# Exercise 11.4

## Answer 1E.

(A) Let $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is Convergent.

If $a_n > b_n$ for all $n$, then we cannot say anything about $\sum a_n$ since $a_n > b_n$ and $b_n$ is conversant then $\sum a_n$ can be convergent or divergent.

(B) If $a_n < b_n$ for all $n$ and $\sum b_n$ is convergent then $\sum a_n$ is also convergent by comparison test.

## Answer 2E.

(A) Let $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is divergent.

If $a_n > b_n$ for all n, then $\sum a_n$ is also divergent by comparison test.

(B) If $a_n < b_n$ for all n, then we can not say anything about $\sum a_n$. It can be convergent or divergent.

## Answer 3E.

The series is $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$.

Determine whether the series converges or diverges.

Apply limit comparison test:

Suppose that $\sum a_n, \sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is a finite number and $c > 0$, then either the series converge or both diverge.

Let $a_n = \frac{n}{2n^3+1}, b_n = \frac{1}{n^2}$

Consider the expression,

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{n}{2n^3+1}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty}\frac{\frac{n}{n\left(2n^2+\frac{1}{n}\right)}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty}\frac{\frac{1}{n^2\left(2+\frac{1}{n^3}\right)}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty}\frac{1}{2+\frac{1}{n^3}}$$

$$= \frac{1}{2} > 0$$

This implies $\lim_{n\to\infty}\frac{a_n}{b_n}$ is a fixed number and greater than 0.

Also, the series $\sum b_n = \sum \frac{1}{n^2}$ converges, because it is in the form $\sum \frac{1}{n^p}$, where $p > 1$.

By comparison test, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ converges.

## Answer 4E.

Consider the series,

$$\sum_{n=2}^{\infty}\frac{n^3}{n^4-1}$$

To determine the convergence of the series, use Limit Comparison Test.

Limit Comparison Test:

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n\to\infty}\frac{a_n}{b_n}=c$, where $c$ is a finite number and $c>0$, then both the series converge or diverge

Suppose $\sum_{n=2}^{\infty}\frac{n^3}{n^4-1}=\sum a_n$

Then, we have

$$a_n=\frac{n^3}{n^4-1}>0$$

Consider $b_n$ by taking the highest power of $n$ from the numerator and the denominator of $a_n$

$$\begin{aligned}b_n&=\frac{n^3}{n^4}\\&=\frac{1}{n}>0\end{aligned}$$

Then $\sum a_n$ and $\sum b_n$ are series with positive terms, because each $a_n$, and $b_n$ is positive for all $n\ge 2$

Now consider the following limit.

$$\begin{aligned}c&=\lim_{n\to\infty}\frac{a_n}{b_n}\\&=\lim_{n\to\infty}\frac{n^3}{n^4-1}\cdot n\\&=\lim_{n\to\infty}\frac{n^4}{n^4\left(1-\frac{1}{n^4}\right)}\\&=\lim_{n\to\infty}\frac{1}{1-\frac{1}{n^4}}\\&=\frac{1}{1-0}\\&=1>0\end{aligned}$$

Therefore, $c>0$, and this is a finite number.

The auxiliary series $\sum \frac{1}{n^p}$ converges if $p > 1$, and diverges if $p \leq 1$

So the series, $\sum b_n = \sum \frac{1}{n^1}$ is diverges as $p = 1$

Hence by Limit Comparison Test, the series $\sum_{n=2}^{\infty} \frac{n^3}{n^4 - 1}$ is diverges.

## Answer 5E.

Consider the series,

$$\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$$

To determine the convergence of the series, use Limit Comparison Test.

Limit Comparison Test:

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$, where $c$ is a finite number and $c > 0$, then both the series converge or diverge

Suppose $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}} = \sum a_n$

Then, we have

$$a_n = \frac{n+1}{n\sqrt{n}} > 0$$

Consider $b_n$ by taking the highest power of $n$ from the numerator and the denominator of $a_n$

$$b_n = \frac{n}{n\sqrt{n}}$$
$$= \frac{1}{\sqrt{n}} > 0$$

Then $\sum a_n$ and $\sum b_n$ are series with positive terms, because each $a_n$, and $b_n$ is positive for all $n \geq 1$

## Answer 6E.

Consider the series,

$$\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$$

To determine the convergence of the series, use Limit Comparison Test.

Limit Comparison Test:

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where *c* is a finite number and $c > 0$, then both the series converge or diverge

Suppose $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}} = \sum a_n$

Then, we have

$$a_n = \frac{n-1}{n^2\sqrt{n}} > 0$$

Consider $b_n$ by taking the highest power of *n* from the numerator and the denominator of $a_n$

$$b_n = \frac{n}{n^2\sqrt{n}}$$
$$= \frac{1}{n\sqrt{n}} > 0$$

Then $\sum a_n$ and $\sum b_n$ are series with positive terms, because each $a_n$, and $b_n$ is positive for all $n \geq 1$

Now consider the following limit.

$$c = \lim_{n\to\infty} \frac{a_n}{b_n}$$
$$= \lim_{n\to\infty} \frac{n-1}{n^2\sqrt{n}} \cdot n\sqrt{n}$$
$$= \lim_{n\to\infty} \frac{n-1}{n}$$
$$= \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)$$
$$= 1 - 0$$
$$= 1 > 0$$

Therefore, $c > 0$, and this is a finite number.

The auxiliary series $\sum \frac{1}{n^p}$ converges if $p > 1$, and diverges if $p \leq 1$

So the series, $\sum b_n = \sum \frac{1}{n^{\frac{3}{2}}}$ is converges as $p = \frac{3}{2} > 1$

Hence by Limit Comparison Test, the series $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$ is converges.

## Answer 7E.

Consider the series,

$$\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$$

To determine the convergence of the series, use Limit Comparison Test.

Limit Comparison Test:

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is a finite number and $c > 0$, then both the series converge or diverge

Suppose $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n} = \sum a_n$

Then, we have

$$a_n = \frac{9^n}{3+10^n} > 0$$

Take,

$$b_n = \frac{9^n}{10^n}$$
$$= \left(\frac{9}{10}\right)^n > 0$$

Then $\sum a_n$ and $\sum b_n$ are series with positive terms, because each $a_n$, and $b_n$ is positive for all $n \geq 1$

Now consider the following limit.

$$c = \lim_{n\to\infty} \frac{a_n}{b_n}$$
$$= \lim_{n\to\infty} \frac{9^n}{3+10^n} \cdot \frac{10^n}{9^n}$$
$$= \lim_{n\to\infty} \frac{10^n}{3+10^n}$$
$$= \lim_{n\to\infty} \frac{10^n}{10^n\left(\frac{3}{10^n}+1\right)}$$
$$= \lim_{n\to\infty} \frac{1}{\frac{3}{10^n}+1}$$
$$= \frac{1}{0+1}$$
$$= 1$$

Therefore, $c > 0$, and this is a finite number.

