$\Rightarrow 4x\left(x^2+y^2\right) + 4y\left(x^2+y^2\right).\frac{dy}{dx} = y + \frac{xdy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left(4x^2y + 4y^3 - x\right) = y - 4x^3 - 4xy^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - 4x^3 - 4xy^2}{4x^2y + 4y^3 - x}$$

28. Let I = $\int_0^{\pi} \frac{x}{1+\sin x}\,dx$ ……………..(i)

[By property of definite integrals]

$$I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}\,dx \qquad \text{using } \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{\pi - x}{1+\sin x}\,dx$$

$$\Rightarrow I = \pi\int_0^{\pi} \frac{1}{1+\sin x}dx - \int_0^{\pi} \frac{x}{1+\sin x}\,dx$$

$$\Rightarrow I = \pi\int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x}dx - I \ldots \left(\text{using (i)}\right)$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \frac{1-\sin x}{\cos^2 x}\,dx$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \left(\sec^2 x - \tan x.\sec x\right)dx$$

$$\Rightarrow 2I = \pi\left[\tan x - \sec x\right]_0^{\pi}$$

2I=π [(tanπ – secπ) – (tan0 – sec0)]

2I= π [0 – (– 1) – (0 – 1)] = 2π

I = π

OR

Consider the given integral

$$I = \int_0^{\frac{\pi}{2}} 2\sin x\cos x\tan^{-1}\left(\sin x\right)dx$$

Let t = sin x

$\Rightarrow$ dt = cos x dx

When $x = \frac{\pi}{2}$, $t = 1$

When $x = 0$, $t = 0$

Now, $\int 2\sin x\cos x\tan^{-1}(\sin x)\,dx$

$= \int 2t\tan^{-1}t\,dt$

$= \left[\tan^{-1}t\right]\int 2t\,dt - \int\left[\frac{d}{dt}.\left(\tan^{-1}t\right)\int 2t\,dt\right]dt$

$= \left[\tan^{-1}t\right]\left[2.\frac{t^2}{2}\right] - \int\left(\frac{1}{1+t^2}\times 2.\frac{t^2}{2}\right)dt$

$= t^2\tan^{-1}t - \int\frac{t^2}{1+t^2}dt$

$= t^2\tan^{-1}t - \int\left[1 - \frac{1}{1+t^2}\right]dt$

$= t^2\tan^{-1}t - t + \tan^{-1}t$

$\therefore I = \int_0^{\frac{\pi}{2}} 2\sin x\cos x\tan^{-1}(\sin x)\,dx$

$= \left[t^2\tan^{-1}t - t + \tan^{-1}t\right]_0^1$

$= \left[1^2\tan^{-1}1 - 1 + \tan^{-1}1\right] - \left[0^2\tan^{-1}0 - 0 + \tan^{-1}0\right]$

$= \left[1\times\frac{\pi}{4} - 1 + \frac{\pi}{4}\right] - 0$

$= \frac{\pi}{4} - 1 + \frac{\pi}{4}$

$= \frac{\pi}{2} - 1$

29. Let E_1, E_2 and E_3 be the events of a driver being a scooter driver, car driver and truck driver respectively. Let A be the event that the person meets with an accident.

There are 2000 insured scooter drivers, 4000 insured car drivers and 6000 insured truck drivers.

Total number of insured vehicle drivers = 2000 + 4000 + 6000 = 12000

$\therefore P\left(E_1\right) = \frac{2000}{12000} = \frac{1}{6}, P\left(E_2\right) = \frac{4000}{12000} = \frac{1}{3}, P\left(E_3\right) = \frac{6000}{12000} = \frac{1}{2}$

Also, we have:

$P(A|E_1) = 0.01 = \frac{1}{100}$

$P(A|E_2) = 0.03 = \frac{3}{100}$

$P(A|E_3) = 0.15 = \frac{15}{100}$

Now, the probability that the insured person who meets with an accident is a scooter driver is $P(E_1|A)$.
Using Bayes' theorem, we obtain:

$$P(E_1|A) = \frac{P(E_1) \times P(A|E_1)}{P(E_1) \times P(A|E_1) + P(E_2) \times P(A|E_2) + P(E_3) \times P(A|E_3)}$$

$$= \frac{\frac{1}{6} \times \frac{1}{100}}{\frac{1}{6} \times \frac{1}{100} + \frac{1}{3} \times \frac{3}{100} + \frac{1}{2} \times \frac{15}{100}}$$

