True False of Complex Numbers

Q. 1. For complex number $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we write $z_1 \cap z_2$, if $x_1 \leq x_2$ and $y_1 \leq y_2$. Then for all complex numbers z with $1 \cap z$, we have $\frac{1-z}{1+z} \cap 0$ (1981 - 2 Marks)

Sol. Let z = x + iy

then $1 \cap z \Rightarrow 1 \leq x$ & $0 \leq y$ (by def .)

Consider

$$\frac{1-z}{1+z} = \frac{1-(x+iy)}{1+(x+iy)}$$

$$= \frac{(1-x)-iy}{(1+x)+iy} \times \frac{(1+x)-iy}{(1+x)-iy}$$

$$= \frac{1-x^2-y^2}{(1+x)^2+y^2} - \frac{iy(1-x+1+x)}{(1+x)^2+y^2}$$

$$= \frac{1-x^2-y^2}{(1+x)^2+y^2} - \frac{2iy}{(1+x)^2+y^2}$$

$$\frac{1-z}{1+z} \cap 0 \Rightarrow \frac{1-x^2-y^2}{(1+x)^2+y^2} \leq 0$$

and $\frac{-2y}{(1+x)^2+y^2} \leq 0$

$\Rightarrow 1 - x^2 - y^2 \leq 0$ and $-2y \leq 0$

$\Rightarrow x^2 + y^2 \geq 1$ and $y \geq 0$

which is true as $x \geq 1$ & $y \geq 0$

$\therefore$ The given statement is true $\forall z \in C$.

Q. 2. If the complex numbers, Z_1, Z_2 and Z_3 represent the vertices of an equilateral triangle such that $|Z_1| = |Z_2| = |Z_3|$ then $Z_1 + Z_2 + Z_3 = 0$. (1984 - 1 Mark)

Sol. As $|z_1| = |z_2| = |z_3|$

$\therefore$ z_1, z_2, z_3 are equidistant from origin.

Hence O is the circumcentre of ΔABC.

But according to question ΔABC is equilateral and we know that in an equilateral Δ circumcentre and centriod coincide.

$\therefore$Centriod of $\Delta ABC = 0$

$$\Rightarrow \frac{z_1 + z_2 + z_3}{3} = 0 \Rightarrow z_1 + z_2 + z_3 = 0$$

$\therefore$ Statement is true.

Q. 3. If three complex numbers are in A.P. then they lie on a circle in the complex plane. (1985 - 1 Mark)

Sol. If z_1, z_2, z_3 are in A.P. then, $\frac{z_1 + z_3}{2} = z_2$

$\Rightarrow$ z_2 is mid pt. of line joining z_1 and z_3.

$\Rightarrow$ z_1, z_2, z_3 lie on a st. line

$\therefore$ Given statement is false

Q. 4. The cube roots of unity when represented on Argand diagram form the vertices of an equilateral triangle. (1988 - 1 Mark)

Sol. $\because$ Cube roots of unity are 1, $\frac{-1+i\sqrt{3}}{2}, \frac{-1-\sqrt{3}}{2}$

$\therefore$ Vertices of triangle are

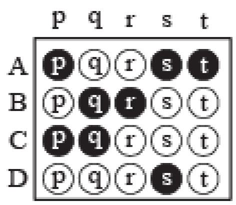
$A(1, 0)$, $B\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$, $C\left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$

$\Rightarrow AB = BC = CA$

$\therefore \Delta$ is equilateral.

Match the following of Complex Numbers

DIRECTIONS (Q. 1 and 2) : Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in ColumnII are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example : If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.



Q. 1. $z \neq 0$ is a complex number (1992 - 2 Marks)

Column I	**Column II**
(A) Rez = 0	**(p) $\mathrm{Rez}^2 = 0$**
(B) Argz = $\frac{\pi}{4}$	**(q) $\mathrm{Imz}^2 = 0$**
	(r) $\mathrm{Rez}^2 = \mathrm{Imz}^2$

Ans. $z \neq 0$ Let $z = a + ib$ $\mathrm{Re}(z) = 0 \Rightarrow z = ib$

$\Rightarrow z^2 = -b^2$

$\therefore$ Im (z)2 = 0

$\therefore$ (A) corresponds to (q)

$\mathrm{Arg}\,\frac{\pi}{4} = \Rightarrow a = b \Rightarrow z = a + ia$

$z^2 = a^2 - a^2 + 2ia^2$; $z^2 = 2ia^2 \Rightarrow \mathrm{Re}(z)^2 = 0$

$\therefore$ (B) corresponds to (p).

Q. 2. Match the statements in Column I with those in Column II. (2010) [Note : Here z takes values in the complex plane and Im z and Re z denote , respectively, the imaginary part and the real part of z.]

Column I	**Column II**
The set of points z satisfying $|\mathbf{z} - \mathbf{i}|\mathbf{z}|| = |\mathbf{z} + \mathbf{i}|\mathbf{z}||$ is contained in or equal to	(p) an ellipse with eccentricity $\frac{4}{5}$
(B) The set of points z satisfying $|z + 4| + |z - 4| = 10$ is contained in or equal to	(q) the set of points z satisfying Im z = 0
(C) If $|w| = 2$, then the set of points $\mathbf{z} = \mathbf{w} - \frac{1}{w}$ is contained in or equal to	(r) the set of points z satisfying $|\text{Im } z| \le 1$
(D) If $|w| = 1$, then the set of points $\mathbf{z} = \mathbf{w} + \frac{1}{w}$ is contained in or equal to	(s) the set of points z satisfying $|\text{Re } z| < 2$
	(t) the set of points z satisfying $|z| \le 3$

Ans. (A) → (q, r) $|z - i|z|| = |z + i|z||$

⇒ z is equidistant from two points (0, |z|) and (0,– |z|) which lie on imaginary axis.

∴ z must lie on real axis ⇒ Im (z)=0 also $|I_m(z)| \le 1$

(B) → p

Sum of distances of z from two fined points (–4, 0) and (4, 0) is 10 which is greater than 8.

