11. Exponents and Logarithms

How do Logarithms Work?

Logarithms

"The Logarithm of a given number to a given base is the index of the power to which the base must be raised in order to equal the given number."

Logarithm of a number is the exponent by which another fixed number (the base) must be raised to produce that number.

If $x = b^y$, then $y = \log_b(x)$

(y is the logarithm of x to base b)

$1000 = 10 \times 10 \times 10 = 10^3$, its written in logarithm as $3 = \log_{10}(1000)$

The Logarithmic Function

The logarithmic function of x is defined as $f(x) = \log_a x$ where $a > 0, a \neq 1$

Laws of Logarithms		Note:
Logarithm of a product (Product Law):	$\log_a xy = \log_a x + \log_a y$	$\log_a 1 = 0$
Logarithm of a quotient (Quotient Law):	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$	$\log_a a = 1$
Logarithm of a power (Power Law):	$\log_a x^m = m\log_a x$	

If a > 0 and ≠ 1 then logarithm of a positive number N is defined as the index x of that power of 'a' which equals N i.e.,

$$\log_a N = x \text{ iff } a^x = N \Rightarrow a^{\log_a N} = N, a > 0, a \neq 1 \text{ and } N > 0$$

It is also known as fundamental logarithmic identity.
Its domain is (0, ∞) and range is R. a is called the base of the logarithmic function.

When base is 'e' then the logarithmic function is called **natural** or **Napierian logarithmic function** and when base is 10, then it is called common logarithmic function.

Characteristic and mantissa

1. The integral part of a logarithm is called the characteristic and the fractional part is called mantissa.

$$\log_{10} N = \underset{\text{Characters tics}}{\underset{\downarrow}{\text{integer}}} + \underset{\text{Mantissa}}{\underset{\downarrow}{\text{fraction}(+ve)}}$$

2. The mantissa part of log of a number is always kept positive.
3. If the characteristics of $\log_{10}$ N be n, then the number of digits in N is (n+1).
4. If the characteristics of $\log_{10}$ N be (− n) then there exists (n − 1) number of zeros after decimal part of N.

Properties of logarithms

Let m and n be arbitrary positive numbers such that a > 0, a ≠ 1, b > 0, b ≠ 1 then

$$(1)\ \log_a a = 1,\ \log_a 1 = 0$$

$$(2)\ \log_a b.\log_b a = 1 \Rightarrow \log_a b = \frac{1}{\log_b a}$$

$$(3)\ \log_c a = \log_b a.\ \log_c b \text{ or } \log_c a = \frac{\log_b a}{\log_b c}$$

$$(4)\ \log_a(mn) = \log_a m + \log_a n$$

$$(5)\ \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

$$(6)\ \log_a m^n = n\log_a m \qquad (7)\ a^{\log_a m} = m$$

$$(8)\ \log_a\left(\frac{1}{n}\right) = -\log_a n \qquad (9)\ \log_{a^\beta} n = \frac{1}{\beta}\log_a n$$

$$(10)\ \log_{a^\beta} n^\alpha = \frac{\alpha}{\beta}\log_a n\ ,\ (\beta \neq 0)$$

$$(11)\ a^{\log_c b} = b^{\log_c a}\ ,\ (a, b, c > 0 \text{ and } c \neq 1)$$

Logarithmic inequalities

$$(1) \text{ If } a > 1, p > 1 \Rightarrow \log_a p > 0$$

$$(2) \text{ If } 0 < a < 1, p > 1 \Rightarrow \log_a p < 0$$

$$(3) \text{ If } a > 1,\ 0 < p < 1 \Rightarrow \log_a p < 0$$

$$(4) \text{ If } p > a > 1 \Rightarrow \log_a p > 1$$

$$(5) \text{ If } a > p > 1 \Rightarrow 0 < \log_a p < 1$$

$$(6) \text{ If } 0 < a < p < 1 \Rightarrow 0 < \log_a p < 1$$

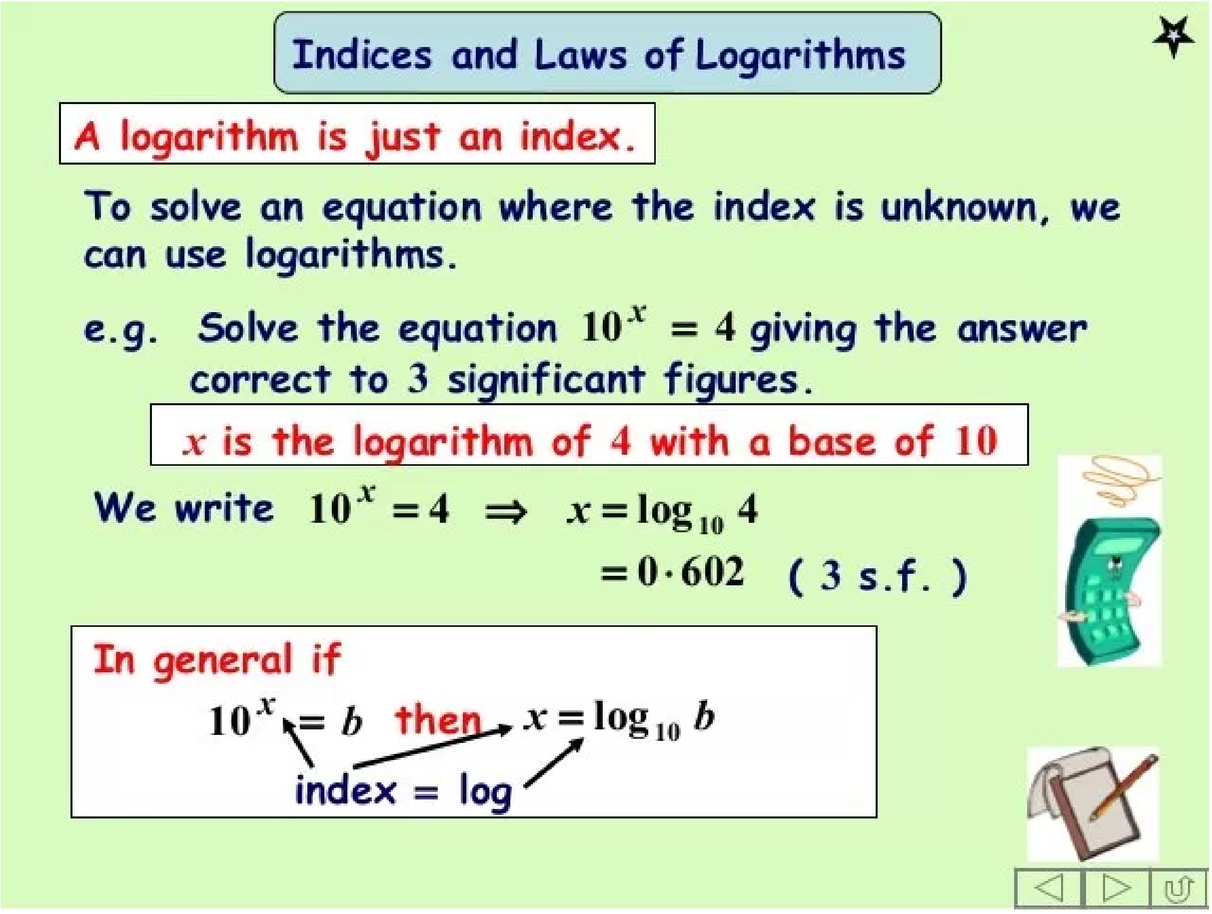
$$(7) \text{ If } 0 < p < a < 1 \Rightarrow \log_a p > 1$$