## Answer 8E.

To determine the series converges or diverges, consider the series

$$\sum_{n=1}^{\infty} \frac{6^n}{5^n - 1}.$$

The Limit Comparison Test states that, if $\sum a_n$ and $\sum b_n$ are series with positive terms, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c,$$

where $c$ is a finite number and $c > 0$, then either both series converge or both diverge.

Let $a_n = \frac{6^n}{5^n - 1}$ and $b^n = \frac{6^n}{5^n}$.

Compute the limit value of $\frac{a_n}{b_n}$ as $n \to \infty$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{x\to\infty} \frac{\frac{6^n}{5^n - 1}}{\frac{6^n}{5^n}} \\
&= \lim_{x\to\infty} \frac{6^n}{5^n - 1} \cdot \frac{5^n}{6^n} \\
&= \lim_{x\to\infty} \frac{5^n}{5^n - 1} \\
&= \lim_{x\to\infty} \frac{5^n}{5^n\left(1 - \frac{1}{5^n}\right)}
\end{aligned}$$

The limit of $\frac{a_n}{b_n}$ as $n \to \infty$ is,

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{x\to\infty} \frac{5^n}{5^n\left(1 - \frac{1}{5^n}\right)} \\
&= \lim_{x\to\infty} \frac{1}{\left(1 - \frac{1}{5^n}\right)} \\
&= \frac{1}{1 - 0} \qquad \text{Because as } n \to \infty, \frac{1}{5^n} \to 0 \\
&= 1
\end{aligned}$$

The series $\sum \frac{6^n}{5^n} = \sum \left(\frac{6}{5}\right)^n$ is a geometric series with common ratio

$r = \frac{6}{5} > 1$

Recall the result that, a geometric series is divergent if $|r| \ge 1$

So, the sereis $\sum b_n = \frac{6^n}{5^n}$ is divergent.

Since $\sum b_n$ divergent and $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, by using limit comparison test it follows that

the series $\sum a_n$ is also divergent.

That is, $\sum_{n=1}^{\infty} \frac{6^n}{5^n - 1}$ is $\boxed{\text{divergent series}}$

## Answer 9E.

To determine the series converges or diverges, consider the series

$\sum_{k=1}^{\infty} \frac{\ln k}{k}$.

On expanding the series with respect to $k$, it becomes

$$\sum_{k=1}^{\infty} \frac{\ln k}{k} = \frac{\ln 1}{1} + \frac{\ln 2}{2} + \sum_{k=3}^{\infty} \frac{\ln k}{k}$$

Take the series $\sum_{k=3}^{\infty} \frac{\ln k}{k}$

Use comparison test to determine the series converges or diverges.

**The Comparison test:**

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all $n$, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all $n$, then $\sum a_n$ is also divergent.

Since $\ln k > 1 \quad \forall k \geq 3$.

It follows that

$$\frac{\ln k}{k} > \frac{1}{k} \quad \forall k \geq 3$$

Let $a_k = \frac{\ln k}{k}$ and $b_k = \frac{1}{k}$

Since each term of the series $\sum_{k=3}^{\infty} \frac{\ln k}{k}$ is greater than the corresponding terms of the series $\sum_{k=3}^{\infty} \frac{1}{k}$ which is $(p-\text{series with } p=1)$ divergent, therefore, the series $\sum_{k=3}^{\infty} \frac{\ln k}{k}$ is divergent by **comparison test**.

Recall that, convergence or divergence of a series does not change by adding finite number of terms to it.

Since $\sum_{k=3}^{\infty} \frac{\ln k}{k}$ is divergent series, so

$\frac{\ln 1}{1} + \frac{\ln 2}{2} + \sum_{k=3}^{\infty} \frac{\ln k}{k}$ is divergent series.

That is, $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ is $\boxed{\text{divergent series}}$.

Answer 10E.

Given series $\sum_{k=1}^{\infty} \frac{k \sin^2 k}{1+k^3}$

We have

$$1+k^3 > k^3$$

$$\Rightarrow \frac{1}{1+k^3} < \frac{1}{k^3}$$

$$\Rightarrow \frac{k \sin^2 k}{1+k^3} < \frac{k}{k^3} \quad (\because \sin^2 k \leq 1)$$

$$\Rightarrow \frac{k \sin^2 k}{1+k^3} < \frac{1}{k^2}$$

We know that $\sum \frac{1}{k^2}$ is convergent (p-series with p=2).

Thus the given series $\sum_{k}^{\infty} \frac{k \sin^2 k}{1+k^3}$ is convergent by the Companion Test.

Answer 11E.

To determine the series converges or diverges, consider the series

$$\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}}$$

Use comparison test to determine the series converges or diverges.

**The Comparison test:**

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all $n$, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all $n$, then $\sum a_n$ is also divergent.

Since $k^3+4k+3 > k^3 \quad \forall k \ge 1$.

It follows that

$$\sqrt{k^3+4k+3} > \sqrt{k^3} \quad \forall k \ge 1$$
$$\frac{1}{\sqrt{k^3+4k+3}} < \frac{1}{\sqrt{k^3}} \quad \forall k \ge 1$$
$$\frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}} < \frac{\sqrt[3]{k}}{\sqrt{k^3}} \quad \forall k \ge 1$$
$$\frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}} < \frac{k^{\frac{1}{3}}}{k^{\frac{3}{2}}} \quad \forall k \ge 1$$

Continuation to the above

$$\frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}} < \frac{k^{\frac{1}{3}}}{k^{\frac{3}{2}}} \quad \forall k \ge 1$$
$$\frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}} < \frac{1}{k^{\frac{3}{2}-\frac{1}{3}}} \quad \forall k \ge 1$$
$$\frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}} < \frac{1}{k^{\frac{9-2}{6}}} \quad \forall k \ge 1$$
$$\frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}} < \frac{1}{k^{\frac{7}{6}}} \quad \forall k \ge 1$$

Let $a_k = \frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}}$ and $b_k = \frac{1}{k^{\frac{7}{6}}}$

Now, $\sum b_k = \sum \frac{1}{k^{\frac{7}{6}}}$

Since $\sum \frac{1}{k^{\frac{7}{6}}}$ is $\left(p\text{-series with } p = \frac{7}{6} > 1\right)$ convergent,

so $\sum b_k$ is convergent.

