CHAPTER

9

Probability

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- Problems on Bayes's Theorem

SOME DEFINITIONS

Experiment

An operation which results in some well-defined outcomes is called an experiment.

Random Experiment

An experiment whose outcome cannot be predicted with certainty is called a random experiment. In other words, if an experiment is performed many times under similar conditions and the outcome each time is not the same, then this experiment is called a random experiment.

An experiment whose outcome can be foretold before is not a random experiment. For example, when a stone is thrown upwards it is sure that the stone will fall downward. Therefore, throwing a stone upward is not a random experiment.

Examples:

(i) 'Tossing a fair coin' is a random experiment because if we toss a coin either a head or a tail will come up. But if we toss a coin again and again the outcome each time will not be the same.

(ii) 'Throwing an unbiased die' is a random experiment because when a die is thrown we cannot say with certainty which one of the numbers 1, 2, 3, 4, 5 and 6 will come up.

Sample Space

The set of all possible outcomes of a random experiment is called the sample space for that experiment. It is usually denoted by S.

Examples:

(i) When a coin is tossed either a head or a tail will come up. If H denotes the occurrence of head and T denotes the occurrence of tail, then sample space is $S = \{H, T\}$.

(ii) When two coins are tossed, sample Space is given by $S = \{(H, H), (H, T), (T, H), (T, T)\}$, where (H, H) denotes the occurrence of head on the first coin and occurrence of head on the second coin. Similarly, (H, T) denotes the occurrence of head on the first coin and occurrence of tail on the second coin.

(iii) When a die is thrown any one of the numbers 1, 2, 3, 4, 5 and 6 will come up. Therefore, sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Here, 1 denotes the occurrence of 1, 2 denotes the occurrence of 2 and so on.

(iv) When two balls are drawn from a bag containing three red and two black balls, the sample space is given by $S = \{(R_1, R_2), (R_1, R_3), (R_2, R_3), (B_1, B_2), (R_1, B_1), (R_1, B_2), (R_2, B_1) (R_2, B_2), (R_3, B_1), (R_3, B_2)\}$.

Event

Consider the experiment of tossing a coin two times. An associated sample space is $S = \{HH, HT, TH, TT\}$.

Now suppose that we are interested in those outcomes which correspond to the occurrence of exactly one head. We find that HT and TH are the only elements of S corresponding to the occurrence of this happening (event). These two elements form the set $E = \{HT, TH\}$.

We know that the set E is a subset of the sample space S. Similarly, we find the following correspondence between events and subsets of S.

Description of events	Corresponding subset of 'S'
Number of tails is exactly 2	$\{TT\}$
Number of tails is at least 1	$\{HT, TH, TT\}$
Number of heads is at most 1	$\{HT, TH, TT\}$
Second toss is not head	$\{HT, TT\}$
Number of tails is at most 2	$\{HH, HT, TH, TT\}$
Number of tails is more than 2	φ

When a die is thrown, sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Let $A = \{1, 3, 5\}$, where A is the event of occurrence of an odd number; $B = \{5, 6\}$, where B is the event of the occurrence of a number greater than 4; $C = \{2, 3, 5\}$, where C is the event of occurrence of a prime number.

Note:

- *Sample space S plays the same role as the universal set for all problems related to the particular experiment.*
- *ϕ is also a subset of S which is called an impossible event.*
- *S is also a subset of S which is called a sure event or a certain event.*

The above discussion suggests that a subset of sample space is associated with an event and an event is associated with a subset of sample space. In the light of this we define an event as follows.

Simple Event or Elementary Event

If an event E has only one sample point of a sample space, it is called a *simple* (or *elementary*) *event*.

Examples:

(i) When a coin is tossed, sample space is $S = \{H, T\}$. Let $A = \{H\}$ be the event of occurrence of head and $B = \{T\}$ be the event of occurrence of tail. Here, A and B are simple events.

(ii) When a die is thrown, sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Let $A = \{5\}$ be the event of occurrence of 5 and $B = \{2\}$ be the event of occurrence of 2. Here, A and B are simple events.

Mixed Event or Compound Event or Composite Event

A subset of the sample space S which contains more than one element is called a mixed event. For example, in the experiment of 'tossing a coin thrice,' the events

E: 'Exactly one head appeared'

F: 'At least one head appeared'

G: 'At most one head appeared', etc.

are all compound events.

The subsets of S associated with these events are

$E = \{HTT, THT, TTH\}$

$F = \{HTT, THT, TTH, HHT, HTH, THH, HHH\}$

$G = \{TTT, THT, HTT, TTH\}$

Each of the above subsets contains more than one sample point, hence they are all compound events.

Trial

When an experiment is repeated under similar conditions and it does not give the same result each time but may result in any one of the several possible outcomes, the experiment is called a trial and the outcomes are called cases. The number of times the experiment is repeated is called the number of trials. For example, one toss of a coin is a trial when the coin is tossed 5 times and one throw of a die is a trial when the die is thrown 4 times.

ALGEBRA OF EVENTS

Complementary Event

For every event A, there corresponds another event A' called the complementary event to A. It is also called the *event 'not A'*. For example, take the experiment 'of tossing three coins'. An associated sample space is $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$. Let $A = \{HTH, HHT, THH\}$ be the event 'only one tail appears'.

Clearly for the outcome HTT, the event A has not occurred. But we may say that the event 'not A' has occurred. Thus, with every outcome which is not in A, we say that 'not A' occurs.

Thus, the complementary event 'not A' to the event A is

$A' = \{HHH, HTT, THT, TTH, TTT\}$

or

$A' = \{\omega : \omega \in S \text{ and } \omega \notin A\} = S - A$

The Event 'A or B'

Recall that union of two sets A and B denoted by $A \cup B$ contains all those elements which are either in A or in B or in both. When the sets A and B are two events associated with a sample space, then '$A \cup B$' is the event 'either A or B or both'. This event '$A \cup B$' is also called 'A or B'.

The Event 'A and B'

We know that intersection of two sets $A \cap B$ is the set of those elements which are common to both A and B, i.e., which belong to both 'A and B'. If A and B are two events, then the set $A \cap B$ denotes the event 'A and B'.

The Event 'A but not B'

We know that $A - B$ is the set of all those elements which are in A but not in B. Therefore, the set $A - B$ may denote the event 'A but not B'. We know that $A - B = A - A \cap B$.

DIFFERENT TYPES OF EVENTS

Equally Likely Events

Cases (outcomes) are said to be equally likely when we have no reason to believe that one is more likely to occur than the other. Thus, when an unbiased die is thrown all the six faces 1, 2, 3, 4, 5 and 6 are equally likely to come up. Similarly, when an unbiased coin is tossed occurrence of head and tail are equally likely cases.

Exhaustive Cases (Events)

Consider the experiment of throwing a die. We have $S = \{1, 2, 3, 4, 5, 6\}$. Let us define the following events:

A: 'a number less than 4 appears'

B: 'a number greater than 2 but less than 5 appears',

C: 'a number greater than 4 appears'

Then $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{5, 6\}$. We observe that

$A \cup B \cup C = \{1, 2, 3\} \cup \{3, 4\} \cup \{5, 6\} = S$.

Such events A, B and C are called exhaustive events. In general, if $E_1, E_2, ..., E_n$ are n events of a sample space S and if $S = E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i$, then $E_1, E_2, ..., E_n$ are called *exhaustive events*. In other words, events $E_1, E_2, ..., E_n$ are said to be exhaustive if at least one of them necessarily occurs whenever the experiment is performed.

Mutually Exclusive Events

In the experiment of rolling a die, the sample space is given by $S = \{1, 2, 3, 4, 5, 6\}$.

Consider events, A 'an odd number appears' and B 'an even number appears'.

Clearly the event A excludes the event B and vice versa.

In other words, there is no outcome which ensures the occurrence of events A and B simultaneously.

Here, $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$. Clearly $A \cap B = \varphi$, i.e., A and B are disjoint sets.

In general, two events A and B are called *mutually exclusive* events if the occurrence of any one of them excludes the occurrence of the other event, i.e., if they can not occur simultaneously.

In this case, the sets A and B are disjoint.

Again in the experiment of rolling a die, consider the events A 'an odd number appears' and event B 'a number less than 4 appears'.

Obviously, $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$. Therefore, A and B are not mutually exclusive events.

Example 9.1 **A coin is tossed three times, consider the following events.**

***A*: 'no head appears'**

***B*: 'exactly one head appears'**

***C*: 'at least two heads appear'**

Do they form a set of mutually exclusive and exhaustive events?

Sol. The sample space of the experiment is
$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Events A, B and C are given by

$A = \{TTT\}$

$B = \{HTT, THT, TTH\}$

$C = \{HHT, HTH, THH, HHH\}$

Now,

$$A \cup B \cup C = \{TTT, HTT, THT, TTH, HHT, HTH, THH, HHH\} = S$$

Therefore, A, B and C are exhaustive events. Also, $A \cap B = \phi$, $A \cap C = \phi$ and $B \cap C = \phi$. Therefore, the events are pair-wise disjoint, i.e., they are mutually exclusive. Hence, A, B and C form a set of mutually exclusive and exhaustive events.

AXIOMATIC APPROACH TO PROBABILITY

Axiomatic approach is another way of describing probability of an event. In this approach, some axioms or rules are depicted to assign probabilities.

Let S be the sample space of a random experiment. The probability P is a real valued function whose domain is the power set of S and range is the interval $[0,1]$ satisfying the following axioms:

(i) For any event E, $P(E) \geq 0$

(ii) $P(S) = 1$

(iii) If E and F are mutually exclusive events, then

$$P(E \cup F) = P(E) + P(F)$$

It follows from (iii) that $P(\phi) = 0$.

Let S be a sample space containing outcomes $\omega_1, \omega_2, \ldots, \omega_n$, i.e., $S = \{\omega_1, \omega_2, \ldots, \omega_n\}$.

It follows from the axiomatic definition of probability that

(i) $0 \leq P(\omega_i) \leq 1$ for each $\omega_i \in S$

(ii) $P(\omega_1) + P(\omega_2) + \cdots + P(\omega_n) = 1$

(iii) For any event A, $P(A) = \Sigma P(\omega_i)$, $\omega_i \in A$.

MATHEMATICAL OR CLASSICAL DEFINITION OF PROBABILITY

Let S be the sample space. Then the probability of occurrence of an event E is denoted by $P(E)$ and is defined as

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of elements in } E}{\text{number of elements in } S}$$

$$= \frac{\text{number of cases favourable to event } E}{\text{total number of cases}}$$

Examples:

1. When a die is rolled, sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

   Let A be the event of occurrence of an odd number, i.e., $\{1, 3, 5\}$ and B be the event of occurrence of a number greater than 4, i.e., $\{5, 6\}$. Then,

   $$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

   $$P(B) = \frac{n(B)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

2. When one ball is drawn at random from a bag containing 3 black balls and 4 red balls (balls of the same colour being identical or different), then sample space is given by

   $S = \{B_1, B_2, B_3, R_1, R_2, R_3, R_4\}$

   $\therefore$ $n(S) = 7$

   Here, the three black balls may be denoted by B_1, B_2 and B_3 even if they are identical because while finding probability only number of black and red balls are to be taken into account. Let E be the event of occurrence of a red ball. Then, $E = \{R_1, R_2, R_3, R_4\}$

   $\therefore n(E) = 4$

   Now,

   $$P(E) = \frac{n(E)}{n(S)} = \frac{4}{7}$$

3. When two coins are tossed, sample space is given by

   $S = \{HH, HT, TH, TT\}$. Let E be the event of occurrence of one head and one tail. Then $E = \{HT, TH\}$. Now,

   $$P(E) = \frac{n(E)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

Value of Probability of Occurrence of an Event

Let S be the sample space and E be an event. Then

$$\phi \subseteq E \subseteq S$$

$\therefore$ $n(\phi) \leq n(E) \leq n(S)$

$\Rightarrow$ $0 \leq n(E) \leq n(S)$

$$\Rightarrow \frac{0}{n(S)} \leq \frac{n(E)}{n(S)} \leq \frac{n(S)}{n(S)} \quad \text{[Dividing by } n(S)]$$

$$\Rightarrow 0 \leq P(E) \leq 1 \quad \left[\because P(E) = \frac{n(E)}{n(S)}\right]$$

Thus, if ϕ is the impossible event, then

$$P(\phi) = \frac{n(\phi)}{n(S)} = \frac{0}{n(S)} = 0$$

and if S is the sure event, then

$$P(S) = \frac{n(S)}{n(S)} = 1$$

If E be Any Event and E' be the Complement of Event E, then $P(E) + P(E') = 1$.

Proof:

Let S be the sample space. Then

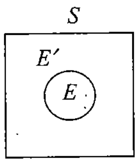

Fig. 9.1

$$n(E) + n(E') = n(S)$$

$$\therefore \frac{n(E)}{n(S)} + \frac{n(E')}{n(S)} = 1 \text{ or } P(E) + P(E') = 1$$

Odds in Favour and Odds Against an Event

Let S be the sample space and E be an event. Let E' denote the complement of event E. Then

(i) odds in favour of event E is

$$\frac{n(E)}{n(E')} = \frac{\text{number of cases favourable to event } E}{\text{number of cases against event } E}$$

(ii) odds against an event E is

$$\frac{n(E')}{n(E)} = \frac{\text{number of cases against the event } E}{\text{number of cases favourable to event } E}$$

Note:

- *Odds in favour of event E is given by*

$$\frac{n(E)}{n(E')} = \frac{n(E)/n(S)}{n(E')/n(S)} = \frac{P(E)}{P(E')}$$

and odds against event E is

$$\frac{n(E')}{n(E)} = \frac{P(E')}{P(E)}$$

- *If any one of P(E), odds in favour of E, and odds against E is given, then other two can be determined.*

Examples:

(i) If $P(E) = 2/7$, then odds in favour of E is 2/5 and odds against E is 5/2.

(ii) If odds against E is 3/11, then odds in favour of E is 11/3 and $P(E)$ is 11/4.

(iii) If odds in favour of E is 3/8, then odds against E is 8/3 and $P(E) = 3/11$.

Example 9.2 **Find the probability of getting more than 7 when two dice are rolled.**

Sol.

Sum of two dice	Result on two dice	Number of cases
8	(4,4), (3,5), (5,3), (2,6), (6,2)	5
9	(4,5), (5,4), (3,6), (6,3)	4
10	(5,5), (4,6), (6,4)	3
11	(5,6), (6,5)	2
12	(6,6)	1

Required probability is

$$P(8) + P(9) + P(10) + P(11) + P(12)$$

$$= \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{15}{36} = \frac{5}{12}$$

Example 9.3 **A card is drawn at random from a pack of cards. What is the probability that the drawn card is neither a heart nor a king.**

Sol. There are 13 heart cards and 4 king cards of which one is heart king. Therefore, the required probability is

$$1 - P(\text{drawn card is either a heart or a king})$$

$$1 - \frac{16}{52} = \frac{36}{52} = \frac{9}{13}$$

Example 9.4 **A determinant is chosen at random from the set of all determinant of order 2 with elements 0 or 1 only. Find the probability that the determinant chosen is non-zero.**

Sol. A determinant of order 2 is of the form

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

It is equal to $ad - bc$.

The total number of ways of choosing a, b, c and d is

$2 \times 2 \times 2 \times 2 = 16$.

Now, $\Delta \neq 0$ if and only if either $ad = 1$, $bc = 0$ or $ad = 0$, $bc = 1$. But $ad = 1$, $bc = 0$ if $a = d = 1$ and one of b, c is zero. Therefore, $ad = 1$, $bc = 0$ in three cases. Similarly, $ad = 0$, $bc = 1$ in three cases.

Therefore, the required probability is 6/16 = 3/8.

Example 9.5 **A dice is rolled three times, find the probability of getting a larger number than the previous number each time.**

Sol. Exhaustive number of cases is $6^3 = 216$. Now if a larger number appears than the previous number each time, all the three numbers are distinct. Now three numbers can be selected from six numbers in $^6C_3$ ways and there is only one way in which three selected numbers can appear. Hence, probability is 20/216 = 5/54.

Example 9.6 **An integer is chosen at random and squared. Find the probability that the last digit of the square is 1 or 5.**

Sol. The last digit of square will be 1 or 5 only when the integer is 1, 5 or 9. Therefore, required probability is 3/10.

Example 9.7 **If a coin be tossed n times then find the probability that the head comes odd times.**

Sol. Total number of cases is 2^n.

If head comes odd times (1, 3, 5, ..., n times), then favourable number of ways is $^nC_1 + {}^nC_3 + {}^nC_5 + \cdots) = 2^{n-1}$. Therefore, the required probability is

$$\frac{2^{n-1}}{2^n} = \frac{1}{2}$$

Example 9.8 **Find the probability that a leap year will have 53 Fridays or 53 Saturdays.**

Sol. There are 366 days in a leap year, in which there are 52 weeks and two days. The combination of 2 days can be

Sunday–Monday

Monday–Tuesday

Tuesday–Wednesday

Wednesday–Thursday

Thursday–Friday

Friday–Saturday

Saturday–Sunday

$$P(\text{53 Fridays}) = \frac{2}{7}; P(\text{53 Saturdays}) = \frac{2}{7}$$

$$P(\text{53 Fridays and 53 Saturdays}) = \frac{1}{7}$$

$\therefore$ P(53 Fridays or Saturdays)

$= P(\text{53 Fridays}) + P(\text{53 Saturdays})$

$-P(\text{53 Fridays and Saturdays})$

$$= \frac{2}{7} + \frac{2}{7} - \frac{1}{7} = \frac{3}{7}$$

Example 9.9 **A mapping is selected at random from the set of all the mappings of the set $A = \{1, 2, \ldots, n\}$ into itself. Find the probability that the mapping selected is an injection.**

Sol. Total number of functions from A to itself is n^n. Out of these functions, $n!$ functions are injective mappings (one–one and onto). So, required probability is

$$\frac{n!}{n^n} = \frac{(n-1)!}{n^{n-1}}$$

Example 9.10 **Two integers x and y are chosen with replacement out of the set {0, 1, 2, 3, ..., 10}. Then find the probability that $|x - y| > 5$.**

Sol. Since x and y each can take values from 0 to 10, so the total number of ways of selecting x and y is $11 \times 11 = 121$. Now,

$$|x - y| > 5 \Rightarrow x - y < -5 \text{ or } x - y > 5$$

When

$x - y > 5$, we have following cases:

Value of x	Value of y	Number of cases
6	0	1
7	0, 1	2
8	0, 1, 2	3
9	0, 1, 2, 3	4
10	0, 1, 2, 3, 4	5
	Total number of cases	15

Similarly, we have 15 cases for $x - y < -5$. There are 30 pairs of values of x and y satisfying these two inequalities. So, favourable number of ways is 30. Hence, required probability is 30/121.

Example 9.11 **Find the probability that the birthdays of six different persons will fall in exactly two calendar months.**

Sol. Since anyone's birthday can fall in one of 12 months, so total number of ways is 12^6.

Now, any two months can be chosen in ${}^{12}C_2$ ways. The six birthdays can fall in these two months in 2^6 ways. Out of these 2^6 ways there are two ways when all the six birthdays fall in one month. So, favourable number of ways is ${}^{12}C_2 \times (2^6 - 2)$. Hence, required probability is

$$\frac{{}^{12}C_2 \times (2^6 - 2)}{12^6} = \frac{12 \times 11 \times (2^5 - 1)}{12^6} = \frac{341}{12^5}$$

Example 9.12 **If 10 objects are distributed at random among 10 persons, then find the probability that at least one of them will not get anything.**

Sol. Since each object can be given to any one of 10 persons. So, 10 objects can be distributed among 10 persons in 10^{10} ways. Thus, the total number of ways is 10^{10}.

The number of ways of distribution in which each one gets only one thing is 10!.

So, the number of ways of distribution in which at least one of them does not get anything is $10^{10} - 10!$.

Hence, required probability is $10^{10} - 10!/10^{10}$.

Example 9.13 **Twelve balls are distributed among three boxes. Find the probability that the first box will contain three balls.**

Sol. Since each ball can be put into any one of the three boxes. So, the total number of ways in which 12 balls can be put into three boxes is 3^{12}. Out of 12 balls, 3 balls can be chosen in ${}^{12}C_3$ ways for first box. Now, remaining 9 balls can be put in the remaining 2 boxes in 2^9 ways. So, the total number of ways in which 3 balls are put in the first box and the remaining balls in other two boxes is ${}^{12}C_3 \times 2^9$. Hence, required probability is ${}^{12}C_3 \times 2^9/3^{12}$.

Example 9.14 **Two numbers a and b are chosen at random from the set of first 30 natural numbers. Find the probability that $a^2 - b^2$ is divisible by 3.**

Sol. The total number of ways of choosing two numbers out of 1, 2, 3, ..., 30 is ${}^{30}C_2 = 435$. So, exhaustive number of cases is 435.

Since $a^2 - b^2$ is divisible by 3 if either a or b or both are divisible by 3 or none of a and b is divisible by 3. Thus, favourable number of cases is ${}^{10}C_2 + {}^{20}C_2 = 235$. Hence, the required probability is 235/435 = 47/87.

Example 9.15 **Find the probability that the 3 N's come consecutively in the arrangement of the letters of the word 'CONSTANTINOPLE'.**

Sol. The total number of arrangement is of the letters of the word 'CONSTANTINOPLE' is (14)!/3! 2! 2!

Since 3 N's are consecutive, then considering all the 3 N's as single letter, the total number of arrangements is (12)!/2! 2!

Therefore, required probability is

$$\frac{(12)!/(2!\,2!)}{(14)!/(3!\,2!\,2!)} = \frac{3!}{14 \times 13} = \frac{3}{91}$$

Example 9.16 **Out of $3n$ consecutive integers, three are selected at random. Find the probability that their sum is divisible by 3.**

Sol. Let the $3n$ consecutive integers be $x, x + 1, x + 2, \ldots, x + 3n - 1$, where x is the starting integer. These $3n$ integers can be classified as

$$x, x + 3, x + 6, \ldots, x + 3n - 3$$
$$x + 1, x + 4, x + 7, \ldots, x + 3n - 2$$
$$x + 2, x + 5, x + 8, \ldots, x + 3n - 1$$

Each of these three rows contains 'n' numbers. If we take three numbers out of $3n$ numbers, obviously their sum shall be divisible by 3 only if either all the three numbers are from the same row or all the three numbers are from different rows.

The number of ways that the three numbers are from the same row is ${}^3C_1\, {}^nC_3 = 3\,{}^nC_3$ and the number of ways the three numbers are from different rows is $n \times n \times n = n^3$. Hence, favourable number of ways that the sum of the three numbers is divisible by 3 is $3\,{}^nC_3 + n^3$. Also the total number of ways of selecting three numbers out of $3n$ numbers is ${}^{3n}C_3$. Therefore, the required probability is

$$\frac{3 \times {}^nC_3 + n^3}{{}^{3n}C_3} = \frac{3n^2 - 3n + 2}{(3n - 1)(3n - 2)}$$

Example 9.17 **A die is loaded so that the probability of a face i is proportional to i, i = 1, 2, ..., 6. Then find the probability of an even number occurring when the die is rolled.**

Sol. Since the probabilities of the faces are proportional to the numbers on them, we can take the probabilities of faces 1, 2,..., 6 as $k, 2k, \ldots, 6k$, respectively. Since one of the faces must occur, we have

$$k + 2k + 3k + 4k + 5k + 6k = 1 \text{ or } k = 1/21$$

Therefore, the probability of occurence of an even number is $2k + 4k + 6k = 12k = 12/21 = 4/7$.

Example 9.18 **A card is drawn from a pack of 52 cards. A gambler bets that it is a spade or an ace. What are the odds against his winning this bet?**

Sol. Probability of the card being a spade or an ace is 16/52 = 4/13. Hence, odds in favour are 4:9. So odds against his winning are 9:4.

Concept Application Exercise 9.1

1. There are n letters and n addressed envelopes. Find the probability that all the letters are not kept in the right envelope.
2. Find the probability of getting a total of 5 or 6 in a single throw of two dice.
3. Two integers are chosen at random and multiplied. Find the probability that the product is an even integer.
4. If out of 20 consecutive whole numbers two are chosen at random, then find the probability that their sum is odd.
5. A bag contains 3 red, 7 white and 4 black balls. If three balls are drawn from the bag, then find the probability that all of them are of the same colour.
6. An ordinary cube has four blank faces, one face marked 2 and one face marked 3. Then find the probability of obtaining a total of exactly 12 in 5 throws.
7. If the letters of the word 'REGULATIONS' be arranged at random, find the probability that there will be exactly, four letters between the R and the E.
8. A five digit number is formed by the digits 1, 2, 3, 4, 5 without repetition. Find the probability that the number formed is divisible by 4.
9. Five persons entered the lift cabin on the ground floor of an 8-floor house. Suppose that each of them, independently and with equal probability, can leave the cabin at any floor beginning with the first. Find out the probability of all five persons leaving at different floors.
10. Two friends A and B have equal number of daughters. There are three cinema tickets which are to be distributed among the daughters of A and B. The probability that all the tickets go to the daughters of A is 1/20. Find the number of daughters each of them have.
11. A bag contains 12 pairs of socks. Four socks are picked up at random. Find the probability that there is at least one pair.
12. There are eight girls among whom two are sisters, all of them are to sit on a round table. Find the probability that the two sisters do not sit together.
13. A bag contains 50 tickets numbered 1, 2, 3, ..., 50 of which five are drawn at random and arranged in ascending order of magnitude ($x_1 < x_2 < x_3 < x_4 < x_5$). Find the probability that $x_3 = 30$.
14. A pack of 52 cards is divided at random into two equal parts. Find the probability that both parts will have an equal number of black and red cards.

ADDITION THEOREM OF PROBABILITY

If A and B be any two events in a sample space S, then the probability of occurrence of at least one of the events A and B is given by $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof:

From set theory, we know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Dividing both sides by $n(S)$, we get

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

or $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Note:

- *If A and B are mutually exclusive events, then* $A \cap B = \phi$ *and hence* $P(A \cap B) = 0$.
  $\therefore P(A \cup B) = P(A) + P(B)$
- *Two events A and B are mutually exclusive if and only if* $P(A \cup B) = P(A) + P(B)$.
- $1 = P(S) = P(A \cup A') = P(A) + P(A')$ $\quad [\because A \cap A' = \phi]$
  or $P(A') = 1 - P(A)$
  If A, B and C are any three events in a sample space S, then
  $P(A \cup B \cup C) = P(A) + P(B) + P(C)$
  $-P(A \cap B) - P(B \cap C) - P(A \cap C)$
  $+ P(A \cap B \cap C)$
- *If A, B, C are mutually exclusive events, then*
  $A \cap B = \phi, B \cap C = \phi, A \cap C = \phi, A \cap B \cap C = \phi$
  $\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C)$
- *If A and B are any two events, then*
  $(A - B) \cap (A \cap B) = \phi$
  and $A = (A - B) \cap (A \cap B)$
  $\therefore P(A) = P(A - B) + P(A \cap B)$
  $= P(A \cap B') + P(A \cap B)$
  or $P(A) - P(A \cap B) = P(A - B) = P(A \cap B)$
  $[\because A - B = A \cap B']$
  Similarly, $P(B) - P(A \cap B) = P(B - A)$
  $= P(B \cap A')$

General Form of Addition Theorem of Probability

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

Note: *For any number of (finite or infinite) mutually exclusive events* $E_1, E_2, E_3, \ldots, E_n$,

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = P(E_1) + P(E_2) + \cdots + P(E_n)$$

Example 9.19 **The probability that at least one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2, then find $P(\bar{A}) + P(\bar{B})$.**

Sol. It is given that $P(A \cup B) = 0.6$ and $P(A \cap B) = 0.2$. Therefore,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$\Rightarrow 0.6 = P(A) + P(B) - 0.2$$
$$\Rightarrow P(A) + P(B) = 0.8$$
$$\Rightarrow 1 - P(\bar{A}) + 1 - P(\bar{B}) = 0.8$$
$$\Rightarrow P(\bar{A}) + P(\bar{B}) = 1.2$$

Example 9.20 **The probabilities of three events A, B and C are $P(A) = 0.6$, $P(B) = 0.4$ and $P(C) = 0.5$. If $P(A \cup B) = 0.8$, $P(A \cap C) = 0.3$, $P(A \cap B \cap C) = 0.2$ and $P(A \cup B \cup C) \geq 0.85$, then find the range of $P(B \cap C)$.**

Sol. We have,

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.6 + 0.4 - 0.8 = 0.2$$

Also,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap C) + P(A \cap B \cap C) - P(A \cap B) - P(B \cap C)$$
$$\Rightarrow P(B \cap C) = 1.2 - P(A \cup B \cup C) \quad (1)$$

Now,

$$0.85 \leq P(A \cup B \cup C) \leq 1$$
$$\therefore \text{ Eq. (1)} \Rightarrow 0.2 \leq P(B \cap C) \leq 0.35$$

Example 9.21 **Given two events A and B. If odds against A are as 2:1 and those in favour of $A \cup B$ are as 3:1, then find the range of $P(B)$.**

Sol. Clearly $P(A) = 1/3$, $P(A \cup B) = 3/4$. Now,

$$P(B) \leq P(A \cup B)$$
$$\Rightarrow P(B) \leq 3/4 \quad (1)$$

Also,

$$P(B) = P(A \cup B) - P(A) + P(A \cap B)$$
$$\geq 3/4 - 1/3$$
$$= \frac{5}{12} \quad [P(A \cap B) \geq 0]$$

Example 9.22 **If A and B are two events, then which of the following does not represent the probability of at most one of A, B occurs**

a. $1 - P(A \cap B)$
b. $P(\bar{A}) + P(\bar{B}) - P(\bar{A} + \bar{B})$
c. $P(\bar{A}) + P(\bar{B}) + P(A \cap B) - 1$
d. $P(A \cap \bar{B}) + P(\bar{A} \cap B) - P(\bar{A} \cap \bar{B})$

Sol. Required probability is

$$P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B)$$

So, option (a) is correct. Again,

$$P(\bar{A} \cup \bar{B}) = P\bar{A} + P\bar{B} - P(\bar{A} \cap \bar{B}) \quad \text{[by addition theorem]}$$

So, option (b) is correct. Again,

$$P(\bar{A} \cup \bar{B}) = P\bar{A} + P\bar{B} - P(\bar{A} \cap \bar{B})$$
$$= P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B})$$
$$= P(\bar{A}) + P(\bar{B}) - \{1 - P(A \cup B)\}$$
$$= P(\bar{A}) + P(\bar{B}) + P(A \cup B) - 1$$

So, option (c) is correct. Finally,

$$P(\bar{A} \cup \bar{B}) = P[(A \cap \bar{B}) \cup (\bar{A} \cap B) + (\bar{A} \cap \bar{B})]$$
$$= P(A \cap \bar{B}) + P(\bar{A} \cap B) + P(\bar{A} \cap \bar{B})$$

[$\because A \cap \bar{B}$, $\bar{A} \cap B$ and $\bar{A} \cap \bar{B}$ are mutually exclusive events]

Example 9.23 **A box contains 6 nails and 10 nuts. Half of the nails and half of the nuts are rusted. If one item is chosen at random, then find the probability that it is rusted or is a nail.**

Sol. The total number of nails and nuts is 6 + 10 = 16.

$$P(R) = \frac{1}{2} \quad (R \text{ stands for rusted})$$
$$P(N) = \frac{6}{16} \quad (N \text{ stands for nails})$$
$$P(R \cap N) = \frac{3}{16} \quad [\because 3 \text{ nails are rusted out of 6 nails}]$$
$$P(R \cup N) = P(R) + P(N) - P(R \cap N)$$
$$= \frac{1}{2} + \frac{6}{16} - \frac{3}{16}$$
$$= \frac{8+6-3}{16} = \frac{11}{16}$$

Example 9.24 **If $P(A \cup B) = 3/4$ and $P(\bar{A}) = 2/3$, then find the value of $P(\bar{A} \cap B)$.**

Sol. Since $\bar{A} \cap B$ and A are mutually exclusive events such that

$$A \cup B = (\bar{A} \cap B) \cup A$$
$$\Rightarrow P(A \cup B) = P(\bar{A} \cap B) + P(A)$$
$$\Rightarrow \frac{3}{4} = P(\bar{A} \cap B) + 1 - \frac{2}{3}$$
$$\Rightarrow P(\bar{A} \cap B) = \frac{5}{12}$$

Example 9.25 **Let A, B, C be three events. If the probability of occurring exactly one event out of A and B is $1 - a$, out of B and C is $1 - 2a$, out of C and A is $1 - a$ and that of occurring three events simultaneously is a^2, then prove that probability that at least one out of A, B, C will occur greater than 1/2.**

Sol. P(exactly one event out of A and B occurs)

$$= P[A \cap B') \cup (A' \cap B)]$$
$$= P(A \cup B) - P(A \cap B)$$
$$= P(A) + P(B) - 2P(A \cap B)$$
$$\therefore P(A) + P(B) - 2P(A \cap B) = 1 - a \quad (1)$$

Similarly,

$$P(B) + P(C) - 2P(B \cap C) = 1 - 2a \quad (2)$$
$$P(C) + P(A) - 2P(C \cap A) = 1 - a \quad (3)$$
$$P(A \cap B \cap C) = a^2 \quad (4)$$

Now,

$$P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)$$
$$-P(C\cap A)+P(A\cap B\cap C)$$
$$=\frac{1}{2}[P(A)+P(B)-2P(B\cap C)+P(B)+P(C)$$
$$-2P(B\cap C)+P(C)+P(A)-2P(C\cap A)]$$
$$+P(A\cap B\cap C)$$
$$=\frac{1}{2}[1-a+1-2a+1-a]+a^2$$

[Using Eqs. (1), (2), (3) and (4)]

$$=\frac{3}{2}-2a+a^2$$
$$=\frac{1}{2}+(a-1)^2>\frac{1}{2}$$

Concept Application Exercise 9.2

1. Three students A and B and C are in a swimming race. A and B have the same probability of winning and each is twice as likely to win as C. Find the probability that B or C wins. Assume no two reach the winning point simultaneously.
2. If A and B are events such that $P(A\cup B)=3/4$, $P(A\cap B)=1/4$ and $P(A^c)=2/3$, then find
   (a) $P(A)$
   (b) $P(B)$
   (c) $P(A\cap B^c)$
   (d) $P(A^c\cap B)$
3. If $P(A\cap B)=\frac{1}{2}, P(\bar{A}\cap\bar{B})=\frac{1}{3}, P(A)=p, P(B)=2p$, then find the value of p.
4. The probabilities of three mutually exclusive events are 2/3, 1/4 and 1/6. Is this statement correct?
5. The probability that at least one of A and B occurs is 0.6. If A and B occur simultaneously with probability 0.3, then find the value of $P(A')+P(B')$.
6. In a class of 125 students 70 passed in Mathematics, 55 in Statistics and 30 in both. Then find the probability that a student selected at random from the class has passed in only one subject.
7. In a certain population, 10% of the people are rich, 5% are famous and 3% are rich and famous. Then find the probability that a person picked at random from the population is either famous or rich but not both.