$$(8) \text{ If } \log_m a > b \Rightarrow \begin{cases} a > m^b, \text{ if } m > 1 \\ a < m^b, \text{ if } 0 < m < 1 \end{cases}$$

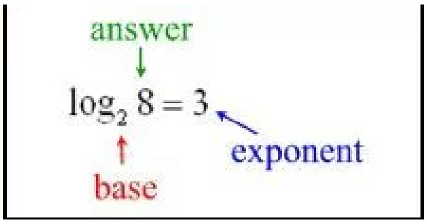
$$(9)\ \log_m a < b \Rightarrow \begin{cases} a < m^b, \text{ if } m > 1 \\ a > m^b, \text{ if } 0 < m < 1 \end{cases}$$

(10) $\log_p a > \log_p b \Rightarrow a \geq b$ if base p is positive and >1 or $a \leq b$ if base p is positive and < 1 *i.e.,* $0 < p < 1$.
In other words, if base is greater than 1 then inequality remains same and if base is positive but less than 1 then the sign of inequality is reversed.

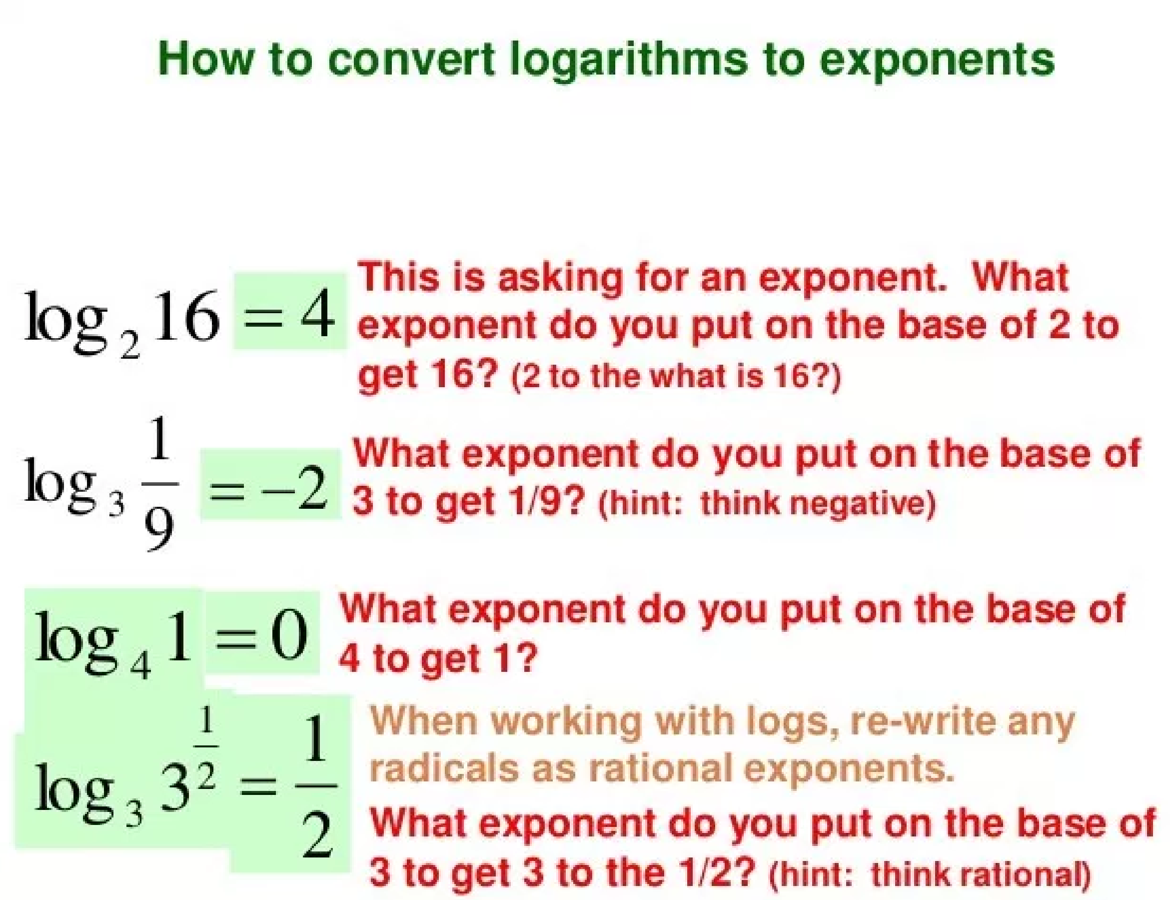
Logarithmic Expressions



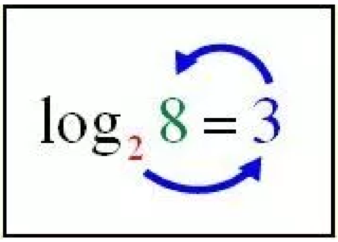
A logarithm is an exponent.