As

$$\sum a_k < \sum b_k \quad \forall k \geq 1$$

and $\sum b_k$ is convergent, so by comparison test it follows that

$$\sum a_k = \sum \frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}}$$

is convergent series.

That is, $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}}$ is $\boxed{\text{convergent series}}$ by comparison test.

## Answer 12E.

To determine the series converges or diverges, consider the series

$$\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$$

Use comparison test to determine the series converges or diverges.

**The Comparison test:**

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all $n$, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all $n$, then $\sum a_n$ is also divergent.

Since $(k+1)(k^2+4)^2 > k^5 \quad \forall k \geq 1$,

It follows that

$$\frac{1}{(k+1)(k^2+4)^2} < \frac{1}{k^5} \qquad \forall k \geq 1$$

$$\frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2} < \frac{2k^3}{k^5} \qquad \forall k \geq 1 \quad \text{since } (2k-1)(k^2-1) < 2k^3 \quad \forall k \geq 1$$

$$\frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2} < \frac{2}{k^2} \qquad \forall k \geq 1$$

Let $a_k = \dfrac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$ and $b_k = \dfrac{2}{k^2}$

Now, $\sum b_k = \sum \dfrac{2}{k^2}$

$= 2\sum \dfrac{1}{k^2}$

Since $\sum \dfrac{1}{k^2}$ is $(p-\text{series with } p>1)$ convergent,

so $\sum b_k$ is convergent.

As

$$\sum a_k < \sum b_k \quad \forall k \geq 1$$

and $\sum b_k$ is convergent, so by comparison test it follows that

$$\sum a_k = \sum \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$$

is convergent series.

That is, $\displaystyle\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$ is $\boxed{\text{convergent series}}$ by comparison test.

## Answer 13E.

Consider the series,

$$\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^{1.2}}$$

To determine the convergence of the series, use Comparison Test.

Comparison Test:

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms such that $a_n \le b_n$

If the series $\sum b_n$ converges, then the series $\sum a_n$ also converges.

Also, if the series $\sum b_n$ diverges, then the series $\sum a_n$ diverges

For all $n \ge 1$, we have

$$\tan^{-1} n < 2$$

$$\frac{\tan^{-1} n}{n^{1.2}} < \frac{2}{n^{1.2}}$$

Suppose $a_n = \frac{\tan^{-1} n}{n^{1.2}}$, and $b_n = \frac{2}{n^{1.2}}$

The auxiliary series $\sum \frac{1}{n^p}$ converges if $p > 1$, and diverges if $p \le 1$

So the series, $\sum \frac{1}{n^{1.2}}$ is converges as $p = 1.2 > 1$, and hence $\sum b_n = \sum \frac{2}{n^{1.2}}$ converges.

So, by Comparison Test, the series $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^{1.2}}$ also converges.

## Answer 14E.

We have the series $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$

We use the limit comparison Test with

$$a_n = \frac{\sqrt{n}}{n-1} \qquad b_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$$

We have $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n-1} \cdot \frac{\sqrt{n}}{1}$

$$= \lim_{n\to\infty} \frac{n}{n-1}$$

$$= \lim_{n\to\infty} \frac{1}{1-1/n}$$

$$= \frac{1}{1-0} = 1 > 0$$

Since this limit exists and $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ is divergent $\left(\text{p-series with } p = \frac{1}{2} < 1\right)$

Therefore by the Limit comparison test the given series $\boxed{\text{diverges}}$.

Answer 15E.

Given series $\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n - 2}$

We have

$$3^n - 2 < 3^n$$
$$\Rightarrow \frac{1}{3^n - 2} > \frac{1}{3^n}$$
$$\Rightarrow \frac{4^{n+1}}{3^n - 2} > \frac{4^n}{3^n} \quad \left(\because 4^{n+1} > 4^n\right)$$
$$\Rightarrow \frac{4^{n+1}}{3^n - 2} > \left(\frac{4}{3}\right)^n$$

We know that $\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$ is divergent, (geometric series with $r = \frac{4}{3} > 1$).

Thus the given series $\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n - 2}$ is divergent by the Companion Test.

Answer 16E.

Given series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4 + 1}}$

We have

$$3n^4 + 1 > n^4$$
$$\Rightarrow \sqrt[3]{3n^4 + 1} > \sqrt[3]{n^4}$$
$$\Rightarrow \frac{1}{\sqrt[3]{3n^4 + 1}} < \frac{1}{\sqrt[3]{n^4}}$$

We know that $\sum \frac{1}{n^{\frac{4}{3}}}$ is convergent (p-series with $p = \frac{4}{3} > 1$)

Thus the given series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4 + 1}}$ is convergent by the Companion Test.

Answer 17E.

We have the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$

We use the limit comparison test with

$$a_n = \frac{1}{\sqrt{n^2 + 1}}, \quad b_n = \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$

We have $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\sqrt{n^2}}{\sqrt{n^2+1}}$

$$=\lim_{n\to\infty}\frac{1}{\sqrt{1+1/n^2}}$$

$$=\frac{1}{1+0}=1>0$$

Since this limit exists, and $\sum b_n=\sum\frac{1}{n}$ diverges (harmonic series) Therefore

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+1}}$ also $\boxed{\text{diverges}}$ by the limit comparison test.

## Answer 18E.

We have the series $\sum_{n=1}^{\infty}\frac{1}{2n+3}$

Take $a_n=\frac{1}{2n+3}$, $b_n=\frac{1}{2n}$

Then $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{2n}{2n+3}$

$$=\lim_{n\to\infty}\frac{2}{2+3/n}$$

$$=\frac{2}{2+0}=1>0$$

Since this limit exists and $\sum\frac{1}{2n}$ diverges (constant multiple of harmonic series)

Therefore $\sum_{n=1}^{\infty}\frac{1}{2n+3}$ $\boxed{\text{diverges}}$ by the limit comparison test.

## Answer 19E.

Consider the series,

$$\sum_{n=1}^{\infty}\frac{1+4^n}{1+3^n}$$

To determine the convergence of the series, use Limit Comparison Test.