INDEPENDENT EVENTS

Two or more events are said to be independent if occurrence or non-occurrence of any of them does not affect the probability of occurrence or non-occurrence of other events.

In other words, two or more events are said to be independent if occurrence or non-occurrence of any of them does not influence the occurrence or non-occurrence of other events.

Examples:

(i) When a coin is tossed twice, the event of occurrence of head in the first throw and the event of occurrence of head in the second throw are independent events.

(ii) When two cards are drawn out of a full pack of 52 playing cards with replacement (the first card drawn is put back in the pack and then the second card is drawn), then the event of occurrence of a king in the first draw and the event of occurrence of a king in the second draw are independent events because the probability of drawing a king in the second draw is 4/52 whether a king is drawn in the first draw or not. But if the two cards are drawn without replacement, then the two events are not independent.

(iii) Let a bag contain 3 red and 2 black balls. Two balls are drawn one by one with replacement. Let A be the event of drawing a red ball in first draw and B be the event of drawing a black ball in the second draw. Then, $P(A)$ = 2/5 when a red bal is drawn in the first draw and $P(B)$ = 2/5 when a black ball is drawn in the first draw. Here probability of occurrence of event B is not affected by occurrence or non-occurrence of event A. Hence, A and B are independent events. But if the two balls are drawn one by one without replacement, then probability of occurrence of a black ball in the second draw when a red ball has been drawn in the first draw is 2/4. Probability of occurrence of a black ball in the second draw when a red ball is not drawn in the first draw is 1/4 (after a black ball is drawn there are only four balls in the bag out of which one is a black ball). Here, the events of drawing a ball in the first draw and the event of drawing a black ball in the second draw are not independent.

COMPOUND AND CONDITIONAL PROBABILITY

Compound Events

When two or more events occur together, their joint occurrence is called a compound event.

Examples:

Drawing a red and black ball from a bag containing 5 red and 6 black balls when two balls are drawn from the bag is a compound event. Compound events are of two types: (i) independent events and (ii) dependent events.

Conditional Probability

Let A and B be any two events and $B\neq\phi$. Then $P(A/B)$ denotes the conditional probability of occurrence of event A when B has already occurred.

Examples:

(i) Let a bag contain 2 red balls and 3 black balls. One ball is drawn from the bag and this ball is not replaced in the bag. Then, a second ball is drawn from the bag.

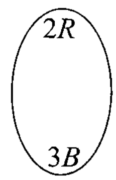

Fig. 9.2

Let B denote the events of occurrence of a red ball in the first draw and A denote the event of occurrence of a black ball in the second draw. When a red ball has been drawn in the first draw, the number of balls left is 4 and out of these four balls one is a red ball and three are black balls. Therefore, the probability of occurrence of a black ball in the second draw when a red ball has been drawn in the first draw is $P(A/B) = 3/4$.

(ii) When a die is thrown, sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Let the event of occurrence of a number greater than 4 be $\{5, 6\}$. The event of occurrence of an odd number is $B = \{1, 3, 5\}$. Then $P(A/B)$ is the probability of occurrence of a number greater than 4, when an odd number has occurred. Here it is known that an odd number has occurred, i.e., one of 1, 3 and 5 has occurred. Out of these three numbers 1, 3 and 5, only 5 is greater than 4. Hence, here when an odd number has occurred, total number of cases is only 3 (not 6) and favourable number of cases is 1 (because out of 1, 3, 5, only 5 is greater than 4).

$$\therefore \quad P(A/B) = \frac{1}{2} = \frac{n(A \cap B)}{n(B)}$$

Clearly, while finding $P(A/B)$, B works as the sample space and $A \cap B$ works as the event.

In the example given above, the probability of occurrence of an odd number when a number greater than 4 has occurred is

$$P(B/A) \frac{n(B \cap A)}{n(A)} = \frac{1}{2}$$

Here, $A = \{5, 6\}$, $B \cap A = \{5\}$. Therefore, $n(A) = 2$ and $n(B \cap A) = 1$

Note: *$P(A/B)$ may or may not be equal to $P(B/A)$.*

Multiplication of Probability (Theorem of Compound Probability)

If A and B are any two events, then $P(A \cap B) = P(B).\ P\left(\frac{A}{B}\right)$, where $B \neq \phi$ and $P\left(\frac{A}{B}\right)$ denotes the probability of occurrence of event A when B has already occurred.

Proof:

Let S be the sample space. In case of occurrence of event A when B has already occurred, B works as the sample space and $A \cap B$ works as the event. Therefore,

$$P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}} = \frac{P(A \cap B)}{P(B)}$$

$$\therefore \quad P(A \cap B) = P(B)\, P(A/B) \tag{1}$$

If A and B are independent events, then probability of occurrence of event A is not affected by occurrence or nor occurrence of event B. Therefore,

$$P\left(\frac{A}{B}\right) = P(A)$$

Hence from Eq. (1), $P(A \cap B) = P(B)\, P(A)$

Thus, $P(A \cap B) = P(A)\, P(B)$ where A and B are independent events.

Note:

1. *If A and B are two events associated with a random experiment, then $P(A \cap B = P(A).P(B/A)$, if $P(A) \neq 0$ or $P(A \cap B) = P(B).P(A/B)$, if $P(B) \neq 0$.*
2. *I Extension of multiplication theorem: If $A_1, A_2, \ldots, A_n$ are n events related to a random experiment, then $P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1)P(A_2/A_1)P(A_3/A_1 \cap A_2) \ldots P(A_n/A_1 \cap A_2 \cap \ldots \cap A_{n-1})$*
   where $P(A_i/A_1 \cap A_2 \cap \ldots \cap A_{n-1})$ represents the conditional probability of the event A_i, given that the events $A_1, A_2, \ldots, A_{i-1}$ have already happened.
3. *Two events A and B are independent if and only if $P(A \cap B)$ or $P(AB) = P(A)\, P(B)$.*
4. *If A, B and C are any three independent events, then $P(A \cap B \cap C) = P([A \cap (B \cap C)] = P(A) \times P(B \cap C) = P(A)\, P(B)\, P(C)$.*
5. *General case: if $A_1, A_2, \ldots, A_n$ be any n events none of which is an impossible events, then*
   $P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)\, P(A_2/A_1)\, P(A_3/A_1A_2) \cdots P(A_n/A_1A_2A_{n-1})$.
   If $A_1, A_2, \ldots, A_n$ are independent events, then
   $P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)\, P(A_2) \cdots P(A_n)$.
6. *Probability of at least one of the n independent events: If $p_1, p_2, p_3, \ldots, p_n$ be the probabilities of happening of n independent events $A_1, A_2, A_2, \ldots, A_n$ respectively, then*
   (i) Probability of happening none of them
   $$= P(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \ldots \cap \bar{A}_n)$$
   $$= P(\bar{A}_1).P(\bar{A}_2).P(\bar{A}_3) \ldots P(\bar{A}_n)$$
   $$= (1-p_1)(1-p_2)(1-p_3) \ldots (1-p_n)$$
   (ii) Probability of happening at least one of them
   $$= P(A_1 \cup A_2 \cup A_3 \ldots \cup A_n)$$
   $$= 1 - P(\bar{A}_1)P(\bar{A}_2)P(\bar{A}_3) \ldots P(\bar{A}_n)$$
   $$= 1-(1-p_1)(1-p_2)(1-p_3) \ldots (1-p_n)$$
   (iii) Probability of happening of first event and not happening of the remaining
   $$= P(A_1)P(\bar{A}_2)P(\bar{A}_3) \ldots P(\bar{A}_n)$$
   $$= p_1(1-p_2)(1-p_3) \ldots (1-p_n)$$

Complementation Rule

If A and B are two independent events, then

$P(A \cup B) = 1 - P(A')\, P(B')$

Proof:

$P(A \cup B) = 1 - P(A \cup B)'$

$= 1 - P(A' \cap B')$ [by De Morgan's law]

$= 1 - P(A') \times P(B)$

[Since A, B are independent events, therefore A' and B' are independent events. Hence, $P(A' \cap B') = P(A')\, P(B')$.]

Note:

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ *[always valid]*
2. $P(A \cup B) = 1 - P(A')\, P(B')$ *[valid only when A and B are independent]*
3. *If* $A_1, A_2, \ldots, A_n$ *are independent events, then* $P(A_1 \cup A_2 \cup \cdots \cup A_n) = 1 - P(A_1') \times P(A_2') \cdots P(A_n')$

Theorems on Independent Events

1. The events A and ϕ are independent.
2. The events A and S (sample space) are independent.
3. If A and B are independent events, then
   - **(i)** A and B' are independent events
   - **(ii)** A' and B are independent events
   - **(iii)** A' and B' are independent events

Proof: Given, A and B are independent events, therefore

$P(A \cap B) = P(A)\, P(B)$ (1)

(i)

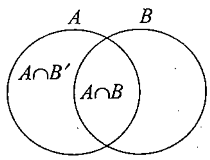

Fig. 9.3

Now, $A = (A \cap B) \cup (A \cap B')$

$\therefore$ $P(A) = P(A \cap B) + P(A \cap B')$ [$\because$ $A \cap B$ and $A \cap B'$ are mutually exclusive events]

$= P(A)\, P(B) + P(A \cap B')$

$\Rightarrow$ $P(A \cap B') = P(A) - P(A)P(B) = P(A)\,[1 - P(B)]$

$= P(A)\, P(B')$ [$\because$ $1 - P(B) = P(B')$]

Hence, A and B' are independent events.

(ii)

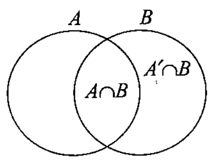

Fig. 9.4

$A' \cap B$ and $A \cap B$ are mutually exclusive events and $(A' \cap B) \cup (A \cap B) = B$.

$\therefore$ $P[(A' \cap B) \cup (A \cap B)] = P(B)$

$\Rightarrow$ $P(A' \cap B) + P(A \cap B) = P(B)$

$\Rightarrow$ $P(A' \cap B) = P(B) - P(A \cap B)$

$= P(B) - P(A)\, P(B)$ [From Eq. (1)]

$= P(B)(1 - P(A))$ [$\because$ $P(A') = 1 - P(A)$]

$= P(B)\, P(A')$

$= P(A')\, P(B)$

Hence, A and B are independent events.

(iii) $P(A' \cap B') = P(A \cup B)'$

$= 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

[By addition theorem of probability]

$= 1 - P(A) - P(B) + P(A)\, P(B)$

[From Eq. (1)]

$= 1 - P(A) - P(B)[1 - P(A)]$

$= [1 - P(A)][1 - P(B)]$

$= P(A')\, P(B')$

Hence, A' and B' are mutually exclusive events.

4. If A and B are two events such that $B \neq \phi$, then $P(A/B) + P(A/B) = 1$.

Proof: $P(A/B) + P(A'/B)$

$$= \frac{P(A \cap B)}{P(B)} + \frac{P(A' \cap B)}{P(B)}$$

$$= \frac{P(A \cap B) + P(A' \cap B)}{P(B)}$$

$$= \frac{P(B)}{P(B)} = 1$$ [$\because$ $(A \cap B) \cup (A' \cap B) = B$ and $A \cap B$ and $A' \cap B$ are mutually exclusive]

5. If A and B be two events such that $A \neq \phi$, then $P(B) = P(A)P(B/A) + P(A')P(B/A')$.

Proof: $P(A)P(B/A) + P(A')\, P(B/A')$

$= P(A \cap B) + P(A' \cap B)$

$= P[(A \cap B) \cup (A' \cap B)$

$= P[(A \cap B) \cup (A' \cap B)]$

[$\because$ $A \cap B$ and $A' \cap B$ are mutually exclusive]

$= P(B)$ [$\because (A \cap B) \cup (A' \cap B) = B$]

Example 9.26 **A fair coin is tossed repeatedly. If tail appears on first four tosses, then find the probability of head appearing on fifth toss.**

Sol. Since the trials are independent so the probability that head appears on the fifth toss does not depend upon previous results of the tosses. Hence, required probability is equal to probability of getting head, i.e., 1/2.

Example 9.27 **If a dice is thrown twice, then find the probability of getting 1 in the first throw only.**

Sol. Probability of getting 1 in first throw, $P(A) = 1/6$. Probability of not getting 1 in second throw, $P(B) = 5/6$.

Both are independent events, so the required probability is

$$P(A \cap B) = P(A)\, P(B) = \frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$$

Example 9.28 **Find the probability of getting at least one tail in 4 tosses of a coin.**

Sol. Probability of getting head in each toss of coin is 1/2.

Then, probability of getting 4 heads in 4 tosses is

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^4$$

Therefore, required probability is

$$1 - P(\text{no coin shows tail}) = 1 - \left(\frac{1}{2}\right)^4 = \frac{15}{16}$$

Example 9.29 **Three persons work independently on a problem. If the respective probabilities that they will solve it are 1/3, 1/4 and 1/5, then find the probability that none can solve it.**

Sol. Required probability is

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)=\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}=\frac{2}{5}$$

Example 9.30 **The probability of hitting a target by three marksmen are 1/2, 1/3 and 1/4. Then find the probability that one and only one of them will hit the target when they fire simultaneously.**

Sol. Here, $P(A) = 1/2$, $P(B) = 1/3$, $P(C) = 1/4$. Hence, required probability is

$$P(A)P(\bar{B})P(\bar{C})+P(\bar{A})P(B)P(\bar{C})+P(\bar{A})P(\bar{B})P(C)$$

$$=\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{3}{4}\right)+\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)$$

$$=\frac{11}{24}$$

Example 9.31 **An electrical system has open-closed switches S_1, S_2 and S_3 as shown in Fig. 9.5.**

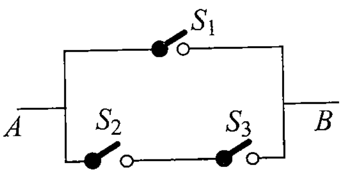

Fig 9.5

The switches operate independently of one another and the current will flow from A to B either if S_1 is closed or if both S_2 and S_3 are closed. If $P(S_1) = P(S_2) = P(S_3) = 1/2$, then find the probability that the circuit will work.

Sol. $P(S_1) = P(S_2) = P(S_3) = 1/2$

Let E be the event that the current will flow. Then,

$$P(E) = P((S_2 \cap S_3) \text{ or } S_1)$$
$$= P(S_2 \cap S_3) + P(S_3) - P(S_1 \cap S_2 \cap S_3)$$
$$=\frac{1}{4}+\frac{1}{2}-\frac{1}{8}=\frac{5}{8}$$

Example 9.32 **A bag contains 3 white, 3 black and 2 red balls. One by one three balls are drawn without replacing them, then find the probability that the third ball is red.**

Sol. Let R stand for drawing red ball, B for drawing black ball and W for drawing white ball. Then, the required probability is

$$P(WWR) + P(BBR) + P(WBR) + P(BWR) + P(WRR) + P(BRR) + P(RWR) + P(RBR)$$

$$=\frac{3\times2\times2}{8\times7\times6}+\frac{3\times2\times2}{8\times7\times6}+\frac{3\times3\times2}{8\times7\times6}+\frac{3\times3\times2}{8\times7\times6}+\frac{3\times2\times1}{8\times7\times6}+\frac{3\times2\times1}{8\times7\times6}+\frac{2\times3\times1}{8\times7\times6}+\frac{2\times3\times1}{8\times7\times6}$$

$$=\frac{2}{56}+\frac{2}{56}+\frac{3}{56}+\frac{3}{56}+\frac{1}{56}+\frac{1}{56}+\frac{1}{56}+\frac{1}{56}$$

$$=\frac{1}{4}$$

Example 9.33 **The unbiased dice is tossed until a number greater than 4 appears. What is the probability that an even number of tosses is needed?**

Sol. Probability of success is $2/6 = 1/3 = p$. Probability of failure is $1 - 1/3 = 2/3 = q$. Probability that success occurs in even number of tosses is

$$P(FS) + P(FFFS) + P(FFFFFS) + \cdots$$

$$= pq + q^3p + q^5p + \cdots = \frac{pq}{1-q^2}=\frac{2}{5}$$

Example 9.34 **'X' speaks truth in 60% and 'Y' in 50% of the cases. Find the probability that they contradict each other narrating the same incident.**

Sol. Here, $P(X) = 3/5$, $P(Y) = 1/2$. Therefore, required probability is

$$P(X)P(\bar{Y})+P(\bar{X})P(Y)=\left(\frac{3}{5}\right)\left(1-\frac{1}{2}\right)+\left(1-\frac{3}{5}\right)\left(\frac{1}{2}\right)$$
$$=\frac{3}{5}\times\frac{1}{2}+\frac{2}{5}\times\frac{1}{2}=\frac{1}{2}$$

Example 9.35 **The odds against a certain event are 5:2 and the odds in favour of other event, independent of the former, are 6:5. Then find the probability that at least of the events will happen.**

Sol. $P\{\text{First event does not happen}\}=\frac{5}{5+2}=\frac{5}{7}$

$$P\{\text{Second event does not happen}\}=\frac{5}{5+6}=\frac{5}{11}$$

$$\therefore\ P\{\text{Both the events fail to happen}\}=\frac{5}{7}\times\frac{5}{11}=\frac{25}{77}$$

Therefore, the probability that at least one of the events will happen is

$$1-P(\text{none of two happens})=1-\frac{25}{77}=\frac{52}{77}$$

Example 9.36 **If four whole numbers taken at random are multiplied together, then find the probability that the last digit in the product is 1, 3, 7 or 9.**

Sol. There are 10 digits 0, 1, 2, ..., 9 any of which can occur in any number at the last place, i.e., at the unit place. It is obvious that if the last digit in any of the four numbers is 0, 2, 4, 5, 6, 8, then the product of any of such four numbers will not give a number having its last digit as 1, 3, 7, 9. Hence, it is necessary that the last digit in each of the four numbers must be any of the four digits 1, 3, 7, 9.

Thus, for each of the four numbers, the number of ways of selection of the last digit is 10. Favourable number of ways of selection of the last digit is 4.

Therefore, the probability that the last digit be any of the four numbers 1, 3, 7, 9 is 4/10 = 2/5.

Hence, the required probability that the last digit in each of the four numbers is 1, 3, 7, 9 so that the last digit in their product is 1, 3, 7, 9 is $(2/5)^4 = 16/625$.

Example 9.37 **A rifleman is firing at a distance target and hence only 10% chance of hitting it. Find the number of rounds, he must fire in order to have more than 50% chance of hitting it at least once.**

Sol. We have,

$$p = \frac{10}{100} = \frac{1}{10}$$

$$\therefore q = \frac{9}{10}$$

By the given condition,

$$1 - q^n > \frac{1}{2}$$

$$\Rightarrow q^n < \frac{1}{2}$$

$$\Rightarrow \left(\frac{9}{10}\right)^n < \frac{1}{2}$$

which is possible if n is at least 7.

$\therefore \quad n = 7$

Example 9.38 **One of 10 keys open the door. If we try the keys one after another, then find the following:**

a. the probability that the door is opened on the first attempt.

b. the probability that the door is opened on the second attempt.

c. the probability that the door is opened on the 10th attempt.

Sol. Since out of 10 keys only one can open the door, so the door will open in first attempt with probability 1/10.

Now if he fails in first attempt which has the probability 9/10, then he will attempt next time with a different key. So, the probability that the door will open in second attempt is 9/10 × 1/9 = 1/10.

Probability that door will open only in 10th attempt is equal to the probability that the door will not open in first nine attempts which is equal to

$$\frac{9}{10}\times\frac{8}{9}\times\frac{7}{8}\times\cdots\times\frac{2}{3}\times\frac{1}{2}=\frac{1}{10}$$

as each time he tries with new key and keeps away the key which does not open the door.

Example 9.39 **A bag contains 'W' white and 3 black balls. Balls are drawn one by one without replacement till all the black balls are drawn. Then find the probability that this procedure for drawing the balls will come to an end at the r^{th} draw.**

Sol. Procedure of drawing the balls has to end at the r^{th} draw.

Exactly 2 black balls must have been drawn up to $(r-1)^{th}$ draw. Now probability of drawing exactly 2 black balls up to $(r-1)^{th}$ draw is

$$\frac{{}^3C_2\,{}^WC_{r-3}}{{}^{W+3}C_{r-1}} = \frac{\frac{3!}{2!1!}\,\frac{W!}{(r-3)!(W-r+3)!}}{\frac{(W+3)!}{(r-1)!(W-r+4)!}}$$

$$= \frac{3(r-1)(r-2)(W-r+4)}{(W+3)(W+2)(W+1)}$$

At the end of $(r-1)^{th}$ draw, we would be left with 1 black and $(W-r+3)$ white balls. Hence, the probability of drawing the black ball at the r^{th} draw is $1/(W-r+4)$. Therefore, the probability of required event is

$$\frac{3(r-1)(r-2)(W-r+4)}{(W+3)(W+2)(W+1)(W-r+4)}$$

$$= \frac{3(r-1)(r-2)}{(W+3)(W+2)(W+1)}$$

Example 9.40 **If A and B are two independent events, the probability that both A and B occur is 1/8 and the probability that neither of them occurs is 3/8. Find the probability of the occurrence of A.**

Sol. We have,

$$P(A\cap B) = \frac{1}{8} \text{ and } P(\bar{A}\cap\bar{B}) = \frac{3}{8}$$

$$\therefore \quad P(A)P(B) = \frac{1}{8} \text{ and } P(\bar{A})P(\bar{B}) = \frac{3}{8}$$

[$\because$ A and B are independent]

Now,

$$P(\bar{A}\cap\bar{B}) = \frac{3}{8} \Rightarrow 1 - P(A\cup B) = \frac{3}{8}$$

$$\Rightarrow 1 - (P(A) + P(B) - P(A\cap B)) = \frac{3}{8}$$

$$\Rightarrow 1 - (P(A) + P(B)) + \frac{1}{8} = \frac{3}{8}$$

$$\Rightarrow P(A) + P(B) = \frac{3}{4}$$

The quadratic equation whose roots are $P(A)$ and $P(B)$ is

$$x^2 - x\{P(A) + P(B)\} + P(A)\,P(B) = 0$$

$$\Rightarrow x^2 - \frac{3}{4}x + \frac{1}{8} = 0$$

$\Rightarrow\ 8x^2 - 6x + 1 = 0$

$\Rightarrow\ x = \frac{1}{2}, \frac{1}{4}$

BINOMIAL TRIALS AND BINOMIAL DISTRIBUTION

Consider a random experiment whose outcomes can be classified as success or failure. It means that experiment results in only two outcomes E_1(success) or E_2(failure). Further assume that experiment can be repeated several times, probability of success or failure in any trial are p and $q(p + q = 1)$ and do not vary from trial to trial and finally different trials are independent. Such a, experiment is called binomial experiment and trials are said to be binomial trials. For instance, tossing of a fair coin several times, each time outcome would be either a success (say occurrence of head) or failure (say occurrence of tail).

A probability distribution representing the binomial trials is said to binomial distribution. Let us consider a binomial experiment which has been repeated 'n' times. Let the probability of success and failure in any trial be p and q, respectively in these n-trials. Now number of ways of choosing 'r' success is in 'n' trials is ${}^nC_r$. Probability of 'r' successes and $(n - r)$ failures is $p^r q^{n-r}$. Thus, probability of having exactly r successes is ${}^nC_r\, p^r q^{n-r}$.

Let 'X' be a random variable representing the number of successes. Then,

$$P(X = r) = {}^nC_r\, p^r q^{n-r}\ (r = 0, 1, 2, \ldots, n)$$

Remark

- *Probability of utmost 'r' successes in n trials is* $\sum_{\lambda=0}^{r} {}^nC_\lambda p^\lambda q^{n-\lambda}$.
- *Probability of at least 'r' successes in n trials is* $\sum_{\lambda=r}^{n} {}^nC_\lambda p^\lambda q^{n-\lambda}$.
- *Probability of having first success at the r^{th} trial is* $p\, q^{r-1}$.

Example 9.41 **A die is thrown 7 times. What is the chance that an odd number turns up (i) exactly 4 times, (ii) at least 4 times?**

Sol. Probability of success is 3/6 = 1/2.

$\therefore\ p = \frac{1}{2}$ and $q = \frac{1}{2}$

(i) For exactly 4 successes, the required probability is

$${}^7C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^3 = \frac{35}{128}$$

(ii) For at least 4 successes, the required probability is

$${}^7C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^3 + {}^7C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^2 + {}^7C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^1 + {}^7C_7\left(\frac{1}{2}\right)^7$$

$$= \frac{35}{128} + \frac{21}{128} + \frac{7}{128} + \frac{1}{128}$$

$$= \frac{64}{128}$$

$$= \frac{1}{2}$$

Example 9.42 ***A* and *B* play a series of games which cannot be drawn and *p*, *q* are their respective chances of winning a single game. What is the chance that *A* wins *m* games before *B* wins *n* games.**

Sol. For this to happen, A must win at least m out of the first $m + n - 1$ games. Therefore, the required probability is

$${}^{m+n-1}C_m p^m q^{n-1} + {}^{m+n-1}C_{m+1}\, p^{m+1} q^{n-2} + \cdots + {}^{m+n-1}C_{m+n-1} p^{m+n-1}$$

Example 9.43 **An experiment succeeds twice as often as it fails. Then find the probability that in the next 6 trials, there will be at least 4 successes.**

Sol. Let p be its probability of success and q that of failure. Then $p = 2q$. Also, $p + q = 1$. It gives $p = 2/3$ and $q = 1/3$.

$P\{4$ successes in the 6 trials$\}$

$$= {}^6C_4 p^4 q^2 = {}^6C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2 \quad (1)$$

$$P\{5 \text{ successes in the 6 trials}\} = {}^6C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right) \quad (2)$$

$$P\{6 \text{ successes in the 6 trials}\} = {}^6C_6\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^0 \quad (3)$$

Therefore, the probability that there are at least 4 successes is

P {either 4 or 5 or 6 successes}

$$= {}^6C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2 + {}^6C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right) + {}^6C_6\left(\frac{2}{3}\right)^6$$

$$= \frac{496}{729}$$

Example 9.44 **What is the probability of guessing correctly at least 8 out of 10 answers on a true–false examination?**

Sol. $$p(X \geq 8) = {}^{10}C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 + {}^{10}C_9\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^1 + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$$

$$= \left(\frac{1}{2}\right)^{10} [{}^{10}C_2 + {}^{10}C_1 + {}^{10}C_0]$$

$$= \left(\frac{1}{2}\right)^{10} [45+10+1]$$

$$= \frac{56}{8\times 2^7}$$

$$= \frac{7}{128}$$

PROBLEMS ON CONDITIONAL PROBABILITY

Example 9.45 **Two dice are thrown. What is the probability that the sum of the numbers appearing on the two dice is 11, if 5 appears on the first?**

Sol. Since we are given that 5 appears on first die, so to get sum of 11, 6 must be one on the second and hence, the required probability is 1/6.

Example 9.46 **Three coins are tossed. If one of them shows tail, then find the probability that all three coins show tail.**

Sol. Let E be the event in which all three coins show tail and F be the event in which a coin shows tail. Therefore, $F = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$ and $E = \{TTT\}$. Hence, the required probability is

$$P(E/F) = \frac{P(E\cap F)}{P(E)} = \frac{1}{7}$$

Example 9.47 **One dice is thrown three times and the sum of the thrown numbers is 15. Find the probability for which number 4 appears in first throw.**

Sol. We have to find the bounded probability to get sum of 15 when 4 appears first. Let the event of getting a sum of 15 of three thrown number is A and the event of appearing 4 is B. So, we have to find $P(A/B)$. But

$$\text{P}(A/B) = \frac{n(A\cap B)}{n(B)}$$

Where $n(A \cap B)$ and $n(B)$, respectively, denote the number of digits in $A \cap B$ and B. Now $n(B) = 36$, because first throw is of 4. So another two throws stop by $6 \times 6 = 36$ types. Three dice have only two throws which starts from 4 and give a sum of 15, i.e., (4, 5, 6). So, $n(A \cap B) = 2$, $n(B) = 36$.

$$\therefore \quad P\left(\frac{A}{B}\right) = \frac{2}{36} = \frac{1}{18}$$

Example 9.48 **A box contains 10 mangoes out of which 4 are rotten. Two mangoes are taken out together. If one of them is found to be good, then find the probability that the other is also good.**

Sol. Let A be the event that the first mango is good and B be the event that the second one is good. Then, required probability is

$$P(B/A) = \frac{P(A\cap B)}{P(A)}$$

Now, probability that both mangoes are good is

$$P(A\cap B) = \frac{^6C_2}{^{10}C_2}$$

Probability that first mango is good is

$$P(A) = \frac{^6C_2}{^{10}C_2} + \frac{^6C_1\times {}^4C_1}{^{10}C_2}$$

Hence,

$$P(B/A) = \frac{^6C_2}{^6C_2 + {}^6C_1 \times {}^4C_1} = \frac{15}{15+24} = \frac{5}{13}$$

Example 9.49 **If two events A and B are such that $P(A^c) = 0.3$, $P(B) = 0.4$ and $P(A\cap B^c) = 0.5$, then find the value of $P[B/(A \cup B^c)]$.**

Sol.
$$P[B/(A\cup B^c)] = \frac{P(B\cap(A\cup B^c))}{P(A\cup B^c)}$$

$$= \frac{P(A\cap B)}{P(A)+P(B^c)-P(A\cap B^c)}$$

$$= \frac{P(A)-P(A\cap B^c)}{P(A)+P(B^c)-P(A\cap B^c)}$$

$$= \frac{0.7-0.5}{0.8} = \frac{1}{4}$$

Concept Application Exercise 9.3

1. A coin is tossed three times.
   Event A: two heads appear
   Event B: last should be head
   Then identify events A and B: independent or dependent.
2. Two cards are drawn one by one randomly from a pack of 52 cards. Then find the probability that both of them are king.
3. A coin is tossed and a dice is rolled. Find the probability that the coin shows the head and the dice shows 6.
4. The probability of happening an event A in one trial is 0.4. Find the probability that the event A happens at least once in three independent trials.
5. A fair coin is tossed n times. If the probability that head occurs 6 times is equal to the probability that head occurs 8 times, then find the value of n.
6. A, B, C in order cut a pack of cards replacing them after each cut on the condition that the first who cuts a spade shall win the prize. Find their respective chances.

7. In a bag, here are 6 balls of which 3 are white and 3 are black. They are drawn successively (i) without replacement, (ii) with replacement. What is the chance that the colours are alternate? It has been supposed that the number of balls drawn remains the same, i.e., six even with replacement.

8. The odds against a certain event is 5:2 and the odds in favour of another event is 6:5. If both the events are independent, then find the probability that at least one of the events will happen.

9. A pair of unbiased dice are rolled together till a sum of 'either 5 or 7' is obtained. Then find the probability that 5 comes before 7.

10. A man and a woman appear in an interview for two vacancies in the same post. The probability of man's selection is 1/4 and that of the woman's selection is 1/3. What is the probability that none of them will be selected?

11. A bag contains 5 white and 3 black balls. 4 balls are successively drawn out and not replaced. What is the probability that they are alternately of different colors.

12. The probability that Krishna will be alive 10 years hence is 7/15 and that Hari will be alive is 7/10. What is the probability that both Krishna and Hari will be dead 10 years hence?

13. A binary number is made up of 8 digits. Suppose that the probability of an incorrect digit appearing is p and that the errors in different digits are independent of each other. Then find the probability of forming an incorrect number.

14. A bag contains a white and b black balls. Two players, A and B alternately draw a ball from the bag, replacing the ball each time after the draw till one of them draws a white ball and wins the game. A begins the game. If the probability of A winning the game is three times that of B, then find the ratio a:b.

15. The probability of India winning a test match against West Indies is 1/2. Assuming independence from match to match find the probability that in a match series India's second win occurs at the third test.

16. In a single throw of two dice what is the probability of obtaining a number greater 7, if 4 appears on the first dice?

17. A coin is tossed three times in succession. If E is the event that there are at least two heads and F is the event in which first throw is a head, then find $P(E/F)$.

18. A die is thrown 4 times. Find the probability of getting at most two 6.

19. The probability that a student is not a swimmer is 1/5. Then find the probability that out of 5 students exactly 4 are swimmer.

20. A and B are two candidates seeking admission in IIT. The probability that A is selected is 0.5 and the probability that A and B are selected is at most 0.3. Is it possible that the probability of B getting selected is 0.9?

BAYES'S THEOREM

Partition of a Set

Consider a sample space 'S'. Let $A_1, A_2, \ldots, A_n$ be the set of mutually exclusive and exhaustive events of sample space S.