In the example shown above, 3 is the exponent to which the base 2 must be raised to create the answer of 8, or 2^3 = = 8. In this example, 8 is called the anitlogarithm base 2 of 3.



Try to remember the "spiral" relationship between the values as shown at the right. Follow the arrows starting with base 2 to get the equivalent exponential form.



Logarithms with base 10 are called common logarithms.
When the base is not indicated, base 10 is implied.

Logarithms with base e are called natural logarithms.
Natural logarithms are denoted by ln.

Operation	Laws of exponents	Laws of logs
Multiplication	$x^m \cdot x^n = x^{m+n}$	$\log(a \cdot b) = \log(a) + \log(b)$
Division	$\frac{x^m}{x^n} = x^{m-n}$	$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$
Exponentiation	$(x^m)^n = x^{mn}$	$\log(a^n) = n \cdot \log(a)$ One of the most useful properties of logs
Zero property	$x^0 = 1$	$\log(1) = 0$
Inverse	$x^{-1} = \frac{1}{x}$	$\log(x^{-1}) = \log\left(\frac{1}{x}\right) = -\log(x)$

On the graphing calculator:

Origins of Change of Base Formula:

$\log_b a = x$	Set $= x$.
$b^x = a$	Convert to exponential form.
$\log b^x = \log a$	Take common log of both sides.
$x \log b = \log a$	Use power rule.
$x = \frac{\log a}{\log b}$	Divide by $\log b$.

Change of Base Formula:

$$\log_b a = \frac{\log a}{\log b} = \frac{\ln a}{\ln b}$$

The base 10 logarithm is the log key.
The base e logarithm is the ln key.
To enter a logarithm with a different base,
use the Change of Base Formula:

$$\log_b x = \frac{\log x}{\log b}$$

Properties of Logs:

Using the properties of exponents, we can arrive at the properties of logarithms.

Properties of Exponents:

$$b^n \cdot b^m = b^{n+m}$$

$$\frac{b^n}{b^m} = b^{n-m}$$

$$\left(b^n\right)^m = b^{nm}$$

Let's see the connection:

Let $x = \log_b m$ and $y = \log_b n$

Using exponential form:

$b^x = m$ and $b^y = n$

Multiplication gives $mn = b^x b^y = b^{x+y}$

Returning to logarithmic form:

$$\log_b(mn) = x + y$$

$$\log_b(mn) = \log_b m + \log_b n$$

Similar investigations lead to the other logarithmic properties.

Properties of Logs:

$$\log_b(m \cdot n) = \log_b m + \log_b n$$

$$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$$

$$\log_b m^r = r \log_b m$$

Also: $\log_b b = 1$ and $\log_b 1 = 0$

These log properties remain the same when working with the **natural log**:

$$\ln(ab) = \ln a + \ln b$$

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

$$\ln a^b = b \ln a$$

Remember: $\ln 1 = 0$ and $\ln e = 1$

Examples:

1.	Write in exponential form: $\log_2 64 = 6$	Answer: $2^6 = 64$
2.	Write in logarithmic form: $3^{-1} = \frac{1}{3}$	Answer: $\log_3\left(\frac{1}{3}\right) = -1$
3.	Evaluate: $\log_4 1$	Answer: $4^{?}=1;\ ? = 0$ If using your calculator, remember to use the change of base formula and enter log 1 / log 4.
4.	What is the value of x? $\log_2 x = 5$	Answer: $2^5 = x$; $x = 32$
5.	Write in expanded form: $\log \frac{a\sqrt{b}}{c^6}$ (Apply the "properties of logs" rules.)	Answer: $\log\left(a\sqrt{b}\right) - \log c^6$ $\log a + \log\sqrt{b} - \log c^6$ $\log a + \log b^{\frac{1}{2}} - \log c^6$ $\log a + \frac{1}{2}\log b - 6\log c$
6.	Write in expanded form: $\ln\sqrt{\sin x \cdot \cos x}$	Answer: $\ln(\sin x \cdot \cos x)^{\frac{1}{2}}$ $\frac{1}{2}\ln(\sin x \cdot \cos x)$ $\frac{1}{2}\left(\ln(\sin x) + \ln(\cos x)\right)$
7.	Express as a single logarithm: $(\log x + 4\log y) - 5\log z$ (Apply the "properties of logs" rules in reverse.)	Answer: $\log\left(\frac{xy^4}{z^5}\right)$
8.	Express as a single logarithm: $\frac{1}{2}[(4\ln a + \ln b) - 4\ln c]$	Answer: $\ln\sqrt{\frac{a^4 b}{c^4}}$
9.	Using properties of logs, show that $\ln 4 = \ln\left(\frac{1}{4}\right)^{-1}$	Answer: $\ln\left(\frac{1}{4}\right)^{-1}$ $= -1(\ln 1 - \ln 4)$ $= -(0 - \ln 4)$ $= \ln 4$
10.	Using properties of logs, solve for x: $\log_3 x = \log_3 4 + \log_3 7$	Answer: $\log_3 4 + \log_3 7$ $= \log_3(4 \cdot 7) = \log_3 28$ $x = 28$

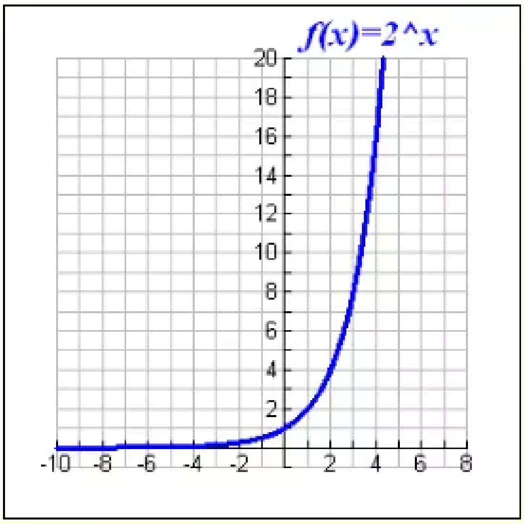
Exponential Functions

Definition:

The exponential function with base b is defined by f⁻(x) = b^x

where b > 0 , b≠1

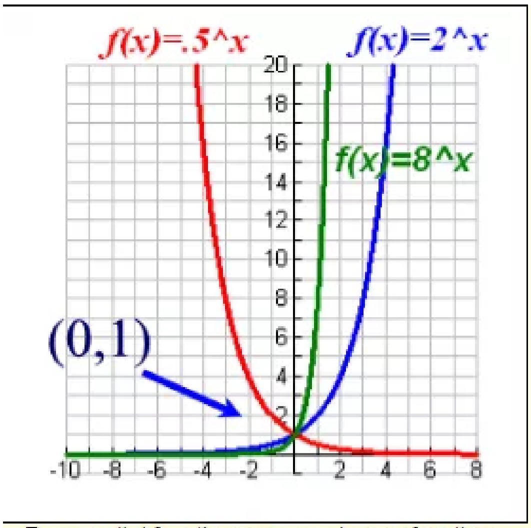
x	$f(x) = y$
-2	$2^{-2} = 1/4$
-1	$2^{-1} = 1/2$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$



Most exponential graphs resemble this same shape.
This graph is very, very small on its left side and is extremely close to the x-axis. As the graph progresses to the right, it starts to grow faster and faster and shoots off the top of the graph very quickly, as seen at the right.

In a straight line, the “rate of change” remains the same across the graph. In these graphs, the “rate of change” increases or decreases across the graphs.

Characteristics:



Exponential functions are one-to-one functions.

Such exponential graphs of the form f(x) = b^x have certain characteristics in common:

Exponential functions are one-to-one functions. • graph crosses the y-axis at (0,1)
• when b > 1, the graph increases

• when 0 < b < 1, the graph decreases

- the domain is all real numbers
- the range is all positive real numbers (never zero)
- graph passes the vertical line test – it is a function
- graph passes the horizontal line test – its inverse is also a function.
- graph is asymptotic to the x-axis – gets very, very close to the x-axis but does not touch it or cross it.

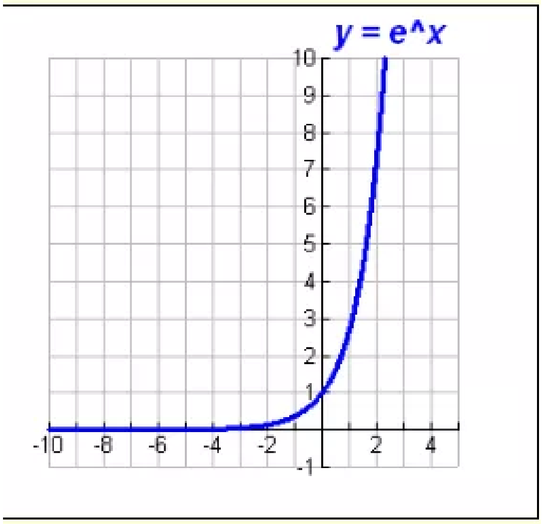
Natural Exponential Function:

The function defined by f (x) = e^x is called the natural exponential function.

(e is an irrational number, approximately 2.71828183, named after the 18th century Swiss mathematician, Leonhard Euler .)

Notice how the characteristics of this graph are similar to those seen above.

This function is simply a "version" of
where b >1.



Inverse of $f(x) = e^x$:

Since $f(x) = e^x$ is a one-to-one function, we know that its inverse will also be a function.

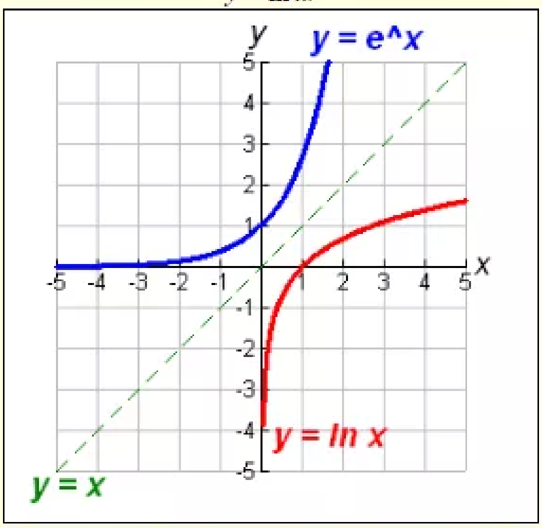
But what is the equation of the inverse of $f(x) = e^x$?

To solve for an inverse algebraically:

$y = e^x$	• set the equation equal to y
$x = e^y$	• swap the x and y
$y = \log_e x$	• solve for y by rewriting in log form
$y = \ln x$	• log base e is called the natural log, ln x.

The inverses for other exponential functions are found in this same manner.

Graphing the inverse of the natural exponential function by reflecting it over the identity function, $y = x$, shows the natural logarithmic function, $y = \ln x$.



Notice how **(0,1)** from $f(x) = e^x$ becomes **(1,0)** for $y = \ln x$. The coordinates switch places between a graph and its inverse.

Other Exponential Inverses:

Graph	Inverse	Graph	Inverse
$y = 2^x$	$y = \log_2 x$	$y = 0.5^x$	$y = \log_{0.5} x$