Limit Comparison Test:

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n\to\infty}\frac{a_n}{b_n}=c$, where $c$ is a finite number and $c>0$, then both the series converge or diverge

Suppose $\sum_{n=1}^{\infty}\frac{1+4^n}{1+3^n}=\sum a_n$

Then, we have

$$a_n=\frac{1+4^n}{1+3^n}>0$$

Consider $b_n$ by taking the highest power of *n* from the numerator and the denominator of $a_n$

$$b_n = \frac{4^n}{3^n}$$
$$= \left(\frac{4}{3}\right)^n > 0$$

Then $\sum a_n$ and $\sum b_n$ are series with positive terms, because each $a_n$, and $b_n$ is positive for all $n \geq 1$

Now consider the following limit.

$$c = \lim_{n\to\infty} \frac{a_n}{b_n}$$
$$= \lim_{n\to\infty} \frac{1+4^n}{1+3^n} \cdot \frac{3^n}{4^n}$$
$$= \lim_{n\to\infty} \frac{4^n\left(\frac{1}{4^n}+1\right)}{3^n\left(\frac{1}{3^n}+1\right)} \cdot \frac{3^n}{4^n}$$
$$= \lim_{n\to\infty} \frac{\frac{1}{4^n}+1}{\frac{1}{3^n}+1}$$
$$= \frac{0+1}{0+1}$$
$$= 1$$

Therefore, $c > 0$, and this is a finite number.

The geometric series $\sum r^n$ converges if $|r| < 1$, and diverges if $r \geq 1$

So the series, $\sum b_n = \sum \left(\frac{4}{3}\right)^n$ diverges as $r = \frac{4}{3} > 1$

Hence by Limit Comparison Test, the series $\sum_{n=1}^{\infty} \frac{1+4^n}{1+3^n}$ is diverges.

## Answer 20E.

Consider the series,

$$\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$$

Use Limit comparison test, to decide the convergence of this series.

**Limit comparison test**:

If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are series with positive terms, and if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, where $c$ is finite, then both the series converges or diverges

Suppose $\sum_{n=1}^{\infty}\frac{n+4^n}{n+6^n}=\sum_{n=1}^{\infty}a_n$

Then, we get

$a_n=\frac{n+4^n}{n+6^n}>0$, for all $n$

Suppose that, $b_n=\frac{4^n}{6^n}>0$

Now we find the value of the limit,

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{n+4^n}{n+6^n}}{\frac{4^n}{6^n}}$$

$$=\lim_{n\to\infty}\frac{n+4^n}{n+6^n}\cdot\frac{6^n}{4^n}$$

$$=\lim_{n\to\infty}\frac{4^n\left(\frac{n}{4^n}+1\right)}{6^n\left(\frac{n}{6^n}+1\right)}\cdot\frac{6^n}{4^n}$$

$$=\lim_{n\to\infty}\frac{\frac{n}{4^n}+1}{\frac{n}{6^n}+1}$$

$$=\frac{\lim_{n\to\infty}\frac{n}{4^n}+1}{\lim_{n\to\infty}\frac{n}{6^n}+1}$$

Use L-Hospital's rule, $\lim_{n\to\infty}\frac{f(n)}{g(n)}=\lim_{n\to\infty}\frac{f'(n)}{g'(n)}$ if $\lim_{n\to\infty}\frac{f(n)}{g(n)}=\frac{\infty}{\infty}$, and use $\frac{d}{dx}\left(a^x\right)=a^x\log a$, to get the value of the limit.

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim_{n\to\infty}\frac{1}{4^n\cdot\log 4}+1}{\lim_{n\to\infty}\frac{1}{6^n\cdot\log 6}+1}$$

$$=\frac{\frac{1}{\infty}+1}{\frac{1}{\infty}+1}$$

$$=\frac{1}{1}$$

$$=1>0$$

And, consider the series,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{4^n}{6^n}$$

The geometric series $\sum_{n=1}^{\infty} r^n$ converges if $0 \le r < 1$, and diverges if $r \ge 1$.

So the series $\sum_{n=1}^{\infty} \frac{4^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges as $r = \frac{2}{3} < 1$.

So by Limit comparison test, the series $\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$ also converges.

Answer 21E.

Consider the series,

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1}$$

Use Limit comparison test, to decide the convergence of this series.

**Limit comparison test**:

If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are series with positive terms, and if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, where $c$ is finite, then both the series converges or diverges.

Suppose $\sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1} = \sum_{n=1}^{\infty} a_n$

Then, we get

$$a_n = \frac{\sqrt{n+2}}{2n^2+n+1} > 0, \text{ for all } n$$

Suppose that,

$$\begin{aligned} b_n &= \frac{\sqrt{n}}{n^2} \\ &= \frac{1}{n\sqrt{n}} > 0 \end{aligned}$$

Now we find the value of the limit,

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{\sqrt{n+2}}{2n^2+n+1}}{\frac{1}{n\sqrt{n}}}$$

$$=\lim_{n\to\infty}\frac{\sqrt{n+2}}{2n^2+n+1}\cdot n\sqrt{n}$$

$$=\lim_{n\to\infty}\frac{\sqrt{n}\sqrt{1+\frac{2}{n}}}{n^2\left(2+\frac{1}{n}+\frac{1}{n^2}\right)}\cdot n\sqrt{n}$$

$$=\lim_{n\to\infty}\frac{n^2\sqrt{1+\frac{2}{n}}}{n^2\left(2+\frac{1}{n}+\frac{1}{n^2}\right)}$$

$$=\frac{\sqrt{1+\frac{2}{\infty}}}{2+\frac{1}{\infty}+\frac{1}{\infty}}$$

$$=\frac{\sqrt{1+0}}{2+0+0}$$

$$=\frac{1}{2}>0$$

And, consider the series,

$$\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$$

The auxiliary series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p>1$.

So the series $\sum_{n=1}^{\infty}\frac{1}{n^p}=\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ converges as $p=\frac{3}{2}>1$.

So by Limit comparison test, the series $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{\sqrt{n+2}}{2n^2+n+1}$ also converges.

Answer 22E.

We have the series $\sum_{n=1}^{\infty} \frac{n+2}{(n+1)^3}$

Take $a_n = \frac{n+2}{(n+1)^3}$, $b_n = \frac{n}{n^3}$

$= \frac{1}{n^2}$

And $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n+2)}{(n+1)^3} n^2$

$= \lim_{n\to\infty} \frac{(n^3+2n^2)}{(n+1)^3}$

$= \lim_{n\to\infty} \frac{(1+2/n)}{(1+1/n)^3}$

$= \frac{1+0}{(1+0)^3} = 1 > 0$

Since this limit exists and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$)

Therefore given series also converges by the limit comparison test

Answer 23E.

The Limit Comparisin Test:

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where $c$ is a finite number and $c > 0$, then either

both series converge or both diverge.