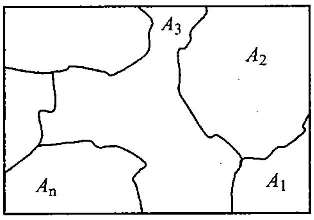

Fig. 9.6

These events $A_1, A_2, \ldots, A_n$ are said to partition the sample space S. We have,

$$A_i \cap A_j = \phi \text{ for } i \neq j,\ 1 \leq i, j \leq n$$

and $\sum_{i=1}^{n} P(A_i) = 1$

Bayes's Theorem

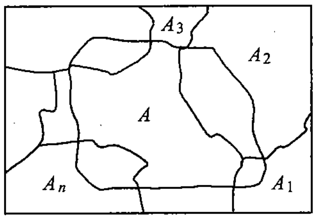

Fig. 9.7

If $A_1, A_2, A_3, \ldots, A_n$ be n mutually exclusive and exhaustive events and A is an event which occurs together (in conjunction) with either of A_i, i.e., if $A_1, A_2, \ldots, A_n$ form a partition of the sample space S and A be any event, then

$$p(A_k/A) = \frac{P(A_k)P(A/A_k)}{P(A_1)P(A/A_1) + P(A_2)P(A/A_2) + \cdots + P(A_n)P(A/A_m)}$$

Proof:

Since $A_1, A_2, \ldots, A_n$ form a partition of S, therefore

(i) $A_1, A_2, \ldots, A_n$ are non-empty

(ii) $A_i \cap A_j = \phi$ for $i \neq j$

(iii) $S = A_1 \cup A_2 \cup \cdots \cup A_n$

Now,

$$A = A \cap S = A \cap (A_1 \cup A_2 \cup \cdots \cup A_n)$$
$$= (A \cap A_1) \cup (A \cap A_2) \cup \cdots \cup (A \cap A_n) \quad (1)$$

Since $A_1, A_2, \ldots, A_n$ are disjoint sets, therefore $A \cap A_1, A \cap A_2, \ldots, A \cap A_n$ are also disjoint. Therefore, from (1), by addition theorem,

$$P(A) = P(A \cap A_1) + P(A \cap A_2) + \cdots + P(A \cap A_n) \quad (2)$$

Now,

$$P(A_k/A) = \frac{P(A_k \cap A)}{P(A)}$$

$$= \frac{P(A_k \cap A)}{P(A_1 \cap A) + P(A_2 \cap A) + \cdots + P(A_n \cap A)}$$

$$= \frac{P(A_k)P(A/A_k)}{P(A_1)P(A/A_1)+P(A_2)P(A/A_2)+\cdots+P(A_n)P(A/A_n)}$$

$[\because P(A \cap B) = P(A)P(B/A) = P(B)P(A/B)]$

Note:

1. *If $A_1, A_2, \ldots, A_n$ form a partition of S and A be any event then from Eq. (2),*

   $P(A) = P(A_1)\, P(A/A_1) + P(A_2)\, P(A/A_2) + \cdots + P(A_n)\, P(A/A_n)$

   $[\because P(A_i \cap A) = P(A_i)\, P(A/A_i)]$

2. *If $P(A_1) = P(A_2) = \cdots = P(A_n)$, then by Bayes's theorem,*

   $$P(A_k/A) = \frac{P(A/A_k)}{P(A/A_1) + P(A/A_2) + \cdots + P(A/A_n)}$$

3. *The probabilities $P(A_1), P(A_2), \ldots, P(A_n)$ which are known before the experiment takes place are called a priori probabilities and $P(A_i/A)$ are called a posteriori probabilities.*

4. ***Special case of Bayes's theorem:***

   If $A_1, A_2, \ldots, A_n$ form a partition of an event A, then

   $$P(A_k/A) = \frac{P(A_k)}{P(A_1) + P(A_2) + \cdots + P(A_n)}$$

   Proof:

   Since $A_1, A_2, \ldots, A_n$ form a partition of A, therefore

   (i) $A_1, A_2, \ldots, A_n$ are non-empty

   (ii) they are pairwise disjoint, i.e., no two of $A_1, A_2, \ldots, A_n$ have any common element

   (iii) $A = A_1 \cup A_2 \cup \cdots \cup A_n$

   From (i), (ii) and (iii), it is clear that $A \cap A_1, A \cap A_2, \ldots, A \cap A_n$ are non-empty pair-wise disjoint (they are mutually exclusive) and

   $A = (A \cap A_1) \cup (A \cap A_2) \cup \cdots \cup (A \cap A_n)$

   Hence by addition theorem of probability,

   $P(A) = P(A \cap A_1) + P(A \cap A_2) + \cdots + P(A \cap A_n)$ (3)

   Now,

   $$P(A_k/A) = \frac{P(A_k \cap A)}{P(A)} \qquad \left[\because P(A/B) = \frac{P(A \cap B)}{P(B)}\right]$$

   $$= \frac{P(A_k \cap A)}{P(A \cap A_1) + P(A \cap A_2) + \cdots + P(A \cap A_n)} \quad \textit{[From Eq. (1)]}$$

   $$= \frac{P(A_k)P(A/A_k)}{P(A_1)P(A/A_1) + P(A_2)P(A/A_2) + \cdots + P(A_n)P(A/A_n)}$$

   $[\because P(A \cap B) = P(A)\, P(B/A)]$

   $$= \frac{P(A_k)}{P(A_1) + P(A_2) + \cdots + P(A_n)}$$

   $[\because A_1, A_2, \ldots, A_n$ are subsets of A $\therefore$ $P(A/A_i) = 1$ for $i = 1, 2, 3, \ldots, n]$

PROBLEMS ON TOTAL PROBABILITY THEOREM

Example 9.50 **The probability that certain electronic component fails when first used is 0.10. If it does not fail immediately, the probability that it lasts for one year is 0.99. Find the probability that a new component will last for one year.**

Sol. Probability that the electronic component fails when first used is $P(F) = 0.10$. Therefore,

$P(F') = 1 - P(F) = 0.90$

Let E be the event that a new component will last for one year. Then,

$$P(E) = P(F)P\left(\frac{E}{F}\right) + P(F')P\left(\frac{E}{F'}\right)$$

[total probability theorem]

$= 0.10 \times 0 + 0.90 \times 0.99 = 0.891$

Example 9.51 **There are two bags, one of which contains 3 black and 4 white balls while the other contains 4 black and 3 white balls. A die is cast. If the face 1 or 3 turns up, a ball is taken from the first bag, and if any other face turn up, a ball is chosen from the second bag. Find the probability of choosing a black ball.**

Sol. Let E_1 be the event that a ball is drawn from first bag, E_2 the event that a ball is drawn from the second bag and E the event a black ball is drawn. Then we have

$P(E) = P(E_1)\, P(E/E_1) + P(E_2)\, P(E/E_2)$

$$= \frac{1}{3} \times \frac{3}{7} \times \frac{2}{3} \times \frac{4}{7} = \frac{11}{21}$$

Example 9.52 **A bag contains $n + 1$ coins. It is known that one of these coins shows heads on both sides, whereas the other coins are fair. One coin is selected at random and tossed. If the probability that toss results in heads is 7/12, then find the value of n.**

Sol. Let E_1 denote an event when a coin with two heads is selected and E_2 an event when a fair coin is selected. Let A be the event when the toss results in heads. Then, $P(E_1)$

$= 1/(n+1)$, $P(E_2) = 1/(n+1)$, $P(A/E_1) = 1$ and $P(A/E_2) = 1/2$.

$\therefore\ P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2)$

$$\Rightarrow \frac{7}{12} = \frac{1}{n\times 1}\times 1 + \frac{n}{n+1}\times\frac{1}{2}$$

$\Rightarrow\ 12 + 6n = 7n + 7$

$\Rightarrow\ n = 5$

Example 9.53 **An urn contains 6 white and 4 black balls. A fair die is rolled and that number of balls are chosen from the urn. Find the probability that the balls selected are white.**

Sol. Let A_i denote the event that the number i appears on the die and let E denote the event that only white balls are drawn. Then,

$$P(A_i) = \frac{1}{6} \text{ for } i = 1, 2, \ldots, 6$$

$$P(E/A_i) = \frac{{}^6C_i}{{}^{10}C_i},\ i = 1, 2, \ldots, 6$$

Then, the required probability is

$$P(E) = P\left(\bigcup_{i=1}^{6}(E\cap A_i)\right)$$

$$= \sum_{i=1}^{6} P(E\cap A_i)$$

$$= \sum_{i=1}^{6} P(A_i)P(E/A_i)$$

$$= \frac{1}{6}\left[\frac{6}{10}+\frac{15}{45}+\frac{20}{120}+\frac{15}{210}+\frac{6}{252}+\frac{1}{210}\right] = \frac{1}{3}$$

PROBLEMS ON BAYES'S THEOREM

Example 9.54 **A pack of playing cards was found to contain only 51 cards. If the first 13 cards, which are examined, are all red, what is the probability that the missing card is black?**

Sol. Let A_1 be the event that black card is lost, A_2 be the event that red card is lost and let A denote occurrence of first 13 cards which are examined and are found to be all red. Then, we have to find $P(A_1/A)$. We have $P(A_1) = P(A_2) = 1/2$. Also, $P(A/A_1) = {}^{26}C_{13}/{}^{51}C_{13}$ and $P(A/A_2) = {}^{23}C_{13}/{}^{51}C_{13}$. Then by Bayes's rule,

$$P(A_1/A) = \frac{P(A_1)P(A/A_1)}{P(A_1)P(A/A_1) + P(A_2)P(A/A_2)}$$

$$= \frac{\frac{1}{2}\frac{{}^{26}C_{13}}{{}^{51}C_{13}}}{\frac{1}{2}\frac{{}^{26}C_{13}}{{}^{51}C_{13}} + \frac{1}{2}\frac{{}^{25}C_{13}}{{}^{51}C_{13}}}$$

$$= \frac{{}^{26}C_{13}}{{}^{26}C_{13} + {}^{25}C_{13}} = \frac{2}{2+1} = \frac{2}{3}$$

Example 9.55 **The chances of defective screws in three boxes A, B, C are 1/5, 1/6, 1/7, respectively. A box is selected at random and a screw drawn from it at random is found to be defective. Then, find the probability that it came from box A.**

Sol. Let E_1, E_2 and E_3 denote the events of selecting boxes A, B, C, respectively, and A be the event that a screw selected at random is defective. Then,

$$P(E_1) = P(E_2) = P(E_3) = 1/3$$

$$\therefore\ P(A/E_1) = \frac{1}{5},\ P(A/E_2) = \frac{1}{6},\ P(A/E_3) = \frac{1}{7}$$

By Bayes's rule, the required probability is

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}$$

$$= \frac{\frac{1}{3}\times\frac{1}{5}}{\frac{1}{3}\times\frac{1}{5}+\frac{1}{3}\times\frac{1}{6}+\frac{1}{3}\times\frac{1}{7}} = \frac{42}{107}$$

Example 9.56 **In an entrance test, there are multiple choice questions. There are four possible answers to each question, of which one is correct. The probability that a student knows the answer to a question is 90%. If he gets the correct answer to a question, then find the probability that he was guessing.**

Sol. We define the following events:

A_1: He knows the answer

A_2: He does not know the answer

E: He gets the correct answer

Then, $P(A_1) = 9/10$, $P(A_2) = 1 - 9/10 = 1/10$, $P(E/A_1) = 1$, $P(E/A_2) = 1/4$.

Therefore, the required probability is

$$P(A_2/E) = \frac{P(A_2)P(E/A_2)}{P(A_1)P(E/A_1)+P(A_2)P(E/A_2)}$$

$$= \frac{\frac{1}{10}\times\frac{1}{4}}{\frac{9}{10}\times 1+\frac{1}{10}\times\frac{1}{4}} = \frac{1}{37}$$

Example 9.57 **Each of the 'n' urns contains 4 white and 6 black balls. The $(n+1)^{th}$ urn contains 5 white and 5 black balls. One of the $n+1$ urns is chosen at random and two balls are drawn from it without replacement. Both the balls turn out to be black. If the probability that the $(n+1)^{th}$ urn was chosen to draw the balls is 1/16, then find the value of n.**

Sol. Let E_1 denote the event that one of the first n urns is chosen and E_2 denote the event that $(n+1)^{th}$ urn is selected. A denotes the event that two balls drawn are black. Then,

$P(E_1) = n/(n+1)$, $p(E_2) = 1/(n+1)$, $P(A/E_1) = {}^6C_2/{}^{10}C_2$

$= 1/3$ and $P(A/E_2) = {}^5C_2/{}^{10}C_2 = 2/9$.

Using Bayes's theorem, the required probability is

$$P(E_2/A) = \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$\Rightarrow \frac{1}{16} = \frac{\left(\frac{1}{n+1}\right)\frac{2}{9}}{\left(\frac{n}{n+1}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{n+1}\right)\left(\frac{2}{9}\right)} = \frac{2}{3n+2}$$

$\Rightarrow n = 10$

Example 9.58 **Die A has 4 red and 2 white faces whereas die B has 2 red and 4 white faces. A coin is flipped once. If it shows a head, the game continues by throwing die A; if it shows tail, then die B is to be used. If the probability that die A is used is 32/33 when it is given that red turns up every time in first n throws, then find the value of n.**

Sol. Let R be the event that a red face appears in each of the first n throws.

E_1: Die A is used when head has already fallen

E_2: Die B is used when tail has already fallen

$$\therefore \quad P(R/E_1) = \left(\frac{2}{3}\right)^n \text{ and } P\left(\frac{R}{E_2}\right) = \left(\frac{1}{3}\right)^n$$

As per the given condition,

$$\frac{P(E_1)P(R/E_1)}{P(E_1)P(R/E_1) + P(E_2)P(R/E_2)} = \frac{32}{33}$$

$$\Rightarrow \frac{1/2(2/3)^n}{\frac{1}{2}\left(\frac{2}{3}\right)^n + \frac{1}{2}\left(\frac{1}{3}\right)^n} = \frac{32}{33}$$

$$\Rightarrow \frac{2^n}{2^n+1} = \frac{32}{33}$$

$\Rightarrow n = 5$

Example 9.59 **A bag contain n balls out of which some balls are white. If probability that a bag contains exactly i white ball is proportional to i^2. A ball is drawn at random from the bag and found to be white, then find the probability that bag contains exactly 2 white balls.**

Sol. We have,

$P(A_i) = ki^2$

$\Rightarrow 1 = k\Sigma n^2$

$$\Rightarrow k = \frac{1}{\Sigma n^2}$$

$$\therefore \quad P(A_i) = \frac{6i^2}{n(n+1)(2n+1)}$$

Let event B denote that the ball drawn is white. Then,

$$\therefore \quad P(B) = \frac{6}{n(n+1)(2n+1)}\left[1^2\frac{1}{n} + 2^2\frac{2}{n} + \cdots + n^2\frac{n}{n}\right]$$

$$= \frac{3(n+1)}{2(2n+1)}$$

$$P(A_2/B) = \frac{\left(\frac{6}{n(n+1)}\right)\left(\frac{2^3}{n}\right)}{\frac{2}{3(n+1)}}$$

Concept Application Exercise 9.4

1. An unbiased coin is tossed. If the result is a head, a pair of unbiased dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well-shuffled pack of 11 cards numbered 2, 3, 4, ..., 12 is picked and the number on the card is noted. What is the probability that the noted number is either 7 or 8?
2. A number is selected at random from the first twenty-five natural numbers. If it is a composite number then it is divided by 5. But if it is not a composite number, it is divided by 2. Find the probability that there will be no remainder in the division.
3. A real estate man has eight master keys to open several new homes. Only one master key will open any given house. If 40% of these homes are usually left unlocked, what is the probability that the real estate man can get into a specific home if he selects three master keys at random before leaving the office?
4. A card from a pack of 52 cards is lost. From the remaining cards, two cards are drawn and are found to be spades. Find the probability that missing card is also a spade.
5. Consider a sample space 'S' representing the adults in a small town who have completed the requirements for a collage degree. They have been categorized according to sex and employment as follows:

	Employed	Unemployed
Male	460	40
Female	140	260

An employed person is selected at random. Find the probability that chosen one is male.

6. A man has coins A, B, C. A is unbiased; the probability that a head will result when B is tossed is 2/3; and the probability that a head will result when C is tossed is 1/3. If one of the coins chosen at random, is tossed three times, giving two heads and one tail, then find the probability that the chosen coin was A.
7. A and B are two independent witnesses (i.e., there is no collusion between them) in a case. The probability that A will speak the truth is x and the probability that B will speak the truth is y. A and B agree in a certain statement. Find the probability that the statement is true.
8. An urn contains five balls. Two balls are drawn and are found to be white. Find the probability that all the balls are white.

EXERCISES

Subjective Type

Solutions on page 9.42

1. Let $A = \{0, 5, 10, 15, \ldots, 195\}$. Let B be any subset of A with atleast 15 elements. What is the probability that B has at least one pair of elements whose sum is divisible by 15?

2. A bag contains n white and n black balls, all of equal size. Balls are drawn at random. Find the probability that there are both white and black balls in the draw and that the number of white balls is greater than that of black balls by 1.

3. There are two bags each containing 10 books all having different titles but of the same size. A student draws out books from first bag as well as from the second bag. Find the probability that the different between the books drawn from the two bags does not exceed 2.

4. Suppose A and B shoot independently until each hits his target. They have probabilities 3.5 and 5/7 of hitting the targets at each shot. Find the probability that B will require more shots than A.

5. From an urn containing a white and b black balls, k balls are drawn and laid aside, their colour unnoted. Then one more ball is drawn. Find the probability that it is white assuming that $k < a, b$.

6. Of three independent events, the chance that only the first occurs is a, the chance that only the second occurs is b and the chance of only third is c. Show that the chances of three events are, respectively, $a/(a + x)$, $b/(b + x)$, $c/(c + x)$, where x is a root of the equation $(a + x)(b + x)(c + x) = x^2$.

7. Two natural numbers x and y are chosen at random. What is the probability that $x^2 + y^2$ is divisible by (i) 5 and (ii) 7.

8. If m things are distributed among a men and b women, show that the chance that the number of things received by man is

$$\frac{1}{2}\frac{(b+a)^m-(b-a)^m}{(b+a)^m}.$$

9. $8n$ players $P_1, P_2, P_3, \ldots, P_{8n}$ play a knock out tournament. It is known that all the players are of equal strength. The tournament is held in three rounds where the players are paired at random in each round. If it is given that P_1 wins in the third round. Find the probability that P_2 loses in the second round.

10. A tennis match of best of 5 sets is played by two players 'A' and 'B'. The probability that first set is won by A is 1/2 and if he loses the first, then probability of his winning the next set is 1/4 otherwise it remains same. Find the probability that A wins the match.

11. A and B participate in a tournament of 'best of 7 games'. It is equally likely that either A wins or B wins or the game ends in a draw. What is the probability that A wins the tournament.

12. Fourteen numbered balls (i.e., 1, 2, 3, …, 14) are divided in three groups randomly. Find the probability that sum of the numbers on the balls, in each group, is odd.

13. Let $P(x)$ denote the probability of the occurrence of event x. Plot all those point $(x, y) = (P(A), P(B))$ in a plane which satisfies the conditions,

$P(A \cup B) \geq 3/4$ and $1/8 \leq P(A \cap B) \leq 3/8$

14. Two players P_1 and P_2 are playing the final of a chess championship, which consists of a series of matches. Probability of P_1 winning a match is 2/3 and that of P_2 is 1/3. The winner will be the one who is ahead by 2 games as compared to the other player and wins at least 6 games. Now, if the player P_2 wins first four matches, find the probability of P_1 wining the championship.

15. Consider a game played by 10 people in which each flips a fair coin at the same time. If all but one of the coins comes up the same then the odd persons wins (e.g. if there are nine tails and one head then head wins). If such a situation does not occur, the players flips again. Find the probability that game is settled on or after n^{th} toss.

16. A bag contains 'n' balls, one of which is white. The probability that A and B speak truth are P_1 and P_2, respectively. One ball is drawn from the bag and A and B both assert that it is white. Find the probability that drawn ball is actually white.

17. A bag contains a total of 20 books on physics and mathematics. Any possible combination of books is equally likely. Ten books are chosen from the bag and it is found that it contains 6 books of mathematics. Find out the probability that the remaining books in the bag contains 2 books on mathematics.

18. A coin is tossed $(m + n)$ times $(m > n)$. Show that the probability of at least m consecutive heads is $n+2/2^{m+1}$.

Objective Type

Solutions on page 9.45

Each question has four choices a, b, c and d, out of which *only one* is correct. Find the correct answer.

1. Given two events A and B. If odds against A are as 2:1 and those in favour of $A \cup B$ are as 3:1, then

   a. $1/2 \leq P(B) \leq 3/4$ **b.** $5/12 \leq P(B) \leq 3/4$
   c. $1/4 \leq P(B) \leq 3/5$ **d.** none of these

2. The probability that a marksman will hit a target is given as $1/5$. Then the probability that at least once hit in 10 shots is

   a. $1-(4/5)^{10}$ **b.** $1/5^{10}$
   c. $1-(1/5^{10})$ **d.** $(4/5)^{10}$

3. A six-faced dice is so biased that it is twice as likely to show an even number as an odd number when thrown. It is thrown twice, the probability that the sum of two numbers thrown is even is

   a. 1/12 **b.** 1/6
   c. 1/3 **d.** 5/9

4. A draws a card from a pack of n cards marked 1, 2, …, n. The card is replaced in the pack and B draws a card. Then the probability that A draws a higher card than B is

a. $(n+1)2n$ b. $1/2$
c. $(n-1)\,2n$ d. none of these

5. A student appears for tests I, II and III. The student is successful if he passes either in tests I and II or tests I and III. The probabilities of the student passing in tests I, II and III are, respectively, p, q and 1/2. If the probability that the student is successful is 1/2, then $p(1 + q) =$
a. 1/2 b. 1
c. 3/2 d. 3/4

6. A problem in mathematics is given to three students A, B, C and their respective probability of solving the problem is 1/2, 1/3 and 1/4. Probability that the problem is solved is
a. 3/4 b. 1/2
c. 2/3 d. 1/3

7. The probability that in a family of 5 members, exactly two members have birthday on Sunday is
a. $(12\times 5^3)\,7^5$ b. $(10\times 6^2)\,7^5$
c. 2/5 d. $(10\times 6^3)\,7^5$

8. Three houses are available in a locality. Three persons apply for the houses. Each applies for one houses without consulting others. The probability that all three apply for the same houses is
a. 1/9 b. 2/9
c. 7/9 d. 8/9

9. The numbers 1, 2, 3, ..., n are arrange in random order. The probability that the digits 1, 2, 3, ..., k ($k < n$) appear as neighbours in that order is
a. $1/n!$ b. $k!/n!$
c. $(n-k)!\,n!$ d. $(n-k+1)!\,n!$

10. A die is thrown a fixed number of times. If probability of getting even number 3 times is same as the probability of getting even number 4 times, then probability of getting even number exactly once is
a. 1/6 b. 1/9
c. 5/36 d. 7/128

11. A pair of four dice is thrown independently three times. The probability of getting a score of exactly 9 twice is
a. 8/9 b. 8/729
c. 8/243 d. 1/729

12. Five horses are in a race. Mr. A selects two of the horses at random and bets on them. The probability that Mr. A selected the winning horse is
a. 3/5 b. 1/5
c. 2/5 d. 4/5

13. Let A and B be two events such that $P(\overline{A\cup B}) = 1/6$, $P(A\cap B) = 1/4$ and $P(\overline{A}) = 1/4$ where $\overline{A}$ stands for complment of event A. Then events A and B are
a. equally likely but not independent
b. equally likely and mutually exclusive
c. mutually exclusive and independent
d. independent but not equally likely

14. Words from the letters of the word PROBABILITY are formed by taking all letters at a time. The probability that both B's are not together and both I's are not together is
a. 52/55 b. 53/55
c. 54/55 d. none of these

15. A class consists of 80 students, 25 of them are girls and 55 are boys. If 10 of them are rich and the remaining are poor and also 20 of them are intelligent, then the probability of selecting an intelligent rich girl is
a. 5/128 b. 25/128
c. 5/512 d. none of these

16. Let A, B, C, D be independent events such that $P(A)=1/2$, $P(B) = 1/3$, $P(C) = 1/5$ and $P(D) = 1/6$. Then the probability that none of A, B, C and D occurs
a. 1/180 b. 1/45
c. 1/18 d. none of these

17. A sample space consists of 3 sample points with associated probabilities given as $2p$, p^2, $4p - 1$. Then the value of p is
a. $p=\sqrt{11}-3$ b. $\sqrt{10}-3$
c. $\frac{1}{4}<p<\frac{1}{2}$ d. none

18. South African cricket captain lost the toss of a coin 13 times out of 14. The chance of this happening was
a. $7/2^{13}$ b. $1/2^{13}$
c. $13/2^{14}$ d. $13/2^{13}$

19. Events A and C are independent. If the probabilities relating A, B and C are $P(A) = 1/5$, $P(B) = 1/6$; $P(A\cap C) = 1/20$; $P(B\cup C) = 3/8$. Then
a. events B and C are independent
b. events B and C are mutually exclusive
c. events B and C are neither independent nor mutually exclusive
d. events B and C are equiprobable

20. There are only two women among 20 persons taking part in a pleasure trip. The 20 persons are divided into two groups, each group consisting of 10 persons. Then the probability that the two women will be in the same group is
a. 9/19 b. 9/38
c. 9/35 d. none

21. Let A and B be two events. Suppose $P(A) = 0.4$, $P(B) = p$ and $P(A\cup B) = 0.7$. The value of p for which A and B are independent is
a. 1/3 b. 1/4
c. 1/2 d. 1/5

22. A man has 3 pairs of black socks and 2 pairs of brown socks kept together in a box. If he dressed hurriedly in the dark, the probability that after he has put on a black sock, he will then put on another black sock is

a. 1/3 b. 2/3
c. 3/5 d. 2/15

23. Five different games are to be distributed among 4 children randomly. The probability that each child get atleast one game is
a. 1/4 b. 15/64
c. 21/64 d. none of these

24. A drawer contains 5 brown socks and 4 blue socks well mixed. A man reaches the drawer and pulls out socks at random. What is the probability that they match?
a. 4/9 b. 5/8
c. 5/9 d. 7/12

25. Two dices are rolled one after the other. The probability that the number on the first is smaller than the number on the second is
a. 1/2 b. 7/18
c. 3/4 d. 5/12

26. A natural number is chosen at random from the first 100 natural numbers. The probability that $x+\frac{100}{x}>50$ is
a. 1/10 b. 11/50
c. 11/20 d. none of these

27. A four figure number is formed of the figures 1, 2, 3, 5 with no repetitions. The probability that the number is divisible by 5 is
a. 3/4 b. 1/4
c. 1/8 d. none of these

28. Twelve balls are distributed among three boxes. The probability that the first box contains three balls is
a. $\frac{110}{9}\left(\frac{2}{3}\right)^{10}$ b. $\frac{9}{110}\left(\frac{2}{3}\right)^{10}$
c. $\frac{{}^{12}C_3}{12^3}\times 2^9$ d. $\frac{{}^{12}C_3}{3^{12}}$

29. A box contains 2 black, 4 white and 3 red balls. One ball is drawn at random from the box and kept aside. From the remaining balls in the box, another ball is drawn at random and kept aside the first. This process is repeated till all the balls are drawn from the box. The probability that the balls drawn are in the sequence of 2 black, 4 white and 3 red is
a. 1/1260 b. 1/7560
c. 1/126 d. none of these

30. A cricket club has 15 members, of whom only 5 can bowl. If the names of 15 members are put into a box and 11 are drawn at random, then the probability of getting an eleven containing at least 3 bowlers is
a. 7/13 b. 6/13
c. 11/15 d. 12/13

31. *A* speaks truth in 60% cases and *B* speaks truth in 70% cases. The probability that they will say the same thing while describing a single event is
a. 0.56 b. 0.54
c. 0.38 d. 0.94

32. Three integers are chosen at random from the first 20 integers. The probability that their product is even is
a. 2/19 b. 3/29
c. 17/19 d. 4/29

33. There are 20 cards. Ten of these cards have the letter '*I*' printed on them and the other 10 have the letter '*T*' printed on them. If three cards are picked up at random and kept in the same order, the probability of making word IIT is
a. 4/27 b. 5/38
c. 1/8 d. 9/80

34. The chance of an event happening is the square of the chance of a second event but the odds against the first are the cube of the odds against the second. The chances of the events are
a. 1/9, 1/3 b. 1/16, 1/4
c. 1/4, 1/2 d. none of these

35. If a party of n persons sit at a round table, then the odds against two specified individuals sitting next to each other are
a. $2:(n-3)$ b. $(n-3):2$
c. $(n-2):2$ d. $2:(n-2)$

36. The sum of two positive quantities is equal to $2n$. The probability that their product is not less than 3/4 times their greatest product is
a. 3/4 b. 1/2
c. 1/4 d. none of these

37. A bag contains an assortment of blue and red balls. If two balls are drawn at random, the probability of drawing two red balls is five times the probability of drawing two blue balls. Furthermore, the probability of drawing one ball of each colour is six times the probability of drawing two blue balls. The number of red and blue balls in the bag is
a. 6, 3 b. 3, 6
c. 2, 7 d. none of these

38. Dialing a telephone number an old man forgets the last two digits remembering only that these are different dialed at random. The probability that the number is dialed correctly is
a. 1/45 b. 1/90
c. 1/100 d. none of these

39. The probability that a teacher will give an unannounced test during any class meeting is 1/5. If a student is absent twice, then the probability that the student will miss at least one test is
a. 4/5 b. 2/5
c. 7/25 d. 9/25

40. A box contains tickets numbered from 1 to 20. Three tickets are drawn from the box with replacement. The probability that the largest number on the tickets is 7 is
a. 2/19 b. 7/20
c. $1-(7/20)^3$ d. none of these

41. An unbiased coin is tossed n times. If the probability that head occurs 6 times is equal to the probability that head occurs 8 times, then $n =$

a. 7 b. 14
c. 16 d. 19

42. One ticket is selected at random from 100 tickets numbered 00, 01, 02, ..., 98, 99. If x_1 and x_2 denotes the sum and product of the digits on the tickets, then $P(x_1 = 9/x_2 = 0)$ is equal to

a. 2/19 b. 19/100
c. 1/50 d. none of these

43. Let A and B be two events such that $P(A \cap B') = 0.20$, $P(A' \cap B) = 0.15$, $P(A' \cap B') = 0.1$, then $P(A/B)$ is equal to

a. 11/14 b. 2/11
c. 2/7 d. 1/7

44. A and B toss a fair coin each simultaneously 50 times. The probability that both of them will not get tail at the same toss is

a. $(3/4)^{50}$ b. $(2/7)^{50}$
c. $(1/8)^{50}$ d. $(7/8)^{50}$

45. Cards are drawn one-by-one at random from a well-shuffled pack of 52 playing cards until 2 aces are obtained from the first time. The probability that 18 draws are required for this is

a. 3/34 b. 17/455
c. 561/15925 d. none of these

46. A father has 3 children with at least one boy. The probability that he has 2 boys and 1 girl is

a. 1/4 b. 1/3
c. 2/3 d. None of these

47. Two players toss 4 coins each. The probability that they both obtain the same number of heads is

a. 5/256 b. 1/16
c. 35/128 d. none of these

48. In a game called 'odd man out' m $(m > 2)$ persons toss a coin to determine who will buy refreshments for the entire group. A person who gets an outcome different from that of the rest of the members of the group is called the odd man out. The probability that there is a loser in any game is

a. $1/2m$ b. $m/2^{m-1}$
c. $2/m$ d. none of these

49. If a is an integer lying in $[-5, 30]$, then the probability that the graph of $y = x^2 + 2(a + 4)x - 5a + 64$ is strictly above the x-axis is

a. 1/6 b. 7/36
c. 2/9 d. 3/5

50. $2n$ boys are randomly divided into two subgroups containing n boys each. The probability that the two tallest boys are in different groups is

a. $n/(2n - 1)$ b. $(n - 1)/(2n - 1)$
c. $(n - 1)/4n^2$ d. none of these

51. A bag contains some white and some black balls, all combinations of balls being equally likely. The total number of balls in the bag is 10. If three balls are drawn at random without replacement and all of them are found to be black, the probability that the bag contains 1 white and 9 black balls is

a. 14/55 b. 12/55
c. 2/11 d. 8/55

52. Three ships A, B and C sail from England to India. If the ratio of their arriving safely are 2:5, 3:7 and 6:11, respectively, then the probability of all the ships for arriving safely is

a. 18/595 b. 6/17
c. 3/10 d. 2/7

53. The probability of solving a question by three students are 1/2, 1/4, 1/6, respectively. Probability of question being solved will be

a. 33/48 b. 35/48
c. 31/48 d. 37/48

54. Let A, B, C be three mutually independent events. Consider the two statements S_1 and S_2.

S_1: A and $B \cup C$ are independent.
S_2: A and $B \cap C$ are independent.