∴ z traces an ellipse with 2a = 10 and 2ae =8

⇒ e= 4/5

(C) → (p, s, t)

Let $\omega = 2(\cos\theta + i \sin\theta)$

then z = ω - $\frac{1}{\omega}$ (cosθ + i sinθ) - $\frac{1}{2}$(cosθ + i sinθ)

⇒ x + iy $\frac{3}{2}\cos\theta + i\frac{5}{2}\sin\theta$

Here $|z| = \sqrt{\frac{9+25}{4}} = \sqrt{\frac{34}{4}} \le 3$ and $|R_e(z)| \le 2$

Also x = $\frac{3}{2}\cos\theta, y = \frac{5}{2}\sin\theta \Rightarrow \frac{4x^2}{9} + \frac{4y^2}{25} = 1$

Which is an ellipse with e = $\sqrt{1-\frac{9}{25}} = \frac{4}{5}$

(D) → (q,r, s,t)

Let ω = cosθ + i sinq then z = 2 cosθ ⇒ Imz=0

Also z ≤ 3 and | Im(z) |≤ 1, | R_e (z) |≤ 2

DIRECTIONS (Q. 3) : Following question has matching lists. The codes for the list have choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

Q. 3. Let z_k = $\left(\frac{2k\pi}{10}\right)$ + sin $\left(\frac{2k\pi}{10}\right)$: k=1,2,……,9. **(JEE Adv. 2014)**

List-I	List-II
P. For each z_k there exists as z_j such that $z_k . z_j = 1$	1. True
Q. There exists a k ∈ {1, 2,…,9} such that $z_1 . z = z_k$ has no solution z in the set of complex numbers	2. False
R. $\frac{\|1-z_1\|\|1-z_2\|....\|1-z_9\|}{10}$ equals	3. 1
S. $1-\sum_{k=1}^{9}\cos\left(\frac{2k\pi}{10}\right)$ equals	4. 2

	P	Q	R	S		P	Q	R	S
(a)	**1**	**2**	**4**	**3**	**(b)**	**2**	**1**	**3**	**4**
(c)	**1**	**2**	**3**	**4**	**(d)**	**2**	**1**	**4**	**3**

Ans. (c)

(P) → (1) : $z_k = \cos\frac{2k\pi}{10} + i\sin\frac{2k\pi}{10}$, k = 1 to 9

$\therefore\ z_k = e^{i\frac{2k\pi}{10}}$

Now $z_k.z_j = 1 \Rightarrow z_j = \frac{1}{z_k} = e^{-i\frac{2k\pi}{10}} = \overline{z_k}$

We know if z_k is 10^{th} root of unity so will be $\overline{z}_k$.

∴ For every z_k, there exist $z_i = \overline{z}_k$.

Such that $z_k . z_j = z_k . \overline{z}_k = 1$

Hence the statement is true.

(Q) → (2) $z_1 = z k \Rightarrow z = \frac{z_k}{z_1}$ for $z_1 \neq 0$

∴ We can always find a solution to $z_1.z = z_k$

Hence the statement is false.

(R) → (3) : We know $z^{10} - 1 = (z - 1)(z - z_1) \ldots (z - z_9)$

$\Rightarrow (z - z_1)(z - z_2) \ldots (z - z_9) = \frac{z^{10}-1}{z-1}$

$= 1 + z + z^2 + \ldots z^9$

For z = 1 we get

$(1 - z_1)(1 - z_2) \ldots (1 - z_9) = 10$

$\therefore \frac{|1-z_1||1-z_2|\ldots|1-z_9|}{10} = 1$

(S) → (4) : 1, Z_1, Z_2, ... Z_9 are 10^{th} roots of unity.

$\therefore Z^{10} - 1 = 0$

From equation $1 + Z_1 + Z_2 + ... + Z_9 = 0$

$Re(1) + Re(Z_1) + Re(Z_2) + ... + Re(Z_9) = 0$

$\Rightarrow Re(Z_1) + Re(Z_2) + ... Re(Z_9) = -1$

$$\Rightarrow \sum_{K=1}^{9} \cos\frac{2k\pi}{10} = -1 \Rightarrow 1 - \sum_{K=1}^{9} \cos\frac{2k\pi}{10} = 2$$

Hence (c) is the correct option.

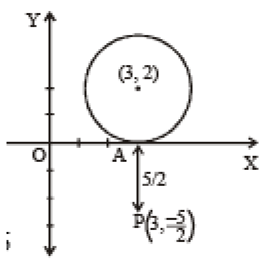
Integar Type ques of Complex Numbers

Q. 1. If z is any complex number satisfying $|z - 3 - 2i| \le 2$, then the minimum value of $|2z - 6 + 5i|$ is (2011)

Sol. Given $|z - 3 - 2i| < 2$ which represents a circular region with centre (3, 2) and radius 2.

Now $|2z - 6 + 5i| = 2\left|z - \left(3 - \frac{5}{2}i\right)\right|$

$= 2 \times$ distance of z from P (where Z lies in or on the circle)



Also min distance of z from P $= \frac{5}{2}$

$\therefore$ Minimum value of $|2z - 6 + 5i| = 5$

Q. 2. Let $\omega = e^{\frac{i\pi}{3}}$, and a, b, c, x, y, z be non-zero complex numbers such that (2011)

$a + b + c = x$

$a + b\omega + c\omega^2 = y$

$a + b\omega^2 + c\omega = z$

Then the value of $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is

Sol. The expression may not attain integral value for all a, b, c.
If we consider a = b = c then

$x = 3a, y = a(1 + \omega + \omega^2) = a(1 + i\sqrt{3})$

$Z = a\,(1 + \omega^2 + \omega) = a(1 + i\sqrt{3})$

$\Rightarrow |x|^2 + |y|^2 + |z|^2 = 9\,|a|^2 + 4\,|a|^2 + 4\,|a|^2 = 17\,|a|^2$

$\Rightarrow \dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = \dfrac{17}{3}$ (which is not an integer)