$$= \frac{\frac{1}{6}}{\frac{1}{6} + 1 + \frac{15}{2}}$$

$$= \frac{1}{6} \times \frac{6}{52}$$

$$= \frac{1}{52}$$

30. Let the equation of plane be $ax + by + cz + d = 0$ .... (1)
Since the plane passes through the point A (0, 0, 0) and B(3, −1, 2), we have
$a \times 0 + b \times 0 + c \times 0 + d = 0$
$\Rightarrow d = 0$ .... (2)
Similarly for point B (3, −1, 2), $a \times 3 + b \times (-1) + c \times 2 + d = 0$
$3a - b + 2c = 0$ (Using, $d = 0$) ...(3)

Given equation of the line is $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$

We can also write the above equation as $\frac{x-4}{1} = \frac{y-(-3)}{-4} = \frac{x-(-1)}{7}$

The required plane is parallel to the above line.
Therefore, $a \times 1 + b \times (-4) + c \times 7 = 0$
$\Rightarrow a - 4b + 7c = 0$ ... (4)
Cross multiplying equations (3) and (4), we obtain:

$$\frac{a}{(-1)\times 7 - (-4)\times 2} = \frac{b}{2 \times 1 - 3 \times 7} = \frac{c}{3 \times (-4) - 1 \times (-1)}$$

$$\Rightarrow \frac{a}{-7+8} = \frac{b}{2-21} = \frac{c}{-12+1}$$

$$\Rightarrow \frac{a}{1} = \frac{b}{-19} = \frac{c}{-11} = k$$

$$\Rightarrow a = k, b = -19k, c = -11k$$

Substituting the values of a, b and c in equation (1), we obtain the equation of plane as:

$kx - 19ky - 11kz + d = 0$

$\Rightarrow k(x - 19y - 11z) = 0 \qquad \text{(From equation(2))}$

$\Rightarrow x - 19y - 11z = 0$

So, the equation of the required plane is x – 19y – 11z = 0

31. $f(x) = \begin{cases} a\sin\frac{\pi}{2}(x+1), x \le 0 \\ \frac{\tan x - \sin x}{x^3}, x > 0 \end{cases}$

The given function f is defined for all x ∈ **R**.

It is known that a function f is continuous at x = 0, if $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\left[a\sin\frac{\pi}{2}(x+1)\right] = a\sin\frac{\pi}{2} = a(1) = a$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0}\frac{\tan x - \sin x}{x^3} = \lim_{x\to 0}\frac{\frac{\sin x}{\cos x} - \sin x}{x^3}$$

$$= \lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3\cos x} = \lim_{x\to 0}\frac{\sin x . 2\sin^2\frac{x}{2}}{x^3\cos x}$$

$$= 2\lim_{x\to 0}\frac{1}{\cos x}\times\lim_{x\to 0}\frac{\sin x}{x}\times\lim_{x\to 0}\left[\frac{\sin\frac{x}{2}}{x}\right]^2$$

$$= 2\times 1\times 1\times\frac{1}{4}\times\lim_{\frac{x}{2}\to 0}\left[\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right]^2$$

$$= 2\times 1\times 1\times\frac{1}{4}\times 1 = \frac{1}{2}$$

Now, $f(0) = a\sin\frac{\pi}{2}(0+1) = a\sin\frac{\pi}{2} = a\times 1 = a$

Since f is continuous at x = 0, $a = \frac{1}{2}$

Part C

32. Let the number of rackets and the number of bats to be made be x and y respectively.

The given information can be tabulated as below:

	Tennis Racket	Cricket Bat
Machine Time (h)	1.5	3
Craftsman's Time (h)	3	1

In a day, the machine time is not available for more than 42 hours.

$\therefore$ $1.5x + 3y \leq 42$

In a day, the craftsman's time cannot be more than 24 hours.

$\therefore$ $3x + y \leq 24$

Let the total profit be Rs. Z.

The profit on a racket is Rs. 20 and on a bat is Rs. 10.