Then

a. both S_1 and S_2 are true b. only S_1 is true
c. only S_2 is true d. neither S_1 nor S_2 is true

55. If the papers of 4 students can be checked by any one of the 7 teachers, then the probability that all the 4 papers are checked by exactly 2 teachers is

a. 2/7 b. 12/49
c. 32/343 d. none of these

56. A and B play a game of tennis. The situation of the game is as follows: if one scores two consecutive points after a deuce, he wins, if loss of a point is followed by win of a point, it is deuce. The chance of a server to win a point is 2/3. The game is at deuce and A is serving. Probability that A will win the match is (serves are changed after each game)

a. 3/5 b. 2/5
c. 1/2 d. 4/5

57. A coin is tossed $2n$ times. The chance that the number of times one gets head is not equal to the number of times one gets tails is

a. $\frac{(2n!)}{(n!)^2}\left(\frac{1}{2}\right)^{2n}$ b. $1 - \frac{(2n!)}{(n!)^2}$

c. $1 - \frac{(2n!)}{(n!)^2}\frac{1}{4^n}$ d. none of these

58. The probability that a bulb produced by a factory will fuse after 150 days if used is 0.50. What is the probability that out of 5 such bulbs none will fuse after 150 days of use

a. $1 - (19/20)^5$ b. $(19/20)^5$
c. $(3/4)^5$ d. $90\,(1/4)^5$

59. If $\bar{E}$ and $\bar{F}$ are the complementary events of events E and F, respectively, and if $0 < P(F) < 1$, then

a. $P(E/F) + P(\bar{E}/F) = 1/2$ b. $P(E/F) + P(E/\bar{F}) = 1$
c. $P(\bar{E}/F) + P(E/\bar{F}) = 1$ d. $P(E/\bar{F}) + P(\bar{E}/\bar{F}) = 1$

60. In a certain town, 40% of the people have brown hair, 25% have brown eyes and 15% have both brown hair and brown eyes. If a person selected at random from the town, has brown hair, the probability that he also has brown eyes is

a. 1/5 b. 3/8
c. 1/3 d. 2/3

61. A letter is known to have come either from LONDON or CLIFTON; on the postmark only the two consecutive letters ON are legible. The probability that it came from LONDON is

a. 1/17 b. 12/17
c. 17/30 d. 3/5

62. In a game a coin is tossed $2n + m$ times and a player wins if he does not get any two consecutive outcomes same for atleast $2n$ times in a row. The probability that player wins the game is

a. $\frac{m+2}{2^{2n}+1}$ b. $\frac{2n+2}{2^{2n}}$
c. $\frac{2n+2}{2^{2n+1}}$ d. $\frac{m+2}{2^{2n}}$

63. Words from the letters of the word PROBABILITY are formed by taking all letters at a time. The probability that both B's are not together and both I's are not together is

a. 52/55 b. 53/55
c. 54/55 d. none of these

64. The probabilities of winning a race by three persons A, B and C are 1/2, 1/4, and 1/4, respectively. They run two races. The probability of A winning the second race when B wins the first race is

a. 1/3 b. 1/2
c. 1/4 d. 2/3

65. A die is rolled 4 times. The probability of getting a larger number than the previous number each time is

a. 17/216 b. 5/432
c. 15/432 d. none of these

66. Four die are thrown simultaneously. The probability that 4 and 3 appear on two of the die given that 5 and 6 have appeared on other two die is

a. 1/6 b. 1/36
c. 12/151 d. none of these

67. A fair coin is tossed 5 times. then the probability that no two consecutive heads occur is

a. 11/32 b. 15/32
c. 13/32 d. none of these

68. A $2n$ digit number starts with 2 and all its digits are prime, then the probability that the sum of all 2 consecutive digits of the number is prime is

a. 4×2^{3n} b. 4×2^{-3n}
c. 2^{3n} d. none of these

69. The numbers (a, b, c) are selected by throwing a dice thrice, then the probability that (a, b, c) are in A.P. is

a. 1/12 b. 1/6
c. 1/4 d. none of these

70. In a n-sided regular polygon, the probability that the two diagonal chosen at random will intersect inside the polygon is

a. $\frac{2\,{}^nC_2}{{}^{({}^nC_2-n)}C_2}$ b. $\frac{{}^{n(n-1)}C_2}{{}^{({}^nC_2-n)}C_2}$
c. $\frac{{}^nC_4}{{}^{({}^nC_2-n)}C_2}$ d. none of these

71. A three-digit number is selected at random from the set of all three-digit numbers. The probability that the number selected has all the three digits same is

a. 1/9 b. 1/10
c. 1/50 d. 1/100

72. Two numbers a, b are chosen from the set of integers 1, 2, 3, ..., 39. Then probability that the equation $7a - 9b = 0$ is satisfied is

a. 1/247 b. 2/247
c. 4/741 d. 5/741

73. Two numbers x and y are chosen at random (without replacement) from amongst the numbers 1, 2, 3, ..., 2004. The probability that $x^3 + y^3$ is divisible by 3 is

a. 1/3 b. 2/3
c. 1/6 d. 1/4

74. One mapping is selected at random from all mappings of the set $S = \{1, 2, 3, ..., n\}$ into itself. If the probability that the mapping is one-one is 3/32, then the value of n is

a. 2 b. 3
c. 4 d. none of these

75. A bag contains 20 coins. If the probability that thebag contains exactly 4 biased coin is 1/3 and that of exactly 5 biased coin is 2/3, then the probability that all the biased coin are sorted out from the bag in exactly 10 draws is

a. $\frac{5}{10}\frac{{}^{16}C_6}{{}^{20}C_9} + \frac{1}{11}\frac{{}^{15}C_5}{{}^{20}C_9}$ b. $\frac{2}{33}\left[\frac{{}^{16}C_6 + 5\,{}^{15}C_5}{{}^{20}C_9}\right]$
c. $\frac{5}{33}\frac{{}^{16}C_7}{{}^{20}C_9} + \frac{1}{11}\frac{{}^{15}C_6}{{}^{20}C_9}$ d. none of these

76. Cards are drawn one by one without replacement from a pack of 52 cards. The probability that 10 cards will precede the first ace is

a. 241/1456 b. 164/4165
c. 451/884 d. none of these

77. If any four numbers are selected and they are multiplied, then the probability that the last digit will be 1, 3, 5 or 7 is

a. 4/625 b. 18/625
c. 16/625 d. none of these

78. Four numbers are multiplied together. Then the probability that the product will be divisible by 5 or 10 is

a. 369/625 **b.** 399/625
c. 123/625 **d.** 133/625

79. A coin is tossed 7 times. Then the probability that at least 4 consecutive heads appear is

a. 3/16 **b.** 5/32
c. 3/16 **d.** 1/8

80. If odds against solving a question by three students are 2:1, 5:2 and 5:3, respectively, then probability that the question is solved only by one student is

a. 31/56 **b.** 24/56
c. 25/56 **d.** none of these

81. An unbiased coin is tossed 6 times. The probability that third head appears on the sixth trial is

a. 5/16 **b.** 5/32
c. 5/8 **d.** 5/64

82. There are two urns A and B. Urn A contains 5 red, 3 blue and 2 white balls, urn B contains 4 red, 3 blue and 3 white balls. An urn is choosen at random and a ball is drawn. Probability that the ball drawn is red is

a. 9/10 **b.** 1/2
c. 11/20 **d.** 9/20

83. Three critics review a book. Odds in favour of the book are 5:2, 4:3 and 3:4, respectively, for the three critics. The probability that majority are in favour of the book is

a. 35/49 **b.** 125/343
c. 164/343 **d.** 209/343

84. Let A and B are events of an experiment and $P(A) = 1/4$, $P(A \cup B) = 1/2$, then value of $P(B/A^c)$ is

a. 2/3 **b.** 1/3
c. 5/6 **d.** 1/2

85. The probability that an automobile will be stolen and found within one week is 0.0006. The probability that an automobile will be stolen is 0.0015. The probability that a stolen automobile will be found in one week is

a. 0.3 **b.** 0.4
c. 0.5 **d.** 0.6

86. A pair of numbers is picked up randomly (without replacement) from the set {1, 2, 3, 5, 7, 11, 12, 13, 17, 19}. The probability that the number 11 was picked given that the sum of the numbers was even is nearly

a. 0.1 **b.** 0.125
c. 0.24 **d.** 0.18

87. An unbiased cubic die marked with 1, 2, 2, 3, 3, 3 is rolled 3 times. The probability of getting a total score of 4 or 6 is

a. 16/216 **b.** 50/216
c. 60/216 **d.** none of these

88. A bag contains 3 red and 3 green balls and a person draws out 3 at random. He then drops 3 blue balls into the bag and again draws out 3 at random. The chance that the 3 later balls being all of different colours is

a. 15% **b.** 20%
c. 27% **d.** 40%

89. A bag contains 20 coins. If the probability that bag contains exactly 4 biased coin is 1/3 and that of exactly 5 biased coin is 2/3, then the probability that all the biased coin are sorted out from the bag in exactly 10 draws is

a. $\frac{5}{33}\frac{^{16}C_6}{^{20}C_9}+\frac{1}{11}\frac{^{15}C_5}{^{20}C_9}$ **b.** $\frac{2}{33}\left[\frac{^{16}C_6+5\ ^{15}C_5}{^{20}C_9}\right]$

c. $\frac{5}{33}\frac{^{16}C_7}{^{20}C_9}+\frac{1}{11}\frac{^{15}C_6}{^{20}C_9}$ **d.** none of these

90. A doctor is called to see a sick child. The doctor knows (prior to the visit) that 90% of the sick children in that neighbourhood are sick with the flu, denoted by F, while 10% are sick with the measles, denoted by M. A well-known symptom of measles is a rash, denoted by R. The probability of having a rash for a child sick with the measles is 0.95. However, occasionally children with the flu also develop a rash, with conditional probability 0.08. Upon examination the child, the doctor finds a rash. Then what is the probability that the child has the measles?

a. 91/165 **b.** 90/163
c. 82/161 **d.** 95/167

91. A dice is thrown six times, it being known that each time a different digit is shown. The probability that a sum of 12 will be obtained in the first three throws is

a. 5/24 **b.** 25/216
c. 3/20 **d.** 1/12

92. A composite number is selected at random from the first 30 natural numbers and it is divided by 5. The probability that there will be a remainder is

a. 14/19 **b.** 5/19
c. 5/6 **d.** 7/15

93. Let E be an event which is neither a certainty nor an impossibility. If probability is such that $P(E) = 1 + \lambda + \lambda^2$ and $P(E') = (1 + \lambda)^2$ in terms of an unknown λ. Then $P(E)$ is equal to

a. 1 **b.** 3/4
c. 1/4 **d.** none of these

94. A student can solve 2 out of 4 problems of mathematics, 3 out of 5 problem of physics and 4 out of 5 problems of chemistry. There are equal number of books of math, physics and chemistry in his shelf. He selects one book randomly and attempts 10 problems from it. If he solves the first problem, then the probability that he will be able to solve the second problem is

a. 2/3 **b.** 25/38
c. 13/21 **d.** 14/23

95. A fair die is tossed repeatedly. A wins if it is 1 or 2 on two consecutive tosses and B wins if it is 3, 4, 5 or 6 on two consecutive tosses. The probability that A wins if the die is tossed indefinitely is

a. 1/3　　b. 5/21
c. 1/4　　d. 2/5

96. Whenever horses a, b, c race together, their respective probabilities of winning the race are 0.3, 0.5 and 0.2, respectively. If they race three times the probability that 'the same horse wins all the three races, and the probability that a, b, c each wins one race are, respectively

a. 8/50, 9/50　　b. 16/100, 3/100
c. 12/50, 15/50　　d. 10/50, 8/50

97. Five different games are to be distributed among 4 children randomly. The probability that each child get atleast one game is

a. 1/4　　b. 15/64
c. 21/64　　d. none of these

98. Forty teams play a tournament. Each team plays every other team just once. Each game results in a win for one team. If each team has a 50% chance of winning each game, the probability that at the end of the tournament, every team has won a different number of games is

a. 1/780　　b. $40!/2^{780}$
c. $40!/3^{780}$　　d. none of these

99. If three squares are selected at random from chessboard, then the probability that they form the letter 'L' is

a. $196/{}^{64}C_3$　　b. $49/{}^{64}C_3$
c. $36/{}^{64}C_3$　　d. $98/{}^{64}C_3$

100. A bag has 10 balls. Six balls are drawn in an attempt and replaced. Then another draw of 5 balls is made from the bag. The probability that exactly two balls are common to both the draw is

a. 5/21　　b. 2/21
c. 7/21　　d. 3/21

101. There are 3 bags. Bag 1 contains 2 red and $a^2 - 4a + 8$ black balls, bag 2 contains 1 red and $a^2 - 4a + 9$ black balls and bag 3 containa 3 red and $a^2 - 4a + 7$ black balls. A ball is drawn at random from at random chosen bag. Then the maximum value of probability that it is a red ball is

a. 1/3　　b. 1/2
c. 2/9　　d. 4/9

102. The probability that a random chosen three-digit number has exactly 3 factors is

a. 2/225　　b. 7/900
c. 1/800　　d. none of these

103. Let p, q be chosen one by one from the set $\{1, \sqrt{2}, \sqrt{3}, 2, e, \pi\}$ with replacement. Now a circle is drawn taking (p, q) as its centre. Then the probability that at the most two rational points exist on the circle is (rational points are those points whose both the coordinates are rational)

a. 2/3　　b. 7/8
c. 8/9　　d. none of these

104. An event X can take place in conjuction with any one of the mutually exclusive and exhaustive events A, B and C. If A, B, C are equiprobable and the probability of X is 5/12, and the probability of X taking place when A has happened is 3/8, while it is 1/4 when B has taken place, then the probability of X taking place in conjuction with C is

a. 5/8　　b. 3/8
c. 5/24　　d. none of these

105. On a Saturday night, 20% of all drivers in U.S.A. are under the influence of alcohol. The probability that a driver under the influence of alcohol will have an accident is 0.001. The probability that a sober driver will have an accident is 0.0001. If a car on a Saturday night smashed into a tree, the probability that the driver was under the influence of alcohol is

a. 3/7　　b. 4/7
c. 5/7　　d. 6/7

106. A purse contains 2 six-sided dice. One is a normal fair die, while the other has two 1's, two 3's, and two 5's. A die is picked up and rolled. Because of some secret magnetic attraction of the unfair die, there is 75% chance of picking the unfair die and a 25% chance of picking a fair die. The die is rolled and shows up the face 3. The probability that a fair die was picked up is

a. 1/7　　b. 1/4
c. 1/6　　d. 1/24

107. Five different marbles are placed in 5 different boxes randomly. Then the probability that exactly two boxes remain empty is (each box can hold any number of marbles)

a. 2/5　　b. 12/25
c. 3/5　　d. none of these

108. There are 10 prizes, five A's, three B's and two C's, placed in identical sealed envelopes for the top 10 contestants in a mathematics contest. The prizes are awarded by allowing winners to select an envelope at random from those remaining. When the 8th contestant goes to select the prize, the probability that the remaining three prizes are one A, one B and one C is

a. 1/4　　b. 1/3
c. 1/12　　d. 1/10

109. Thirty-two players ranked 1 to 32 are playing in a knockout tournament. Assume that in every match between any two players, the better-ranked player wins, the probability that ranked 1 and ranked 2 players are winner and runner up, respectively, is

a. 16/31　　b. 1/2
c. 17/31　　d. none of these

110. Three integers are chosen at random from the set of first 20 natural numbers. The chance that their product is a multiple of 3 is

a. 194/285　　b. 1/57
c. 13/19　　d. 3/4

111. A car is parked among N cars standing in a row, but not at either end. On his return, the owner finds that exactly 'r' of the N places are still occupied. The probability that the places neighouring his car are empty is

a. $\dfrac{(r-1)!}{(N-1)!}$ b. $\dfrac{(r-1)!\ (N-r)!}{(N-1)!}$

c. $\dfrac{(N-r)(N-r-1)}{(N+1)(N+2)}$ d. $\dfrac{{}^{N-r}C_2}{{}^{N-1}C_2}$

112. One ticket is selected at random from 100 tickets numbered 00, 01, 02, ..., 99. Suppose A and B are the sum and product of the digit found on the ticket. Then $P((A = 7)/(B = 0))$ is given by

a. 2/13 **b.** 2/19

c. 1/50 **d.** none of these

113. A fair coin is tossed 100 times. The probability of getting tails 1, 3, ..., 49 times is

a. 1/2 **b.** 1/4

c. 1/8 **d.** 1/16

114. A pair of unbiased dice is rolled together till a sum of either 5 or 7 is obtained. The probability that 5 comes before 7 is

a. 2/5 **b.** 3/5

c. 4/5 **d.** none of these

115. If n integers taken at random are multiplied together, then the probability that the last digit of the product is 1, 3, 7 or 9 is

a. $2^n/5^n$ **b.** $4^n - 2^n/5^n$

c. $4^n/5^n$ **d.** none of these

116. If A and B each toss three coins. The probability that both get the same number of heads is

a. 1/9 **b.** 3/16

c. 5/16 **d.** 3/8

117. Let A be a set containing n elements. A subset P of the set A is chosen at random. The set A is reconstructed by replacing the elements of P, and another subset Q of A is chosen at random. The probability that $P \cap Q$ contains exactly $m(m < n)$ elements is

a. $3^{n-m}/4^n$ **b.** ${}^nC_m \times 3^m/4^n$

c. ${}^nC_m \times 3^{n-m}/4^n$ **d.** none of these

118. A fair die is thrown 20 times. The probability that on the 10th throw, the fourth six appears is

a. ${}^{20}C_{10} \times 5^6/6^{20}$ **b.** $120 \times 5^7/6^{10}$

c. $84 \times 5^6/6^{10}$ **d.** none of these

119. If p is the probability that a man aged x will die in a year, then the probability that out of n men $A_1, A_2, \ldots, A_n$ each aged x, A_1 will die in an year and be the first to die is

a. $1-(1-p)^n$ **b.** $(1-p)^n$

c. $1/n\,[1-(1-p)^n]$ **d.** $1/n\,(1-p)^n$

120. A bag contains n white and n black balls. Pairs of balls are drawn without replacement until the bag is empty. The probability that each pair consists of one white and one black ball is

a. $1/{}^{2n}C_n$ **b.** $2n/{}^{2n}C_n$

c. $2n/n!$ **d.** $2n/(2n!)$

121. A man alternately tosses a coin and throws a die beginning with the coin. The probability that he gets a head in the coin before he gets a 5 or 6 in the dice is

a. 3/4 **b.** 1/2

c. 1/3 **d.** none of these

122. There are 3 bags which are known to contain 2 white and 3 black, 4 white and 1 black, and 3 white and 7 black balls, respectively. A ball is drawn at random from one of the bags and found to be a black ball. Then the probability that it was drawn from the bag containing the most black ball is

a. 7/15 **b.** 5/19

c. 3/4 **d.** None of these

123. Consider $f(x) = x^3 + ax^2 + bx + c$. Parameters a, b, c are chosen, respectively, by throwing a die three times. Then the probability that $f(x)$ is an increasing function is

a. 5/36 **b.** 8/36

c. 4/9 **d.** 1/3

124. A fair coin is tossed 10 times. Then the probability that two heads do not occur consecutively is

a. 7/64 **b.** 1/8

c. 9/16 **d.** 9/64

125. If a and b are chosen randomly from the set consisting of numbers 1, 2, 3, 4, 5, 6 with replacement. Then the probability that $\lim\limits_{x\to 0}[(a^x + b^x)/2]^{2/x} = 6$ is

a. 1/3 **c.** 1/4

c. 1/9 **d.** 2/9

126. An artillery target may be either at point I with probability 8/9 or at point II with probability 1/9. We have 55 shells, each of which can be fired either rat point I or II. Each shell may hit the target, independent of the other shells, with probability 1/2. Maximum number of shells must be fired at point I to have maximum probability is

a. 20 **b.** 25

c. 29 **d.** 35

127. An urn contains 3 red balls and n white balls. Mr. A draws two balls together from the urn. The probability that they have the same colour is 1/2. Mr. B draws one balls form the urn, notes its colour and replaces it. He then draws a second ball from the urn and finds that both balls have the same colour is 5/8. The possible value of n is

a. 9 **b.** 6

c. 5 **d.** 1

128. A hat contains a number of cards with 30% white on both sides, 50% black on one side and white on the other side, 20% black on both sides. The cards are mixed up, and a single card is drawn at random and placed on the table. Its upper side shows up black. The probability that its other side is also black is

a. 2/9 **b.** 4/9

c. 2/3 **d.** 2/7

129. All the jacks, queens, kings and aces of a regular 52 cards deck are taken out. The 16 cards are thoroughly shuffled and my opponent, a person who always tells the truth, simultaneously draws two cards at random and says, 'I hold at least one ace'. The probability that he holds two aces is

a. 2/8 b. 4/9
c. 2/3 d. 1/9

130. Mr. A lives at origin on the Cartesian plane and has his office at (4, 5). His friend lives at (2, 3) on the same plane. Mr. A can go to his office travelling one block at a time either in the $+y$ or $+x$ direction. If all possible paths are equally likely then the probability that Mr. A passed his friends house is (shortest path for any event must be considered)

a. 1/2 b. 10/21
c. 1/4 d. 11/21

Multiple Correct Answers Type

Solutions on page 9.60

Each question has four choices a, b, c and d, out of which *one or more* answers are correct.

1. If A and B are two independent events such that $P(A) = 1/2$ and $P(B) = 1/5$, then

a. $P(A \cup B) = 3/5$ b. $P(A/B) = 1/4$
c. $P(A/A \cup B) = 5/6$ d. $P(A \cap B/\bar{A} \cup \bar{B}) = 0$

2. Let A and B be two events such that $P(A \cap B') = 0.20$, $P(A' \cap B) = 0.15$ and $P(A$ and B both fail$) = 0.10$. Then

a. $P(A/B) = 2/7$ b. $P(A) = 0.3$
c. $P(A \cup B) = 0.55$ d. $P(B/A) = 1/2$

3. If A and B are two events such that $P(A) = 3/4$ and $P(B) = 5/8$, then

a. $P(A \cup B) \geq 3/4$ b. $P(A' \cap B) \leq 1/4$
c. $3/8 \leq P(A \cap B) \leq 5/8$ d. $3/8 \leq P(A \cap B) \leq 5/8$

4. If A and B are two mutually exclusive events, then

a. $P(A) \leq P(\bar{B})$ b. $P(A) > P(B)$
c. $P(B) \leq P(\bar{A})$ d. $P(A) > P(B)$

5. The probability that a married man watches a certain TV show is 0.4 and the probability that a married woman watches the show is 0.5. The probability that a man watches the show, given that his wife does, is 0.7. Then

a. the probability that married couple watchs the show is 0.35
b. the probability that a wife watches the show given that her husband does is 7/8
c. the probability that atleat one person of a married couple will watch the show is 0.55
d. none of these

6. A and B are two events defined as follows:
A: It rains today with $P(A) = 40\%$
B: It rains tomorrow with $P(B) = 50\%$
Also, P(it rains today and tomorrow) = 30%
Also, E_1: $P((A \cap B)/(A \cup B))$ and E_2: $P(\{(A \cap \bar{B})$or $(B \cup \bar{A})\}/(A \cup B))$. Then which of the following is/are true?

a. A and B are independent
b. $P(A/B) < P(B/A)$
c. E_1 and E_2 are equiprobable
d. $P(A/(A \cup B)) = P(B/(A \cup B))$

7. Two numbers are randomly selected and multiplied. Consider two events E_1 and E_2 defined as
E_1: Their product is divisible by 5
E_2: Unit's places in their product is 5
Which of the following statement is/are correct?

a. E_1 is twice as likely to occur as E_2
b. E_1 and E_2 are disjoint
c. $P(E_2/E_1) = 1/4$
d. $P(E_1/E_2) = 1$

8. Probability if n heads in $2n$ tosses of a fair coin can be given by

a. $\prod_{r=1}^{n}\left(\frac{2r-1}{2r}\right)$ b. $\prod_{r=1}^{n}\left(\frac{n+r}{2r}\right)$

c. $\sum_{r=0}^{n}\left(\frac{{}^nC_r}{2^n}\right)^2$ d. $\frac{\sum_{r=0}^{n}({}^nC_r)^2}{\left(\sum_{r=0}^{2n}{}^{2n}C_r\right)}$

9. The probability that a 50-year-old man will be alive at 60 is 0.83 and the probability that a 45-year-old woman will be alive at 55 is 0.87. Then

a. the probability that both will be alive is 0.7221
b. at least one of them will alive is 0.9779
c. at least one of them will alive is 0.8230
d. the probability that both will be alive is 0.6320

10. Which of the following statement is/are correct?

a. Three coins are tossed once. At least two of them must land the same way. No mater whether they land heads or tails, the third coin is equally likely to land either the same way or oppositely. So, the chance that all the three coins land the same way is 1/2.

b. Let $0 < P(B) < 1$ and $P(A/B) = P(A/B^C)$. Then A and B are independent.

c. Suppose an urn contains 'w' white and 'b' black balls and a ball is drawn from it and is replaced along with 'd' additional balls of the same colour. Now a second ball is drawn from it. The probability that the second drawn ball is white is independent of the value of 'd'.

d. A, B, C simultaneously satisfy
$P(ABC) = P(A)\, P(B)\, P(C)$
$P(AB\bar{C}) = P(A)\, P(B)\, P(\bar{C})$
$P(A\bar{B}C) = P(A)\, P(\bar{B})\, P(C)$
$P(A{-}BC) = P(\bar{A})\, P(B)\, P(C)$
Then A, B, C are independent.

11. A bag initially contains 1 red and 2 blue balls. An experiment consisting of selecting a ball at random, noting its colour and replacing it together with an additional ball of the same colour. If three such trials are made, then

a. probability that at least one blue ball is drawn is 0.9

b. probability that exactly one blue ball is drawn is 0.2

c. probability that all the drawn balls are red given that all the drawn balls are of same colour is 0.2

d. probability that atleast one red ball is drawn is 0.6

12. $P(A) = 3/8$; $P(B) = 1/2$; $P(A \cup B) = 5/8$, which of the following do/does hold good?

a. $P(A^C/B) = 2P(A/B^C)$ **b.** $P(B) = P(A/B)$

c. $15\ P(A^C/B^C) = 8P(B/A^C)$ **d.** $P(A/B^C) = (A \cap B)$

13. The chance of an event happening is the square of the chance of a second event but the odds against the first are the cube of the odds against the second. The chances of the events are

a. $p_1 = 1/9$ **b.** $p_1 = 1/16$

c. $p_2 = 1/3$ **d.** $p_2 = 1/4$

14. A bag contains b blue balls and r red balls. If two balls are drawn at random, the probability of drawing two red balls is five times the probability of drawing two blue balls. Furthermore, the probability of drawing one ball of each colour is six times the probability of drawing two blue balls. Then

a. $b + r = 9$ **b.** $br = 18$

c. $|b - r| = 4$ **d.** $b/r = 2$

15. In a precision bombing attack, there is a 50% chance that any one bomb will strike the target. Two direct hits are required to destroy the target completely. The number of bombs which should be dropped to give a 99% chance or better of completely destroying the target can be

a. 12 **b.** 11

c. 10 **d.** 13

16. If A and B are two events, the probability that exactly one of them occurs is given by

a. $P(A) + P(B) - 2P(A \cap B)$ **b.** $P(A \cap \bar{B}) + P(\bar{A} \cap B)$

c. $P(A \cup B) - P(A \cap B)$ **d.** $P(\bar{A}) + P(\bar{B}) - 2P(\bar{A} \cap \bar{B})$

17. If A and B are two events, then which one of the following is/are always true?

a. $P(A \cap B) \geq P(A) + P(B) - 1$

b. $P(A \cap B) \leq P(A)$

c. $P(A' \cap B') \geq P(A') + P(B') - 1$

d. $P(A \cap B) = P(A)\ P(B)$

18. If A and B are two independent events such that $P(A) = 1/2$, $P(B) = 1/5$, then

a. $P(A/B) = 1/2$ **b.** $P\left(\dfrac{A}{A \cup B}\right) = \dfrac{5}{6}$

c. $P\left(\dfrac{A \cap B}{A' \cup B'}\right) = 0$ **d.** none of these

19. If A and B are two independent events such that $P(\bar{A} \cap B) = 2/15$ and $P(A \cap \bar{B}) = 1/6$, then $P(B)$ is

a. 1/5 **b.** 1/6

c. 4/5 **d.** 5/6

20. Two buses A and B are scheduled to arrive at a town central bus station at noon. The probability that bus A will be late is 1/5. The probability that bus B will be late is 7/25. The probability that the bus B is late given that bus A is late is 9/10. Then,

a. probability that neither bus will be late on a particular day is 7/10

b. probability that bus A is late given that bus B is late is 18/28

c. probability that at least one bus is late is 3/10

d. probability that at least one bus is in time is 4/5

21. If p and q are chosen randomly from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} with replacement, then the probability that the roots of the equation $x^2 + px + q = 0$

a. are real is 33/50 **b.** are imaginary is 19/50

c. are real and equal is 3/50 **d.** are real and distinct is 3/5

22. Two numbers are chosen from {1, 2, 3, 4, 5, 6, 7, 8} one after another without replacement. Then the probability that

a. the smaller value of two is less than 3 is 13/28

b. the bigger value of two is more than 5 is 9/14

c. product of two number is even is 11/14

d. none of these

23. A fair coin is tossed 99 times. Let X be the number of times heads occurs. Then $P(X = r)$ is maximum when r is

a. 49 **b.** 52

c. 51 **d.** 50

24. If the probability of choosing an integer 'k' out of $2m$ integers 1, 2, 3, …, $2m$ is inversely proportional to k^4 $(1 \leq k \leq m)$. If x_1 is the probability that chosen number is odd and x_2 is the probability that chosen number is even, then

a. $x_1 > 1/2$ **b.** $x_1 > 2/3$

c. $x_2 < 1/2$ **d.** $x_2 < 2/3$

Reasoning Type

Solutions on page 9.64

Each question has four choices a, b, c, and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1.

b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1.

c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE.

d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE.

1. Statement 1: For events A and B of sample space if $P\left(\dfrac{A}{B}\right) \geq P(A)$, then $P\left(\dfrac{B}{A}\right) \geq P(B)$.

Statement 2: $P\left(\dfrac{A}{B}\right) = \dfrac{P(A \cap B)}{P(B)}$ $(P(B) \neq 0)$.

2. A fair die is rolled once.
   Statement 1: The probability of getting a composite number is 1/3.
   Statement 2: There are three possibilities for the obtained number: (i) the number is a prime number, (ii) the number is a composite number and (iii) the number is 1. Hence probability of getting a prime number is 1/3.
3. **Statement 1:** If $P(A) = 0.25$, $P(B) = 0.50$ and $P(A \cap B) = 0.14$, then the probability that neither A nor B occurs is 0.39.
   Statement 2: $\overline{(A \cup B)} = \bar{A} \cup \bar{B}$.
4. Let A and B be two events such that $P(A) = 3/5$ and $P(B) = 2/3$. Then
   Statement 1: $4/15 \le P(A \cap B) \le 3/5$.
   Statement 2: $2/5 \le P(A/B)) \le 9/10$.
5. **Statement 1:** If A, B, C be three mutually independent events, then A and $B \cup C$ are also independent events.
   Statement 2: Two events A and B are independent if and only if $P(A \cap B) = P(A)\, P(B)$.
6. **Statement 1:** Out of 5 tickets consecutively numbered, three are drawn at random. The chance that the numbers on them are in A.P. is 2/15.
   Statement 2: Out of $2n + 1$ tickets consecutively numbered, three are drawn at random, the chance that the numbers on them are in A.P. is $3n/(4n^2 - 1)$.
7. Let A and B be two event such that $P(A \cup B) \ge 3/4$ and $1/8 \le P(A \cap B) \le 3/8$.
   Statement 1: $P(A) + P(B) \ge 7/8$.
   Statement 2: $P(A) + P(B) \le 11/8$.
8. **Statement 1:** The probability of drawing either an ace or a king from a pack of card in a single draw is 2/13.
   Statement 2: For two events A and B which are not mutually exclusive, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
9. Let A and B be two independent events.
   Statement 1: If $P(A) = 0.4$ and then $P(A \cup \bar{B}) = 0.9$, then $P(B)$ is 1/6.
   Statement 2: If A and B are independent, then $P(A \cap B) = P(A)P(B)$.
10. Consider an event for which probability of success is 1/2.
    Statement 1: Probability that in n trials, there are r successes where $r = 4k$ and k is an integer is
    $$\frac{1}{4} + \frac{1}{2^{n/2+1}}\cos\left(\frac{n\pi}{4}\right)$$
    Statement 2: ${}^nC_0 + {}^nC_4 + {}^nC_8 + \cdots = 2^{n/2} \sin\left(\frac{n\pi}{4}\right)$.
11. **Statement 1:** If A and B are two events such that $0 < P(A)$, $P(B) < 1$, then $P(A/\bar{B}) + P(\bar{A}/\bar{B}) = 3/2$.
    Statement 2: If A and B are two events such that $0 < P(A)$, $P(B) < 1$, then
    $P(A/B) = P(A \cap B)/P(B)$ and $P(\bar{B}) = P(A \cap \bar{B}) + P(\bar{A} \cap \bar{B})$
12. **Statement 1:** If a fair coin is tossed 15 times, then the probability of getting head as many times in the first ten throws as in the last five is 3003/32768.
    Statement 2: Sum of the series ${}^mC_r\,{}^nC_0 + {}^mC_{r-1}\,{}^nC_1 + \cdots + {}^mC_0\,{}^nC_r = {}^{m+n}C_r$.
13. **Statement 1:** If $A = \{2, 4, 6\}$, $B = \{1, 2, 3\}$ where A and B are the events of numbers occurring on a dice, then $P(A) + P(B) = 1$.
    Statement 2: If $A_1, A_2, A_3, \ldots, A_n$ are all mutually exclusive events, then $P(A_1) + P(A_2) + \cdots + P(A_n) = 1$.
14. **Statement 1:** There are 4 addressed envelopes and 4 letters for each one of them. The probability that no letter is mailed in its correct envelops is 3/8.
    Statement 2: The probability that all letters are not mailed in their correct envelope is 23/24.
15. Let A and B be two independent events.
    Statement 1: If $P(A) = 0.3$ and $P(A \cup \bar{B}) = 0.8$, then $P(B)$ is 2/7.
    Statement 2: $P(\bar{E}) = 1 - P(E)$, where E is any event.

Linked Comprehension Type

Solutions on page 9.66

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

In a class of 10 students, probability of exactly i students passing an examination is directly proportional to i^2. Then answer the following questions:

1. The probability that exactly 5 students passing an examination is
   a. 1/11 **b.** 5/77
   c. 25/77 **d.** 10/77
2. If a student is selected at random, then the probability that he has passed the examination is
   a. 1/7 **b.** 11/35
   c. 11/14 **d.** none of these
3. If a students selected at random is found to have passed the examination, then the probability that he was the only student who has passed the examination is
   a. 1/3025 **b.** 1/605
   c. 1/275 **d.** 1/121

For Problems 4–6

A shoping mall is running a scheme: 'Each packet of detergent "SURF" contains a coupon which bears letter of the word "SURF", if a person buys at least four packets of detergent "SURF", and produce all the letters of the word "SURF", then he gets one free packet of detergent.

4. If a person buys 8 such packets at a time, then number of different combinations of coupon he has
   a. 4^8 **b.** 8^4
   c. ${}^{11}C_3$ **d.** ${}^{12}C_4$

5. If person buys 8 such packets, then the probability that he gets exactly one free packets is

a. 7/33
b. 102/495
c. 13/55
d. none of these

6. If a person buys 8 such packets, then the probability that he gets two free packets is

a. 1/7
b. 1/5
c. 1/42
d. 1/165

For Problems 7–9

In an objective paper, there are two sections of 10 questions each. For 'section 1', each question has 5 options and only one option is correct and 'section 2' has 4 options with multiple answers and marks for a question in this section is awarded only if he ticks all correct answers. Marks for each question in 'section 1' is 1 and in 'section 2' is 3. (There is no negative marking.)