Note : However if $\omega = e^{i(2\frac{\pi}{3})}$ then the value of expression can be evaluated as follows

$$\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = \frac{x\bar{x} + y\bar{y} + z\bar{z}}{|a|^2 + |b|^2 + |c|^2}$$

$$(a+b+c)(\bar{a}+\bar{b}+\bar{c}) + (a+b\omega+c\omega^2)(\bar{a}+\bar{b}\omega^2+\bar{c}\omega) +$$

$$= \frac{(a+b\omega^2+c\omega)(\bar{a}+\bar{b}\omega+\bar{c}\omega^2)}{|a|^2+|b|^2+|c|^2}$$

$$= \frac{3|a|^2+3|b|^2+3|c|^2+(a\bar{b}+\bar{a}b+b\bar{c}+\bar{b}c+a\bar{c}+\bar{a}c)(1+\omega+\omega^2)}{|a|^2+|b|^2+|c|^2}$$

$= 3 \qquad (\because 1 + \omega + \omega^2 = 0)$

Q. 3. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. **The value of the expression**

$$\frac{\sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3} |\alpha_{4k-1} - \alpha_{4k-2}|}$$ **is** **(JEE Adv. 2015)**

Sol. $\alpha_k = \cos\frac{k\pi}{7} + i\sin\frac{k\pi}{7} = e^{\frac{i\pi k}{7}}$

$$\alpha_{k+1} - \alpha_k = e^{\frac{i\pi(k+1)}{7}} - e^{\frac{i\pi k}{7}} = e^{\frac{i\pi k}{7}}(e^{i\pi/7} - 1)$$

$$|\alpha_{k+1} - \alpha_k| = |e^{i\pi/7} - 1|$$

$$\Rightarrow \sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k| = 12|e^{i\pi/7} - 1|$$

Similarly $\sum_{k=1}^{3}\left|\alpha_{4k-1}-\alpha_{4k-2}\right| = 3\left|e^{i\pi/7}-1\right|$

$\therefore$

$$\frac{\sum_{k=1}^{12}\left|\alpha_{k+1}-\alpha_k\right|}{\sum_{k=1}^{3}\left|\alpha_{4k-1}-\alpha_{4k-2}\right|}=4$$

Fill Ups of Complex Numbers

Q. 1. If the expression (1987 - 2 Marks)

$$\frac{\left[\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)+i\tan(x)\right]}{\left[1+2\,i\,\sin\left(\frac{x}{2}\right)\right]}$$

is real, then the set of all possible values of x is ....

Ans. $2n\pi,\ n\pi+\frac{\pi}{4}$

Sol.

$$\text{Let } z=\frac{\sin x/2+\cos x/2+i\tan x}{1+2i\sin x/2}$$

$$=\frac{(\sin x/2+\cos x/2+i\tan x)(1-2i\sin x/2)}{(1+2i\sin x/2)(1-2i\sin x/2)}$$

$$=\frac{[\sin x/2+\cos x/2+2\sin x/2\tan x) +i(\tan x-2\sin^2 x/2-2\sin x/2\cos x/2)]}{(1+4\sin^2 x/2)}$$

But ATQ, $I_m(z)$ = 0 (as z is real)

$$\Rightarrow \quad \tan x-2\sin\frac{x}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)=0$$

$$\Rightarrow \quad \frac{\sin x}{\cos x}-2\sin^2 x/2-2\sin x/2\cos x/2=0$$

$$\Rightarrow \quad \frac{\sin x}{\cos x}-(1-\cos x)-\sin x=0$$

$$\Rightarrow \quad \sin x\left[\frac{1}{\cos x}-1\right]-[1-\cos x]=0$$

$$\Rightarrow \left(\frac{1-\cos x}{\cos x}\right)\sin x-[1-\cos x]=0$$

$$\Rightarrow (1-\cos x)\left(\frac{\sin x}{\cos x}-1\right)=0$$

$\Rightarrow \cos x = 1 \Rightarrow \mathbf{x} = 2n\pi$ and

$\tan x = 1 \Rightarrow x = n\pi + \pi/4$

$\therefore x = 2n\pi,\ n\pi + \pi/4$

Q. 2. For any two complex numbers z_1, z_2 and any real number a and b. (1988 - 2 Marks) $| az_1 - bz_2 |^2 + | bz_1 + az_2 |^2$ = ...

Ans. $(a^2 + b^2)(| z_1 |^2 + | z_2 |^2)$

Sol.

$$| az_1 - bz_2 |^2 + | bz_1 + az_2 |^2$$
$$= \imath^2 | z_1 |^2 + b^2 | z_2 |^2 - 2ab\ \mathrm{Re}(z_1\bar{z}_2) + b^2 | z_1 |^2$$

$$+ a^2 | z_2 |^2 + 2ab\ \mathrm{Re}\ (z_1\bar{z}_2)$$

$$= (a^2 + b^2)\ (| z_1 |^2 + | z_2 |^2)$$

Q. 3. If a, b, c, are the numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then a = ...and b = ... (1989 - 2 Marks)

Ans. $2-\sqrt{3},\ 2-\sqrt{3}$

KEY CONCEPT : $| z_1 - z_2 |$= distance between two points represented by z_1 and z_2.

As $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, therefore

$|z_1 - z_3| = |z_2 - z_3| = |z_1 - z_2|$

$| a + i | = | 1 + bi | = | (a - 1) + i (1 - b) |$

$\Rightarrow a^2 + 1 = 1 + b^2 = (a - 1)^2 + (1 - b)^2$

$\Rightarrow a^2 = b^2 = a^2 + b^2 - 2a - 2b + 1$

$\Rightarrow a = b$ ....(1)

($\therefore$ a, b > 0 $\quad \therefore$ a$\neq$ – b) and

$b^2 - 2a - 2b + 1 = 0$

or $a^2 - 2a - 2b + 1 = 0$ ....(2)

$\Rightarrow a^2 - 2a - 2a + 1 = 0$ [$\therefore$ a = b]