We have the series $\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$

Take $a_n = \frac{5+2n}{(1+n^2)^2}$

$b_n = \frac{2n}{n^4}$

$= \frac{2}{n^3}$

And $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{5+2n}{\left(1+n^2\right)^2}\cdot\frac{n^3}{2}$

$$=\lim_{n\to\infty}\frac{1}{2}\frac{\left(\frac{5}{n}+2\right)}{\left(\frac{1}{n^2}+1\right)^2}$$

$$=\frac{1}{2}\times 2$$

$$=1>0$$

So the limit exists and is positive.

Now the series $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}\frac{2}{n^3}$

$$=2\sum_{n=1}^{\infty}\frac{1}{n^3}$$ converges ($p$-sereis ,with $p$=3>1)

Therefore $\sum_{n=1}^{\infty}\frac{5+2n}{\left(1+n^2\right)^2}$ also **converges** by the limit compression test.

## Answer 24E.

We have the series $\sum_{n=1}^{\infty}\frac{n^2-5n}{n^3+n+1}$

Take $a_n=\frac{n^2-5n}{n^3+n+1}$, $b_n=\frac{n^2}{n^3}=\frac{1}{n}$

And $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^2-5n}{n^3+n+1}\times n$

$$=\lim_{n\to\infty}\frac{1-5/n}{1+\frac{1}{n^2}+\frac{1}{n^3}}$$

$$=\frac{1-0}{1+0}=1>0$$

Since this limit exists and $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent (harmonic series)

Therefore $\sum_{n=1}^{\infty}\frac{n^2-5n}{n^3+n+1}$ also **diverges** by the limit comparison test.

## Answer 25E.

Given series $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n^2}$

We have

$$n^3+n^2 < 2n^3$$

$$\Rightarrow \frac{1}{n^3+n^2} > \frac{1}{2n^3}$$

$$\Rightarrow \frac{\sqrt{n^4+1}}{n^3+n^2} > \frac{\sqrt{n^4}}{2n^3} \quad \left(n^4+1 > n^4\right)$$

$$\Rightarrow \frac{\sqrt{n^4+1}}{n^3+n^2} > \frac{1}{2n}$$

We know that $\frac{1}{2}\sum \frac{1}{n}$ is divergent (p-series with $p=1$).

Thus the given series $\sum_{n=1}^{\infty} \frac{\sqrt[1]{n^4+1}}{n^3+n^1}$ is divergent by the Companion Test.

## Answer 26E.

To determine the series converges or diverges, consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$$

Use comparison test to determine the series converges or diverges.

**The Comparison test:**

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all $n$, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all $n$, then $\sum a_n$ is also divergent.

Since $n^2-1 > n \quad \forall n \ge 2$.

It follows that

$$\sqrt{n^2-1} > \sqrt{n} \qquad \forall n \ge 2$$

$$n\sqrt{n^2-1} > n\sqrt{n} \qquad \forall n \ge 2$$

$$\frac{1}{n\sqrt{n^2-1}} < \frac{1}{n\sqrt{n}} \qquad \forall n \ge 2$$

Continuation to the above

$$\frac{1}{n\sqrt{n^2-1}} < \frac{1}{n\sqrt{n}} \quad \forall n \ge 2$$

$$\frac{1}{n\sqrt{n^2-1}} < \frac{1}{n \cdot n^{\frac{1}{2}}} \quad \forall n \ge 2$$

$$\frac{1}{n\sqrt{n^2-1}} < \frac{1}{n^{\frac{3}{2}}} \quad \forall n \ge 2$$

Let $a_n = \frac{1}{n\sqrt{n^2-1}}$ and $b_n = \frac{1}{n^{\frac{3}{2}}}$

Now, $\sum b_n = \sum \frac{1}{n^{\frac{3}{2}}}$

Since $\sum \frac{1}{n^{\frac{3}{2}}}$ is convergent $\left(p-\text{series with } p = \frac{3}{2} > 1\right)$.

so $\sum b_n$ is convergent.

As

$$\sum a_n < \sum b_n \quad \forall n \ge 2$$

and $\sum b_n$ is convergent, so by **comparison test** it follows that

$$\sum a_n = \sum \frac{1}{n\sqrt{n^2-1}}$$

is convergent series.

That is, $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ is $\boxed{\text{convergent series}}$ by comparison test.

Answer 27E.

We have to find that the series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^2 e^{-n}$ is convergent or divergent.

Since $\frac{1}{n} \le 1$

$$\Rightarrow 1+\frac{1}{n} \le 2$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^2 \le 4$$

Then $\frac{\left(1+\frac{1}{n}\right)^2}{e^n} \le \frac{4}{e^n}$

Since $\sum_{n=1}^{\infty} \frac{4}{e^n}$ is a geometric series with $a = \frac{4}{e}$ *and* $r = \frac{1}{e}$

Since $|r| = \frac{1}{e} < 1$ so this series is convergent.

Then by comparison test, the series $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n}$ *is* $\boxed{convergent}$

## Answer 28E.

Consider the series,

$$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}$$

Use Limit comparison test, to decide the convergence of this series.

**Limit comparison test**:

If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are series with positive terms, and if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, where $c$ is finite, then both the series converges or diverges.

Suppose $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n} = \sum_{n=1}^{\infty} a_n$

Then, we get

$a_n = \frac{e^{\frac{1}{n}}}{n} > 0$, for all $n$

Suppose that, $b_n = \frac{1}{n} > 0$

Now we find the value of the limit,

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{e^{\frac{1}{n}}}{n}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{e^{\frac{1}{n}}}{n} \cdot n \\ &= e^{\frac{1}{\infty}} \\ &= e^0\end{aligned}$$

$= 1 > 0$

And, consider the series,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

The auxiliary series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \le 1$.

So the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges as $p = 1$.

So by Limit comparison test, the series $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}$ also diverges.

## Answer 29E.

Consider the series,

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

The objective is to determine whether the given series is convergent or divergent.

Use the ratio test to check the convergence of the series.

**The Ratio Test:**

For a series $\sum a_n$, suppose the sequence of ratios $\frac{|a_{n+1}|}{|a_n|}$ has a limit:

$$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = L.$$

If $L < 1$, then $\sum a_n$ converges.

If $L > 1$, or if $L$ is infinite, then $\sum a_n$ diverges.

If $L = 1$, the test does not tell us anything about the convergence of $\sum a_n$.