7. If a candidate attempts only two questions by guessing, one from 'section 1' and one from 'section 2', the probability that he scores in both questions is

a. 74/75
b. 1/25
c. 1/15
d. 1/75

8. If a candidate in total attempts 4 questions all by guessing, then the probability of scoring 10 marks is

a. $1/15(1/15)^3$
b. $4/5(1/15)^3$
c. $1/5(14/15)^3$
d. none of these

9. The probability of getting a score less than 40 by answering all the questions by guessing in this paper is

a. $(1/75)^{10}$
b. $1-(1/75)^{10}$
c. $(74/75)^{10}$
d. none of these

For Problems 10–12

There are two die A and B both having six faces. Die A has three faces marked with 1, two faces marked with 2 and one face marked with 3. Die B has one face marked with 1, two faces marked with 2 and three faces marked with 3. Both dices are thrown randomly once. If E be the event of getting sum of the numbers appearing on top faces equal to x and let $P(E)$ be the probability of even E, then

10. $P(E)$ is maximum when x equal to

a. 5
b. 3
c. 4
d. 6

11. $P(E)$ is minimum when x equals to

a. 3
b. 4
c. 5
d. 6

12. When $x = 4$, then $P(E)$ is equal to

a. 5/9
b. 6/7
c. 7/18
d. 8/19

For Problems 13–15

A JEE aspirant estimates that she will be successful with an 80% chance if she studies 10 hours per day, with a 60% chance if she studies 7 hours per day and with a 40% chance if she studies 4 hours per day. She further believes that she will study 10 hours, 7 hours and 4 hours per day with probabilities 0.1, 0.2 and 0.7, respectively.

13. The chance she will be successful is

a. 0.28
b. 0.38
c. 0.48
d. 0.58

14. Given that she is successful, the chance that she studied for 4 hours is

a. 6/12
b. 7/12
c. 8/12
d. 9/12

15. Given that she does not achieve success, the chance she studied for 4 hour is

a. 18/26
b. 19/26
c. 20/26
d. 21/26

For Problems 16–18

Let S and T are two events defined on a sample space with probabilities $P(S) = 0.5$, $P(T) = 0.69$, $P(S/T) = 0.5$

16. Events S and T are

a. mutually exclusive
b. independent
c. mutually exclusive and independent
d. neither mutually exclusive nor independent

17. The value of $P(S \text{ and } T)$ is

a. 0.3450
b. 0.2500
c. 0.6900
d. 0.350

18. The value of $P(S \text{ or } T)$ is

a. 0.6900
b. 1.19
c. 0.8450
d. 0

For Problems 19–21

An amoeba either splits into two or remains the same or eventually dies out immediately after completion of every second with probabilities, respectively, 1/2, 1/4 and 1/4. Let the initial amoeba be called as mother amoeba and after every second, the amoeba, if it is distinct from the previous one, be called as 2nd, 3rd, ... generations.

19. The probability that immediately after completion of 2 s all the amoeba population dies out is

a. 9/32
b. 11/32
c. 1/2
d. 3/32

20. The probability that after 2 s exactly 4 amoeba are alive is

a. 1/16
b. 1/8
c. 1/4
d. 1/2

21. The probability that amoeba population will be maximum after completion of 3 s is

a. $1/2^7$
b. $1/2^6$
c. $1/2^8$
d. none of these

For Problems 22–24

A cube having all of its sides painted is cut by two horizontal, two vertical and other two planes so as to form 27 cubes all having the same dimensions. Of these cubes, a cube is selected at random.

22. The probability that the cube selected has none of its sides painted is

a. 1/9 b. 1/27
c. 1/18 d. 5/54

23. The probability that the cube selected has two sides painted is

a. 1/9 b. 4/9
c. 8/27 d. none of these

24. The total number of cubes having at least one of its sides painted is

a. 8 b. 53
c. 49 d. 26

For Problems 25–27

There are some experiments in which the outcomes cannot be identified discretely. For example, an ellipse of eccentricity $2\sqrt{2}/3$ is inscribed in a circle and a point within the circle is chosen at random. Now, we want to find the probability that this point lies outside the ellipse. Then, the point must lie in the shaded region shown in Fig. 9.8. Let the radius of the circle be a and length of minor axis of the ellipse be $2b$. Given that

$$1-\frac{b^2}{a^2}=\frac{8}{9} \Rightarrow \frac{b^2}{a^2}=\frac{1}{9}$$

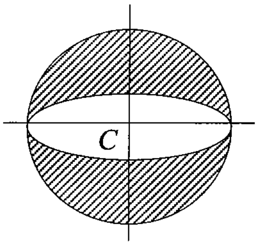

Fig. 9.8

Then, the area of circle serves as sample space and area of the shaded region represents the area for favourable cases. Then, required probability is

$$p=\frac{\text{area of shaded region}}{\text{area of circle}}$$
$$=\frac{\pi a^2-\pi ab}{\pi a^2}$$
$$=1-\frac{b}{a}$$
$$=1-\frac{b}{a}$$
$$=1-\frac{1}{3}$$
$$=\frac{2}{3}$$

Now answer the following questions.

25. A point is selected at random inside a circle. The probability that the point is closer to the centre of the circle than to its circumference is

a. 1/4 b. 1/2
c. 1/3 d. $1/\sqrt{2}$

26. Two persons A and B agree to meet at a place between 5 and 6 pm. The first one to arrive waits for 20 min and then leave. If the time of their arrival be independent and at random, then the probability that A and B meet is

a. 1/3 b. 1/3
c. 2/3 d. 5/9

27. If points x, y are chosen randomly from the intervals [0, 2] and [0, 1], respectively, then the probability that $y \le x^2$ is

a. 1/2 b. 2/3
c. 3/4 d. 1/4

For Problems 28–30

If the squares of a 8 × 8 chessboard are painted either red or black at random.

28. The probability that not all the squares in any column are alternating in colour is

a. $(1-1/2^7)^8$ b. $1/2^{56}$
c. $1-1/2^7$ d. none of these

29. The probability that the chessboard contains equal number of red and black squares is

a. $\frac{^{64}C_{32}}{2^{64}}$ b. $\frac{64!}{322^{64}!}$
c. $\frac{2^{32}-1}{2^{64}}$ d. none of these

30. The probability that all the squares in any column are of same colour and that of a row are of alternating colour is

a. $1/2^{64}$ b. $1/2^{63}$
c. 1/2 d. none of these

For Problems 31–33

Two fair dice are rolled. Let $P(A_i) > 0$ denote the event that the sum of the faces of the dice is divisible by i.

31. Which one of the following events is most probable?

a. A_3 b. A_4
c. A_5 d. A_6

32. For which one of the following pairs (i, j) are the events A_i and A_j independent?

a. (3, 4) b. (4, 6)
c. (2, 3) d. (4, 2)

33. The number of all possible ordered pairs (i, j) for which the events A_i and A_j are independent is

a. 6 b. 12
c. 13 d. 25

For Problems 34–36

A player tosses a coin and scores one point for every head and two points for every tail that turns up. He plays on until his score reaches or passes n. P_n denotes the probability of getting a score of exactly n.

34. The value of P_n is equal to

a. $(1/2)[P_{n-1}+P_{n-2}]$ b. $(1/2)[2P_{n-1}+P_{n-2}]$
c. $(1/2)[P_{n-1}+2P_{n-2}]$ d. none of these

35. The value of $P_n + (1/2)\, P_{n-1}$ is equal to

a. 1/2 b. 2/3
c. 1 d. none of these

36. Which of the following is not true?

a. $P_{100} > 2/3$ b. $P_{101} < 2/3$
c. $P_{100}, P_{101} > 2/3$ d. none of these

For Problems 37–39

The probability that a family has exactly n children is αp^n, $n \geq 1$. All sex distributions of n children in a family have the same probability.

37. The probability that a family contains exactly k boys is (where $k \geq 1$)

a. $\alpha p^k(1-p)^{-k-1}$ b. $2\alpha p^k(2-p)^{-k-1}$
c. $2\alpha p^k(2-p)^{-k}$ d. $2\alpha p^{k-1}(2-p)^{-k-1}$

38. The probability that a family includes at least one boy is

a. $\dfrac{\alpha^2 p}{(2-p)(1-p)}$ b. $\dfrac{\alpha p^2}{(2-p)(1-p)}$
c. $\dfrac{\alpha p}{(2-p)(1-p)}$ d. $\dfrac{2\alpha p}{(2-p)(1-p)}$

39. Given that a family includes at least one boy, show that the probability that there are two or more boys is

a. $p/(2-p)$ b. $p/(1-p)$
c. $p/(2-p)^2$ d. $p/(1-p)^2$

Matrix-Match Type

Solutions on page 9.69

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct matches are a→p, a→s, b→q, b→r, c→p, c→q and d→s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	(p)	(q)	(r)	(s)
b	(p)	(q)	(r)	(s)
c	(p)	(q)	(r)	(s)
d	(p)	(q)	(r)	(s)

1. An urn contains four black and eight white balls. Three balls are drawn from the urn without replacement. Three events are defined on this experiment.

A: Exactly one black ball is drawn.

B: All balls are drawn are of the same colour.

C: Third drawn ball is black.

Match the entries of column I with none, one or more entries of column II.

Column I	Column II
a. The events A and B are	p. mutually exclusive
b. The events B and C are	q. independent
c. The events C and A are	r. neither independent nor mutually exclusive
d. The events A, B and C are	s. exhaustive

2.

Column I	Column II
a. If the probability of getting at least one head is at least 0.8 is n trials then value of n can be	p. 2
b. One mapping is selected at random from all mappings of the set $s = \{1, 2, 3, \ldots, n\}$ into itself. If the probability that the mapping being one-one is 3/32, then find the value of n is	q. 3
c. If m is selected at random from set $\{1, 2, \ldots, 10\}$ and the probability that the quadratic equation $2x^2 + 2mx + m + 1 = 0$ has real roots is k, then value of $5k$ is more than	r. 4
d. A man firing at a distant target as 20% chance of hitting the target in one shoot. If P be the probability of hitting the target in 'n' attempts, where $20P^2 - 13P + 2 \leq 0$, then the ratio of maximum and minimum value of n is less than	s. 5

3.

Column I	Column II
a. The probability of a bomb hitting a bridge is 1/2. Two direct hits are needed to destroy it. The number of bombs required so that the probability of the bridge being destroyed is greater than 0.9 can be	p. 4
b. A bag contains 2 red, 3 white and 5 black balls, a ball is drawn its colour is noted and replaced. The number of times, a ball can be drawn so that the probability of getting a red ball for the first time is at least 1/2	q. 6
c. A drawer contains a mixture of red socks and blue socks, at most 17 in all. It so happens that when two socks are selected randomly without replacement, there is a probability of exactly 1/2 that both are red or both are blue. Then number of red socks in the drawer can be	r. 7
d. There are two red, two blue, two white and certain number (greater than 0) of green socks in a drawer. If two socks are taken at random from the drawer without replacement, the probability that they are of the same colour is 1/5, then the number of green socks are	s. 10

4. Let A and B are two independent events such that $P(A) = 1/3$ and $P(B) = 1/4$.

Column I	Column II
a. $P(A \cup B)$ is equal to	p. 1/12
b. $P(A/A \cup B)$ is equal to	q. 1/2
c. $P(B/A' \cap B')$ is equal to	r. 2/3
d. $P(A'/B)$ is equal to	s. 0

5. A bag contains some white and some black balls, all combinations being equally likely. The total number of balls in the bag is 12. Four balls are drawn at random from the bag at random without replacement.

Column I	Column II
a. Probability that all the four balls are black is equal to	**p.** 14/33
b. If the bag contains 10 black and 2 white balls then the probability that all four balls are black is equal to	**q.** 1/3
c. If all the four balls are black, then the probability that the bag contains 10 black balls is equal to	**r.** 70/429
d. Probability that two balls are black and two are white is	**s.** 13/165

6.

Column I	Column II
a. Six different balls are put in three different boxes, none being empty. The probability of putting the balls equal number is	**p.** 20/27
b. Six letters are posted in three letter boxes. The probability that no letter box remains empty is	**q.** 1/6
c. Two persons A and B throw two dice each. If A throw a sum of 9, then the probability of B throwing a sum greater than A is	**r.** 1/3
d. If A and B are independent and $P(A) = 0.3$ and $P(A \cup \bar{B}) = 0.8$, then $P(B)$ is equal to	**s.** 2/7

7. An urn contains r red balls and b black balls.

Column I	Column II
a. If the probability of getting two red balls in first two draws (without replacement) is 1/2, then value of r can be	**p.** 10
b. If the probability of getting two red balls in first two draws (without replacement) is 1/2 and b is an even number, then r can be	**q.** 3
c. If the probability of getting exactly two red balls in four draws (with replacement) from the urn is 3/8 and $b = 10$, then r can be	**r.** 8
d. If the probability of getting exactly n red balls in $2n$ draw (with replacement) is equal to probability of getting exactly n black balls in $2n$ draws (with replacement), then the ratio r/b can be	**s.** 2

8. 'n' whole numbers are randomly chosen and multiplied,

Column I	Column II
a. The probability that the last digit is 1, 3, 7 or 9 is	**p.** $\frac{8^n - 4^n}{10^n}$
b. The probability that the last digit is 2, 4, 6, 8 is	**q.** $\frac{5^n - 4^n}{10^n}$
c. The probability that the last digit is 5 is	**r.** $\frac{4^n}{10^n}$
d. The probability that the last digit is zero is	**s.** $\frac{10^n - 8^n - 5^n + 4^n}{10^n}$

Integer Type

Solutions on page 9.72

1. If the probability of a six-digit number N whose six digits are 1, 2, 3, 4, 5, 6 written as random order is divisible by 6 is p, then the value of $1/p$ is.
2. If the probability that the product of the outcomes of three rolls of a fair dice is a prime number is p, then the value of $1/(4p)$ is.
3. If two loaded dice each have the property that 2 or 4 is three times as likely to appear as 1, 3, 5 or 6 on each roll. When two such dice are rolled, the probability of obtaining a total of 7 is p, then the value of $[1/p]$ is, where $[x]$ represents the greatest integer less than or equal to x.
4. An urn contains three red balls and n white balls. Mr. A draws two balls together from the urn. The probability that they have the same colour is 1/2. Mr. B draws one balls from the urn, notes its colour and replaces it. He then draws a second ball from the urn and finds that both balls have the same colour is. 5/8. The possible value of n is.
5. Suppose A and B are two events with $P(A) = 0.5$ and $P(A \cup B) = 0.8$. Let $P(B) = p$ if A and B are mutually exclusive and $P(B) = q$ if A and B are independent events, then the value of q/p is.
6. There are two red, two blue, two white and certain number (greater than 0) of green socks in a drawer. If two socks are taken at random from the drawer without replacement, the probability that they are of the same colour is 1/5, then the number of green socks are.
7. A drawer contains a mixture of red socks and blue socks, at most 17 in all. It so happens that when two socks are selected randomly without replacement, there is a probability of exactly 1/2 that both are red or both are blue. The largest possible number of red socks in the drawer that is consistent with this data is.
8. Two different numbers are taken from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. The probability that their sum and positive difference are both multiple of 4 is $x/55$, then x equals.
9. Thirty-two players ranked 1 to 32 are playing in a knockout tournament. Assume that in every match between any two players the better ranked player wins, the probability that ranked 1 and ranked 2 players are winner and runner up

respectively is p, then the value of $[2/p]$ is, where [.] represents the greatest integer function

10. Five different games are to be distributed among four children randomly. The probability that each child get at least one game is p, then the value of $[1/p]$ is, where [.] represents the greatest integer function

11. A die is weighted such that the probability of rolling the face numbered n is proportional to n^2 ($n = 1, 2, 3, 4, 5, 6$). The die is rolled twice, yielding the numbers a and b. The probability that $a < b$ is p, then the value of $[2/p]$ is, where [.] represents the greatest integer function.

12. A bag contains 10 different balls. Five balls are drawn simultaneously and then replaced and then seven balls are drawn. The probability that exactly three balls are common to the two drawn is p, then the value of $12p$ is.

13. If A and B are two events such that $P(A) = 0.6$ and $P(B) = 0.8$, if the greatest value that $P(A/B)$ can have is p, then the value of $8p$ is.

14. A die is thrown three times. The chance that the highest number shown on the die is 4 is p, then the value of $[1/p]$ is where [.] represents greatest integer function is.

15. Two cards are drawn from a well shuffled pack of 52 cards. The probability that one is a heart card and the other is a king is p, then the value of $104p$ is.

Archives

Solutions on page 9.73

Subjective Type

1. Balls are drawn one-by-one without replacement from a box containing 2 black, 4 white and 3 red balls till all the balls are drawn. Find the probability that the balls drawn are in the order 2 black, 4 white and 3 red. **(IIT-JEE, 1978)**

2. Six boys and six girls sit in a row randomly. Find the probability that (i) the six girls sit together, (ii) the boys and girls sit alternately. **(IIT-JEE, 1979)**

3. An anti-aircraft gun can take a maximum of four shots at an enemy plane moving away from it. The probabilities of hitting the plane at the first, second, third and fourth shot are 0.4, 0.3, 0.2 and 0.1, respectively. What is the probability that the gun hits the plane? **(IIT-JEE, 1981)**

4. A and B are two candidates seeking admission in IIT. The probability that A is selected is 0.5 and the probability that A and B are selected is at most 0.3. Is it possible that the probability of B getting selected is 0.9? **(IIT-JEE, 1982)**

5. Cards are drawn one at random from a well-shuffled full pack of 52 playing cards until 2 aces are obtained for the first time. If N is the number of cards required to the drawn, then show that

$$P_r\{N = n\} = \frac{(n-1)(52-n)(51-n)}{50 \times 49 \times 17 \times 13}, \text{ where } 2 < n < 50$$

(IIT-JEE, 1983)

6. Let A, B, C be three events such that $P(A) = 0.3$, $P(B) = 0.4$, $P(C) = 0.8$, $P(A \cap B) = 0.88$, $P(A \cap C) = 0.28$, $P(A \cap B \cap C) = 0.09$. If $P(A \cup B \cup C) \geq 0.75$, then show that $0.23 \leq P(B \cap C) \leq 0.48$.

7. A and B are two independent events. The probability that both A and B occur is 1/6 and the probability that neither of them occurs is 1/3. Find the probability of the occurrence of A. **(IIT-JEE, 1984)**

8. In a certain city, only 2 newspapers A and B are published. It is known that 25% of the city population read A and 20% read B while 8% reads both A and B. It is also known that 30% of those who read A but not B look into advertisement and 40% of those who read B but not A look into advertisements while 50% of those who read both A and B look into advertisements. What is the percentage of the population who read an advertisement? **(IIT-JEE, 1984)**

9. In a multiple choice question, there are four alternative answers of which one or more than one is correct. A candidate will get marks on the question only if he ticks the correct answer. The candidate decides to tick answers at random. If he is allowed up to three chances to answer the question, then find the probability that he will get marks on it. **(IIT-JEE, 1985)**

10. A lot contains 20 articles. The probability that the lot contains exactly two defective articles is 0.4 and the probability that it contains exactly 3 defective articles is 0.6. Articles are drawn from the lot at random one by one without replacement and are tested till all defective articles are found. Then find the probability that the testing procedure ends at the twelfth testing. **(IIT-JEE, 1985)**

11. A man takes a step forward with probability 0.4 and backward with probability 0.6. Then find the probability that at the end of eleven steps he is one step away from the starting point. **(IIT-JEE, 1987)**

12. An urn contains 2 white and 2 black balls. A ball is drawn at random. If it is white, it is not replaced into urn otherwise it is replaced along with another ball of the same colour. The process is repeated. Then find the probability that the third ball drawn is black. **(IIT-JEE, 1987)**

13. A box contains two 50-paise coins, five 25-paise coins and a certain fixed number $N(\geq 2)$ of 10 and 5-paise coins. Five coins are taken out of the box at random. Find the probability that the total value of these 5 coins is less than 1 rupee and 50 paise. **(IIT-JEE, 1988)**

14. Suppose the probability for A to win a game against B is 0.4. If A has an option of playing either a 'best of 3 games' or a 'best of 5 games' match against B, which option should be chosen so that the probability of his winning the match is higher? (No game ends in a draw.) **(IIT-JEE, 1989)**

15. A is a set containing n elements. A subset P of A is chosen at random. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen at random. Find the probability that P and Q have no common elements. **(IIT-JEE, 1990)**

16. In a test, an examinee either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he makes a guess is 1/3 and the probability that he copies the answer is 1/6. The probability that his answer is correct given that he copied it is 1/8. Find the probability that he knew the answer to the question, given that he correctly answered it.

17. A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random, one at a time, with replacement. The events A, B, C are defined as follows:

A: the first bulb is detective

B: the second bulb is non-defective

C: the two bulbs are both defective or both non-defective

Determine whether

(i) A, B, C are pair-wise independent

(ii) A, B, C are independent **(IIT-JEE, 1992)**

18. Numbers are selected at random, one at a time, from the two-digit numbers 00, 01, 02 ..., 99 with replacement. An event E occurs if the only product of the two digits of a selected number is 18. If four numbers are selected, find the probability that the event E occurs at least 3 times. **(IIT-JEE, 1993)**

19. An unbiased coin is tossed. If the result is a head, a pair of unbiased dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well-shuffled pack of 11 cards numbered 2, 3, 4, ...,12 is picked and the number on the card is noted. What is the probability that the noted number is either 7 or 8? **(IIT-JEE, 1994)**

20. In how many ways three girls and nine boys can be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls should sit together in a back row on adjacent seats? Now, if all the seating arrangements are equally likely, what is the probability of 3 girls sitting together in a back row on adjacent seats? **(IIT-JEE, 1996)**

21. Sixteen players S_1, S_2, ..., S_{16} play in a tournament. They are divided into eight pairs at random. From each pair a winner is decided on the basis of a game played between the two players of the pair. Assume that all the players are of equal strength. **(IIT-JEE, 1997)**

a. Find the probability that the player S_1 is among the eight winners.

b. Find the probability that exactly one of the two players S_1 and S_2 is among the eight winners.

22. If p and q are chosen randomly from the set {1, 2, 3, 4,5, 6, 7, 8, 9, 10} with replacement, determine the probability that the roots of the equation $x^2 + px + q = 0$ are real. **(IIT-JEE, 1997)**

23. Three players, A, B and C, toss a coin cyclically in that order (i.e., is A, B, C, A, B, C, A, B, ...) till a head shows. Let p be the probability that the coin shows a head. Let α, β and γ be, respectively, the probabilities that A, B and C gets the first head. Prove that $\beta = (1 - p)\alpha$. Determine α, β and γ (in terms of p). **(IIT-JEE, 1998)**

24. Eight players P_1, P_2, ..., P_8 play a knock-out tournament. It is known that whenever the players P_i and P_j play, the player P_i will win if $i < j$. Assuming that the players are paired at random in each round, what is the probability that the player P_4 reaches the final? **(IIT-JEE, 1999)**

25. A coin has probability p of showing head when tossed. It is tossed n times. Let P_n denote the probability that no two (or more) consecutive heads occur. Prove that $p_1 = 1$, $p_2 = 1 - p^2$ and $p_n = (1 - p)\, p_{n-1} + p(1 - p)\, p_{n-2}$ for all $n \geq 3$. **(IIT-JEE, 2000)**

26. An urn contains m white and n black balls. A ball is drawn at random and is put back into the urn along with k additional balls of the same colour as that of the ball drawn. A ball is again drawn at random. What is the probability that the ball drawn now is white? **(IIT-JEE, 2001)**

27. An unbiased die, with faces numbered 1, 2, 3, 4, 5, 6 is thrown n times and the list of n numbers showing up is noted. What is the probability that, among the numbers 1, 2, 3, 4, 5, 6, only three numbers appear in this list? **(IIT-JEE, 2001)**

28. A box contains N coins, m of which are fair and the rest are biased. The probability of getting a head when a fair coin is tossed is 1/2, while it is 2/3 when a biased coin is tossed. A coin is drawn from the box at random and is tossed twice. The first time it shows head and the second time it shows tail. What is the probability that the coin drawn is fair? **(IIT-JEE, 2002)**

29. For a student to qualify, he must pass at least two out of three exams. The probability that he will pass the first exam is p. If he fails in one of the exams then the probability of his passing in the next exam is $p/2$, otherwise it remains the same. Find the probability that he will qualify. **(IIT-JEE, 2003)**

30. A is targeting to B, B and C are targeting to A. Probability of hitting the target by A, B and C are 2/3, 1.2 and 1/3, respectively. If A is hit, then find the probability that B hits the target and C does not. **(IIT-JEE, 2003)**

31. A and B are two independent events. C is an event in which exactly one of A or B occurs. Prove that $P(C) \geq P(A \cup B)$ $P(A \cap B)$. **(IIT-JEE, 2004)**

32. A box contains 12 red and 6 white balls. Balls are drawn from the bag one at a time without replacement. If in 6 draws, there are at least 4 white balls, find the probability that exactly one white ball is drawn in the next two draws. (Binomial coefficients can be left as such.) **(IIT-JEE, 2004)**

33. A person goes to office either by car, scooter, bus or train, the probability of which being 1/7, 3/7, 2/7 and 1/7, respectively. Probability that he reaches office late, if he takes car, scooter, bus or train is 2/9, 1/9, 4/9 and 1/9 respectively. Given that he reached office in time, then what is the probability that he travelled by a car. **(IIT-JEE, 2005)**

Objective Type

Fill in the blanks

1. For a biased die, the probability for the different face to turn up are given below:

Face	1	2	3	4	5	6
Probability	0.1	0.32	0.21	0.15	0.05	0.17

This die is tossed and you are told that either face 1 or face 2 has turned up. Then the probability that it is face 1 is ______. **(IIT-JEE, 1981)**

2. $P(A \cup B) = P(A \cap B)$ if and only if the relation between $P(A)$ and $P(B)$ is ______. **(IIT-JEE, 1985)**

3. A box contains 100 tickets numbered 1, 2, ..., 100. Two tickets are chosen at random. It is given that the maximum number on the two chosen tickets is not more than 10. The minimum number on them is 5 with probability ______. **(IIT-JEE, 1985)**

4. If $(1 + 3p)/3$, $(1 - p)/4$ and $(1 - 2p)/2$ are the probabilities of three mutually exclusive events, then the set of all values of p is ______. **(IIT-JEE, 1986)**

5. Urn A contains 6 red and 4 black balls and urn B contains 4 red and 6 black balls. One ball is drawn at random from urn P and placed in urn Q. Then one ball is drawn at random from urn Q and placed in urn A. If one ball is now drawn at random from urn P, the probability that it is found to be red is ______. **(IIT-JEE, 1988)**

6. A pair of fair dice is rolled together till a sum of either 5 or 7 is obtained. Then the probability that 5 comes before 7 is ______. **(IIT-JEE, 1989)**

7. Let A and B be two events such that $P(A) = 0.3$ and $P(A \cup B) = 0.8$. If A and B are independent events, then $P(B) =$ ______. **(IIT-JEE, 1990)**

8. Three faces of a fair die are yellow, two faces red and one blue. The die is tossed three times. The probability that the colours, yellow, red and blue, appear in the first, second and the third tosses, respectively, is ______. **(IIT-JEE, 1992)**

9. If two events A and B are such that $P(A^C) = 0.3$, $P(B) = 0.4$ and $P(A \cap B^C) = 0.5$, then $P[B/(A \cup B^C)] =$ ______. **(IIT-JEE, 1994)**

10. Three numbers are chosen at random without replacement from {1, 2, ..., 10}. The probability that the minimum of the chosen numbers is 3, or their maximum is 7, is ______. **(IIT-JEE, 1997)**

True or false

1. If the letters of the word 'ASSASSIN' are written down at random in a row, the probability that no two S's occur together is 1/35. **(IIT-JEE, 1983)**

2. If the probability for A to fail in an examination is 0.2 and that for B is 0.3, then the probability that either A or B fails is 0.5. **(IIT-JEE, 1989)**

Multiple choice questions with one correct answer

1. Two fair dice are tossed. Let x be the event that the first die shows an even number and y be the event that the second die shows an odd number. The two events x and y are
   a. mutually exclusive
   b. independent and mutually exclusive
   c. dependent
   d. none of these **(IIT-JEE, 1979)**

2. Two events A and B have probabilities 0.25 and 0.50, respectively. The probability that both A and B occur simultaneously is 0.14. Then the probability that neither A nor B occurs is
   a. 0.39 **b.** 0.25
   c. 0.11 **d.** none of these
   (IIT-JEE, 1980)

3. The probability that an event A happens in one trial of an experiment is 0.4. Three independent trials of the experiment are performed. The probability that the event A happens at least once is
   a. 0.936 **b.** 0.784
   c. 0.904 **d.** none of these
   (IIT-JEE, 1980)

4. If A and B are two events such that $P(A) > 0$ and $P(B) \neq 1$, then $P(\bar{A}/\bar{B})$ is equal to

   (Here $\bar{A}$ and $\bar{B}$ are complements of A and B, respectively.)

   a. $1 - P\left(\frac{A}{B}\right)$ **b.** $1 - P\left(\frac{\bar{A}}{B}\right)$

   c. $\frac{1 - P(A \cup B)}{P(\bar{B})}$ **d.** $\frac{P(\bar{A})}{P(B)}$

   (IIT-JEE, 1982)

5. Fifteen coupons are numbered 1, 2, 3, ..., 15. Seven coupons are selected at random one at a time with replacement. The probability that the largest number appearing on selected coupon is 9 is
   a. $(9/16)^6$ **b.** $(8/15)^7$
   c. $(3/5)^7$ **d.** none of these
   (IIT-JEE, 1983)

6. One-hundred identical coins, each with probability, p, of showing up heads are tossed once. If $0 < p < 1$ and the probability of heads showing on 50 coins is equal to that of heads showing on 51 coins, then the value of p is
   a. 1/2 **b.** 49/101
   c. 50/101 **d.** 51/101
   (IIT-JEE, 1988)

7. India plays two matches each with West Indies and Australia. In any match, the probabilities of India getting points 0, 1 and 2 are 0.45, 0.05 and 0.50, respectively. Assuming that the outcomes are independent, the probability of India getting at least 7 points is
   a. 0.8750 **b.** 0.0875
   c. 0.0625 **d.** 0.0250 **(IIT-JEE, 1982)**

8. An unbiased die with faced marked 1, 2, 3, 4, 5 and 6 is rolled four times. Out of four face values obtained, the probability that the minimum face value is not less than 2 and the maximum face value is not greater than five is then

a. 16/81 **b.** 1/81
c. 80/81 **d.** 65/81 **(IIT-JEE, 1993)**

9. The probability of India winning a test match against West Indies is 1/2. Assuming independence from match to match, the probability that in a five match series India's second win occurs at third test is

a. 1/8 **b.** 1/4
c. 1/2 **d.** 2/3 **(IIT-JEE, 1995)**

10. Three of the six vertices of a regular hexagon are chosen at random. The probability that the triangle with three vertices is equilateral is

a. 1/2 **b.** 1/5
c. 1/10 **d.** 1/20 **(IIT-JEE, 1995)**

11. For the three events A, B and C, P(exactly one of the events A or B occurs) = P(exactly one of the two events B or C occurs) = P(exactly one of the events C or A occurs) = p and P(all the three events occur simultaneously) = p^2, where $0 < p < 1/2$. Then the probability of at least one of the three events A, B and C occurring is

a. $\frac{3p+2p^2}{2}$ **b.** $\frac{p+3p^2}{4}$
c. $\frac{p+3p^2}{2}$ **d.** $\frac{3p+2p^2}{4}$
(IIT-JEE, 1996)

12. If the integers m and n are chosen at random between 1 and 100, then the probability that a number of the form $7^m + 7^n$ is divisible by 5 equals

a. 1/4 **b.** 1/7
c. 1/8 **d.** 1/49 **(IIT-JEE, 1999)**

13. Two numbers are selected randomly from the set $S = \{1, 2, 3, 4, 5, 6\}$ without replacement one by one. The probability that minimum of the two numbers is less than 4 is

a. 1/15 **b.** 14/15
c. 1/5 **d.** 4/5 **(IIT-JEE, 2003)**

14. If $P(B) = 3/4$, $P(A \cap B \cap \bar{C}) = 1/3$ and $P(\bar{A} \cap B \cap \bar{C}) = 1/3$, then $P(B \cap C)$ is

a. 1/12 **b.** 1/6
c. 1/15 **d.** 1/9 **(IIT-JEE, 2003)**

15. If three distinct numbers are chosen randomly from the first 100 natural numbers, then the probability that all three of them are divisible by both 2 and 3 is

a. 4/25 **b.** 4/35
c. 4/33 **d.** 4/1155
(IIT-JEE, 2004)

16. A six-faced fair dice is shown until 1 comes. Then the probability that 1 comes in even number of trials is

a. 5/11 **b.** 5/6
c. 6/11 **d.** 1/6 **(IIT-JEE, 2005)**

17. Three identical dice are rolled. The probability that the same number will appear on each of them is

a. 1/6 **b.** 1/36
c. 1/18 **d.** 3/28 **(IIT-JEE, 1984)**

18. A box contains 24 identical balls of which 12 are white and 12 are black. The balls are drawn at random from the box one at a time with replacement. The probability that a white ball is drawn for the 4th time on the 7th draw is

a. 5/64 **b.** 27/32
c. 5/32 **d.** 1/2 **(IIT-JEE, 1984)**

19. Let A, B, C be three mutually independent events. Consider the two statements S_1 and S_2.

S_1: A and $B \cup C$ are independent
S_2: A and $B \cap C$ are independent
Then,

a. both S_1 and S_2 are true **b.** only S_1 is true
c. only S_2 is true **d.** neither S_1 nor S_2 is true
(IIT-JEE, 1994)

20. One Indian and four American men and their wives are to be seated randomly around a circular table. Then, the conditional probability that the Indian man is seated adjacent to his wife given that each American man is seated adjacent to his wife is

a. 1/2 **b.** 1/3
c. 2/5 **d.** 1/5 **(IIT-JEE, 2007)**

21. Let E^c denote the complement of an event E. Let E, F, G be pairwise independent events with $P(G) > 0$ and $P(E \cap F \cap G) = 0$. Then $P(E^c \cap F^c/G)$ equals

a. $P(E^c) + P(F^c)$ **b.** $P(E^c) - P(F^c)$
c. $P(E^c) - P(F)$ **d.** $P(E) - P(F^c)$
(IIT-JEE, 2007)

22. A signal which can be green or red with probability $\frac{2}{3}$ and $\frac{1}{5}$ respectively, is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is $\frac{3}{4}$. If the signal received at station B is green, then the probability that the original signal was green is

(a) $\frac{3}{5}$ (b) $\frac{6}{7}$
(c) $\frac{20}{23}$ (d) $\frac{9}{20}$ **(IIT-JEE, 2010)**