$\Rightarrow a^2 - 4a + 1 = 0$

$$\Rightarrow \quad a = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3} \quad \text{But } 0 < a, b < 1$$

$$\therefore \quad a = 2 - \sqrt{3} \qquad \text{also } b = 2 - \sqrt{3}$$

Q. 4. ABCD is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy BD = 2AC. If the points D and M represent the complex numbers 1 + i and 2 - i respectively, then A represents the complex number ....or.......... (1993 - 2 Marks)

Ans. $3 - \frac{i}{2}$ or $1 - \frac{3}{2}i$

Sol :

If we see the problem as in co-ordinate geometry we have D ≡ (1,1) and M≡ (2, – 1)

We know that diagonals of rhombus bisect each other at 90°

$\therefore$ AC passes through M and is ^ to BD

$\therefore$ Eq. of AC in symmetric form can be written as

$$\frac{x-2}{2/\sqrt{5}} = \frac{y+1}{1/\sqrt{5}} = r$$

Now for pt. A, as

$$AM = \frac{1}{2}DM = \frac{1}{2}\sqrt{(2-1)^2 + (-1-1)^2} = \sqrt{5}/2$$

Putting $r = \pm\sqrt{5}/2$ *we get*,

$$\frac{x-2}{2/\sqrt{5}} = \frac{y+1}{1/\sqrt{5}} = \pm\sqrt{5}/2$$

$$\Rightarrow \quad x = \pm 1 + 2, y = \pm\frac{1}{2} - 1$$

$$\Rightarrow \quad x = 3 \quad \text{or} \quad 1, y = \frac{-1}{2} \text{ or } \frac{-3}{2}$$

$\therefore$ Pt. A is $3 - i/2$ or $1 - (3/2)i$

Q. 5. Suppose Z_1, Z_2, Z_3 are the vertices of an equilateral triangle inscribed in the circle |Z| = 2. If $Z_1 = 1 + i\sqrt{3}$ then Z_2 = ..., Z_3 = ... (1994 - 2 Marks)

Ans. -2, $1 - i\sqrt{3}$

Let z_1, z_2,z_3 be the vertices A, B and C respectively of equilateral ΔABC, inscribed in a circle | z | = 2, centre (0, 0) rasius = 2

Given $\quad z_1 = 1 + i\sqrt{3}$

$$z_2 = e^{\frac{2\pi i}{3}} z_1$$

$$= \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)(1 + i\sqrt{3})$$

$$= \frac{-1-3}{2} = -2$$

and $z_3 = e^{4(\pi/3)i} z_1$

$$= \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)(1 + i\sqrt{3})$$

$$= \left(\frac{-1 - i\sqrt{3}}{2}\right)(1 + i\sqrt{3}) = \frac{-1 - 2i\sqrt{3} + 3}{2} = 1 - i\sqrt{3}$$

Q. 6. The value of the expression

$1 \cdot (2-\omega)(2-\omega^2)+2 \cdot (3-\omega)(3-\omega^2)+....+(n-1).(n-\omega)(n-\omega^2)$,

where w is an imaginary cube root of unity, is..... (1996 - 2 Marks)

Ans. $\frac{1}{4}$ n (n - 1)(n^2 + 3n + 4)

Sol:

rth term of the given series,

= r [(r +1) −ω](r +1) −w2]

$= r\,[(r+1)^2 - (\omega+\omega^2)(r+1) + \omega^3]$

$= r\,[(r+1)^2 - (-1)(r+1) + 1]$

$= r\,[(r^2 + 3r + 3] = r^3 + 3r^2 + 3r$

Thus, sum of the given series,

$$= \sum_{r=1}^{(n-1)} (r^3 + 3r^2 + 3r)$$

$$= \frac{1}{4}(n-1)^2 n^2 + 3.\frac{1}{6}(n-1)\ (n)\ (2n-1)\ + 3.\frac{1}{2}(n-1)n$$

$$= (n-1)\ (n)\left[\frac{1}{4}(n-1)\,n + \frac{1}{2}\ (2n-1) + \frac{3}{2}\right]$$

$$= \frac{1}{4}(n-1)\ n[n^2 - n + 4n - 2 + 6]$$

$$= \frac{1}{4}(n-1)\,n[n^2 + 3n + 4]$$

Subjective questions of Complex Numbers

Q. 1. Express $\frac{1}{1-\cos\theta+2i\sin\theta}$ in the form x + iy. (1978)

Ans. $=\left(\frac{1}{5+3\cos\theta}\right)+\left(\frac{-2\cot\theta/2}{5+3\cos\theta}\right)i$

Sol.

$$\frac{1}{1-\cos\theta+2i\sin\theta}$$

$$=\frac{1}{2\sin^2\theta/2+4i\sin\theta/2\cos\theta/2}=\frac{1}{2\sin\theta/2}$$

$$\left[\frac{\sin\theta/2-2i\cos\theta/2}{(\sin\theta/2+2i\cos\theta/2)(\sin\theta/2-2i\cos\theta/2)}\right]$$

$$=\frac{1}{2\sin\theta/2}\left[\frac{\sin\theta/2-2i\cos\theta/2}{(\sin^2\theta/2+4\cos^2\theta/2}\right]$$

$$=\frac{1}{2\sin\theta/2}\left[\frac{2\sin\theta/2-4i\cos\theta/2}{1-\cos\theta+4+4\cos\theta}\right]$$

$$=\frac{2}{2\sin\theta/2}\left[\frac{2\sin\theta/2-2i\cos\theta/2}{5+3\cos\theta}\right]$$

$$=\left(\frac{1}{5+3\cos\theta}\right)+\left(\frac{-2\cot\theta/2}{5+3\cos\theta}\right)i$$

which is of the form X + iY.

Q. 2. If x = a + b, y = aγ + bβ and z = aβ + bγ where γ and b are the complex cube roots of unity, show that xyz = $a^3 + b^3$. (1978)

Ans. Sol. As b and **γ** are the complex cube roots of unity therefore,

let **β** = ω and **γ** = ω^2

so that $\omega + \omega^2 + 1 = 0$ and $\omega^3 = 1$.