Let $a_n = \frac{1}{n!}$

Then $a_{n+1} = \frac{1}{(n+1)!}$

Find the ratio:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{(n+1)!} \cdot \frac{n!}{1}\right|$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n!}{(n+1)n!}\right|$$

$$= \frac{1}{n+1}$$

Now take the limit:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{n+1}$$

$$= \lim_{n\to\infty}\frac{1}{n\left(1+\frac{1}{n}\right)}$$

$$= 0 < 1$$

Because the limit $L = 0 < 1,$

Thus by the Ratio Test, the given series is **converges to 0**.

## Answer 30E.

Consider the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$.

To find whether the series is convergent or divergent, use the comparison test.

The comparison test, suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \le b_n$ for all $n$, then $\sum a_n$ is also convergent.

The expanding of the series is,

$$a_1 = 1$$

$$a_2 = \frac{1.2}{2.2}$$

......

$$a_n = \frac{1.2.3.4......n}{n.n.n.n......n}$$

$$a_n = \frac{1}{n}\left(\frac{2.3.4......n}{n.n.n......n}\right)$$

$$= \frac{1}{n}k \text{ where } k = \frac{2.3.4......n}{n.n.n......n}$$

Each term in parenthesis is less than or equal to 1 where $k<1$ because the numerator is less than the denominator.

We can write it as $0<a_n \le \frac{1}{n} \le \frac{n!}{n^n}$.

The value of the limit $\sum a_n$ is,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{n}\right)$$
$$=0 \qquad a_n \to 0 \text{ as } n\to\infty$$

Hence the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ is $\boxed{\text{convergent}}$ from the comparison test.

## Answer 31E.

We have $a_n = \sin(1/n)$

Let $b_n = 1/n$

Then using limit comparison test

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin(1/n)}{(1/n)}$$

Let $\frac{1}{n}=\theta$ so $\theta \to 0$ as $n\to\infty$

Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

So $\lim_{n\to\infty}\frac{a_n}{b_n} = 1 > 0$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n}$ is a harmonic series which is divergent.

So the series $\sum_{n=1}^{\infty}\sin(1/n)$ is also $\boxed{\text{divergent}}$

## Answer 32E.

We take $b_n = \frac{1}{n}$

Then
$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{n^{1+1/n}}}{1/n}$$
$$= \lim_{n\to\infty}\frac{n}{n^{1+1/n}}$$
$$= \lim_{n\to\infty}\frac{1}{n^{1/n}}$$

Now let $y = x^{1/x} \Leftrightarrow \ln y = \frac{1}{x} \ln x$

Taking limit as $x \to \infty$, we have

$$\lim_{x\to\infty} (\ln y) = \lim_{x\to\infty} \left( \frac{\ln x}{x} \right)$$

$$= \lim_{x\to\infty} \left( \frac{1/x}{1} \right) \qquad \text{[By L-Hospital's rule]}$$

$$= \lim_{x\to\infty} \left( \frac{1}{x} \right) = 0$$

Therefore $\lim_{x\to\infty} (y) = e^0 = 1$


Now let $y = x^{1/x} \Leftrightarrow \ln y = \frac{1}{x} \ln x$

Taking limit as $x \to \infty$, we have

$$\lim_{x\to\infty} (\ln y) = \lim_{x\to\infty} \left( \frac{\ln x}{x} \right)$$

$$= \lim_{x\to\infty} \left( \frac{1/x}{1} \right) \qquad \text{[By L-Hospital's rule]}$$

$$= \lim_{x\to\infty} \left( \frac{1}{x} \right) = 0$$

Therefore $\lim_{x\to\infty} (y) = e^0 = 1$

## Answer 33E.

Consider the series,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+1}}$$

The objective is to use the sum of the first 10 terms to approximate the sum of the series.

Estimate the series.

Since

$$\frac{1}{\sqrt{n^4+1}} < \frac{1}{\sqrt{n^4}}$$

Or $\frac{1}{\sqrt{n^4+1}} < \frac{1}{n^2}$

The given series is convergent by Comparison Test.

The remainder of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4}}$ is found by the Remainder Estimate for the Integral Test.

$$T_n \le \int_n^{\infty} \frac{1}{\sqrt{x^4}}$$
$$= \int_n^{\infty} \frac{1}{x^2}$$
$$= \frac{1}{n}$$

Therefore, the remainder for the given series satisfies

$$R_n \le T_n \le \frac{1}{n}$$

With $n = 10$

$$R_{10} \le \frac{1}{10}$$
$$= 0.1$$

Using a programmable calculator,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+1}}$$
$$\approx \sum_{n=1}^{10} \frac{1}{\sqrt{n^4+1}}$$
$$\approx \boxed{1.249}$$

With error less than $\boxed{0.1}$.

Answer 34E.

Consider the series,

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$$

Use the following results for check the convergence of this series.

1. The auxiliary series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \le 1$, and converges if $p > 1$.

2. Comparison test: if $a_n < b_n$, for all $n$, and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} b_n$ also converges.

Now we have,

$$\sin^2 n \le 1 \ \ \forall n \ge 1$$

$$\frac{\sin^2 n}{n^3} \le \frac{1}{n^3} \ \ \ldots\ldots \ (1)$$

The series, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges as $p = 3 > 1$

So by comparison test, the series $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$ also converges.

To approximate the sum of the given series, use "Remainder Estimate for the Integral Test".

**Remainder Estimate for the Integral Test**:

Suppose that $f$ is a continuous, positive, decreasing function for $x \ge n$, $f(k) = a_k$, and $\sum a_n$ is convergent. If $T_n = s - s_n$, then

$$T_n \le \int_n^{\infty} f(x)\,dx$$

Here $T_n$ is the remainder for the comparison series $\sum_{n=1}^{\infty} \frac{1}{n^3}$

Suppose $f(x)=\frac{1}{x^3}$, where $x \geq 1$

The function $f(x)=\frac{1}{x^3}$ is defined for $x \geq 1$

And it is continuous, because this is a rational function.

Also, $f(x)=\frac{1}{x^3}>0 \ \forall x \geq 1$

So the function $f$ is positive for $x \geq 1$

Also, we have

$$\frac{1}{x}<\frac{1}{y} \ \forall x>y \geq 1$$
$$\frac{1}{x^3}<\frac{1}{y^3} \ \forall x>y$$
$$f(x)<f(y) \ \forall x>y$$

So the function $f$ is decreasing for $x \geq 1$

And finally, the series $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is convergent.

Therefore the function $f$ satisfies all the conditions of Remainder Estimate for the Integral Test.