23. Let ω be a complex cube root of unity with $\omega \neq 1$. A fair die is thrown three times. If r_1, r_2 and r_3 are the numbers obtained on the die, then the probability that $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$ is

a. 1/18
b. 1/9
c. 2/9
d. 1/36 **(IIT-JEE, 2010)**

Multiple choice questions with one or more than one correct answer

1. If M and N are any two events, the probability that exactly one of them occurs is

   a. $P(M) + P(N) - 2P(M \cap N)$
   b. $P(M) + P(N) - P(M \cap N)$
   c. $P(M^C) + P(N^C) - 2P(M^C \cap N^C)$
   d. $P(M \cap N^C) + P(M^C \cap N)$ **(IIT-JEE, 1984)**

2. A student appears for tests I, II and III. The student is successful if he passes either in tests I and II or tests I and III. The probabilities of the student passing in tests I, II and III are p, q and 1/2, respectively. The probability that the student is successful is then

   a. $p = q = 1$
   b. $p = q = 1/2$
   c. $p = 1, q = 0$
   d. none of these

   (IIT-JEE, 1986)

3. The probability that at least one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2, then $P(\bar{A}) + P(\bar{B})$ is

   (Here $\bar{A}$ and $\bar{B}$ are complements of A and B, respectively.)

   a. 0.4
   b. 0.8
   c. 1.2
   d. none

   (IIT-JEE, 1987)

4. For two given events A and B, $P(A \cap B)$ is

   a. not less than $P(A) + P(B) - 1$
   b. not greater than $P(A) + P(B)$
   c. equal to $P(A) + P(B) - P(A \cup B)$
   d. equal to $P(A) + P(B) + P(A \cup B)$

   (IIT-JEE, 1988)

5. If E and F are independent events such that $0 < P(E) < 1$ and $0 < P(F) < 1$, then

   a. B and F are mutually exclusive
   b. E and F^C (the complement of the event F) are independent
   c. E^C and F^C are independent
   d. $P(E|F) + P(E^C|F) = 1$ **(IIT-JEE, 1989)**

6. For any two events A and B in a sample space,

   a. $P(A/B) \geq \dfrac{P(A) + P(B) - 1}{P(B)}$ $(P(B) \neq 0)$ is always true
   b. $P(A \cap \bar{B}) = P(A) - P(A \cap B)$ does not hold
   c. $P(A \cup B) = 1 - P(\bar{A})\, P(\bar{B})$, if A and B are independent
   d. $P(A \cup B) = 1 - P(\bar{A})\, P(\bar{B})$, if A and B are disjoint

   (IIT-JEE, 1997)

7. E and F are two independent events. The probability that both E and F happen is 1/12 and the probability that neither E nor F happens is 1/2. Then,

   a. $P(E) = 1/3, P(F) = 1/4$
   b. $P(E) = 1/4, P(F) = 1/3$
   c. $P(E) = 1/6, P(F) = 1/2$
   d. $P(E) = 1/2, P(F) = 1/6$ **(IIT-JEE, 1993)**

8. If from each of the three boxes containing 3 white and 1 black, 2 white and 2 black, 1 white and 3 black balls, one ball is drawn at random, then the probability that 2 white and 1 black ball will be drawn is

   a. 13/32
   b. 1/4
   c. 1/32
   d. 3/16 **(IIT-JEE, 1998)**

9. If $\bar{E}$ and $\bar{F}$ are the complementary events of events E and F, respectively, and if $0 < P(F) < 1$, then

   a. $P(E/F) + P(\bar{E}/F) = 1$
   b. $P(E/F) + P(E/\bar{F}) = 1$
   c. $P(\bar{E}/F) + P(E/\bar{F}) = 1$
   d. $P(E/\bar{F}) + P(\bar{E}/\bar{F}) = 1$ **(IIT-JEE, 1998)**

10. There are four machines and it is known that exactly two of them are faulty. They are tested, one by one, in a random order till both the faulty machines are identified. Then the probability that only two tests are needed is

    a. 1/3
    b. 1/6
    c. 1/2
    d. 1/4 **(IIT-JEE, 1998)**

11. If E and F are events with $P(E) \leq P(F)$ and $P(E \cap F) > 0$, then

    a. occurrence of $E \Rightarrow$ occurrence of F
    b. occurrence of $F \Rightarrow$ occurrence of E
    c. non-occurrence of $E \Rightarrow$ non-occurrence of F
    d. none of the above implications holds **(IIT-JEE, 1998)**

12. A fair coin is tossed repeatedly. If the tail appears on first four tosses, then the probability of the head appearing on the fifth toss equals

    a. 1/2
    b. 1/32
    c. 31/32
    d. 1/5 **(IIT-JEE, 1998)**

13. The probabilities that a student passes in Mathematics, Physics and Chemistry are m, p and c, respectively. Of these subjects, the student has a 75% chance of passing in at least one, a 50% chance of passing in at least two and a 40% chance of passing in exactly two. Which of the following relations are true?

    a. $p + m + c = 19/20$
    b. $p + m + c = 27/20$
    c. $pmc = 1/10$
    d. $pmc = 1/4$

    (IIT-JEE, 1999)

14. Seven white balls and three black balls are randomly placed in a row. The probability that no two black balls are placed adjacently equals

a. 1/2 b. 7/15

c. 2/15 d. 1/3 **(IIT-JEE, 1998)**

15. Let $0 < P(A) < 1$, $0 < P(B) < 1$ and $P(A \cup B) = P(A) + P(B) - P(A)\,P(B)$. Then

a. $P(B/A) = P(B) - P(A)$

b. $P(A' - B') = P(A') - P(B')$

c. $P(A \cup B)' = P(A')\,P(B')$

d. $P(A/B) = P(A)$ **(IIT-JEE, 1995)**

16. Let E and F be two independent events. The probability that exactly one of them occurs is $\frac{11}{25}$ and the probability of none of them occurring is $\frac{2}{25}$. If $P(T)$ denotes the probability of occurrence of the event T, then

(a) $P(E) = \frac{4}{5}$, $P(F) = \frac{3}{5}$

(b) $P(E) = \frac{1}{5}$, $P(F) = \frac{2}{5}$

(c) $P(E) = \frac{2}{5}$, $P(F) = \frac{1}{5}$

(d) $P(E) = \frac{3}{5}$, $P(F) = \frac{4}{5}$ **(IIT-JEE, 2011)**

Comprehension

Read the passage given below and answer the questions that follow.

For Problems 1–3

There are n urns, each of these contain $n + 1$ balls. The i^{th} urn contains i white balls and $n + 1 - i$ red balls. Let u_i be the event of selecting i^{th} urn, $i = 1, 2, 3, \ldots, n$ and w be the event of getting a white ball.

1. If $P(u_i) \propto i$, where $i = 1, 2, 3, \ldots, n$, then $\lim_{n\to\infty} P(w)$ equals to

a. 1 b. 2/3

c. 3/4 d. 1/4 **(IIT-JEE, 2005)**

2. If $P(u_i) = c$ (a constant), then $P(u_n/w)$ equals to

a. $2/(n + 1)$ b. $1/(n + 1)$

c. $n/(n + 1)$ d. 1/2 **(IIT-JEE, 2006)**

3. Let $P(u_i) = 1/n$. If n is even and E denotes the event of choosing even numbered urn then the value of $P(w/E)$ is

a. $\frac{n+2}{2n+1}$ b. $\frac{n+2}{2(n+1)}$

c. $\frac{2}{n+1}$ d. $\frac{1}{n+1}$ **(IIT-JEE, 2006)**

For Problems 4–6

A fair die is tossed repeatedly until a 6 is obtained. Let X denote the number of tosses required.

4. The probability that $X = 3$ equals

a. 25/216 b. 25/36

c. 5/36 d. 125/216

(IIT-JEE, 2009)

5. The probability that $X \geq 3$ equals

a. 125/216 b. 25/36

c. 5/36 d. 25/216

6. The conditional probability that $X \geq 6$ given $X > 3$ equals

a. 125/216 b. 25/36

c. 5/36 d. 25/216

For Problems 7–8

Let U_1 and U_2 be two urns such that U_1 contains 3 white and 2 red balls, and U_2 contains only 1 white ball. A fair coin is tossed. If head appears then 1 ball is drawn at random for U_1 and put into U_2. However, if tail appears then 2 balls are drawn at random from U_1 and put into U_2. Now I ball is drawn at random from U_2

(IIT-JEE, 2011)

7. The probability of the drawn ball from U_2 being white is

a. $\frac{13}{30}$ b. $\frac{23}{30}$

c. $\frac{19}{30}$ d. $\frac{11}{30}$

8. Given that the drawn ball from U_2 is white, the probability that head appeared on the coin is

a. $\frac{17}{23}$ b. $\frac{11}{23}$

c. $\frac{15}{23}$ d. $\frac{12}{23}$

Assertion and reason

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and Statement 2 is the correct explanation of Statement 1.

b. Both the statements are TRUE but Statement 2 is NOT the correct explanation of Statement 1.

c. Statements 1 is TRUE and Statement 2 is FALSE.

d. Statements 1 is FALSE and Statement 2 is TRUE.

1. Let $H_1, H_2, \ldots, H_n$ be mutually exclusive and exhaustive events with $P(H_i) > 0$, $i = 1, 2, \ldots, n$. Let E be any other event with $0 < P(E) < 1$.

Statement 1:

$P(H_i/E) > P(E/H_i) \times P(H_i)$ for $i = 1, 2, \ldots, n$.

Statement 2: $\sum_{i=1}^{n} P(H_i) = 1$. **(IIT-JEE, 2007)**

2. Consider the system of equations $ax + by = 0$, $cx + dy = 0$, where $a, b, c, d \in \{0, 1\}$.

Statement 1: The probability that the system of equations has a unique solution is 3/8.

Statement 2: The probability that the system of equations has a solution is 1. **(IIT-JEE, 2008)**

ANSWERS AND SOLUTIONS

Subjective Type

1. The elements of A are all multiples of 5. Sum of every pair of elements of A is divisible by 5. Therefore, we have to find the probability that B has two distinct elements whose sum is divisible by 3.

Let A_0 be the set of elements of A of the form $3k$, i.e., {0, 15, 30, ..., 195}; A_1 be the set of elements of A of the form $3k + 1$, i.e., {10, 25, ..., 190}; A_2 be the set of elements of A of the form $3k + 2$, i.e., {5, 20, 35, 185}. Then $n(A_0) = 14$, $n(A_1) = n(A_2) = 13$.

If B has at least two elements from A_0, then we are done.

If B contains at most one element of A_0, then it must have at least one element from each of A_1 and A_2 for which the sum of these two elements will be divisible by 3. So, the required probability is 1.

2. Let S be the sample space consisting of elements representing balls that can be drawn from the bag containing $2n$ balls (n white + n black). Let E_{ij} be the event representing drawing balls such that number of white balls is greater than that of black balls by one. Then,

$$E = E_{21} \cup E_{32} \cup \cdots \cup E_{n,n-1}$$

$$m(S) = {}^{2n}C_1 + {}^{2n}C_2 + \cdots + {}^{2n}C_{2n} = 2^{2n} - 1$$

$$\begin{aligned} m(E) &= m(E_{21}) + m(E_{32}) + \cdots + m(E_{n,n-1}) \\ &= {}^nC_2\,{}^nC_1 + {}^nC_3\,{}^nC_2 + \cdots + {}^nC_n\,{}^nC_{n-1} \\ &= {}^{2n}C_{n-1} - {}^nC_1\,{}^nC_0 \\ &= {}^{2n}C_{n-1} - n \end{aligned}$$

Hence,

$$P(E) = \frac{{}^{2n}C_{n-1} - n}{2^{2n} - 1}$$

3. Let 'S' be the sample space, A_0 be the event that books drawn from two bags are equal in number, A_1 be the event that number of books drawn from one bag exceed those drawn from another bag by one, and A_2 be the event that number of books drawn from one bag exceed those drawn from other bag by two. Total number of ways is $({}^{10}C_1 + {}^{10}C_2 + \cdots + {}^{10}C_{10})^2 = (2^{10} - 1)^2$. Favourable number of ways for A_0 is $({}^{10}C_1)^2 + ({}^{10}C_2)^2 + \cdots + ({}^{10}C_{10})^2 = {}^{20}C_{10} - 1$.

Favourable number of ways for A_1 is

$$\begin{aligned} &2({}^{10}C_1\,{}^{10}C_2 + {}^{10}C_2\,{}^{10}C_3 + \cdots + {}^{10}C_9\,{}^{10}C_{10}) \\ &= 2({}^{20}C_9 - {}^{10}C_0 \cdot {}^{10}C_1) \\ &= 2({}^{20}C_9 - 10) \end{aligned}$$

Favourable number of ways for A_2 is

$$\begin{aligned} &2({}^{10}C_1\,{}^{10}C_3 + {}^{10}C_2\,{}^{10}C_4 + \cdots + {}^{10}C_8\,{}^{10}C_{10}) \\ &= 2({}^{20}C_8 - {}^{10}C_0\,{}^{10}C_2) \\ &= 2({}^{20}C_8 - 45) \end{aligned}$$

Therefore, the required probability is

$$\frac{2\,{}^{20}C_8 + 2\,{}^{20}C_9 + {}^{20}C_{10} - 111}{(2^{10} - 1)^2}$$

4. Let X be the number of times A shoots at the target to hit it for the first time and Y be the number of times B shoots at the target to hit for the first time. Then,

$$P(X = m) = \left(\frac{2}{5}\right)^{m-1}\left(\frac{3}{5}\right) \text{ and } P(Y = n) = \left(\frac{2}{7}\right)^{n-1}\left(\frac{5}{7}\right)$$

We have,

$$P(Y > X) = \sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty} P(X = m)\,P(Y = n)$$

[$\because X$ and Y are independent]

$$\begin{aligned} &= \sum_{m=1}^{\infty}\left[\left\{\left(\frac{2}{5}\right)^{m-1}\left(\frac{3}{5}\right)\right\}\sum_{n=m+1}^{\infty}\left\{\left(\frac{2}{7}\right)^{n-1}\left(\frac{5}{7}\right)\right\}\right] \\ &= \sum_{m=1}^{\infty}\left(\frac{2}{5}\right)^{m-1}\left(\frac{3}{5}\right)\left\{\frac{5}{7}\cdot\frac{\left(\frac{2}{7}\right)^m}{1 - \frac{2}{7}}\right\} \\ &= \sum_{m=1}^{\infty}\left(\frac{2}{5}\right)^{m-1}\left(\frac{3}{5}\right)\left(\frac{2}{7}\right)^m \\ &= \frac{6}{35}\sum_{m=1}^{\infty}\left(\frac{4}{35}\right)^{m-1} \\ &= \frac{6}{35}\frac{1}{1 - \frac{4}{35}} = \frac{6}{31} \end{aligned}$$

5. Let E_i denote the event that out of the first k balls drawn, i balls are white and A be the event that $(k + 1)^{\text{th}}$ ball drawn is also white. We have to find $P(A)$. Now ways of selecting i white balls from a white balls and $k - i$ black balls from b black balls is ${}^aC_i\,{}^bC_{k-i}$ $(0 \le i \le k)$. Ways to select k balls from $a + b$ balls is ${}^{a+b}C_k$.

$\therefore \quad P(E_i) = ({}^aC_i\,{}^bC_{k-i})/({}^{a+b}C_k),\ 0 \le i \le k$

Also,

$$P(A/E_i) = \frac{{}^{a-i}C_1}{{}^{a+b-k}C_1} = \frac{a-i}{a+b-k} \quad (0 \le i \le k)$$

By the theorem of total probability, we have

$$\begin{aligned} P(A) &= \sum_{i=0}^{k} P(E_i)\,P(A/E_i) \\ &= \sum_{i=0}^{k}\frac{{}^aC_i\,{}^bC_{k-i}}{{}^{a+b}C_k}\,\frac{a-i}{a+b-k} \\ &= \sum_{i=0}^{k}\frac{[(a-i)\,{}^aC_{a-i}]\,{}^bC_{k-i}}{(a+b-k)\,{}^{a+b}C_{a+b-k}} \\ &= \frac{a}{a+b}\sum_{i=0}^{k}\frac{{}^{a-1}C_{a-1-i}\,{}^bC_{k-i}}{{}^{a+b-1}C_{a+b-k-1}} \\ &= \frac{a}{a+b}\sum_{i=0}^{k}\frac{{}^{a-1}C_i\,{}^bC_{k-i}}{{}^{a+b-1}C_k} \end{aligned}$$

$$= \frac{a}{a + b({}^{a+b-1}C_k)} \sum_{i=0}^{k} {}^{a-1}C_i \, {}^{b}C_{k-i}$$

$$= \frac{a}{a + b({}^{a+b-1}C_k)} \, {}^{a+b-1}C_k$$

$$= \frac{a}{a+b}$$

6. Let A, B, C be the three independent events having probability p, q, and r, respectively. Then according to the hypothesis, we have

$$p(1-q)(1-r) = a, \; (1-p)q(1-r) = b \text{ and } (1-p)(1-q)r = c$$

$$\therefore \quad pqr\,[(1-p)(1-q)(1-r)]^2 = abc \text{ or}$$

$$\frac{abc}{pqr}\left[(1-p)(1-q)(1-r)\right]^2 - 1 = x^2 \text{ (say)} \qquad (1)$$

Then,

$$\frac{a}{x} = \frac{p}{1-p}$$

$$\Rightarrow \quad a - ap = px$$

$$\Rightarrow \quad p = a/(a+x)$$

Similarly, $q = b/(b+x)$ and $r = c/(c+x)$. Then from Eq. (1), clearly $(a+x)(b+x)(c+x) = x^2$, i.e., x is a root of the equation $(a+x)(b+x)(c+x) = x^2$.

7. (i) Let r and s be the remainders when x and y are divided by 5, so

$$x = 5p + r, \; y = 5q + s, \; p, q \in N, \; 0 \le r \le 4, \; 0 \le s \le 4$$

$$\therefore \quad x^2 + y^2 = 25(p^2 + q^2) + 10(pr + qs) + r^2 + s^2 = 5k + (r^2 + s^2), \; k \in N$$

Thus, $x^2 + y^2$ will be divisible by 5 if $r^2 + s^2$ is divisible by 5. Therefore, total number of ways is equal to the number of ways of selecting r and s, which is $5^2 = 25$.

For favourable ways, we should have $r^2 + s^2$ is divisible by 5 and $0 \le r^2 + s^2 \le 32$ so $r^2 + s^2 = 0, 5, 10, 20, 25$ or 30. Thus

r	0	1	2	3	4
s	0	2, 3	1, 4	1, 4	2, 3

So number of favourable ways is 9. Hence, the probability is 9/25.

(ii) Similarly, following the above steps, we can find the required result.

8. Suppose that p and q, respectively, denote the probability that a thing goes to a man and to a woman, respectively. Then,

$$p = \frac{a}{a+b} \text{ and } q = \frac{b}{a+b}$$

Now, the probabilities of 0, 1, 2, 3, … things going to man are the first, second, third terms, etc., in the following binomial expansion.

$$(q+p)^m = q^m + {}^mC_1 a^{m-1}p + {}^mC_2 a^{m-2}p^2 + \cdots + p^m \qquad (1)$$

But men are to receive an odd number of things. Hence, the required probability is the sum of even terms in Eq. (1). To obtain the sum of even terms, we write the expansion

$$(q+p)^m = q^m + {}^mC_1 a^{m-1}p + {}^mC_2 a^{m-2}p^2 + \cdots + (-1)^m p^m \qquad (2)$$

Taking the difference of Eq. (2) from Eq. (1), we obtain

$(q+p)^m - (q-p)^m$ = sum of even terms in Eq. (1). Hence, the required probability is

$$\frac{1}{2}[(q+p)^m - (q-p)^m]$$

$$= \frac{1}{2}\left[1 - \left(\frac{b-a}{b+a}\right)^m\right]$$

$$= \frac{1}{2}\frac{(b+a)^m (b-a)^m}{(b+a)^m}$$

9. Let A be the event of P_1 winning in third round and B be the event of P_2 winning in first round but loosing in second round. We have

$$P(A) = \frac{{}^{8n-1}C_{n-1}}{{}^{8n}C_n} = \frac{1}{8}$$

$P(B \cap A)$

= Probability of both P_1 and P_2 winning in first round

× probability of P_1 winning and P_2 losing in second round

× probability of P_1 winning in third round

$$= \frac{{}^{8n-2}C_{4n-2}}{{}^{8n}C_{4n}} \times \frac{{}^{4n-2}C_{2n-1}}{{}^{4n}C_{2n}} \times \frac{{}^{2n-1}C_{n-1}}{{}^{2n}C_n}$$

$$= \frac{n}{4(8n-1)}$$

Hence,

$$P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)}$$

$$= \frac{2n}{8n-1}$$

Alternative Solution:

Probability that P_2 wins in first round given P_1 wins is

$$\frac{{}^{8n-2}C_{4n-2}}{{}^{8n-1}C_{4n-1}} = \frac{4n-1}{8n-1}$$

In second round, probability that P_2 loses in second round given P_1 wins is

$$1 - \frac{2n-1}{4n-1} = \frac{2n}{4n-1}$$

Hence, probability that P_2 loses in second round, given P_1 wins in third round is $2n/(8n-1)$.

10. In the tennis match of best of 5 sets, A can win the match, if score of A against the score of B is (3, 0), (3, 1) or (3, 2).

The probability of A's doing the score of (3, 0) is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

The probability of A's winning by the score of (3, 1) is

P(A wins the first III sets)

+ P(A wins I, loses II and wins III, IV sets)

+ P(A wins sets I, II, loses III and wins set IV) 2

$$= \frac{1}{4}\left(\frac{1}{2}\right)^3 + \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{4} = \frac{3}{32}$$

The probability of A's winning by the score of (3, 2) is

$P(A$ looses I and II sets)
$+ P(A$ looses I and III sets)
$+ P(A$ looses I and IV sets)
$+ P(A$ looses II and III sets)
$+ P(A$ looses II and IV sets)
$+ P(A$ looses III and IV sets)

$$= \frac{1}{2}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)$$
$$+ \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)$$
$$+ \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\frac{1}{4}$$
$$= \frac{3}{128} + \frac{1}{128} + \frac{1}{128} + \frac{3}{128} + \frac{1}{128} + \frac{3}{128}$$
$$= \frac{12}{128}$$

The probability that A wins the match is

$$\frac{1}{8} + \frac{3}{32} + \frac{12}{128} = \frac{16+12+12}{128} = \frac{40}{128} = \frac{5}{16}$$

11. Let the probability that A wins the tournament be x and the probability that B wins the tournament be x. Also let the probability that the game ends in a draw be y so that

$$2x + y = 1 \Rightarrow x = \frac{1-y}{2}$$

Case I: The probability that the game ends in 1 draw, 3 wins and 3 losses is

$$\frac{7!}{3!3!}\left(\frac{1}{3}\right)^7$$

Case II: The probability that the game ends in 3 draws, 2 wins and 2 losses is

$$\frac{7!}{3!2!2!}\left(\frac{1}{3}\right)^7$$

Case III: The probability that the game ends in 5 draws, 1 win and 1 losses is

$$\frac{7}{5!}\left(\frac{1}{3}\right)^7$$

Case IV: The probability that the game ends in all draws is

$$\left(\frac{1}{3}\right)^7$$

$$\therefore \quad y = \frac{1}{3^7}\left(\frac{7!}{3!3!} + \frac{7!}{3!2!2!} + \frac{7!}{5!} + 1\right) = \frac{131}{729}$$

$$\Rightarrow \quad x = \frac{1}{2}\left(1 - \frac{131}{729}\right) = \frac{299}{729}$$

12. Each group should have odd number of odd numbered balls.

Case I: Two groups have three odd-numbered balls and the third group has only one odd-numbered ball. The number of such cases is

$$\frac{7!}{(3!)^2 2!} \times 3^7 \text{ (each even numbered ball has three possibilities)}$$

Case II: Two groups have one odd-numbered ball and the third group has five odd-numbered balls.

The number of such cases is

$$\frac{7!}{5! \times 2!} \times 3^7$$

The total number of cases for dividing 14 balls into three non-empty groups is $(3^{14} - {}^3C_1 2^{14} + {}^3C_2)/3!$ Hence, the required probability is

$$\frac{\left(\frac{7! \times 3!}{(3!)^2 2!} + \frac{7! \times 3!}{5! \times 2!}\right)}{(3^{14} - {}^3C_1 2^{14} + {}^3C_2)} \times 3^7$$

13.

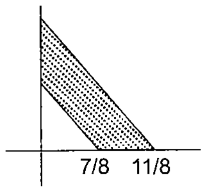

Fig. 9.9

$$P(A \cup B) \geq \frac{3}{4} \text{ and } \frac{1}{8} \leq P(A \cap B) \leq \frac{3}{8}$$

Hence,

$$P(A) + P(B) - P(A \cap B) \geq \frac{3}{4}$$
$$\Rightarrow \quad P(A) + P(B) \geq + P(A \cap B) \geq \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$$
$$\Rightarrow \quad x + y \geq \frac{7}{8}$$

We know that

$$P(A \cap B) \leq 1$$
$$P(A) + P(B) \leq 1 + P(A \cap B) \leq 1 + \frac{3}{8} = \frac{11}{8}$$
$$\Rightarrow \quad x + y \leq \frac{11}{8}$$

The shaded part in the figure is the required region.

14. P_1 can win in the following mutually exclusive ways:

a. P_1 wins the next six matches.

b. P_1 wins five out of next six matches, so that after next six matches scores of P_1 and P_2 are tied up. This tie continues up to next '$2n$' matches ($n \geq 0$) and finally P_1 wins 2 consecutive matches. Now, for case (a), probability is given by $(2/3)^6$ and probability of tie after 6 matches [in case (b)] is

$${}^6C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right) = 6 \times \frac{2^5}{3^6} = \frac{2^6}{3^5}$$

Now probability that scores are still tied up after another next two matches is

$$\frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} = \frac{4}{9}$$

[First match won by P_1 and second by P_2 or first by P_2 and second by P_1.]

Similarly, probability that scores are still tied up after another $2n$ matches is $(4/9)^n$.

Therefore, the total probability of P_1 winning the championship is

$$\left(\frac{2}{3}\right)^6+\frac{2^6}{3^5}\left(\sum_{n=0}^{\infty}\left(\frac{4}{9}\right)^n\left(\frac{2}{3}\right)^2\right)$$

$$=\left(\frac{2}{3}\right)^6+\frac{2^5}{3^5}\left(\frac{2}{3}\right)^2\left(\frac{1}{1-\frac{4}{9}}\right)$$

$$=\frac{17}{5}\left(\frac{2}{3}\right)^6=\frac{1088}{3645}$$

15. (i) The probability that one gets tail and nine get head is

$$^{10}C_1\frac{1}{2}\left(\frac{1}{2}\right)^9$$

(ii) The probability that one gets head and nine get tail is

$$^{10}C_1\frac{1}{2}\left(\frac{1}{2}\right)^9$$

Hence, probability that the game is settled is

$$2\times {}^{10}C_1\frac{1}{2}\left(\frac{1}{2}\right)^9=\frac{5}{2^8}$$

If the game is not settled in First toss, its probability is $1-5/2^8$. If the game is not settled in Second toss, its probability is $(1-5/2^8)^2$. Similarly, the probability that game is not settled in first $n-1$ toss is $(1-5/2^8)^{n-1}$, which is equal to the probability that the game is settled on or after the n^{th} toss.

16. Let A_1 (A_2) be the event that drawn ball is white (non-white) and 'E' be the event that A and B claim that drawn ball is white. Clearly,

$$P(A_1)=\frac{1}{n},\ P(A_2)=\frac{n-1}{n}$$

$$P(E/A_1)=P_1P_2$$

$$P(E/A_2)=(1-P_1)(1-P_2)(n-1)^{-2}$$

[As $n-1$ balls remain in the bag and one of them is white, the chance that 'A' should choose this ball and wrongly assert that it was drawn from the bag is $(1-P_1)/(n-1)$.]

$$\therefore\quad P(E)=P(A_1)P(E/A_1)+P(A_2)P(E/A_2)$$

$$=\frac{P_1P_2}{n}+\frac{(n-1)}{n}\frac{(1-P_1)(1-P_2)}{(n-1)^2}$$

$$=\frac{(n-1)P_1P_2+(1-P_1)(1-P_2)}{n(n-1)}$$

$$\Rightarrow\quad P(A_1/E)=\frac{P(A_1)\,P(E/A_1)}{P(E)}$$

$$=\frac{(n-1)P_1P_2}{(n-1)P_1P_2+(1-P_1)(1-P_2)}$$

17. Let E_i ($i=0, 1, 2, \ldots, 20$) be the event that the bag contains i books on mathematics. Since all these events are equally likely and mutually exclusive and exhaustive, so $P(E_i)=1/21$ ($i=0, 1, 2, \ldots, 20$) and let A be the event that a draw of 10 books contains 6 books on mathematics. Then,

$$P(A)=\sum_{i=0}^{20}P(E_i)\cdot P(A/E_i)$$

$$=\frac{1}{21}\left[\sum_{i=0}^{20}P(A/E_i)\right]$$

$$=\frac{1}{21}\left[\sum_{i=6}^{16}\frac{{}^iC_6\times{}^{20-i}C_4}{{}^{20}C_{10}}\right]$$

Now, we want that the bag should contain 2 more books on mathematics, i.e., E_8 must occur.

$$P(E_8/A)=\frac{P(E_8)P(A/E_8)}{P(A)}$$

$$=\frac{\dfrac{{}^8C_6\times{}^{12}C_4}{{}^{20}C_{10}}}{\sum_{i=6}^{16}\left(\dfrac{{}^iC_6\times{}^{20-i}C_4}{{}^{20}C_{10}}\right)}$$

$$=\frac{{}^8C_6\times{}^{12}C_4}{\sum_{i=6}^{16}({}^iC_6\times{}^{20-i}C_4)}$$

18. Let H and T denote turning up of the head and tail and X denote the turning of head or tail. Then

$$P(H)=P(T)=\frac{1}{2}\text{ and }P(X)=1$$

$P(HH \ldots m$ times) $(XXX \ldots n$ times)

$$=\frac{1}{2}\times\frac{1}{2}\cdots m\text{ times}$$

$$=\frac{1}{2^m}$$

$P[T\{HHH \ldots m \text{ times}\}\{XX \ldots n-1 \text{ times}\}]$

$=P(T)\,P(H)+\cdots+m$ times.

$P(XX \ldots n-1$ times$)=\dfrac{1}{2^{m+1}}$

If the sequence of heads starts with $(r+1)^{th}$ throw, then the first $r-1$ throws may be head or tail but r^{th} throw must be tail and we have

$(XX \ldots (r-1)$ times$)$ T $(HH \ldots m$ times$)$ $(XX \ldots$

$(n-m-r)$ times$)=\dfrac{1}{2^{m+1}}$

Since all the above cases are mutually exclusive, the required probability is

$$\frac{1}{2^m}+\left[\frac{1}{2^{m+1}}+\frac{1}{2^{m+1}}+\cdots+n\text{ times}\right]$$

$$=\frac{1}{2^m}+\frac{n}{2^{m+1}}=\frac{n+2}{2^{m+1}}$$

Objective Type

1. b. $P(A)=\dfrac{1}{3},\ P(A\cup B)=\dfrac{3}{4}$

Now,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \le P(A) + P(B)$$

$$\Rightarrow \frac{3}{4} \le \frac{1}{3} + P(B)$$

$$\Rightarrow \frac{5}{12} \le P(B)$$

Again we have $B \subseteq A \cup B$.

$$\therefore P(B) \le P(A \cup B) = \frac{3}{4}$$

Hence, $5/12 \le P(B) \le 3/4$.

2. a. The probability of hitting a target is $p = 1/5$. Therefore, the probability of not hitting a target is $q = 1 - 1/5 = 4/5$. Hence, the required probability is $1 - (4/5)^{10}$.

3. d. Let the probability for getting an odd number be p. Therefore, the probability for getting an even number is $2p$.

$$\therefore p + 2p = 1 \Rightarrow 3p = 1 \Rightarrow p = \frac{1}{3}$$

Sum of two nunbers is even means either both are odd or both are even. Therefore, the required probability is

$$\frac{1}{3}\times\frac{1}{3}+\frac{2}{3}\times\frac{2}{3} = \frac{1}{9}+\frac{4}{9} = \frac{5}{9}$$

4. c. If A draws card higher than B, then number of favourable cases is $(n-1) + (n-2) + \cdots + 3 + 2 + 1$ (as when B draws card number 1, then A can draw any card from 2 to n and so on). Therefore, the required probability is

$$\frac{\frac{n(n-1)}{2}}{n^2} = \frac{n-1}{2n}$$

5. b. If A, B, C represent events that the student is successful in tests I, II, III, respectively, Then the probability tha the student is successful is

$$P[(A \cap B \cap C') \cup (A \cap B' \cap C) \cup (A \cap B \cap C)]$$

$$= P(A \cap B \cap C') + P(A \cap B' \cap C) + P(A \cap B \cap C)$$

$$= P(A)\,P(B)\,P(C') + P(A)\,P(B')\,P(C) + P(A)\,P(B)\,P(C)$$

[$\therefore A, B, C$ are independent events]

$$= pq\left(1-\frac{1}{2}\right) + p(1-q)\frac{1}{2} + pq\frac{1}{2}$$

$$= pq + \frac{1}{2}p - \frac{1}{2}pq$$

$$= \frac{1}{2}(pq + p)$$

$$\therefore \frac{1}{2}p(1+q) = \frac{1}{2}$$

$$\Rightarrow p(1+q) = 1$$

6. a. $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}, P(C) = \frac{1}{4}$

$$\therefore P(\bar{A}) = 1 - \frac{1}{2} = \frac{1}{2}, P(\bar{B}) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$P(\bar{C}) = 1 - \frac{1}{4} = \frac{3}{4}$$

Therefore, the required probability is

$$1 - P(\bar{A})\,P(\bar{B})\,P(\bar{C}) = 1 - \frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}$$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

7. d. A person can have his/her birthday on any one of the seven days of the week. So 5 persons can have their birthdays in 7^5 ways. Out of 5, three persons can have their birthdays on days other than Sundays in 6^3 ways and other 2 on Sundays. Hence, the required probability is

$$\frac{{}^5C_2 \times 6^3}{7^5} = \frac{10 \times 6^3}{7^5}$$

(Note that 2 persons can be selected out of 5 in ${}^5C_2$ ways.)