$\therefore$ $Z = 20x + 10y$

Thus, the given linear programming problem can be stated as follows:

Maximise $Z = 20x + 10y$ ... (1)

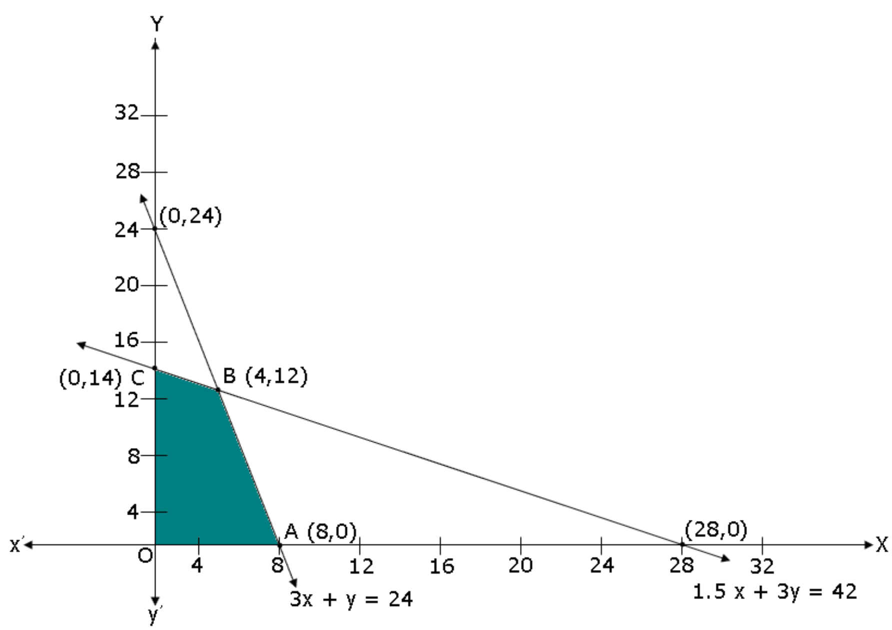
Subject to

$1.5x + 3y \leq 42$ ... (2)

$3x + y \leq 24$ ... (3)

$x, y \geq 0$ ... (4)

The feasible region can be shaded in the graph as below:



The corner points are A(8,0), B(4,12), C(0,14) and 0(0,0).

The values of Z at these corner points are tabulated as follows:

Corner point	Z = 20x + 10y
A(8,0)	160
B(4,12)	200 → Maximum
C(0,14)	140
O(0,0)	0

The maximum value of Z is 200, which occurs at x = 4 and y = 12.
Thus, the factory must produce 4 tennis rackets and 12 cricket bats to earn the maximum profit of Rs. 200.

33. Let the events M, F and G be defined as follows:
M: A male is selected
F: A female is selected
G: A person has grey hair
It is given that the number of males = the number of females

$\therefore$ P (M) = P(F) = $\frac{1}{2}$

Now, P (G/M) = Probability of selecting a grey haired person given that the person is a Male = 5% = $\frac{5}{100}$

Similarly, P (G/F) = 0.25% = $\frac{0.25}{100}$

A grey haired person is selected at random, the probability that this person is a male = P(M|G)

$$= \frac{P(M) \times P(G|M)}{P(M) \times P(G|M) + P(F) \times P(G|F)}$$ [Using Baye's Theorem]

$$= \frac{\frac{1}{2} \times \frac{5}{100}}{\frac{1}{2} \times \frac{5}{100} + \frac{1}{2} \times \frac{0.25}{100}}$$

$$= \frac{\frac{5}{100}}{\frac{5}{100} + \frac{0.25}{100}}$$

$$= \frac{5}{5.25}$$

$$= \frac{20}{21}$$

OR

Consider the following events:

E_1= Getting 5 OR 6 in a single throw of the die

E_2= Getting 1, 2, 3 OR 4 in a single throw of the die

A = Getting exactly 2 heads

We have to find, $P(E_2/A)$.

Since $P(E_2/A) = \dfrac{P(A/E_2)P(E_2)}{P(A/E_1)P(E_1) + P(A/E_2)P(E_2)}$

Now, $P(E_1) = \dfrac{2}{6} = \dfrac{1}{3}$ and $P(E_2) = \dfrac{4}{6} = \dfrac{2}{3}$

Also,

$P(A/E_1) =$ Probability of getting exactly 2 heads when a coin is tossed 3 times $= \dfrac{3}{8}$

And, $P(A/E_2) =$ Probability of getting 2 heads when a coin is tossed 2 times $= \dfrac{1}{4}$