$y = 2^x$ • set the equation equal to y

$x = 2^y$ • swap the x and y

$y = \log_2 x$ • solve for y by rewriting in log form

$y = 0.5^x$ • set the equation equal to y

$x = 0.5^y$ • swap the x and y

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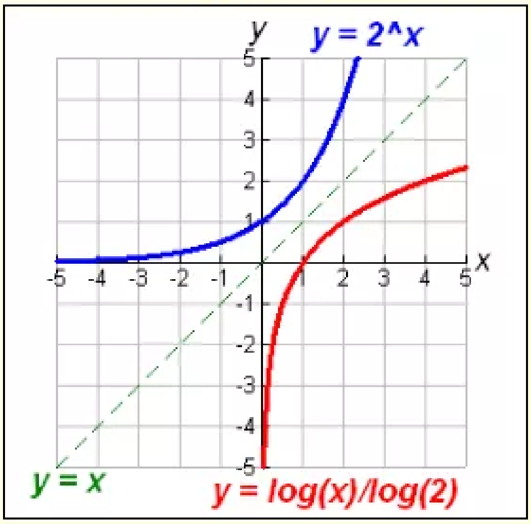
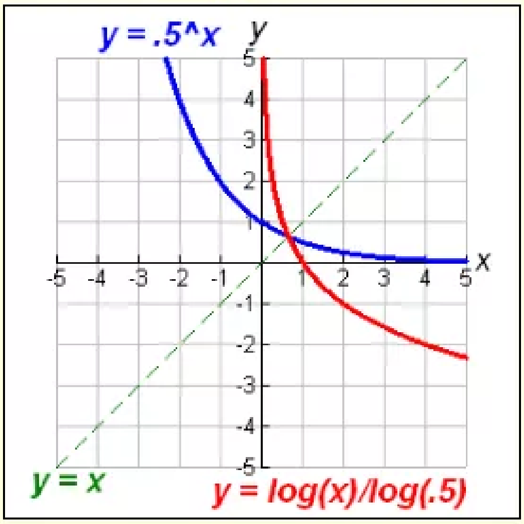




Another view of finding the inverse algebraically:

$y = 2^x$ Set the equation equal to y.

$x = 2^y$ Swap the x and y variables.

$\log x = \log 2^y$ Take the log of both sides.

$\log x = y \log 2$ Apply the rule $\log a^r = r \log a$.

Solve for y.

$\frac{\log x}{\log 2} = y$ Apply the change of base formula: $\log_b a = \log a / \log b$

$\log_2 x = y$

Logarithmic Functions

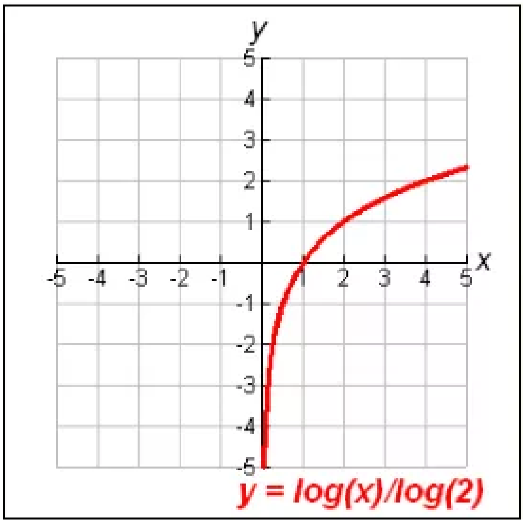
Definition:

The **logarithmic function** is the function $y = \log_b x$, where b is any number such that $b > 0,\ b \neq 1,$ and $x > 0$.

$y = \log_b x$ is equivalent to $x = b^y$

The function is read "log base b of x".

x	$f(x) = y$
1/4	$\log_2 \frac{1}{4} = -2$
1/2	$\log_2 \frac{1}{2} = -1$
1	$\log_2 1 = 0$
2	$\log_2 2 = 1$
3	$\log_2 3 \approx 1.585$
4	$\log_2 4 = 2$



Most logarithmic graphs resemble this same shape.
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Characteristics:

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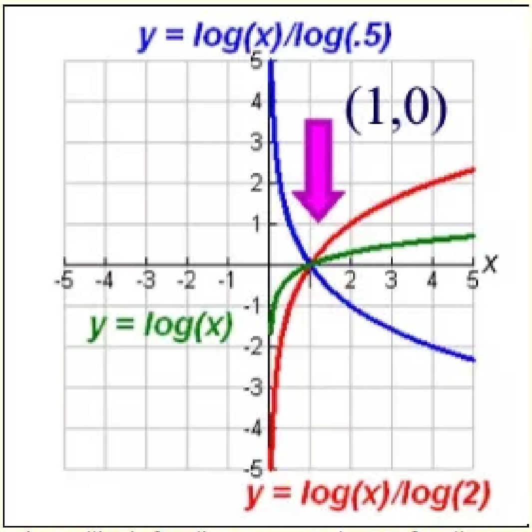

Logarithmic functions are one-to-one functions.

- graph crosses the x-axis at (1, 0)
- when $b > 1$, the graph increases
- when $0 < b < 1$, the graph decreases
- the domain is all positive real numbers (never zero)
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The function defined by

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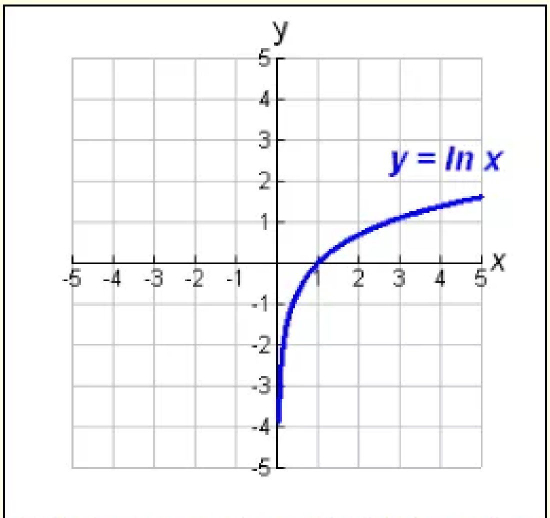
Notice how the characteristics of this graph are similar to those seen above.

This function is simply a "version" of

$y = \log_b x$ where $b > 1$.



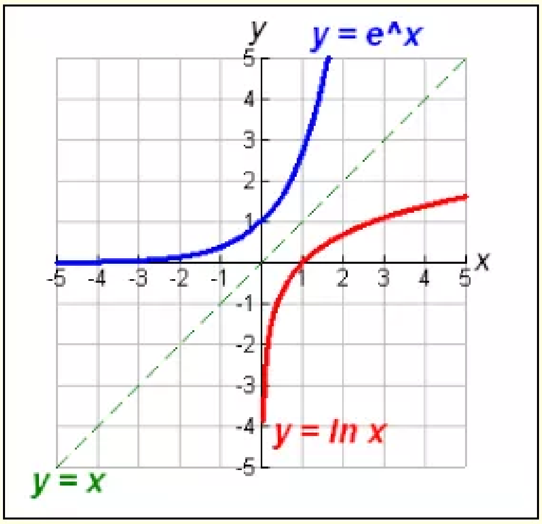
While the graph may "appear" to STOP near -4 on the y-axis, it does NOT stop. It continues extremely close to the y-axis heading to negative infinity.