8. a. Required probability $= \frac{\text{No. of favourable cases}}{\text{Total no. of exhaustive cases}}$

$$= \frac{3}{3\times3\times3} = \frac{1}{9}$$

9. d. The number of ways of arranging n numbers is $n!$ In each order obtained, we must now arrange the digits $1, 2, \ldots, k$ as group and the $n - k$ remaining digits. This can be done in $(n - k + 1)!$ ways. Therefore, the probability for the required event is $(n - k + 1)!/n!$

10. d. According to the given condition

$${}^nC_3\left(\frac{1}{2}\right)^n = {}^nC_4\left(\frac{1}{2}\right)^n$$

where n is the number of times die is thrown.

$$\therefore {}^nC_3 = {}^nC_4 \Rightarrow n = 7$$

Thus, the required probability is

$${}^7C_1\left(\frac{1}{2}\right)^7 = \frac{7}{2^7} = \frac{7}{128}$$

11. c. Possibilities of getting 9 are (5, 4), (4, 5), (6, 3), (3, 6).

$$\therefore p = \frac{4}{36} = \frac{1}{9} \text{ and } q = 1 - \frac{1}{9} = \frac{8}{9}$$

Therefore, the required probability is

$${}^3C_2 q^1 p^2 = (3)\left(\frac{8}{9}\right)\left(\frac{1}{9}\right)^2 = \frac{8}{243}$$

12. c. Out of 5 horses, only one is the wining horse. The probability that Mr. A selected that losing horse is $4/5 \times 3/4$. Therefore, the required probability is

$$1 - \frac{4}{5}\times\frac{3}{4} = 1 - \frac{3}{5} = \frac{2}{5}$$

13. d. We have,

$$P(\overline{A \cup B}) = \frac{1}{4},\ P(A \cap B) = \frac{1}{4}$$

$$P(\bar{A}) = \frac{1}{4} \Rightarrow P(A) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\therefore\quad P(\overline{A \cup B}) = 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$\Rightarrow\quad \frac{1}{6} = 1 - \frac{3}{4} - P(B) + \frac{1}{4}$$

$$\Rightarrow\quad P(B) = 1 - \frac{1}{2} - \frac{1}{6} = \frac{6-3-1}{6} = \frac{2}{6} = \frac{1}{3}$$

Since $P(A \cap B) = P(A)P(B)$ and $P(A) \neq P(B)$, therefore A and B are independent but not equally likely.

14. b. The total number of cases is 11!/2! × 2!. The number of favourable cases is [11!/(2! × 2!)] – 9! Therefore, the required probability is

$$1 - \frac{9! \times 4}{11!} = \frac{53}{55}$$

15. c. Total number of the students is 80. Total number of girls is 25. Total number of boys is 55. There are 10 rich, 70 poor, 20 intelligent students in the class. Therefore, required probability is

$$\underset{(I)}{\frac{1}{4}} \times \underset{(R)}{\frac{1}{8}} \times \underset{(G)}{\frac{25}{80}} = \frac{5}{512}$$

16. b. $P(A' \cap B \cap C' \cap D) = P(A')\, P(B)\, P(C')\, P(D)$

$$= \left(1 - \frac{1}{2}\right)\frac{1}{3}\left(1 - \frac{1}{5}\right)\left(\frac{1}{6}\right)$$

$$= \frac{1}{2} \times \frac{1}{3} \times \frac{4}{5} \times \frac{1}{6} = \frac{1}{45}$$

17. a. $p^2 + 2p + 4p - 1 = 1$ (Exhaustive)

$$p^2 + 6p - 2 = 0$$

$$\Rightarrow\quad p = -3 \pm \sqrt{11}$$

$$\Rightarrow\quad p = \sqrt{11} - 3$$

18. a. L and W can be filled at 14 places in 2^{14} ways.

$$\therefore\quad n(S) = 2^{14}$$

Now 13 L's and 1 W can be arranged at 14 places in 14 ways.

Hence, $n(A) = 14$.

$$\therefore\quad p = \frac{14}{2^{14}} = \frac{7}{2^{13}}$$

19. a. $P(A \cap C) = P(A)\, P(C)$

$$\Rightarrow\quad \frac{1}{20} = \frac{1}{5} P(C)$$

$$\Rightarrow\quad P(C) = \frac{1}{4}$$

Now,

$$P(B \cup C) = \frac{1}{6} + \frac{1}{4} - P(B \cap C)$$

$$\Rightarrow\quad P(B \cup) = \frac{3}{8} - \frac{1}{3} = \frac{1}{24} = \frac{1}{24} = P(B)P(C)$$

Therefore, B and C are independent.

20. a. The number of ways in which 20 people can be divided into two equal groups is

$$n(s) = \frac{20!}{10!\ 10!\ 2!}$$

The number of ways in which 18 people can be divided into groups of 10 and 8 is

$$n(A) = \frac{18!}{10!\ 8!}$$

$$\therefore\quad P(E) = \frac{18!}{10!8!}\ \frac{10!10!2}{20!} = \frac{10 \times 9 \times 2}{20 \times 19} = \frac{9}{19}$$

21. c. $P(A \cup B) = P(A) + P(B) - P(A) \times P(B)$

$$\Rightarrow\quad 0.7 = 0.4 + p - 0.4p$$

$$\therefore\quad 0.6p = 0.3 \Rightarrow p = \frac{1}{2}$$

22. a. $P(B_1) = \dfrac{{}^6C_1}{{}^{10}C_1} = \dfrac{6}{10} = \dfrac{3}{5}$

$$P(B_2 / B_1) = \frac{5}{9}\ (B_2 = \text{black})$$

$$\therefore\quad P(B_1 \cap B_2) = P(B_1)\, P(B_2 / B_1) = \frac{3}{5} \times \frac{5}{9} = \frac{1}{3}$$

23. b. Total number of ways of distribution is 4^5.

$$\therefore\quad n(S) = 4^5$$

Total number of ways of distribution so that each child gets at least one game is

$$4^5 - {}^4C_1\, 3^5 + {}^4C_2\, 2^5 - {}^4C_3 = 1024 - 4 \times 243 + 6 \times 32 - 4$$
$$= 240$$

$$\therefore\quad n(E) = 240$$

Therefore, the required probability is

$$\frac{n(E)}{n(S)} = \frac{240}{4^5} = \frac{15}{64}$$

24. a. Out of 9 socks, 2 can be drawn in ${}^9C_2$ ways. Therefore, the total number of cases is ${}^9C_2$. Two socks drawn from the drawer will match if either both are brown or both are blue. Therefore, favourable number of cases is ${}^5C_2 + {}^4C_2$. Hence, the required probability is

$$\frac{{}^5C_2 + {}^4C_2}{{}^9C_2} = \frac{4}{9}$$

25. d. Consider two events as follows:

A_i: getting number i on first die

B_i: getting a number more than i on second die

The required probability is

$$P(A_1 \cap B_1) + P(A_2 \cap B_2) + P(A_3 \cap B_3) + P(A_4 \cap B_4) + P(A_5 \cap B_5) = \sum_{i=1}^{5} P(A_i \cap B_i) = \sum_{i=1}^{5} P(A_i)\,P(B_i)$$

[$\because$ A_i, B_i are independent]

$$= \frac{1}{6}[P(B_1) + P(B_2) + \cdots + P(B_5)]$$

$$= \frac{1}{6}\left(\frac{5}{6} + \frac{4}{6} + \frac{3}{6} + \frac{2}{6} + \frac{1}{6}\right) = \frac{5}{12}$$

26. d. We have,

$$x + \frac{100}{x} > 50$$

$$\Rightarrow\ x^2 + 100 > 50x$$

$$\Rightarrow\ (x - 25)^2 > 525$$

$$\Rightarrow\ x - 25 < \sqrt{525} \text{ or } x - 25 > \sqrt{525}$$

$$\Rightarrow\ x < 25 - \sqrt{525} \text{ or } 25 + \sqrt{525}$$

As x is a positive integer and $\sqrt{525} = 22.91$, we must have $x \le 2$ or $x \ge 48$. Thus, the favourable number of cases is 2 + 53 = 55. Hence, the required probability is 55/100 = 11/20.

27. b. The total number of ways in which four-figure numbers can be formed is 4! = 24. A number is divisible by 5 if at unit's place we have 5. Therefore, unit's place can be filled in one way and the remaining 3 places can be filled with the other digits in 3! ways. Hence, total number of numbers divisible by 5 is 3! = 6. So, the required probability is 6/24 = 1/4.

28. a. Since each ball can be placed in any one of the 3 boxes, therefore there are 3 ways in which a ball can be placed in any one of the three boxes. Thus, there are 3^{12} ways in which 12 balls can be placed in 3 boxes. The number of ways in which 3 balls out of 12 can be put in the box is ${}^{12}C_3$. The remaining 9 balls can be placed in 2 boxes in 2^9 ways. So, required probability is

$$\frac{{}^{12}C_3}{3^{12}}\, 2^9 = \frac{110}{9}\left(\frac{2}{3}\right)^{10}$$

29. a. The required probability is

$$\frac{2}{9} \times \frac{1}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} \times \frac{1}{4} \times 1 \times 1 \times 1 = \frac{1}{1260}$$

30. d. The total number of ways of choosing 11 players out of 15 is ${}^{15}C_{11}$. A team of 11 players containing at least 3 bowlers can be chosen in the following mutually exclusive ways:

(I) Three bowlers out of 5 bowlers and 8 other players out of the remaining 10 players.

(II) Four bowlers out of 5 bowlers and 7 other players out of the remaining 10 players.

(III) Five bowlers out of 5 bowlers and 6 other players out of the remaining 10 players.

So, required probability is

$$P(\text{I}) + P(\text{II}) + P(\text{III}) = \frac{{}^5C_3 \times {}^{10}C_8}{{}^{15}C_{11}} + \frac{{}^5C_4 \times {}^{10}C_7}{{}^{15}C_{11}} + \frac{{}^5C_5 \times {}^{10}C_6}{{}^{15}C_{11}}$$

$$= \frac{1260}{1365} = \frac{12}{13}$$

31. b. Consider the following events:

A_1: A speaks truth

A_2: B speaks truth

Then, $P(A_1) = 60/100 = 3/5$, $P(A_2) = 70/100 = 7/10$.

For the required event, either both of them should speak the truth or both of them should tell a lie. Thus, the required probability is

$$P((A_1 \cap A_2) \cup (\bar{A}_1 \cap \bar{A}_2)) = P(A_1 \cap A_2) + P(\bar{A}_1 \cap \bar{A}_2)$$

$$= P(A_1)\,P(A_2) + P(\bar{A}_1)\,(\bar{A}_2)$$

$$= \frac{3}{5} \times \frac{7}{10} + \left(1 - \frac{3}{5}\right)\left(1 - \frac{7}{10}\right) = 0.54$$

32. c. The total number of ways in which 3 integers can be chosen from first 20 integers is ${}^{20}C_3$. The product of three integers will be even if at least one of the integers is even. Therefore, the required probability is

1 – Probability that none of the three integers is even

$$= 1 - \frac{{}^{10}C_3}{{}^{20}C_3} = 1 - \frac{2}{19} = \frac{17}{19}$$

33. b. Consider the following events:

A: getting a card with mark I in first draw

B: getting a card with mark I in second draw

C: getting a card with mark T in this draw

Then, the required probability is

$$P(A \cap B \cap C) = P(A)\,P(B/A)\,P(C/A \cap B)$$

$$= \frac{10}{20} \times \frac{9}{19} \times \frac{10}{18} = \frac{5}{38}$$

34. a. Let p_1 and p_2 be the chances of happening of the first and second events, respectively, then according to the given conditions, we have

$$p_1 = p_2^2 \text{ and } \frac{1 - p_1}{p_1} = \left(\frac{1 - p_2}{p_2}\right)^3$$

$$\Rightarrow\ \frac{1 - p_2^2}{p_2^2} = \left(\frac{1 - p_2}{p_2}\right)^3$$

$$\Rightarrow\ p_2(1 + p_2) = (1 - p_2)^2$$

$$\Rightarrow\ p_2=\frac{1}{3}$$

and so

$$p_1=\frac{1}{9}$$

35. b. The total number of ways in which n persons can sit at a round table is $(n-1)!$ So, total number of cases is $(n-1)!$.

Let A and B be two specified persons. Considering these two as one person, the total number of ways in which $n-1$ persons, $n-2$ other persons and one AB can sit at a round table is $(n-2)!$ So, favourable number of cases is $2!(n-2)!$ Thus, the required probability is

$$p=\frac{2!\,(n-2)!}{(n-1)!}=\frac{2}{n-1}$$

Hence, the required odds are $(1-p):p$ or $(n-3):2$.

36. b. Let one of the quantities be x. Then the other is $2n-x$. Their product will be greatest when they are equal, i.e., each is n in which case the product is n^2. According to the proposition,

$$x(2n-x)\geq\frac{3}{4}n^2$$

$$\Rightarrow\ 4x^2-8nx+3n^2\leq 0$$

$$\Rightarrow\ (2x-3n)(2x-n)\leq 0$$

$$\Rightarrow\ \frac{n}{2}\leq x\leq\frac{3}{2}n$$

So, favourable number of cases is $3/2n-n-2=n$. Hence, the required probability is $n/2n=1/2$.

37. a. Let the number of red and blue balls be r and b, respectively. Then, the probability of drawing two red balls is

$$p_1=\frac{{}^rC_2}{{}^{r+b}C_2}=\frac{r(r-1)}{(r+b)(r+b-1)}$$

The probability of drawing two blue balls is

$$p_2=\frac{{}^bC_2}{{}^{r+b}C_2}=\frac{b(b-1)}{(r+b)(r+b-1)}$$

The probability of drawing one red and one blue ball is

$$p_3=\frac{{}^rC_1\,{}^bC_1}{{}^{r+b}C_2}=\frac{2br}{(r+b)(r+b-1)}$$

By hypothesis, $p_1=5p_2$ and $p_3=6p_2$.

$\therefore\ r(r-1)=5b(b-1)$ and $2br=6b(b-1)$

$\Rightarrow\ r=6, b=3$

38. b. There are 10 digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The last two digits can be dialled in ${}^{10}P_2=90$ ways out of which only one way is favourable, thus, the required probability is 1/90.

39. d. The probability that one test is held is $2\times(1\times 5)\times(4\times 5)=8/25$. The probability that test is held on both days is $(1\times 5)\times(1\times 5)=1/25$.

Therefore, the probability that the student misses at least one test is $8/25+1/25=9/25$.

40. d. Let X denote the largest number on the 3 tickets drawn.

Then, $P(X\leq 7)=(7/20)^3$ and $P(X\leq 6)=(6/20)^3$. Then, the required probability is

$$P(X=7)=\left(\frac{7}{10}\right)^3-\left(\frac{6}{20}\right)^3$$

41. b. Let X denote the number of heads in n trials. Then X is a binomial variant with $p=q=1/2$. Therefore,

$$P(X=r)={}^nC_r\left(\frac{1}{2}\right)^n$$

Now,

$$P(X=6)=P(X=8)$$

$$\Rightarrow\ {}^nC_6\left(\frac{1}{2}\right)^n={}^nC_8\left(\frac{1}{2}\right)^n$$

$$\Rightarrow\ {}^nC_6={}^nC_8\Rightarrow n=14$$

42. a. Let the number selected be xy. Then

$$x+y=9,\ 0\leq x,\ y\leq 9$$

and

$$xy=0\Rightarrow x=0,\ y=9$$

or

$$y=0,\ x=9$$

$$P(x_1=9/x_2=0)=\frac{P(x_1=9\cap x_2=0)}{P(x_2=0)}$$

Now,

$$P(x_2=0)=\frac{19}{100}$$

and

$$P(x_1=9\cap x_2=0)=\frac{2}{100}$$

$$\Rightarrow\ P(x_1=9/x_2=0)=\frac{2/100}{19/100}=\frac{2}{19}$$

43. a. $P(A\cap B')=P(A)-P(A\cap B)=0.20$

Also,

$$P(A'\cap B)=P(B)-P(A\cap B)=0.15$$

$$\Rightarrow\ P(A)+P(B)-2P(A-B)=0.35$$

Now,

$$P(A'\cap B')=1-P(A\cup B)$$

$$\Rightarrow\ 0.1=1-P(A)-P(B)+P(A\cap B)$$

$$\Rightarrow\ P(A)+P(B)-P(A\cap B)=0.9$$

$$\Rightarrow\ P(A\cap B)=0.9-0.35=0.55$$

and

$P(A) = 0.75, P(B) = 0.70$

Now,

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.55}{0.70}$$

44. a. For each toss, there are four choices:

(i) A gets head, B gets head

(ii) A gets tail, B gets head

(iii) A gets head, B gets tail

(iv) A gets tail, B gets tail

Thus, exhaustive number of ways is 4^{50}. Out of the four choices listed above, (iv) is not favourable to the required event in a toss. Therefore, favourable number of cases is 3^{50}. Hence, the required probability is $(3/4)^{50}$.

45. c. 18 draws are required for 2 aces means in the first 17 draws, there is one ace and 16 other cards and 18th draw produces an ace. So, the required probability is

$$\frac{{}^{48}C_{16} \times {}^{4}C_{1}}{{}^{52}C_{17}} \times \frac{3}{35} = \frac{561}{15925}$$

46. b. Consider the following events:

A: Father has at least one boy

B: Father has 2 boys and one girl

Then,

A = one boy and 2 girls, 2 boys and one girl, 3 boys and no girl

$A \cap B$ = 2 boys and one girl

Now, the required probability is

$$P(A/B) = \frac{P(A \cap B)}{P(A)} = \frac{1}{3}$$

47. c. The required probability is

$$\left[{}^{4}C_{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{4}\right]^{2} + \left[{}^{4}C_{1}\left(\frac{1}{2}\right)^{1}\left(\frac{1}{2}\right)^{3}\right]^{2} + \left[{}^{4}C_{2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{2}\right]^{2} + \left[{}^{4}C_{3}\left(\frac{1}{2}\right)^{3}\left(\frac{1}{2}\right)^{1}\right]^{2} + \left[{}^{4}C_{4}\left(\frac{1}{2}\right)^{4}\left(\frac{1}{2}\right)^{0}\right]^{2} = \frac{35}{128}$$

48. b. Let A denote the event that there is an odd man out in a game. The total number of possible cases is 2^m. A person is odd man out if he is alone in getting a head or a tail.

The number of ways in which there is exactly one tail (head) and the rest are heads (tails) is ${}^{m}C_{1} = m$. Thus, the number of favourable ways is $m + m = 2m$. Therefore,

$$P(A) = \frac{2m}{2^m} = \frac{m}{2^{m-1}}$$

49. c. $x^2 + 2(a+4)x - 5a + 64 \geq 0$

If $D \leq 0$, then

$$(a+4)^2 - (-5a + 64) < 0$$

$$\Rightarrow a^2 + 13a - 48 < 0$$

$$\Rightarrow (a+16)(a-3) < 0$$

$$\Rightarrow -16 < a < 3 \Leftrightarrow -5 \leq a \leq 2$$

Then, the favorable cases is equal to the number of integers in the interval [–5, 2], i.e., 8.

Total number of cases is equal to the number of integers in the interval [–5, 30], i.e., 36.

Hence, the required probability is 8/36 = 2/9.

50. a. The total number of ways in which $2n$ boys can be divided into two equal groups is

$$\frac{(2n)!}{(n!)^2 2!}$$

Now, the number of ways in which $2n - 2$ boys other than the two tallest boys can be divided into two equal groups is

$$\frac{(2n-2)!}{((n-1)!)^2 2!}$$

Two tallest boys can be put in different groups in ${}^{2}C_{1}$ ways. Hence, the required probability is

$$\frac{2\,\dfrac{(2n-2)!}{((n-1)!)^2 2!}}{\dfrac{(2n)!}{(n!)^2 2!}} = \frac{n}{2n-1}$$

51. a. Let E_i denote the event that the bag contains i black and $(10 - i)$ white balls $(i = 0, 1, 2, \ldots, 10)$. Let A denote the event that the three balls drawn at random from the bag are black. We have,

$$P(E_i) = \frac{1}{11} \ (i = 0, 1, 2, \ldots, 10)$$

$P(A/E_i) = 0$ for $i = 0, 1, 2$ and $P(A/E_i) = {}^{i}C_3/{}^{10}C_3$ for $i \geq 3$

$$\Rightarrow P(A) = \frac{1}{11} \times \frac{1}{{}^{10}C_3}\left[{}^{3}C_3 + {}^{4}C_3 + \cdots + {}^{10}C_3\right]$$

But

$${}^{3}C_3 + {}^{4}C_3 + {}^{5}C_3 + \cdots + {}^{10}C_3 = {}^{4}C_4 + {}^{4}C_3 + {}^{5}C_3 + \cdots + {}^{10}C_3$$

$$= {}^{5}C_4 + {}^{5}C_3 + {}^{6}C_3 + \cdots + {}^{10}C_3$$

$$\vdots$$

$$= {}^{11}C_4$$

$$\Rightarrow p(A) = \frac{1}{11} \times \frac{1}{{}^{10}C_3} \times {}^{11}C_4$$

$$= \frac{\dfrac{11 \times 10 \times 9 \times 8}{4!}}{11 \times \dfrac{10 \times 9 \times 8}{3!}} = \frac{1}{4}$$

$$\therefore \quad P(E_9/A) = \frac{P(E_9)\,P(A/E_9)}{P(A)}$$

$$= \frac{\frac{1}{11} \times \frac{^9C_3}{^{10}C_3}}{\frac{1}{4}}$$

$$= \frac{14}{55}$$

52. a. We have ratio of the ships A, B and C for arriving safely are 2:5, 3:7 and 6:11, respectively. Therefore, the probability of ship A for arriving safely is $2/(2 + 5) = 2/7$.

Similarly, for B the probability is $3/(3 + 7) = 3/10$ and for C the probability is $C = 6/(6 + 11) = 6/17$.

Therefore, the probability of all the ships for arriving safely is $(2/7) \times (3/10) \times (6/17)$ 18/595.

53. a. (i) This question can also be solved by one student.

(ii) This question can be solved by two students simultaneously.

(iii) This question can be solved by three students all together.

$$P(A) = \frac{1}{2}, P(B) = \frac{1}{4}, P(C) = \frac{1}{6}$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - [P(A)\,P(B) + P(B)\,P(C) + P(C)\,P(A)] + [P(A)\,P(B)\,P(C)]$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} - \left[\frac{1}{2} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{2}\right] + \left[\frac{1}{2} \times \frac{1}{4} \times \frac{1}{6}\right]$$

$$= \frac{33}{48}$$

Alternative solution:

We have,

$$P(\bar{A}) = \frac{1}{2}, P(\bar{B}) = \frac{3}{4}, P(\bar{C}) = \frac{5}{6}$$

Then the probability that the problem is not solved is

$$P(\bar{A})\,P(\bar{B})\,P(\bar{C}) = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) = \frac{5}{16}$$

Hence probability that problem is solved is $1 - 5/16 = 11/16$.

54. a. We are given that

$$P(A \cap B) = P(A)\,P(B)$$

$$P(B \cap C) = P(B)P(C)$$

$$P(C \cap A) = P(C)P(A)$$

$$P(A \cap B \cap C) = P(A)(B)P(C)$$

We have,

$$P(A \cap (B \cap C)) = P(A \cap B \cap C) = P(A)P(B)P(C) = P(A)P(B \cap C)$$

$\Rightarrow$ A and $B \cap C$ are indepedent

Therefore, S_2 is true. Also,

$$P[(A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$$

$$= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$$

$$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$$

$$= P(A)P(B) + P(A)\,P(C) - P(A)\,P(B)\,P(C)$$

$$= P(A)[P(B) + P(C) - P(B)P(C)]$$

$$= P(A)[P(B) + P(C) - P(B \cap C)]$$

$$= P(A)P(B \cup C)$$

Therefore, A and $B \cup C$ are independent.

55. d. The total number of ways in which papers of 4 students can be checked by seven teachers is 7^4. The number of ways of choosing two teachers out of 7 is $^7C_2$. The number of ways in which they can check four papers is 2^4. But this includes two ways in which all the papers will be checked by a single teacher. Therefore, the number of ways in which 4 papers can be checked by exactly two teachers is $2^4 - 2 = 14$. Therefore, the number of favourable ways is $(^7C_2)\,(14) = (21)\,(14)$. Thus, the required probability is $(21)(14)/7^4 = 6/49$.

56. c. Let us assume that A wins after n deuces, $n = 0, 1, 2, 3, \ldots$. The probability of a deuce is $(2/3) \times (2/3) + (1/3) \times (1/3) = (5/9)$. [A wins his serve, then B wins his serve or A loses his serve.] So, the probability that 'A' wins game after n deuces is $(5/9)^n \times (2/3) \times (1/3)$. [After n^{th} deuce, A serves and wins, then B serves and loses.] Therefore, the required probability of 'A' winning the game is

$$\sum_{n=0}^{\infty} \left(\frac{5}{9}\right)^n \times \frac{2}{3} \times \frac{1}{3} = \frac{1}{1 - \frac{5}{9}} \times \frac{2}{9} = \frac{1}{2}$$

57. c. The required probability is

1 – Probability of getting equal number of heads and tails

$$= 1 - {}^{2n}C_n \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{2n-n}$$

$$= 1 - \frac{(2n)!}{n!n!}\left(\frac{1}{4}\right)^n$$

$$= 1 - \frac{(2n)!}{(n!)^2} \times \frac{1}{4^n}$$

58. b. Here $p = 19/20$, $q = 1/20$, $n = 5$, $r = 5$. The required probability is

$$^5C_5 \left(\frac{19}{20}\right)^5 \left(\frac{1}{20}\right)^0 = \left(\frac{19}{20}\right)^5$$

59. d. $$P(E/F) + P(\bar{E}/F) = \frac{P(E \cap F) + P(\bar{E} \cap F)}{P(F)}$$

$$= \frac{P\{(E \cap F) \cup (\bar{E} \cap F)\}}{P(F)}$$

[$\because$ $E \cap F$ and $\bar{E} \cap F$ are disjoint]

$$= \frac{P\{(E \cup \bar{E}) \cap F\}}{P(F)} = \frac{P(F)}{P(F)} = 1$$

Similarly, we can show that (b) and (c) are not true while (d) is true.

$$P\left(\frac{E}{\bar{F}}\right)+P\left(\frac{\bar{E}}{\bar{F}}\right)=\frac{P(E\cap\bar{F})}{P(\bar{F})}+\frac{P(\bar{E}\cap\bar{F})}{P(\bar{F})}=\frac{P(\bar{F})}{P(\bar{F})}=1$$

60. b. We have,

$$P(A)=\frac{40}{100},\ P(B)=\frac{25}{100} \text{ and } P(A\cap B)=\frac{15}{100}$$

So,

$$P\left(\frac{B}{A}\right)=\frac{P(A\cap B)}{P(A)}=\frac{15/100}{40/100}=\frac{3}{8}$$

61. b. We define the following events:

A_1: Selecting a pair of consecutive letters from the word LONDON

A_2: Selecting a pair of consecutive letters from the word CLIFTON

E: Selecting a pair of letters 'ON'

Then, $P(A_1\cap E)=2/5$ as there are 5 pairs of consecutive letters out of which 2 are ON and $P(A_2\cap E)=1/6$ as there are 6 pairs of consecutive letters of which 1 is ON. Therefore, the required probability is

$$P\left(\frac{A_1}{E}\right)=\frac{P(A_1\cap E)}{P(A_1\cap E)+P(A_2\cap E)}=\frac{\frac{2}{5}}{\frac{2}{5}+\frac{1}{6}}=\frac{12}{17}$$

62. d. Player should get $(HT, HT, HT, \ldots)$ or $(TH, TH, \ldots)$ at least $2n$ times. If the sequence starts from first place, then the probability is $1/2^{2n}$ and if starts from any other place, then the probability is $1/2^{2n+1}$. Hence, required probability is

$$2\left(\frac{1}{2^{2n}}+\frac{m}{2^{2n+1}}\right)=\frac{m+2}{2^{2n}}$$

63. b. The total number of cases is $11!/2! \times 2!$ The number of favourable cases is $11!/(2! \times 2!) - 9!$ Therefore, required probability is

$$1-\frac{9!\times 4}{11!}=\frac{53}{55}$$

64. b. The probability of winning of A the second race is 1/2 (since both events are independent).

65. b. Given that $n(S) = 6 \times 6 \times 6 \times 6 = 6^4$. The number of favourable ways is ${}^6C_4 = 6 \times 5/2 = 15$. Therefore, the required probability is

$$\frac{15}{6\times 216}=\frac{5}{2\times 216}=\frac{5}{432}$$

66. c. Given that 5 and 6 have appeared on two of the dice, the sample space reduces to $6^4 - 2 \times 5^4 + 4^4$ (inclusion–exclusion principle). Also, the number of favourable cases are $4! = 24$. So, the required probability is $24/302 = 12/151$.

67. c. Let a_n be the number of strings of H and T of length n with no two adjacent H's. Then $a_1 = 2$, $a_2 = 3$. Also,

$$a_{n+2} = a_{n+1} + a_n \quad \text{(since the string must begin with } T \text{ or } HT)$$

So,

$$a_3 = 5,\ a_4 = 8,\ a_5 = 8 + 5 = 13$$

Therefore, the required probability is $13/2^5 = 13/32$.

68. b.

2							

The prime digits are 2, 3, 5, 7. If we fix 2 at first place, then other $2n - 1$ places are filled by all four digits. So the total number of cases is 4^{2n-1}.

Now, sum of 2 consecutive digits is prime when consecutive digits are (2, 3) or (2, 5). Then 2 will be fixed at all alternative places.

2		2		2		2	

So favourable number of cases is 2^n. Therefore, probability is

$$\frac{2^n}{4^{2n-1}}=2^n\,2^{-4n+2}=2^2\,2^{-3n}=\frac{4}{2^{3n}}$$

69. d. Since a, b, c are in A.P., therefore, $2b = a + c$. The possible cases are tabulated as follows.

b	a	c	Number of ways
1	1	1	1
2	2	2	1
2	1	3	6
3	3	3	1
3	1	5	6
3	2	4	6

Total number of ways is 21. So, required probability is $21/216 = 7/72$.

70. c. When 4 points are selected, we get one intersecting point. So, probability is

$$\frac{{}^nC_4}{{}^{({}^nC_2-n)}C_2}$$

71. d. Three-digit numbers are 100, 101, ..., 999. Total number of such numbers is 900. The three-digit numbers (which have all same digits) are 111, 222, 333, ..., 999. Favorable number of cases is 9. Therefore, the required probability is $9/900 = 1/100$.

72. c. Given,

$$7a - 9b = 0 \Rightarrow b = \frac{7}{9}a$$

Hence, number of pairs (a, b) can be (9, 7); (18, 14); (27, 21); (36, 28). Hence, the required probability is $4/{}^{39}C_2 = 4/741$.

73. a. Let $E_1 = 1, 4, 7, \ldots$ (n each)

$E_2 = 2, 5, 8, \ldots$ (n each)

$$E_3 = 3, 6, 9, \ldots \text{ (} n \text{ each)}$$

x and y belong to (E_1, E_2), (E_2, E_1) or (E_3, E_3). So, the required probability is

$$\frac{n^2 + {}^nC_2}{{}^{3n}C_2} = \frac{1}{3}$$

74. c. The total number of mapping is n^n. The number of one-one mapping is ${}^nC_1\,{}^{n-1}C_1 \cdots {}^1C_1 = n!$ Hence, the probability is

$$\frac{n!}{n^n} = \frac{3}{32} = \frac{4!}{4^4}$$

Comparing, we get $n = 4$.

75. b. $P(4 \text{ biased coin}) = \dfrac{1}{3}$

$P(5 \text{ biased coin}) = \dfrac{1}{4}$

Hence, the required probability is

$$\frac{1}{3}\frac{{}^4C_3\,{}^{16}C_6}{{}^{20}C_9} + \frac{2}{3}\frac{{}^5C_4\,{}^{15}C_5}{{}^{20}C_9}\frac{1}{{}^{11}C_1}$$

$$= \frac{2}{33}\left[\frac{{}^{16}C_6 + 5\,{}^{15}C_5}{{}^{20}C_9}\right]$$

76. b. Let event A be drawing 9 cards which are not ace and B be drawing an ace card. Therefore, the required probability is $P(A \cap B) = P(A) \times P(B)$

Now, there are four aces and 48 other cards. Hence,

$$P(A) = \frac{{}^{48}C_9}{{}^{52}C_9}$$

After having drawn 9 non-ace cards, the 10^{th} card must be ace. Hence,

$$P(B) = \frac{{}^4C_1}{{}^{42}C_1} = \frac{4}{42}$$

Hence,

$$P(A \cap B) = \frac{{}^{48}C_9}{{}^{52}C_9}\frac{4}{42}$$

77. c. The total number of digits in any number at the unit's place is 10.

$\therefore \quad n(S) = 10$

If the last digit in product is 1, 3, 5 or 7, then it is necessary that the last digit in each number must be 1, 3, 5 or 7.

$\therefore \quad n(A) = 4$

$\therefore \quad P(A) = \dfrac{4}{10} = \dfrac{2}{5}$

Hence, the required probability is $(2/5)^4 = 16/625$.

78. a. The divisibility of the product of four numbers depends upon the value of the last digit of each number. The last digit of a number can be any one of the 10 digits 0, 1, 2, ..., 9. So, the total number of ways of selecting last digits of four numbers is $10 \times 10 \times 10 \times 10 = 10^4$. If the product of the 4 numbers is not divisible by 5 or 10, then the number of choices for the last digit of each number is 8 (excluding 0 or 5). So, favourable number of ways is 8^4. Therefore, the probability that the product is not divisible by 5 or 10 is $(8/10)^4$. Hence, the required probability is $1 - (8/10)^4 = 369/625$.