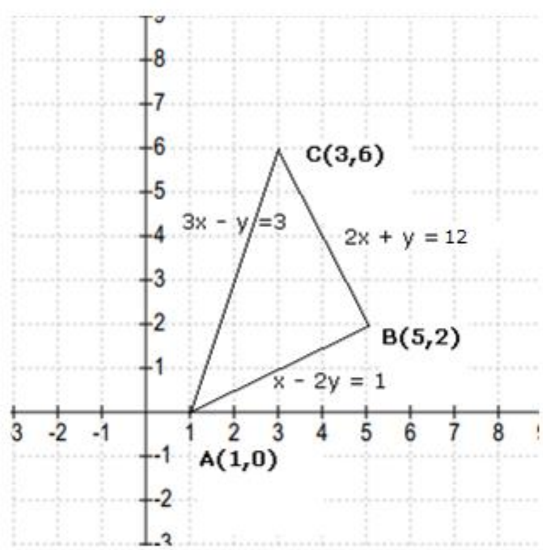
$$\therefore P(E_2/A) = \frac{\frac{1}{4}\times\frac{2}{3}}{\frac{3}{8}\times\frac{1}{3}+\frac{1}{4}\times\frac{2}{3}} = \frac{\frac{1}{6}}{\frac{1}{8}+\frac{1}{6}} = \frac{\frac{1}{6}}{\frac{3+4}{24}} = \frac{1}{6}\times\frac{24}{7} = \frac{4}{7}$$

34. Given equations are:

$3x - y = 3$ ... (1)

$2x + y = 12$ ... (2)

$x - 2y = 1$ ... (3)



To Solve (1) and (2),

(1) + (2) $\Rightarrow 5x = 15 \Rightarrow x = 3$

(2) $\Rightarrow y = 12 - 6 = 6$

Thus (1) and (2) intersect at C(3, 6).

To solve (2) and (3),

(2) – 2(3) $\Rightarrow 5y = 10 \Rightarrow y=2$

(2) $\Rightarrow 2x = 12 - 2 = 10 \Rightarrow x = 5$

Thus (2) and (3) intersect at B(5, 2).

To solve (3) and (1),

2(1) – (3) $\Rightarrow 5x = 5 \Rightarrow x = 1$

(3) $\Rightarrow 1 - 2y = 1 \Rightarrow y = 0$

Thus (3) and (1) intersect at A(1, 0).

$$\text{Area} = \int_1^3 (3x-3)dx + \int_3^5 (12-2x)dx - \int_1^5 \frac{1}{2}(x-1)dx$$

$$= 3\left[\frac{x^2}{2} - x\right]_1^3 + \left[12x - x^2\right]_3^5 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^5$$

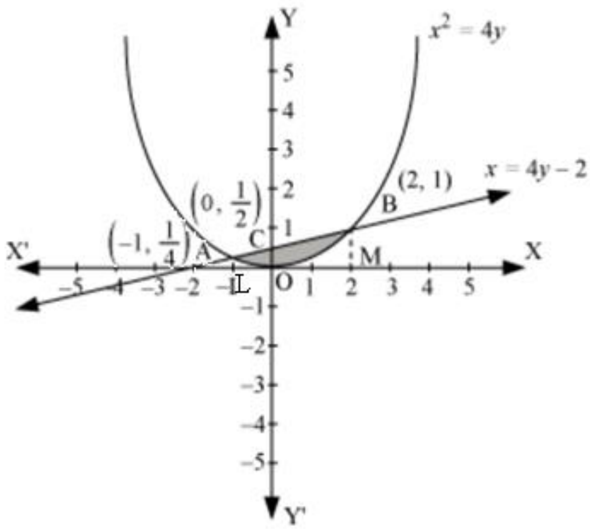
$$= 3\left[\left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right)\right] + \left[(60-25) - (36-9)\right] - \frac{1}{2}\left[\left(\frac{25}{2} - 5\right) - \left(\frac{1}{2} - 1\right)\right]$$

$$= 3\left[\frac{3}{2} + \frac{1}{2}\right] + [35 - 27] - \frac{1}{2}\left[\frac{15}{2} + \frac{1}{2}\right]$$

$$= 6 + 8 - 4 = 10 \text{ sq. units}$$

OR

The shaded area OBAO represents the area bounded by the curve $x^2 = 4y$ and line $x = 4y - 2$.



Let A and B be the points of intersection of the line and parabola.

Co-ordinates of point A are $\left(-1, \frac{1}{4}\right)$. Co-ordinates of point B are (2, 1).