Then xyz = $(a + b)(a\omega^2 + b\omega)(a\omega + b\omega^2)$

$= (a + b)(a^2\omega^3 + ab\omega^4 + ab\omega^2 + b^2\omega^3)$

$= (a + b)(a^2 + ab\omega + ab\omega^2 + b^2)$ (using $\omega^3 = 1$)

$= (a + b)(a^2 + ab(\omega + \omega^2) + b^2)$

$= (a + b)(a^2 - ab + b^2)$ (using $\omega + \omega^2 = -1$)

$= a^3 + b^3$ Hence proved.

Q. 3. If $x + iy = \sqrt{\frac{a+ib}{c+id}}$, prove that $(x^2 + y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$. (1979)

Ans.

Sol. Given $x + iy = \sqrt{\frac{c+ib}{c+id}}$

$\Rightarrow (x + iy)^2 = \frac{a+ib}{c+id}$ ....(1)

Taking conjugate on both sides, we get

$(x - iy)^2 = \frac{a-ib}{c-id}$ ....(2)

Multiply (1) and (2), we get

$(x^2 + y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$

Q. 4. Find the real values of x and y for which the following equation is satisfied $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$ (1980)

Sol. $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$

$\Rightarrow (4 + 2i)x - 6i - 2 + (9 - 7i)y + 3i - 1 = 10i$

$\Rightarrow (4x + 9y - 3) + (2x - 7y - 3)i = 10i$

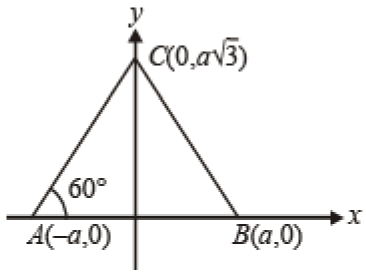
$\Rightarrow 4x + 9y - 3 = 0$ and $2x - 7y - 3 = 10$

On solving these two, we get $x = 3$, $y = -1$

Q. 5. Let the complex number z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let $z_0$be the circumcentre of the triangle.

Then prove that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$. (1981 - 4 Marks)

Sol.



Let us consider the equilateral Δ with each side of length 2a and having two of its vertices on x-axis namely A (–a,0) and B (a, 0), then third vertex C will clearly lie on y-axis s.t.

OC = 2a sin 60°=$a\sqrt{3}$ ∴ C has the co-ordinates (0,$a\sqrt{3}$).

Now in the form of complex numbers if A, B and C are represented by z_1, z_2, z_3 then z_1 = - a ; z_2 = a ; z_3= a_3i As in an equilateral Δ, centriod and circumcentre coincide, we get

Circumcentre, $z_0 = \dfrac{z_1+z_2+z_3}{3}$

$\Rightarrow z_0 = \dfrac{-a+a+a\sqrt{3}\,i}{3} = \dfrac{ia}{\sqrt{3}}$

Now, $z_1^2+z_2^2+z_3^2 = a^2+a^2-3a^2=-a^2$

and $3z_0^2=(ia)^2=-a^2$

∴ Clearly $3z_0^2=z_1^2+z_2^2+z_3^2$

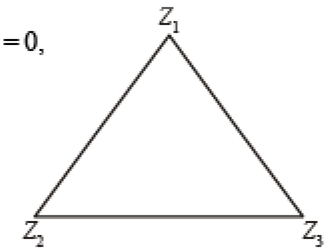
Q. 6. Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only if $z_1^2 + z_2^2 - z_1 z_2 = 0$. (1983 - 3 Marks)

Ans. Sol. We know that if z_1, z_2, z_3 are vertices of an equilateral Δ then

$\dfrac{z_1-z_2}{z_3-z_2} = \dfrac{z_3-z_1}{z_2-z_1}$

Here

$z_3 = 0,$



We get $\frac{z_1 - z_2}{-z_2} = \frac{-z_1}{z_2 - z_1}$

$\Rightarrow -(z_1 - z_2) = z_1 z_2$

$\Rightarrow -z_1^2 - z_2^2 + 2 z_1 z_2 = z_1 z_2 \Rightarrow z_1^2 + z_2^2 - z_1 z_2 = 0.$

Q. 7. If 1, a_1, a_2 ......, a_{n-1} are the n roots of unity, then show that $(1 - a_1)(1 - a_2)(1 - a_3) \ldots (1 - a_{n-1}) = n$ (1984 - 2 Marks)

Sol. 1, a_1, a_2, ... a_{n-1} are the n roots of unity. Clearly above n values are roots of eq. $x_{n-1} = 0$

Therefore we must have (by factor theorem)

$x_{n-1} = (x - 1)(x - a_1)(x - a_2) \ldots (x - a_{n-1}) \ldots (1)$

$\Rightarrow \frac{x^n - 1}{x - 1} = (x - a_1)(x - a_2) \ldots (x - a_{n-1}) \ldots (2)$

Differentiating both sides of eq. (1), we get

$nxn - 1 = (x - a_1)(x - a_2) \ldots (x - a_{n-1}) + (x - 1)(x - a_2) \ldots (x - a_{n-1}) + \ldots + (x - 1)(x - a_1) \ldots (x - a_{n-2})$

For x = 1, we get $n = (1 - a_1)(1 - a_2) \ldots (1 - a_{n-1})$

[All the terms except first contain $(x - 1)$ and hence become zero for x = 1] Proved.

Q. 8. Show that the area of the triangle on the Argand diagram formed by the complex numbers z, iz and z + iz is $\frac{1}{2}|z|^2$ (1986 - 2½ Marks)

Sol. Let A = z = x + iy, B = iz = − y + ix,

C = z + iz = (x − y) + i (x + y)

Now, area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -y & x & 1 \\ x-y & x+y & 1 \end{vmatrix}$

Operating $R_2 - R_1$, $R_3 - R_1$, we get

$\Delta = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -y-x & x-y & 0 \\ -y & x & 0 \end{vmatrix}$

$\frac{1}{2}$ |x(-y - x) +y (x - y)|

$= \frac{1}{2}$ |- xy -x^2 + xy-y^2| = $\frac{1}{2}$ |-x^2-y^2|

$= \frac{1}{2}$ |x^2+y^2|= $\frac{1}{2}$ |z^2| Hence Proved.