Hence, by Remainder Estimate for the Integral Test,

$$T_n \leq \int_n^{\infty} f(x)\,dx$$
$$=\int_n^{\infty}\frac{1}{x^3}dx$$
$$=\int_n^{\infty}x^{-3}dx$$
$$=\lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t \qquad \text{Use} \int x^n dx=\frac{x^{n+1}}{n+1}, n \neq -1$$
$$=\lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2n^2}\right)$$
$$=\frac{1}{2n^2}$$

Therefore, we get

$$T_n \leq \frac{1}{2n^2} \quad \ldots\ldots (2)$$

By (1), and (2), we get the remainder $R_n$ for the given series $\sum_{n=1}^{\infty}\frac{\sin^2 n}{n^3}$ as,

$$R_n \le T_n \le \frac{1}{2n^2}$$

$$R_n \le \frac{1}{2n^2}$$ By transitive property

With $n=10$, we have

$$R_{10} \le \frac{1}{2(10)^2}$$
$$=\frac{1}{200}$$
$$=0.005$$

Therefore,

$R_{10} \le 0.005$ ...... (3)

Also, by using Maple, we get the sun of the series as,

$$\sum_{n=1}^{10}\frac{\sin(n)\cdot\sin(n)}{(n)^3}$$

$$\sin(1)^2 + \frac{1}{8}\sin(2)^2 + \frac{1}{27}\sin(3)^2 + \frac{1}{64}\sin(4)^2 + \frac{1}{125}\sin(5)^2 + \frac{1}{216}\sin(6)^2 + \frac{1}{343}\sin(7)^2 + \frac{1}{512}\sin(8)^2 + \frac{1}{729}\sin(9)^2 + \frac{1}{1000}\sin(10)^2$$

$$\xrightarrow{\text{at 5 digits}}$$

0.83253

Therefore, the sum of the given series using first 10 terms is approximately $\boxed{0.833}$ with an error less than $\boxed{0.005}$ (by using (3)).

Answer 35E.

We know that $\cos^2 n \le 1$ for all $n$. Therefore, $\frac{\cos^2 n}{5^n} \le \frac{1}{5^n}$ for all $n$. Consider

$\sum a_n = \frac{\cos^2 n}{5^n}$ and $\sum b_n = \frac{1}{5^n}$.

Let $s = \sum a_n$ and $t = \sum b_n$. Then, $R_n = s - s_n$ and $T_n = t - t_n$. Since $a_n \le b_n$ for all $n$, we have $R_n \le T_n$. But $T_n = \frac{1}{5^{n+1}} + \frac{1}{5^{n+2}} + \ldots$.

Simplify.

$$T_n = \frac{1}{5^{n+1}}\left[1+\frac{1}{5}+\frac{1}{5^2}+\ldots\right]$$

$$= \frac{1}{5^{n+1}}\left(\frac{1}{1-\frac{1}{5}}\right)$$

$$= \frac{1}{(4)5^n}$$

Find $T_{10}$.

$$T_{10} = \frac{1}{(4)5^{10}}$$

$$= 2.56\times 10^{-8}$$

This means that $R_{10} \le 2.56\times 10^{-8}$.

Now, $\sum_{n=1}^{\infty}\frac{\cos^2 n}{5^n} = \sum_{n=1}^{10}\frac{\cos^2 n}{5^n} \approx 0.073929303$ with error less than $2.56\times 10^{-8}$.

## Answer 36E.

Given series $\sum_{n=1}^{\infty}\frac{1}{3^n+4^n}$

Since $\frac{1}{3^n+4^n} < \frac{1}{2.3^n}$, the given series is convergent by the Comparision Test.

Let

$$s=\sum_{n=1}^{\infty}\frac{1}{3^n+4^n} \quad , \quad s_n=\sum_{i=1}^{n}\frac{1}{3^i+4^i} \quad \text{and.}$$

and $$t=\sum_{n=1}^{\infty}\frac{1}{2.3^n} \quad , \quad t_n=\sum_{i=1}^{n}\frac{1}{2.3^i}$$

let $R_n = s-s_n \; and \; T_n = t-t_n$

Now

$$T_n = t - t_n$$

$$= \frac{\frac{1}{6}}{1-\frac{1}{3}} - \frac{\frac{1}{6}(1-\frac{1}{3^n})}{1-\frac{1}{3}}$$

$$= \frac{1}{4\cdot 3^n}$$

Therefore the reminder $R_n$ for the given series satisfies

$$R_n \le T_n \le \frac{1}{4.3^n}$$

With n = 10, we have

$$R_{10} \le \frac{1}{4.3^{10}} = 4.23\times10^{-6}$$

Now

$$\sum_{n=1}^{10} \frac{1}{3^n+4^n}$$

$$= 0.428571 + 0.04 + 0.010989 + 0.0029674 + 0.0007892$$
$$+ 0.0002072 + 0.0000538 + 0.0000138 + 0.0000035 + 0.0000009$$
$$= 0.197869$$

With error less than $4.23\times10^{-6}$.

## Answer 37E.

We have

$$0.d_1d_2d_3d_4\ldots\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \ldots\ldots$$
$$= \sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

Since $d_n \le 9$

$$\Rightarrow \frac{d_n}{10^n} \le \frac{9}{10^n}$$

Let $a_n = \frac{d_n}{10^n}$ *and* $b_n = \frac{9}{10^n}$

$$\Rightarrow a_n \le b_n$$

Since $\sum_{n=1}^{\infty} b_n$ is a geometric series with $r = \frac{1}{10} < 1$

So $\sum_{n=1}^{\infty} b_n$ is convergent then by comparison test series $\sum_{n=1}^{\infty} \frac{d_n}{10^n}$ is also convergent

Answer 38E.

Since $\frac{1}{n^p \ln n} \le \frac{1}{n^p}$ for n>2

Let $a_n = \frac{1}{n^p \ln n}$ *and* $b_n = \frac{1}{n^p}$

So $a_n \le b_n$

Since $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^p}$ is a p-series, it will be convergent when p >1

So by comparison test, the series $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ will also converge *for* $\boxed{p>1}$

Answer 39E.

We have $a_n \ge 0$

And $\sum a_n$ converges

Then $\lim_{t\to\infty} a_n = 0$

So there exists a number N such that

$|a_n - 0| < 1$ *for all* $n > N$ [*by* defintion of limit]

$\Rightarrow 0 \le a_n < 1$ for all $n > N$

$\Rightarrow 0 \le a_n^{\ 2} < a_n$

Since $\sum a_n$ converges so by comparison test $\sum a_n^{\ 2}$ will also be convergent.

Answer 40E.