Area OBAO = Area OBCO + Area OACO ... (1)

Area OBCO =

$$= \int_0^2 \frac{x+2}{4}dx - \int_0^2 \frac{x^2}{4}dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_0^2 - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{4}\ 2 + 4 - \frac{1}{4}\left[\frac{8}{3}\right]$$

$$= \frac{3}{2} - \frac{2}{3} = \frac{5}{6}$$

Area OACO =

$$= \int_{-1}^0 \frac{x+2}{4}dx - \int_{-1}^0 \frac{x^2}{4}dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x\right]_{-1}^0 - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^0$$

$$= \frac{1}{4}\left[-\frac{-1^{2}}{2} - 2\ -1\right] - \frac{1}{4}\left[-\left(\frac{-1^{3}}{3}\right)\right]$$

$$= \frac{1}{4}\left[-\frac{1}{2} + 2\right] - \frac{1}{4}\left[\frac{1}{3}\right]$$

$$= \frac{3}{8} - \frac{1}{12} = \frac{7}{24}$$

Therefore, required area = $\left(\frac{5}{6}+\frac{7}{24}\right)=\frac{9}{8}$ sq. units

35. We have

$S = 2\pi rh + 2\pi r^2$ ....(i)

$h = \frac{S - 2\pi r^2}{2\pi r}$

$V = \pi r^2 h = \pi r^2\left[\frac{S - 2\pi r^2}{2\pi r}\right] = \frac{1}{2}\left[Sr - 2\pi r^3\right]$

$\frac{dV}{dr} = \frac{1}{2}\left[S - 6\pi r^2\right]$

$\frac{dV}{dr} = 0 \Rightarrow \frac{1}{2}\left[S - 6\pi r^2\right] = 0 \Rightarrow S - 6\pi r^2 = 0 \Rightarrow r = \sqrt{\frac{S}{6\pi}}$

$\frac{d^2V}{dr^2} = -6\pi r$

$\left.\frac{d^2V}{dr^2}\right]_{r=\sqrt{\frac{S}{6\pi}}} = -6\pi r\Big]_{r=\sqrt{\frac{S}{6\pi}}} = -6\pi\sqrt{\frac{S}{6\pi}} < 0$

$\Rightarrow$ When $r = \sqrt{\frac{S}{6\pi}}$; V is maximum.

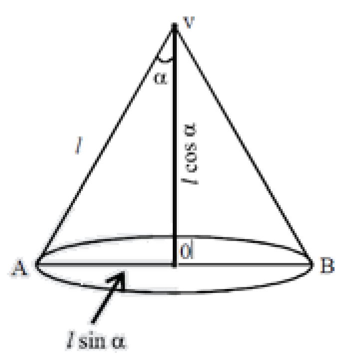
$\Rightarrow S = 6\pi r^2 = 2\pi rh + 2\pi r^2$

$\Rightarrow 4\pi r^2 = 2\pi rh$

$\Rightarrow h = 2r$

$\Rightarrow$ height = diameter of the base.

OR



Volume of cone

$= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(\ell \sin\alpha)^2(\ell\cos\alpha)$

$= \frac{1}{3}\pi\ell^3 \sin^2\alpha\cos\alpha$

$$\frac{dv}{d\alpha} = \frac{\pi l^3}{3}[-\sin^3\alpha + 2\sin\alpha\cos\times\cos\alpha]$$

$$= \frac{\pi l^3 \sin\alpha}{3}(-\sin^2\alpha + 2\cos^2\alpha)$$

For maximum or minimum

$$\frac{dv}{d\alpha} = 0$$

$$\frac{\pi l^3 \sin\alpha}{3}(-\sin^2\alpha + 2\cos^2\alpha) = 0$$

$$\sin\alpha \neq 0$$

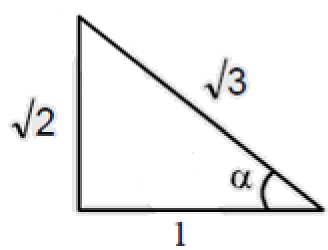
$$2\cos^2\alpha = \sin^2\alpha$$

$$\tan^2\alpha = 2$$

$$\tan\alpha = \sqrt{2}$$

$$\cos\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \cos^{-1}\frac{1}{\sqrt{3}}$$