Q. 9. Let Z_1 = 10 + 6i and Z_2 = 4 + 6i. If Z is any complex number such that the argument of $\frac{(Z-Z_1)}{(Z-Z_2)}$ is $\frac{\pi}{4}$, then prove that $|Z - 7 - 9i| = 3\sqrt{2}$. (1990 - 4 Marks)

Ans. Sol. We are given that $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$

Also $\arg\left(\frac{z-z_1}{z-z_2}\right) = \frac{\pi}{4}$

$\Rightarrow \arg(z - z_1) - \arg(z - z_2) = \frac{\pi}{4}$ = NOTE THIS STEP

$\Rightarrow \arg((x+iy) - (10 + 6i)) - \arg((x+iy) - (4 + 6i)) = \frac{\pi}{4}$

$\Rightarrow \arg[(x - 10) + i(y - 6)] - \arg[(x - 4) + i(y - 6)] = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{y-6}{x-10}\right) - \tan^{-1}\left(\frac{y-6}{x-4}\right) = \frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{y-6}{x-10}-\frac{y-6}{x-4}}{1+\frac{(y-6)^2}{(x-4)(x-10)}}\right)=\frac{\pi}{4}$$

$$\Rightarrow \frac{(x-4)(y-6)-(x-10)(y-6)}{(x-4)(x-10)+(y-6)^2}=\tan\frac{\pi}{4}$$

$\Rightarrow (x-4-x+10)(y-6)=(x-4)(x-10)+(y-6)^2$

$\Rightarrow 6y-36=x^2+y^2-14x-12y+40+36$

$\Rightarrow x^2+y^2-14x-18y+112=0$

$\Rightarrow (x^2-14x+49)+(y^2-18y+81)=18$

$\Rightarrow (x-7)^2+(y-9)^2=(3\sqrt{2})^2$

$\Rightarrow (x+iy)-(7+9i)=3\sqrt{2}$

$\Rightarrow z-(7+9i)=3\sqrt{2}$. Hence Proved.

Q. 10. If $iz^3+z^2-z+i=0$, then show that $|z|=1$. (1995 - 5 Marks)

Sol. Dividing through out by i and knowing that 1/i = -i we get i =-
$z^3-iz^2+iz+1=0$

or $z^2(z-i)+i(z-i)=0$

as $1=-i^2$ or $(z-i)(z^2+i)=0$ $\therefore z=i$

or $z^2=-i$

$\therefore |z|=|i|=1$ or $|z^2|=|z|^2=|-i|=1$

$\Rightarrow |z|=1$

Hence in either case $|z|=1$

Q. 11. If $|Z|\le 1$, $|W|\le 1$, show that

$|Z-W|^2\le(|Z|-|W|)^2+(\text{Arg } Z-\text{Arg} W)^2$ (1995 - 5 Marks)

Ans. Sol. Let $Z=r_1(\cos\theta_1+i\sin\theta_1)$

and $W=r_2(\cos\theta_2+i\sin\theta_2)$

We have $| Z | = r_1$, $| W | = r_2$, Arg $Z = \theta_1$ and

Arg $W = \theta_2$

Since $| Z | \leq 1$, $|W| \leq 1$, it follows that $r_1 \leq$ and $r_2 \leq 1$

We have $Z - W = (r_l \cos\theta_1 - r_2 \cos\theta_2)$

$+i(r_l \sin\theta_1 - r_2 \sin\theta^2)$

$|Z - W|^2 = (r_1 \cos\theta_1 - r_2 \cos\theta_2) + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2$

$= r_1^2 \cos^2\theta_1 + r_2^2 \cos^2\theta_2 - 2 r_1 r_2 \cos\theta_1 \cos\theta_2 + r_1^2 \sin^2\theta_1$

$+ r_2 \sin\theta_2 - 2 r_1 r_2 \sin\theta_1 \sin\theta_2$

$= r_1^2 (\cos^2\theta_1 + \sin^2\theta_1) + r_2^2 (\cos^2\theta_2 + \sin^2\theta_2)$

$- 2 r_1 r_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)$

$= = r_1^2 + r_2^2 - 2 r_1 r^2 \cos(\theta_1 - \theta_2)$

$= (r_1 - r_2)^2 + 2r_1 r_2[1 - \cos(\theta_1 - \theta_2)]$

$= (r1 - r2)^2 + 4 r_1 r_2 \sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)$

$= |r_1 - r_2|^2 + 4 r_1 r_2 \left|\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\right|^2$

$\leq |r_1 - r_2|^2 + 4\left|\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\right|$ $[\because r_1, r_2 \leq 1]$

But $|\sin\theta| \leq |\theta|\ \forall\ \theta \in R$

NOTE THIS STEP

Therefore,

$$|Z - W|^2 \leq |r_1 - r_2|^2 + 4\left|\frac{\theta_1 - \theta_2}{2}\right|^2 \leq |r_1 - r_2|^2 + |\theta_1 - \theta_2|^2$$

Thus $|Z - W|^2 \leq (|Z| - |W|)^2 + (\text{Arg } Z - \text{Arg}W)^2$

12. Find all non-zero complex numbers Z satisfying $\overline{Z} = iZ^2$ (1996 - 2 Marks)

Sol. Let z = x + iy then $\overline{z} = iz^2$

$\Rightarrow x - iy = i(x^2 - y^2 + 2ixy)$

$\Rightarrow x - iy = i(x^2 - y^2) - 2xy$

$\Rightarrow x(1 + 2y) = 0 ; x^2 - y^2 + y = 0$

$\Rightarrow x = 0$ or $y = -\frac{1}{2} \Rightarrow x = 0, \; y = 0, \; 1$

or $y = -\frac{1}{2}, x = \pm\frac{\sqrt{3}}{2}$

For non zero complex number z

$x = 0, y = 1;$

$x = \frac{\sqrt{3}}{2}, y = -\frac{1}{2}; x = \frac{-\sqrt{3}}{2}, y = -\frac{1}{2}$