(A) Let $\sum a_n$ *and* $\sum b_n$ are series with positive terms and $\sum b_n$ be convergent.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

So there exists a number N such that

$\left|\frac{a_n}{b_n} - 0\right| < 1$ for all $n > N$ [By defination of limit]

$\Rightarrow 0 \le \frac{a_n}{b_n} < 1$ for all $n > N$

$\Rightarrow 0 \le a_n < b_n$

Since $\sum b_n$ is convergent as our assumption so by comparison test $\sum a_n$ is also convergent

(B) (i) If $a_n = \frac{\ln n}{n^3}$ *and* $b_n = \frac{1}{n^2}$

Since $\sum b_n = \sum \frac{1}{n^2}$ is a p-series with p>1 so $\sum b_n$ is convergent

Now we calculate $\lim_{n\to\infty} \frac{a_n}{b_n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(\ln n)/n^3}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{\ln n}{n}$$

$$= \lim_{x\to\infty} \frac{\ln x}{x}$$

$$= \lim_{x\to\infty} \frac{(1/x)}{1} \qquad \text{(using L-Hospital rule)}$$

$$= \lim_{x\to\infty} \frac{1}{x} = 0$$

So by part (a) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ is a $\boxed{convergent}$ series

(ii) If $a_n = \frac{\ln n}{\sqrt{n}\, e^n}$ *and* $b_n = \frac{1}{e^n}$

Since $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{e^n}$ is a geometric series with $|r| = 1/e < 1$

So $\sum b_n$ is a convergent series.

Then we find $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\ln n}{\sqrt{n e^n}}}{1/e^n}$

$$= \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{1/x}{\frac{1}{2\sqrt{x}}} \qquad \text{[by L-Hospital rule]}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

So by part (a) $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n e^n}}$ is a convergent series

Answer 41E.

(A) Let $\sum a_n$ converges and $\sum b_n$ diverges

If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$

$$\Rightarrow \lim_{n\to\infty} \frac{b_n}{a_n} = 0$$

So by limit comparison test if $\sum a_n$ converges, $\sum b_n$ must be convergent this is the contradiction. So $\sum a_n$ must diverge

(B)(i) If $a_n = \frac{1}{\ln n}$ and $b_n = \frac{1}{n}$

Since $\sum b_n$ is a harmonic series so it is divergent

Now $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\ln n}$

$= \lim_{x\to\infty} \frac{x}{\ln x}$

$= \lim_{x\to\infty} \frac{1}{1/x}$ [L-Hospital Rule]

$= \lim_{x\to\infty} x = \infty$

So by part (a) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is a $\boxed{\text{divergent}}$ series

(ii) If $a_n = \frac{\ln n}{n}$ and $b_n = \frac{1}{n}$

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n}(n)$

$= \lim_{n\to\infty} (\ln n)$

$= \infty$

Since $\sum b_n$ is divergent so $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is divergent

Answer 42E.

We choose $\sum a_n = \sum \frac{1}{n^2}$

Which is a p-series with $p = 2 > 1$ so $\sum a_n$ is convergent series

And $\sum b_n = \sum \frac{1}{\sqrt{n}}$

Which is also a p-series with $p = \frac{1}{2} < 1$ so $\sum b_n$ diverges.

Now $\lim_{n\to\infty} = \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n^2}{1/\sqrt{n}}$

$= \lim_{n\to\infty} \frac{\sqrt{n}}{n^2}$

$= \lim_{n\to\infty} \frac{1}{n\sqrt{n}}$

$= 0$ $n\sqrt{n} \to \infty$ as $n \to \infty$

So we have $\boxed{\sum a_n = \sum \frac{1}{n^2}}$ *and* $\boxed{\sum b_n = \sum \frac{1}{\sqrt{n}}}$

With $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, where $\sum a_n$ is convergent and $\sum b_n$ is divergent.

## Answer 43E.

We have $a_n > 0$ and $\lim_{n\to\infty} n a_n \neq 0$

$$\Rightarrow \lim_{n\to\infty} n a_n = \lim_{n\to\infty} \frac{a_n}{1/n} > 0$$

Let $b_n = 1/n$, we use the limit comparison test.

Since $\lim_{n\to\infty} n a_n > 0$ so either both series converge or both diverge.

Since $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges because it is a p-series with $p = 1$.

So $\boxed{\sum a_n \text{ must be divergent}}$

## Answer 44E.

Since $\sum a_n$ is convergent and $a_n > 0$

Then $\lim_{n\to\infty} a_n = 0$

We use limit comparison test

We find $\lim_{n\to\infty} \frac{\ln(1+a_n)}{(a_n)}$

Let $x = a_n$

And since $a_n \to 0$ *as* $n \to \infty$ *so* $x \to 0$ *as* $n \to \infty$

$$\text{Then } \lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n} = \lim_{x\to 0} \frac{\ln(1+x)}{x}$$

$$= \lim_{x\to 0} \frac{1/(1+x)}{1} \qquad [\text{by } L-\text{Hosptial rule}]$$

$$= \lim_{x\to 0} \frac{1}{1+x}$$

$$= \frac{1}{1+0}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n} = 1 > 0$$

And since $\sum a_n$ *converges* so $\sum \ln(1+a_n)$ must be convergent.

## Answer 45E.

Since $\sum a_n$ *is convergent* (*given*)

Then $\lim_{n\to\infty} a_n = 0$

Now we use limit comparison test for $\sum \sin(a_n)$

We find $\lim_{n\to\infty} \frac{\sin(a_n)}{a_n}$

For a instant we assume that $x = a_n$

So $x \to 0 \quad as \quad n \to \infty$

Then $\lim_{n\to\infty} \frac{\sin(a_n)}{a_n} = \lim_{x\to 0} \frac{\sin(x)}{x}$

$= 1 > 0 \qquad \left[\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right]$

And since $\sum a_n$ is convergent. So $\sum \sin(a_n)$ must be convergent

$\boxed{\text{Yes}, \sum \sin(a_n)}$ is a convergent series

## Answer 46E.

Since $\sum a_n \, and \, \sum b_n$ both are convergent series.

So $\lim_{n\to\infty} a_n = 0 \qquad \ldots\ldots(1)$

And $\lim_{n\to\infty} b_n = 0 \qquad \ldots\ldots.(2)$

We use extension of limit comparison test for $\sum a_n b_n$

$$\lim_{n\to\infty} \frac{a_n b_n}{b_n} = \lim_{n\to\infty} a_n$$

$$= 0 \qquad [\text{from}(1)]$$

So $\lim_{n\to\infty} \frac{a_n b_n}{b_n} = 0$

And $\sum b_n$ is convergent so $\sum a_n$ must be convergent.