Again diff. w.r.t. α, we get

$$\frac{d^2v}{d\alpha^2} = \frac{1}{3}\pi l^3 \cos^3\alpha(2 - 7\tan^2\alpha)$$

$$\text{at} \quad \cos\alpha = \frac{1}{\sqrt{3}}$$

$$\frac{d^2v}{d\alpha^2} < 0$$

$$\text{V is maximum when } \cos\alpha = \frac{1}{\sqrt{3}} \text{ or } \alpha = \cos^{-1}\frac{1}{\sqrt{3}}$$

36. Let the equation of required plane be $\ell x + my + nz + p = 0$ …… (1)

Plane passes through points (1, 2, 3) and (0, -1, 0)

∴ (1, 2, 3) and (0, -1, 0) satisfies the equation (1)

$\ell + 2m + 3n + p = 0$ …… (2)

$-m + p = 0$

$\Rightarrow p = m$ …… (3)

d.c.'s of line $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z}{-3}$ are 2, 3, −3

d.c.'s normal to plane are ℓ, m, and n normal to the plane will be ⊥ to line

i.e., $2\ell + 3m - 3n = 0$

$\Rightarrow \ell = \frac{3}{2}(n-m)$ …… (4)

From (2) and (3) we have

$\ell + 3m + 3n = 0$

$\ell = -3(m+n)$ …… (5)

From (4) and (5)

$\frac{3}{2}(n-m) = -3(m+n)$

$n - m = -2m - 2n$

$\Rightarrow 3n = -m$ or $m = -3n$

$\ell = -3(-3n + n) = -3 \times -2n$

Using $\ell = 6n$, $m = -3n$ & $p = -3n$ in (1) we have required equation as

$6x - 3y + z - 3 = 0$

OR

Equation of any plane through the intersection of the planes

$2x + y - z = 3$ and $5x - 3y + 4z + 9 = 0$ is

$(2x + y - z - 3) + \lambda(5x - 3y + 4z + 9) = 0$ ... (i)

$\Rightarrow (2 + 5\lambda)x + (1 - 3\lambda)y + (4\lambda - 1)z - (3 - 9\lambda) = 0$

This is parallel to the line

$\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-5}{5}$

$\therefore (2+5\lambda)\times 2 + (1-3\lambda)\times 4 + (4\lambda - 1)\times 5 = 0$

$\Rightarrow 18\lambda + 3 = 0 \Rightarrow \lambda = -\frac{1}{6}$

Substitute for $\lambda = -\frac{1}{6}$ in Eq. (i)

$\left(2-\frac{5}{6}\right)x + \left(1+\frac{1}{2}\right)y + \left(-\frac{2}{3}-1\right)z - \left(3+\frac{3}{2}\right) = 0$

$\Rightarrow 7x + 9y - 10z - 27 = 0.$

Thus the required equation of plane is $7x + 9y - 10z - 27 = 0.$

37. From the given data, we write the following equations:

$$\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 1600$$

$$\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} = 2300$$

$$\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 900$$

From above system, we get:

$3x+2y+z=1600$

$4x+y+3z=2300$

$x+y+z=900$

Thus we get:

$$\begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1600 \\ 2300 \\ 900 \end{pmatrix}$$

This is of the form

$$AX=B, \text{ where } A=\begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B=\begin{pmatrix} 1600 \\ 2300 \\ 900 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 3(1-3)-2(4-3)+1(4-1) = -6-2+3 = -5 \neq 0$$

We need to find A^{-1}:

$C_{11} = -2; C_{12} = -1; C_{13} = 3$

$C_{21} = -1; C_{22} = 2; C_{23} = -1$

$C_{31} = 5; C_{32} = -5; C_{33} = -5$

$$\text{Therefore, adj } A = \begin{pmatrix} -2 & -1 & 3 \\ -1 & 2 & -1 \\ 5 & -5 & -5 \end{pmatrix}^T = \begin{pmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{pmatrix}$$

$$\text{Thus, } A^{-1} = \frac{\text{adj}A}{|A|} = -\frac{1}{5}\begin{pmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{pmatrix}$$

Therefore, $X = A^{-1}B$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{pmatrix}\begin{pmatrix} 1600 \\ 2300 \\ 900 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -2\times 1600 - 1\times 2300 + 5\times 900 \\ -1\times 1600 + 2\times 2300 - 5\times 900 \\ 3\times 1600 - 1\times 2300 - 5\times 900 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -1000 \\ -1500 \\ -2000 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 200 \\ 300 \\ 400 \end{pmatrix}$$

Awards can be given for discipline.