$\therefore \quad z = i, \frac{\sqrt{3}}{2} - \frac{i}{2}, -\frac{\sqrt{3}}{2} - \frac{i}{2}$

Q. 13. Let z_1 and z_2 be roots of the equation $z^2+pz+q = 0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = a \neq 0$ and OA = OB, where O is the origin, prove that

$p^2 = 4q \cos 2\left(\frac{\alpha}{2}\right)$ (1997 - 5 Marks)

Ans. Sol. $z^2 + pz + q = 0$

$z_1 + z_2 = -p, \; z_1 z_2 = q$

By rotation through a in anticlockwise direction

$z_2 = z_1 e^{i\alpha}$ ....(1)

$\frac{z_2}{z_1} = \frac{e^{i\alpha}}{1} = \frac{\cos\alpha + i\sin\alpha}{1}$

Add 1 in both sides to get $z_1 + z_2 = -p$

$$\therefore \frac{z_1+z_2}{z_1} = \frac{1+\cos\alpha + i\sin\alpha}{1} = 2\cos\frac{\alpha}{2}\left[\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right]$$

or $\frac{(z_2+z_1)}{z_1} = 2\cos\frac{\alpha}{2}e^{i\alpha/2}$

On squaring $(z_1 + z_2)^2$ $4\cos^2(\alpha/2)z_1^2.e^{i\alpha}$

$$= 4\cos^2\frac{\alpha}{2}z_1^2.\frac{z_2}{z_1} = 4\cos^2\frac{\alpha}{2}z_1z_2$$

or $p^2 = 4q\ \cos^2\frac{\alpha}{2}$

Q. 14. For complex numbers z and w, prove that $|z|^2 w - |w|^2 z = z - w$ if and only if $z = w$ or $z\bar{w} = 1$. (1999 - 10 Marks)

Sol. Given that z and w are two complex numbers.

To prove $|z|^2 w - |w|^2 z = z - w \Leftrightarrow z = w$ or $z\bar{w} = 1$

First let us consider

$|z|^2 w - |w|^2 z = z - w$ ....(1)

$\Rightarrow z(1 + |w|^2) = w(1 + |z|^2)$

$\Rightarrow \frac{z}{w} = \frac{1+|z|^2}{1+|w|^2} =$ a real number

$\Rightarrow \overline{\left(\frac{z}{w}\right)} = \frac{z}{w} \Rightarrow \frac{\bar{z}}{\bar{w}} = \frac{z}{w}$

$\Rightarrow \bar{z}w = z\bar{w}$ ....(2)

Again from equation (1),

$z\bar{z}w - w\bar{w}z = z - w$

$z(\bar{z}w - 1) - w(\bar{w}z - 1) = 0$

$z(z\bar{w} - 1) - w(z\bar{w} - 1) = 0$ (Using equation (2))

$\Rightarrow (z\bar{w} - 1)(z - w) = 0 \Rightarrow z\bar{w} = 1$ or $z = w$

Conversely if $z = w$ then

L.H.S. of (1) $= |w|^2 \ w - |w|^2 \ w = 0.$

R.H.S. of (1) $= w - w = 0$

$\therefore$ (1) holds

Also if $z\bar{w} = 1$ then

L.H.S. of (1) $= z\bar{z}w - w\bar{w}z$

$= zz\bar{w} - w\bar{w}z$ = z – w = R .H.S. Hence proved.

Q. 15. Let a complex number α, α ≠ 1 , be a root of the equation $z^{p+q} - z^p - z^q + 1 = 0$, where p, q are distinct primes. Show that either $1 + \alpha + \alpha^2 + ... + \alpha^{p-1} = 0$ or $1 + \alpha + \alpha^2 + ... + \alpha^{q-1} = 0$, but not both together. (2002 - 5 Marks)

Sol. The given equation can be written as

$(z^p - 1)(z^q - 1) = 0$

$\therefore z = (1)^{1/p}$ or $(1)^{1/q}$ ....(1)

where p and q are distinct prime numbers.

Hence both the equations will have distinct roots and as

$z \neq 1$, both will not be simultaneously zero for any value of z given by equations in (1)

NOTE THIS STEP

Also $1 + \alpha + \alpha^2 + ... + \alpha^{p-1} = \dfrac{1-\alpha^p}{1-\alpha} = 0 \ (\alpha \neq 1)$

or $1 + \alpha + \alpha^2 + ... + \alpha^p = \dfrac{1-\alpha^q}{1-\alpha} = 0 \ (\alpha \neq 1)$

Because of (1) either $\alpha^p = 1$ and if $\alpha^q = 1$ but not both simultaneously as p and q are distinct primes.

Q. 16. If z_1 and z_2 are two complex numbers such taht $|z_1| < 1 < |z_2|$ then prove that $\left|\frac{1-z_1\bar{z}_2}{z_1-z_2}\right| < 1$. (2003 - 2 Marks)

Sol. Given that $|z_1| < 1 < |z_2|$

Then $\left|\frac{1-z_1\bar{z}_2}{z_1-z_2}\right| < 1$ is true

if $|1-z_1\bar{z}_2| < |z_1 - z_2|$ is true

if $|1-z_1\bar{z}_2| < |z_1 - z_2|^2$ is true

if $(1-z_1\bar{z}_2)\overline{(1-z_1\bar{z}_2)} < (z_1 - z_2)\overline{(z_1-z_2)}$ is true

if $(1-z_1\bar{z}_2)(1-\bar{z}_1 z_2) < (z_1 - z_2)(\bar{z}_1-\bar{z}_2)$

if $1-z_1\bar{z}_2-\bar{z}_1 z_2+z_1\bar{z}_1 \; z_2\bar{z}_2$

$- z_1\bar{z}_1 - z_1\bar{z}_2$ is true

if $1+|z_1|^2|z_2|^2<|z_1|^2+|z_2|^2$ is true

if $(1-|z_1|^2)(1-|z_2|^2) < 0$ is true.

which is obviously true as $|z_1| < 1 < |z_2|$

$\Rightarrow |z_1|^2 < 1 < |z_2|^2$

$\Rightarrow |1-|z_1|^2 > 0$ and $(1-|z_2|^2) < 0$ Hence proved.