Inverse of $y = \log_e x = \ln x$:

Since $y = \log_e x = \ln x$ is a one-to-one function, we know that its inverse will also be a function.

When we graph the inverse of the natural logarithmic function, we notice that we obtain the natural exponential function, $f(x) = e^x$.

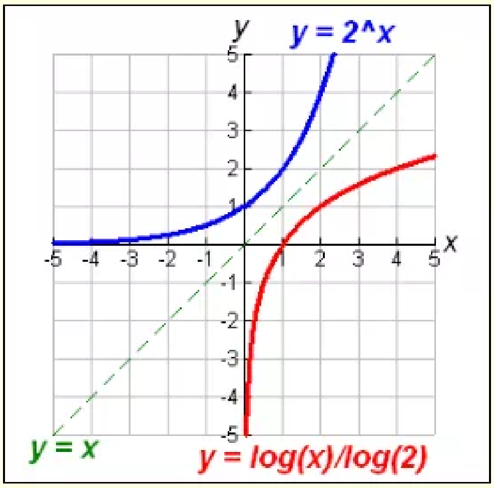
Notice how **(1,0)** from $y = \ln x$ becomes **(0,1)** for $f(x) = e^x$. The coordinates switch places between a graph and its inverse.

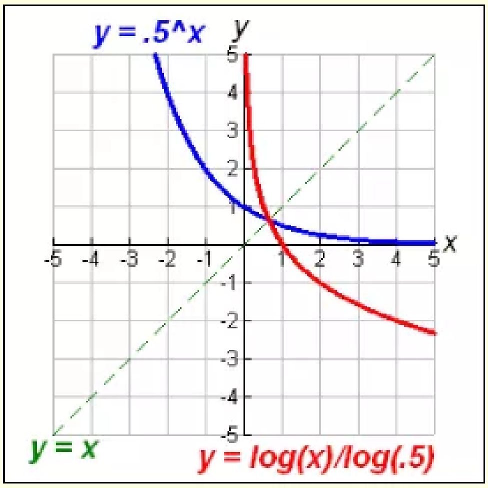


Other Logarithmic Inverses:

Graph	Inverse
$y = \log_2 x$	$y = 2^x$

Graph	Inverse
$y = \log_{0.5} x$	$y = 0.5^x$





Finding the inverse algebraically:

$y = \log_2 x$	Set the equation equal to y.
$x = \log_2 y$	Swap the x and y variables.
$y = 2^x$	Utilize the definition of the log function. (put in exponential form)

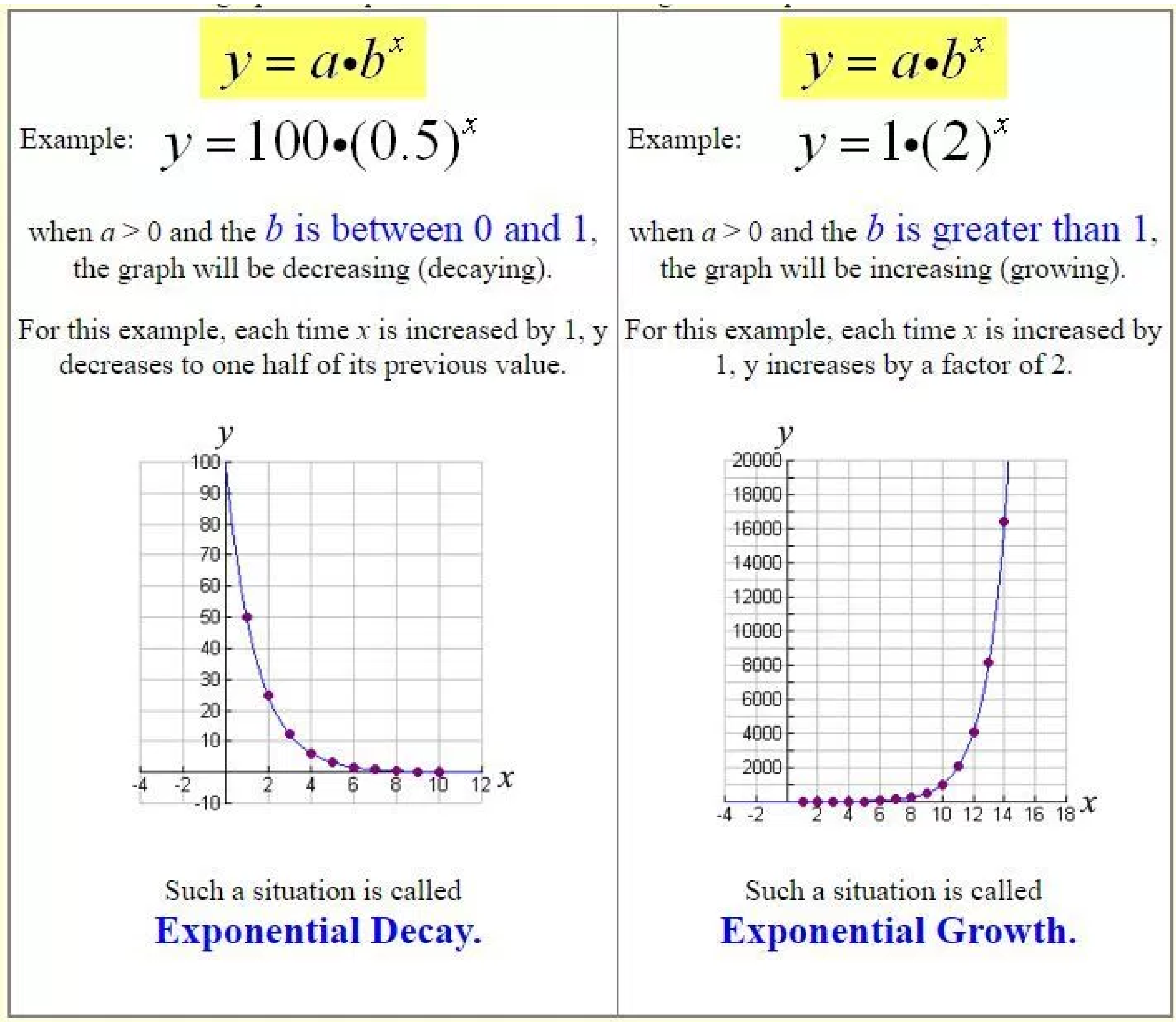
Exponential Growth and Decay

Exponential functions are of the form

$$y = a \cdot b^x$$

Notice: The **variable x is an exponent.** As such, the graphs of these functions are not straight lines. In a straight line, the "rate of change" is the same across the graph. In these graphs, the "rate of change" increases or decreases across the graphs.