Q. 17. Prove that there exists no complex number z such that $|z| < \frac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$ where $|a_r| < 2$. (2003 - 2 Marks)

Sol. Let us consider, $\sum_{r=1}^{n} a_r z^r = 1$ where $|a_r| < 2$

$\Rightarrow a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n = 1$

$\Rightarrow |a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n| = 1$ ....(1)

But we know that $| z_1 + z_2 | \le | z_1 | + |z_2|$

$\therefore$ Using its generalised form, we get

$| a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n |$

$\le | a_1 z | + | a_2 z^2 | + \ldots + | a_n z^n |$

$\Rightarrow 1 \le | a_1| \, | z | + | a_2 | \, |z^2 | + | a_3 | \, | z^3 | + \ldots + | a_n | \, |z^n |$ (Using eqn (1))

But given that $| a_r | < 2 \ \forall \ r = 1(1)^n$

$\therefore 1 < 2 [| z | + | z |^2 + | z |^3 + \ldots + | z |^n]$ [Using $| z^n | = | z |^n$]

$\Rightarrow 1 < 2 \left[\frac{|z|(1-|z|^n)}{1-|z|}\right] \Rightarrow 2\left[\frac{|z|-|z|^{n+1}}{1-|z|}\right] > 1$

$\Rightarrow 2 [| z | - | z |^{n+1}] > 1 - | z |$ $\quad (\because 1 - | z | > 0$ as $| z | < 1/3)$

$\Rightarrow [|z| - |z|^{n+1}] > \frac{1}{2} - \frac{1}{2}|z| \Rightarrow \frac{3}{2}|z| > \frac{1}{2} + |z|^{n+1}$

$\Rightarrow [|z| > \frac{1}{3} + \frac{2}{3}|z|^{n+1} \Rightarrow [|z| > \frac{1}{3}$

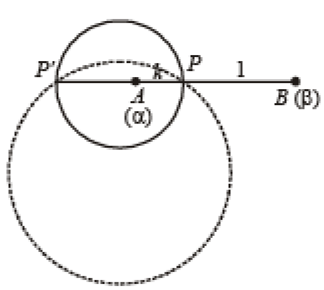
which is a contradiction as given that $[|z| < \frac{1}{3}$

$\therefore$ There exist no such complex number.

Q. 18. Fin d the centre an d radius of circle given by $\left|\frac{z-\alpha}{z-\beta}\right| = k, k \ne 1$

where, $z = x + iy$, $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$ (2004 - 2 Marks)

Ans. Sol. We are given that



$$\left|\frac{z-\alpha}{z-\beta}\right| = k \Rightarrow |z - a| = k\,|z - b|$$

Let pt. A represents complex number α and B that of β, and P represents z. then $|z - \alpha| = k\,|z - \beta|$

⇒ z is the complex number whose distance from A is k times its distance from B. i.e. PA = k PB

⇒ P divides AB in the ratio k : 1 internally or externally (at P').

Then $P\left(\frac{k\beta+\alpha}{k+1}\right)$ and $P'\left(\frac{k\beta-\alpha}{k-1}\right)$

Now through PP' there can pass a number of circles, but with given data we can find radius and centre of that circle for which PP' is diameter.
And hence then centre = mid. point of PP'

$$=\left(\frac{\frac{k\beta+\alpha}{k+1}+\frac{k\beta-\alpha}{k-1}}{2}\right) = \frac{k^2\beta+k\alpha-k\beta-\alpha+k^2\beta-k\alpha+k\beta-\alpha}{2(k^2-1)}$$

$$=\frac{k^2\beta-\alpha}{k^2-1}=\frac{\alpha-k^2\beta}{1-k^2}$$

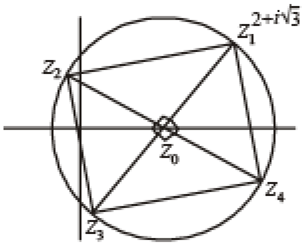
Also radius

$$=\frac{1}{2}|PP'| = \frac{1}{2}\left|\frac{k\beta+\alpha}{k+1}-\frac{k\beta-\alpha}{k-1}\right|$$

$$=\frac{1}{2}\left|\frac{k^2\beta+k\alpha-k\beta-\alpha-k^2\beta+k\alpha-k\beta+\alpha}{k^2-1}\right| = \frac{k\,|\alpha-\beta|}{|1-k^2|}$$

Q. 19. If one the vertices of the square circumscribing the circle $|z-1| = \sqrt{2}$ is $2+\sqrt{3}\,i$. Find the other vertices of thesquare. (2005 - 4 Marks)

Ans. Sol. The given circle is $|z - 1| = \sqrt{2}$ where $z_0 = 1$ is the centre and $\sqrt{2}$ is radius of circle. z_1 is one of the vertex of square inscribed in the given circle.



Clearly z_2 can be obtained by rotating z_1 by an ∠90° in anticlockwise sense, about centre z_0 Thus, $z_2 - z_0 = (z_1 - z_0)\, e^{i\pi/2}$

or $z_2 - 1 = (2 + i\sqrt{3} - 1)i \Rightarrow z_2 = i - \sqrt{3} + 1$

$z_2 = (1 - \sqrt{3}) + i$

Again rotating z_2 by 90° about z_0 we get

$z_3 - z_0 = (z_2 - z_0)\, i$

$\Rightarrow z_3 - 1 = [(1 - \sqrt{3}) + i - 1]\, i = -\sqrt{3}i - 1 \Rightarrow z_3 = -i\sqrt{3}$

and similarly $1 = (-i\sqrt{3} - 1)\, i = \sqrt{3} - i$

$\Rightarrow z_4 = (\sqrt{3} + 1) - i$

Thus the remaining vertices are

$(1 - \sqrt{3}) + i,\ -i\sqrt{3},\ (\sqrt{3} + 1) - i$