Observe how the graphs of exponential functions change based upon the values of a and b:



Many real world phenomena can be modeled by functions that describe how things grow or decay as time passes. Examples of such phenomena include the studies of populations, bacteria, the AIDS virus, radioactive substances, electricity, temperatures and credit payments, to mention a few.

Any quantity that grows or decays by a fixed percent at regular intervals is said to possess **exponential growth** or **exponential decay.**

At the Algebra level, there are two functions that can be easily used to illustrate the concepts of growth or decay in applied situations. When a quantity grows by a fixed percent at regular intervals, the pattern can be represented by the functions,

Growth:	**Decay:**
$y = a(1+r)^x$	$y = a(1-r)^x$

a = initial **amount** before measuring growth/decay

r = growth/decay **rate** (often a percent)

x = number of **time** intervals that have passed

Example: A bank account balance, b, for an account starting with s dollars, earning an annual interest rate, r, and left untouched for n years can be calculated as b = s(1 + r)^n (an exponential growth formula). Find a bank account balance to the nearest dollar, if the account starts with $100, has an annual rate of 4%, and the money left in the account for 12 years.

$$b = s(1+r)^n$$

$$b = 100(1+.04)^{12}$$

$$b = \$160$$

We will now examine rate of growth and decay in a three step process. We will (1) build a chart to examine the data and "see" the growth or decay, (2) write an equation for the function, and (3) prepare a scatter plot of the data along with the graph of the function.

Consider these examples of growth and decay:

Growth:

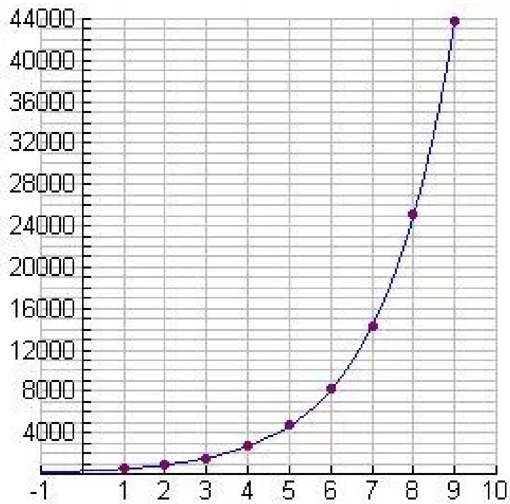
Cell Phone Users In 1985, there were 285 cell phone subscribers in the small town of Centerville. The number of subscribers **increased** by 75% per year after 1985. How many cell phone subscribers were in Centerville in 1994? (Don't consider a fractional part of a person.)

Years	$x = 1$ 1986	2 1987	3 1988	4 1989	5 1990	6 1991	7 1992	8 1993	9 1994
Number of Cell Phone users	498	872	1527	2672	4677	8186	14325	25069	**43871**

There are 43871 subscribers in 1994.

Function: $y = a(1+r)^x$	a = the initial amount before the growth begins r = growth rate x = the number of intervals
$y = 285(1+.75)^x$	as x ranges from 1 to 9 for this problem

The scatter plot of the data table can be prepared by hand or with the use of a graphing calculator. For graphing the function, employ your graphing calculator.



horizontal axis = year (1986 = 1)
vertical axis = number of cell phone users

Growth by doubling:

One of the most common examples of exponential growth deals with bacteria. Bacteria can multiply at an alarming rate when each bacteria splits into two new cells, thus doubling. For example, if we start with only one bacteria which can double every hour, by the end of one day we will have over 16 million bacteria.

End of Hour	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...	24
Bacteria - starting with one	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	...	16777216
Pattern:	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}		2^{24}

At the end of 24 hours, there are 16,777,216 bacteria.

By looking at the pattern, we see that the growth in this situation can be represented as a function:

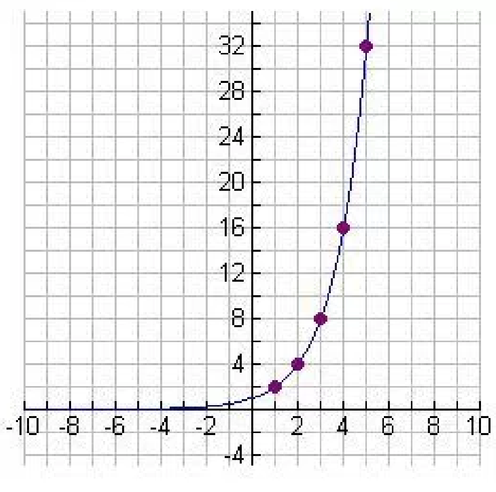
$$y = 2^x$$

Will our formula show this same function? If an amount doubles, the rate of increase is 100%.

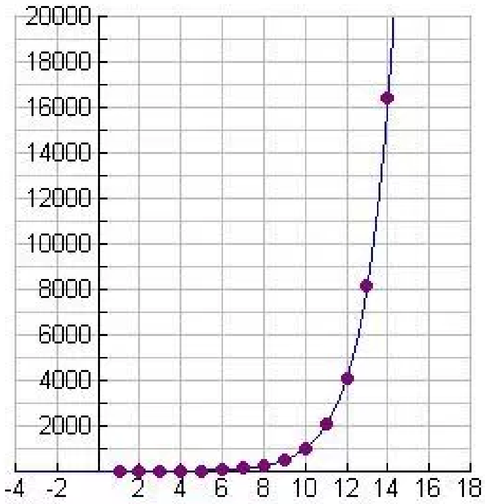
Function: $y = a(1+r)^x$	a = the initial amount before the growth begins r = growth rate x = the number of intervals
$y = 1(1+1.00)^x = 2^x$	as x ranges from 1 to 24 for this problem

horizontal axis = end of hour
vertical axis = number of bacteria

Let's examine the graph of our scatter plot and function. To the left of the origin we see that the function graph tends to flatten, but stays slightly above the x-axis. To the right of the origin the function graph grows so quickly that it is soon off the graph. The rate at which the graph changes increases as time increases.



When we can see larger y-values, we see that the growth still continues at a rapid rate. This is what is meant by the expression "increases exponentially".



Note: In reality, exponential growth does not continue indefinitely. There would, eventually, come a time when there would no longer be any room for the bacteria, or nutrients to sustain them. Exponential growth actually refers to only the early stages of the process and to the manner and speed of the growth.

Decay:

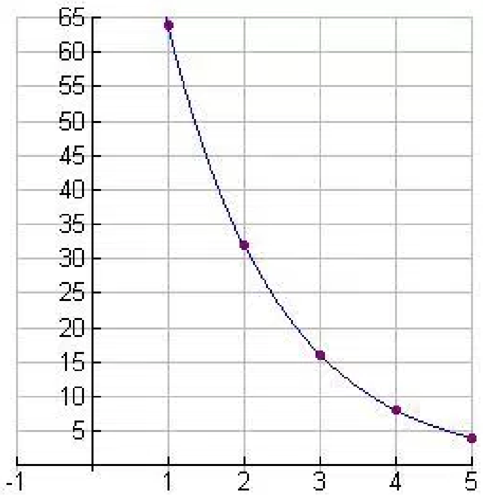
Tennis Tournament Each year the local country club sponsors a tennis tournament. Play starts with 128 participants. During each round, half of the players are eliminated. How many players remain after 5 rounds?

Rounds	1	2	3	4	5
Number of Players left	64	32	16	8	4

There are 4 players remaining after 5 rounds.

Function: $y = a(1-r)^x$	a = the initial amount before the decay begins r = decay rate x = the number of intervals
$y = 128(1-.50)^x$	as x ranges from 1 to 5 for this problem

Notice the shape of this graph compared to the graphs of the growth functions.



horizontal axis = rounds
vertical axis = number of players left

Decay by half-life:

The pesticide DDT was widely used in the United States until its ban in 1972. DDT is toxic to a wide range of animals and aquatic life, and is suspected to cause cancer in humans. The half-life of DDT can be 15 or more years. Half-life is the amount of time it takes for half of the amount of a substance to decay. Scientists and environmentalists worry about such substances because these hazardous materials continue to be dangerous for many years after their disposal.

For this example, we will set the half-life of the pesticide DDT to be 15 years.
Let's mathematically examine the half-life of 100 grams of DDT.

End of Half life cycle	1 15 yrs	2 30 yrs	3 45 yrs	4 60 yrs	5 75 yrs	6 90 yrs	7 105 yrs	8 120 yrs	9 135 yrs	10 150 yrs
Grams of DDT remaining	50	25	12.5	6.25	3.125	1.5625	.78125	.390625	.1953125	.09765625
Pattern:	$\frac{100}{2^1}$	$\frac{100}{2^2}$	$\frac{100}{2^3}$	$\frac{100}{2^4}$	$\frac{100}{2^5}$	$\frac{100}{2^6}$	$\frac{100}{2^7}$	$\frac{100}{2^8}$	$\frac{100}{2^9}$	$\frac{100}{2^{10}}$

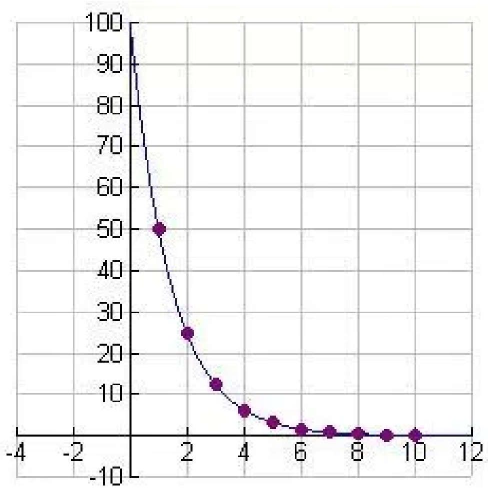
By looking at the pattern, we see that this decay can be represented as a function: $y = \frac{amount}{2^x}$

Function: $y = a(1-r)^x$	a = the initial amount before the decay begins r = decay rate x = the number of intervals
$y = 100(1-.50)^x$	as x ranges from 1 to 10 for this problem

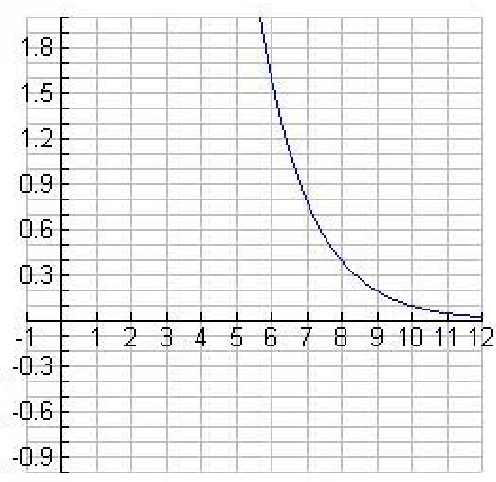
$$y = 100(1-.50)^x = 100\left(\frac{1}{2}\right)^x = 100\left(\frac{1^x}{2^x}\right) = \frac{100}{2^x}$$

horizontal axis = end of half life cycle
vertical axis = grams of DDT remaining

Let's examine the scatter plot and the function. At 0 the y-intercept is 100. To the right of the origin we see that the graph declines rapidly and then tends to flatten, staying slightly above the x-axis. The rate of change decreases as time increases.



When we zoom in on the flattened area of the graph, we see that the graph does stay above the x-axis. This makes sense because we could not have a "negative" number of grams of DDT leftover.



Exponential growth and decay are mathematical changes. The rate of the change continues to either increase or decrease as time passes. In exponential growth, the rate of change increases over time – the rate of the growth becomes faster as time passes. In exponential decay, the rate of change decreases over time – the rate of the decay becomes slower as time passes. Since the rate of change is not constant (the same) across the entire graph, these functions are not straight lines.