79. b. Let H denote the head, T the tail and * any of the head or tail. Then, $P(H) = 1/2$, $P(T) = 1/2$ and $P(*) = 1$. For at least four consecutive heads, we should have any of the following patterns:

	Probability
(i) $HHHH***$	$(1/2)^4 \times 1 = 1/16$
(ii) $THHHH**$	$(1/2)^5 = 1/32$
(iii) $*THHHH*$	$(1/2)^5 = 1/32$
(iv) $**THHHH$	$(1/2)^5 = 1/32$

Since all the above cases are mutually exclusive, the probability of getting at least four consecutive heads (on adding) is $1/16 + 3/32 = 5/32$.

80. c. The probabilities of solving the question by these three students are 1/3, 2/7 and 3/8, respectively.

$$\therefore \quad P(A) = \frac{1}{3};\ P(B) = \frac{2}{7};\ P(C) = \frac{3}{8}$$

Then probability of question solved by only one student is

$$P((A\bar{B}\bar{C} \text{ or } \bar{A}\,B\bar{C} \text{ or } \bar{A}\,\bar{B}\,C)) = P(A)\,P(\bar{B})\,P(\bar{C}) + P(\bar{A})\,P(B)\,P(\bar{C}) + P(\bar{A})\,P(\bar{B})\,P(C)$$

$$= \frac{1}{3}\times\frac{5}{7}\times\frac{5}{8} + \frac{2}{3}\times\frac{2}{7}\times\frac{5}{8} + \frac{2}{3}\times\frac{5}{7}\times\frac{3}{8}$$

$$= \frac{25 + 20 + 30}{168} = \frac{25}{56}$$

81. b. Probability of getting 2 heads in the first 5 trials is

$${}^5C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = \frac{10}{32} = \frac{5}{16}$$

Therefore, the probability that third head appears on the sixth trial is $5/16 \times 1/2 = 5/32$.

82. d. Let A and B, respectively, be the events that urn A and urn B are selected. Let R be the event that the selected ball is red. Since the urn is chosen at random, Therefore,

$$P(A) = P(B) = \frac{1}{2}$$

and

$$P(R) = P(A)P(R/A) + P(B)P(R/B)$$

$$= \frac{1}{2}\times\frac{5}{10} + \frac{1}{2}\times\frac{4}{10}$$

$$= \frac{9}{20}$$

83. d. The probability that the first critic favours the book is

$$P(E_1) = \frac{5}{5+2} = \frac{5}{7}$$

The probability that the second critic favours the book is

$$P(E_2) = \frac{4}{4+3} = \frac{4}{7}$$

The probability that the third critic favours the book is

$$P(E_3) = \frac{3}{3+4} = \frac{3}{7}$$

Majority will be in favour of the book if at least two critics favour the book. Hence, the probability is

$$P(E_1 \cap E_2 \cap \bar{E}_3) + P(E_1 \cap \bar{E}_2 \cap E_3) + P(\bar{E}_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

$$= P(E_1)P(E_2)P(\bar{E}_3) + P(E_1)P(\bar{E}_2)\,P(E_3) + P(\bar{E}_1 \cap E_2 \cap E_3) + P(E_1)P(E_2)P(E_3)$$

$$= \frac{5}{7}\times\frac{4}{7}\times\left(1-\frac{3}{7}\right)+\frac{5}{7}\times\left(1-\frac{4}{7}\right)\times\frac{3}{7} + \left(1-\frac{5}{7}\right)\times\frac{4}{7}\times\frac{3}{7}+\frac{5}{7}\times\frac{4}{7}\times\frac{3}{7}$$

$$= \frac{209}{343}$$

84. b.

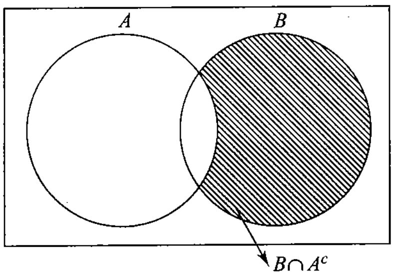

Fig. 9.10

$$P(A) = \frac{1}{4},\ P(A \cup B) = \frac{1}{2}$$

$$P\left(\frac{B}{A^C}\right) = \frac{P(B \cap A^C)}{P(A^C)}$$

$$= \frac{P(A \cup B) - P(A)}{1 - P(A)} \qquad [\because P(A \cup B) + P(B) - P(A \cap B)]$$

$$= \frac{\frac{1}{2}-\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{4}\times\frac{4}{3} = \frac{1}{3}$$

85. b. $P(S \cap F) = 0.0006$, where S is the event that the motor cycle is stolen and F is the event that the moter cycle is found. Therefore,

$$P(S) = 0.0015$$

$$P(F/S) = \frac{P(F \cap S)}{P(S)} = \frac{6\times10^{-4}}{15\times10^{-4}} = \frac{2}{5}$$

86. c. Let A be the event that 11 is picked and B be the event that sum is even. The number of ways of selecting 11 along with one more odd number is $n(A \cap B) = {}^7C_1$.

The number of ways of selecting either two even numbers or selecting two odd numbers is $n(B) = 1 + {}^8C_2$.

$$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{7}{29} = 0.24$$

87. b. Die marked with 1, 2, 2, 3, 3, 3 is thrown 3 times.

$$P(1) = \frac{1}{6},\ P(2) = \frac{2}{6},\ P(3) = \frac{3}{6}$$

$$P(S) = P(4 \text{ or } 6)$$

$$= P(112(3 \text{ cases}) \text{ or } 123\ (6 \text{ cases}) \text{ or } 222)$$

$$= 3\times\frac{1}{6}\times\frac{1}{6}\times\frac{2}{6}+6\frac{1}{6}\times\frac{2}{6}\times\frac{3}{6}+\frac{2}{6}\times\frac{2}{6}\times\frac{2}{6}$$

$$= \frac{6+36+8}{216} = \frac{50}{216} = \frac{25}{108}$$

88. c.

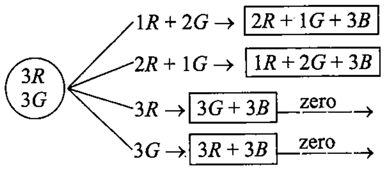

Fig. 9.12

The required probability is

$$\frac{{}^3C_1\,{}^3C_2}{{}^6C_3}\,\frac{{}^2C_1\,{}^1C_1\,{}^3C_1}{{}^6C_3} + \frac{{}^3C_2\,{}^3C_1}{{}^6C_3}\,\frac{{}^1C_1\,{}^2C_1\,{}^3C_1}{{}^6C_3}$$

$$= 2\frac{9}{20}\times\frac{6}{20}$$

$$= \frac{27}{100}$$

89. c. $P(4 \text{ biased coins}) = \frac{1}{3}$

$P(5 \text{ biased coins}) = \frac{1}{4}$

The required probability is

$$\frac{1}{3}\frac{{}^4C_3\,{}^{16}C_6}{{}^{20}C_9}\frac{1}{{}^{11}C_1}+\frac{2}{3}\frac{{}^5C_4\,{}^{15}C_5}{{}^{20}C_9}\frac{1}{{}^{11}C_1}$$

$$=\frac{2}{33}\left[\frac{{}^{16}C_6+5\,{}^{15}C_5}{{}^{20}C_9}\right]$$

90. d. A: Doctor finds a rash

B_1: Child has measles

S: Sick children

$P(S/F) = 0.9$

B_2: Child has flu $\Rightarrow P(B_2) = 9/10$

$P(S/M) = 0.10$

$P(A/B_1) = 0.95$

$P(R/M) = 0.95$

$P(A/B_2) = 0.08$

$P(R/F) = 0.08$

$$P(B_1/A)=\frac{0.1\times 0.95}{0.1\times 0.95+0.9\times 0.08}$$

$$=\frac{0.095}{0.095+0.072}$$

$$=\frac{0.095}{0.167}=\frac{95}{167}$$

91. c. The sum is 12 in the first three throws if they are (1, 5, 6) in any order or (2, 4, 6) in any order or (3, 4, 5) in any order. Therefore, the required probability is

$$\frac{1}{6}\times\frac{1}{5}\times\frac{1}{4}\times 3!+\frac{1}{6}\times\frac{1}{5}\times\frac{1}{4}\times 3!+\frac{1}{6}\times\frac{1}{5}\times\frac{1}{4}\times 3!=\frac{3}{20}$$

(because after throwing 1, in the next throw 1 cannot come, etc.)

92. a. The number of composite numbers in 1 to 30 is $n(S) = 19$.

The number of composite number when divided by 5 leaves a remainder is $n(E) = 14$. Therefore, the required probability is 14/19.

93. b. $P(E) + P(E') = 1 = 1 + \lambda + \lambda^2 + (1 + \lambda)^2$

$\Rightarrow 2\lambda^2 + 3\lambda + 1 = 0$

$\Rightarrow (2\lambda + 1)(\lambda + 1) = 0$

$\Rightarrow \lambda = -1, -\frac{1}{2}$

Then,

$P(E) = 1 + (-1) + (-1)^2 = 1$ (not possible)

$$\Rightarrow P(E)=1+\left(-\frac{1}{2}\right)+\left(-\frac{1}{2}\right)^2=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$$

94. b. Let $P(m)$, $P(p)$, $P(c)$ be the probability of selecting a book of maths, physics and chemistry, respectively. Clearly,

$$P(m)=P(p)=P(c)=\frac{1}{3}$$

Again let $P(s_1)$ and $P(s_2)$ be the probability that he solves the first as well as second problem, respectively. Then,

$$P(s_1)=P(m)\times P\left(\frac{s_1}{m}\right)+P(p)\times P\left(\frac{s_1}{p}\right)+P(c)\times P\left(\frac{s_1}{c}\right)$$

$$\Rightarrow P(s_1)=\frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{3}{5}+\frac{1}{3}\times\frac{4}{5}=\frac{19}{30}$$

Similarly,

$$P(S_2)=\frac{1}{3}\times\left(\frac{1}{2}\right)^2+\frac{1}{3}\times\left(\frac{3}{5}\right)^2+\frac{1}{3}\times\left(\frac{4}{5}\right)^2=\frac{125}{300}$$

$$\Rightarrow P\left(\frac{S_2}{S_1}\right)=\frac{\frac{125}{300}}{\frac{19}{30}}=\frac{25}{38}$$

95. b. Let,

$P(S) = P(1 \text{ or } 2) = 1/3$

$P(F) = P(3 \text{ or } 4 \text{ or } 5 \text{ or } 6) = 2/3$

$P(A \text{ wins}) = P[(S\,S \text{ or } S\,F\,S\,S \text{ or } S\,F\,S\,F\,S\,S \text{ or } \ldots)$

or $(F\,S\,S \text{ or } F\,S\,F\,S\,S \text{ or } \ldots)]$

$$=\frac{\frac{1}{9}}{1-\frac{2}{9}}+\frac{\frac{2}{27}}{1-\frac{2}{9}}$$

$$=\frac{1}{9}\times\frac{9}{7}+\frac{2}{27}\times\frac{9}{7}$$

$$=\frac{1}{7}+\frac{2}{21}=\frac{3+2}{21}=\frac{5}{21}$$

$P(A \text{ winning}) = \frac{5}{21}$, $P(B \text{ winning}) = \frac{16}{21}$

96. a. $P(a) = 0.3$, $P(b) = 0.5$, $P(c) = 0.2$. Hence, a, b, c are exhaustive.

$P(\text{same horse wins all the three races}) = P(aaa \text{ or } bbb \text{ or } ccc)$

$$=(0.3)^3+(0.5)^3+(0.2)^3$$

$$=\frac{27+125+8}{1000}=\frac{160}{1000}$$

$$=\frac{4}{25}$$

$P(\text{each horse wins exactly one race})$

$= P(abc \text{ or } acb \text{ or } bca \text{ or } bac \text{ or } cab \text{ or } cba)$

$= 0.3 \times 0.5 \times 0.2 \times 6 = 0.18 = \frac{9}{50}$

97. b. The total number of ways of distribution is 4^5.

$\therefore\ n(S) = 4^5$

The total number of ways of distribution so that each child gets at least one game is $4^5 - {}^4C_1\, 3^5 + {}^4C_2\, 2^5 - {}^4C_3$.

$\therefore \quad n(E) = 240$

Hence, the required probability is

$$\frac{n(E)}{n(S)} = \frac{240}{4^5} = \frac{15}{64}$$

98. b. Team totals must be 0, 1, 2, ..., 39. Let the teams be $T_1 T_2, \ldots, T_{40}$, so that T_i loses to T_j for $i < j$. In other words, this order uniquely determines the result of every game. There are 40! such orders and 780 games, so 2^{780} possible outcomes for the games. Hence, the probability is $40!/2^{780}$.

99. a. We have,

$n(S) = {}^{64}C_3$

Let 'E' be the event of selecting 3 squares which form the letter 'L'.

The number of ways of selecting squares consisting of 4 unit squares is $7 \times 7 = 49$.

Each square with 4 unit squares form 4 L-shapes consisting of 3 unit squares.

$\therefore \quad n(E) = 7 \times 7 \times 4 = 196$

$$\therefore \quad P(E) = \frac{196}{{}^{64}C_3}$$

100.a. The total number of ways of making the second draw is ${}^{10}C_5$.

The number of draws of 5 balls containing 2 balls common with first draw of 6 balls is ${}^6C_2\, {}^4C_3$. Therefore, the probability is

$$\frac{{}^6C_2\, {}^4C_3}{{}^{10}C_5} = \frac{5}{21}$$

101. a. The required probability is

$$P(A) = \frac{1}{3}\frac{6}{a^2 - 4a + 10}$$

$$(P(A))_{\max} = \frac{1}{3} \times \frac{6}{6} = \frac{1}{3}$$

102. b. A number has exactly 3 factors if the number is squares of a prime number. Squares of 11, 13, 17, 19, 23, 29, 31 are 3-digit numbers. Hence, the required probability is 7/900.

103. c. Suppose, there exist three rational points or more on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Therefore, if (x_1, y_1), (x_2, y_2) and (x_3, y_3) be those three points, then

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad (1)$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_1 + c = 0 \qquad (2)$$

$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \qquad (3)$$

Solving Eqs. (1), (2) and (3), we will get g, f, c as rational. Thus, centre of the circle $(-g, -f)$ is a rational point. Therefore, both the coordinates of the centre are rational numbers. Obviously, the possible values of p are 1, 2. Similarly, the possible values of q are 1, 2. Thus for this case, (p, q) may be chosen in 2×2, i.e., 4 ways. Now, (p, q) can be, without restriction, chosen in 6×6, i.e., 36 ways.

Hence, the probability that at the most two rational points exist on the circle is $(36 - 4)/36 = 32/36 = 8/9$.

104. a. $P(A) = P(B) = P(C)$ and $P(A) + P(B) + P(C) = 1$

$$\therefore \quad P(A) = P(B) = P(C) = \frac{1}{3}$$

Also,

$$P(X) = \frac{5}{12}, P(X/A) = \frac{3}{8}, P(X/B) = \frac{1}{4}$$

We have,

$$P(X) = P(A)\, P(X/A) + P(B)\, P(X/B) + P(C)\, P(X/C)$$

$$\therefore \quad \frac{5}{12} = \frac{1}{3}\left\{\frac{3}{8} + \frac{1}{4} + P(X/C)\right\}$$

$$\Rightarrow \quad P(X/C) = \frac{5}{8}$$

105. c. A: car met with an accident

B_1: driver was alcoholic, $P(B_1) = 1/5$

B_2: driver was sober, $P(B_2) = 4/5$

$P(A/B_1) = 0.001$; $P(A/B_2) = 0.0001$

$$P(B_1/A) = \frac{(0.2)\,(0.001)}{(0.2)\,(0.001) + (0.8)(0.0001)} = 5/7$$

106. a.

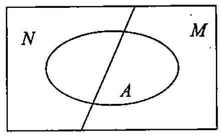

Fig. 9.12

Let N be the event of picking up a normal die; $P(N) = 1/4$. Let M be the event of picking up a magnetic die; $P(M) = 3/4$. Let A be the event that die shows up 3.

$$\therefore \quad P(A) = P(A \cap N) + P(A \cap M)$$

$$= P(N)P(A/N) + P(M)\, P(A/M)$$

$$= \frac{1}{4} \times \frac{1}{6} + \frac{3}{4} \times \frac{2}{3} = \frac{7}{24}$$

$$P(N/A) = \frac{P(N \cap A)}{P(A)} = \frac{(1/4)(1/6)}{7/24} = \frac{1}{7}$$

107. c. We have,

$n(S) = 5^5$

For computing favourable outcomes, 2 boxes which are to remain empty, can be selected in ${}^5C_2$ ways and 5 marbles can be placed in the remaining 3 boxes in groups of 221 or 311 in

$$3!\left[\frac{5!}{2!2!2!} + \frac{5!}{3!2!}\right] = 150 \text{ ways} \Rightarrow n(A) = {}^5C_2 \times 150$$

Hence,

$$P(E) = {}^5C_2 \times \frac{150}{5^5} = \frac{60}{125} = \frac{12}{25}$$

108. a. $n(S) = {}^{10}C_7 = 120$

$$n(A) = {}^5C_4 \times {}^3C_2 \times {}^2C_1$$

$$P(E) = \frac{5\times 3\times 2}{120} = \frac{1}{4}$$

109. a. For ranked 1 and 2 players to be winners and runners up, respectively, they should not be paired with each other in any round. Therefore, the required probability is 30/31 × 14/15 × 6/7 × 2/3 = 16/31.

110. a. The total number of ways of selecting 3 integers from 20 natural numbers is ${}^{20}C_3 = 1140$. Their product is a multiple of 3 means at least one number is divisible by 3. The numbers which are divisible by 3 are 3, 6, 9, 12, 15, 18 and the number of ways of selecting at least one of them is ${}^6C_1 \times {}^{14}C_2 + {}^6C_2 \times {}^{14}C_1 + {}^6C_3 = 776$. Hence, the required probability is 776/1140 = 194/285.

111. d. Since there are r cars in N places, total number of selection of places out of $N-1$ places for $r-1$ cars (excepting the owner's car) is

$${}^{N-1}C_{r-1} = \frac{(N-1)!}{(r-1)!\,(N-r)!}$$

If neighbouring places are empty, then $r-1$ cars must be parked in $N-3$ places. So, the favourable number of cases is

$${}^{N-3}C_{r-1} = \frac{(n-3)!}{(r-1)!\,(N-r-2)!}$$

Therefore, the required probability is

$$\frac{(N-3)!}{(r-1)!\,(N-r-2)!} \times \frac{(r-1)!\,(N-r)!}{(N-1)!}$$

$$= \frac{(N-1)(N-r-1)}{(N-1)(N-2)} = \frac{{}^{N-r}C_2}{{}^{N-1}C_2}$$

112. b. The sum of the digits can be 7 in the following ways: 07, 16, 25, 34, 43, 52, 61, 70.

$$\therefore \quad (A=7) = \{07, 16, 25, 34, 43, 52, 61, 70\}$$

Similarly,

$$(B=0) = \{00, 01, 02, \ldots, 10, 20, 30, \ldots, 90\}$$

Thus,

$$(A=7)\cap(B=0) = \{09, 70\}$$

$$\therefore \quad P((A=7)\cap(B=0)) = \frac{2}{100},\ P((B=0)) = \frac{19}{100}.$$

Hence,

$$P(A=7 \mid B=0) = \frac{P((A=7)\cap(B=0))}{P(B=0)} = \frac{\frac{2}{100}}{\frac{19}{100}} = \frac{2}{19}$$

113. b. Let the probability of getting a tail in a single trial be $p = 1/2$. The number of trials be $n = 100$ and the number of trials in 100 trials be X. We have,

$$P(X=r) = {}^{100}C_r p^r q^{n-r}$$

$$= {}^{100}C_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{100-r}$$

$$= {}^{100}C_r\left(\frac{1}{2}\right)^{100}$$

Now,

$$P(X=1)+P(X=3)+\cdots+P(X=49)$$

$$= {}^{100}C_1\left(\frac{1}{2}\right)^{100} + {}^{100}C_3\left(\frac{1}{2}\right)^{100} + \cdots + {}^{100}C_{49}\left(\frac{1}{2}\right)^{100}$$

$$= ({}^{100}C_1 + {}^{100}C_3 + \cdots + {}^{100}C_{49})\left(\frac{1}{2}\right)^{100}$$

But

$${}^{100}C_1 + {}^{100}C_3 + \cdots + {}^{100}C_{99} = 2^{99}$$

Also,

$${}^{100}C_{99} = {}^{100}C_1$$

$${}^{100}C_{97} = {}^{100}C_3, \ldots\ {}^{100}C_{51} = {}^{100}C_{49}$$

Thus,

$$2({}^{100}C_1 + {}^{100}C_3 + \cdots + {}^{100}C_{49}) = 2^{99}$$

$$\Rightarrow \quad {}^{100}C_1 + {}^{100}C_3 + \cdots + {}^{100}C_{49} = 2^{98}$$

Therefore, probability of required event is

$$\frac{2^{98}}{2^{100}} = \frac{1}{4}\ 2^{98}/2^{100} = 1/4$$

114. a. Let A denote the event that a sum of 5 occurs, B the event that a sum of 7 occurs and C the event that neither a sum of 5 nor a sum of 7 occurs. We have,

$$P(A) = \frac{4}{36} = \frac{1}{9},\ P(B) = \frac{6}{36} = \frac{1}{6}$$

$$P(C) = \frac{26}{36} = \frac{13}{18}$$

Thus, probability that A occurs before B is

$$P[A \text{ or } (C\cap A) \text{ or } (C\cap C\cap A) \text{ or } \ldots]$$

$$= P(A) + P(C\cap A) + P(C\cap C\cap A) + \cdots$$

$$= P(A) + P(C)P(A) + P(C)^2P(A) + \cdots$$

$$= \frac{1}{9} + \left(\frac{13}{18}\right)\times\frac{1}{9} + \left(\frac{13}{18}\right)^2\frac{1}{9} + \cdots$$

$$= \frac{1/9}{1-13/18} = \frac{2}{5}$$

115. a. In any number the last digits can be 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Therefore, last digit of each number can be chosen in 10 ways. Thus, exhaustive number of ways is 10^n. If the last digit be 1, 3, 7 or 9, then none of the numbers can be even or end in 0 or 5. Thus, we have a choice of 4 digits, viz., 1, 3, 7 or 9 with which each of n numbers should end. So favourable number of ways is 4^n. Hence, the required probability is

$$\frac{4^n}{10^n}=\left(\frac{2}{5}\right)^n$$

116. c. The probability that A gets r heads in three tosses of a coin is

$$P(X=r)={}^3C_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{3-r}={}^3C_r\left(\frac{1}{2}\right)^3$$

The probability that A and B both get r heads in three tosses of a coin is

$${}^3C_r\left(\frac{1}{2}\right)^3{}^3C_r\left(\frac{1}{2}\right)^3=({}^3C_r)^2\left(\frac{1}{2}\right)^6$$

Hence, the required probability is

$$\sum_{r=0}^{3}({}^3C_r)^2\left(\frac{1}{2}\right)^6=\left(\frac{1}{2}\right)^6\{1+9+9+1\}=\frac{20}{64}=\frac{5}{16}$$

117. c. We know that the number of subsets of a set containing n elements is 2^n. Therefore, the number of ways of choosing P and Q is

$${}^{2^n}C_1\times{}^{2^n}C_1=2^n\times 2^n=4^n$$

Out of n elements, m elements are chosen and then from the remaining $n-m$ elements either an element belongs to P or Q. But not both P and Q. Suppose P contains r elements from the remaining $n-m$ elements. Then, Q may contain any number of elements from the remaining $(n-m)-r$ elements Therefore, P and Q can be chosen in ${}^{n-m}C_r2^{(n-m)-r}$ ways.

But r can vary from 0 to $n-m$. So, in general the number of ways in which P and Q can be chosen is

$$\left(\sum_{r=0}^{n-m}{}^{n-m}C_r2^{(n-m)-r}\right){}^nC_m=(1+2)^{n-m}\,{}^nC_m={}^nC_m3^{n-m}$$

Hence, the required probability is ${}^nC_m3^{n-m}/4^n$.

118. c. In the first 9 throws, we should have three sixes and six non-sixes; and a six in the 10^{th} throw, and thereafter it does not matter whatever face appears. So, the required probability is

$${}^9C_3\left(\frac{1}{6}\right)^3\times\left(\frac{5}{6}\right)^6\times\frac{1}{6}\times\underbrace{1\times1\times1\times\cdots\times1}_{10\text{ times}}$$

$$=\frac{84\times5^6}{6^{10}}$$

119. c. Let the probability that a man aged x dies in a year p. Thus the probability that a man aged x does not die in a year $=1-p$. The probability that all n men aged x do not die in a year is $(1-p)^n$. Therefore, the probability that at least one man dies in a year is $1-(1-p)^n$. The probability that out of n men, A_1 dies first is $1/n$. Since this event is independent of the event that at least one man dies in a year, hence, the probability that A_1 dies in the year and he is first one to die is $1/n\,[1-(1-p)^n]$.

120. b. The required probability is

$$\frac{n^2}{{}^{2n}C_2}\frac{(n-1)^2}{{}^{2n-2}C_2}\frac{(n-2)^2}{{}^{2n-4}C_2}\cdots\frac{2^2}{{}^4C_2}\frac{1^2}{{}^2C_2}$$

$$=\frac{(1\times2\times3\times4\times\cdots\times(n-1)n)^2}{\frac{(2n)!}{2^n}}=\frac{2^n(n!)^2}{(2n)!}=\frac{2^n}{{}^{2n}C_n}$$

121. a. The probability of getting a head in a single toss of a coin is $p=1/2$ (say). The probability of getting 5 or 6 in a single throw of a die is $q=2/6=1/3$ (say). Therefore, the required probability is

$$p+(1-p)(1-q)p+(1-p)(1-q)(1-p)(1-q)p+\cdots$$

$$=p+(1-p)(1-q)p+(1-p)^2(1-q)^2p+\cdots$$

$$=\frac{p}{1-(1-p)(1-q)}$$

$$=\frac{1/2}{1-1/2\times2/3}=\frac{3}{4}$$

122. a. Consider the following events.

A: ball drawn is black

E_1: bag I is chosen

E_2: bag II is chosen

E_3: bag III is chosen

Then,

$$P(E_1)=(E_2)=P(E_3)=\frac{1}{3}$$

$$P(A/E_1)=\frac{3}{5},\ P(A/E_2)=\frac{1}{5},\ P(A/E_3)=\frac{7}{10}$$

Therefore, the required probability is

$$P(E_3/A)$$

$$=\frac{P(E_3)P(A/E_3)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}=\frac{7}{15}$$

123. c. $f'(x)=3x^2+2ax+9$

$y=f(x)$ is increasing

$\Rightarrow$ $f'(x)\geq0,\ \forall x$ and for $f'(x)=0$ should not form an interval

$\Rightarrow$ $(2a)^2-4\times3\times b\leq0\Rightarrow a^2-3b\leq0$

This is true for exactly 16 ordered pairs (a, b), $1\leq a, b\leq 6$, namely (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 6) and (4, 6). Thus, the required probability is $16/36=4/9$

124. d. Let p_i denote the probability that out of 10 tosses, head occurs i times and no two heads occur consecutively. It is clear that $i>5$.

For $i=0$, i.e., no head, $p_0=1/2^{10}$.

For $i=1$, i.e., one head, $p_1={}^{10}C_1\,(1/2)^1\,(1/2)^9=10/2^{10}$.

Now for $i=2$, we have 2 heads and 8 tails. Then, we have 9 possible places for heads. For example, see the construction:

$$x\,T\,x\,T\,x\,T\,x\,T\,x\,T\,x\,T\,x\,T\,x\,T\,x$$

Here x represents possible places for heads.

$$\therefore\quad p_2={}^9C_2\left(\frac{1}{2}\right)^2(1/2)^8=36/2^{10}$$

Similarly,

$$p_3={}^8C_3/2^{10}=56/2^{10}$$

$p_4 = {}^7C_2/2^{10} = 35/2^{10}$

$p_5 = {}^6C_5/2^{10} = 6/2^{10}$

$\therefore \quad p = p_0 + p_1 + p_2 + p_3 + p_4 + p_5$

$$= \frac{1+10+36+56+35+6}{2^{10}} = \frac{144}{2^{10}} = \frac{9}{64}$$

125. c. Given limit,

$$\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{2}{x}}$$

$$= \lim_{x\to 0}\left(1+\frac{a^x+b^x-2}{2}\right)^{\frac{2}{a^x+b^x-2}\,\lim_{x\to 0}\left(\frac{a^x-1+b^x-1}{x}\right)}$$

$= e^{\log ab} = ab = 6$

Total number of possible ways in which a, b can take values is $6 \times 6 = 36$. Total possible ways are (1, 6), (6, 1), (2, 3), (3, 2). The total number of possible ways is 4. Hence, the required probability is $4/36 = 1/9$.

126. c. Let A denote the event that target is hit when x shells are fired at point I. Let P_1 and P_2 denote the event that the target is at point I and II, respectively. We have $P(P_1) = 8/9$, $P(P_2) = 1/9$, $P(A/P_1) = 1 - (1/2)^x$, $P(A/P_2) = 1 - (1/2)^{55-x}$.

Now from total probability theorem,

$$P(A) = P(P_1)\,P(A/P) + P(P_2)\,P(A/P_2)$$

$$= \frac{1}{9}\left(8 - 8\left(\frac{1}{2}\right)^x + 1 - \left(\frac{1}{2}\right)^{55-x}\right)$$

$$= \frac{1}{9}\left(9 - 8\left(\frac{1}{2}\right)^x - \left(\frac{1}{2}\right)^{55-x}\right)$$

Now,

$$\frac{dP(A)}{dx} = \frac{1}{9}\left(-8\left(\frac{1}{2}\right)^x \ln\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{55-x}\ln\left(\frac{1}{2}\right)\right)$$

(Note that in this step, it is being assumed that $x \in R^+$)

$$= \frac{1}{9}\ln\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{55-x}\left(1 - \left(\frac{1}{2}\right)^{2x-58}\right)$$

> 0 if $x < 29$

< 0 if $x > 29$

Therefore, $P(A)$ is maximum at $x = 29$. Thus, '29' shells must be fired at point I.

127. d. In the first case, the urn constains 3 red and n white balls. The probability that colour of both the balls matches is

$$\frac{{}^3C_2 + {}^nC_2}{{}^{n+3}C_2} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{6 + n(n-1)}{(n+3)(n+2)} = \frac{1}{2}$$

$\Rightarrow \quad 2(n^2 - n + 6) = n^2 + 5n + 6$

$\Rightarrow \quad n^2 - 7n + 6 = 0$

$\Rightarrow \quad n = 1$ or 6 (1)

In the second case,

$$\frac{3}{n+3}\,\frac{3}{n+3} + \frac{n}{n+3}\,\frac{n}{n+3} = \frac{5}{8}$$

Solving, we get

$n^2 - 10n + 9 = 0$

$\Rightarrow \quad n = 9$ or 1. (2)

From Eqs. (1) and (2), we have $n = 1$.

128. b. Let us consider the following events.

A: card shows up black

B_1: card with both sides black

B_2: card with both sides white

B_3: card with one side white and one black

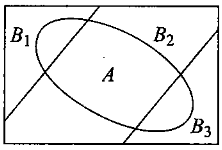

Fig. 9.13

$$P(B_1) = \frac{2}{10},\ P(B_2) = \frac{3}{10},\ P(B_3) = \frac{5}{10}$$

$$P(A/B_1) = 1,\ P(A/B_2) = 0,\ P(A/B_3) = \frac{1}{2}$$

$$P(B_1/A) = \frac{\frac{2}{10}\times 1}{\frac{2}{10}\times 1 + \frac{3}{10}\times(0) + \frac{5}{10}\times\frac{1}{2}} = \frac{4}{4+5} = \frac{4}{9}$$

129. d. A: exactly one ace

B: both aces

E: $A \cup B$

$$P(B/A \cup B)) = \frac{{}^4C_2}{{}^4C_1\,{}^{12}C_1 + {}^4C_2} = \frac{6}{54} = \frac{1}{9}$$

130. b. $n(S) = \dfrac{9!}{4!.5!} = 126$

$n(A) = 0$ to F and F to P

$$= \frac{5!}{2!\times 3!} \times \frac{4}{2!\times 2!} = 60$$

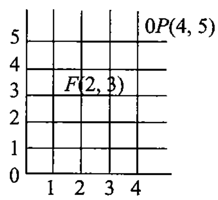

Fig. 9.14

$$\Rightarrow \quad P(A) = \frac{60}{126} = \frac{10}{21}$$

Multiple Correct Answers Type

1. a, c, d.

Since A and B are independent events, therefore,

$$P(A \cap B) = P(A)\,P(B) = \frac{1}{2} \times \frac{1}{5} = \frac{1}{10}$$

$$P(A/B) = P(A) = \frac{1}{2}$$

Now,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{2} + \frac{1}{5} - \frac{1}{10} = \frac{3}{5}$$

Now,

$$P(A/A \cup B) = \frac{P[A \cap (A \cup B)]}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)} = \frac{1/2}{3/5} = \frac{5}{6}$$

$$P[(A \cap B)/(\bar{A} \cup \bar{B})] = P(A \cap B/\overline{(A \cap B)}) = 0$$

2. a, b, c.

We are given that

$$P(A \cap B') = 0.20,\ P(A' \cap B) = 0.15,\ P(A \cap B) = 0.10$$

Now,

$$P(B) = P(A' \cap B) + P(A \cap B) = 0.15 + 0.20 = 0.35$$

$$\therefore\quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.10}{0.35} = \frac{2}{7}$$

$$P(A) = P(A \cap B') + P(A \cap B) = 0.3$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.55$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{1}{3}$$

3. a, b, c, d.

$$A \subseteq A \cup B$$

$$\Rightarrow\quad P(A) \le P(A \cup B) \Rightarrow P(A \cup B) \ge \frac{3}{4}$$

Also,

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$\ge P(A) + P(B) - 1$$

$$= \frac{3}{4} + \frac{5}{8} - 1 = \frac{3}{8}$$

Now,

$$A \cap B \subseteq B$$

$$\Rightarrow\quad P(A \cap B) \le P(B) = \frac{5}{8}$$

$$\therefore\quad \frac{3}{8} \le P(A \cap B) \le \frac{5}{8} \qquad (1)$$

and

$$P(A \cap B') = P(A) - P(A \cap B)$$

$$\Rightarrow\quad \frac{3}{4} - \frac{5}{8} \le P(A \cap B') \le \frac{3}{4} - \frac{3}{8}$$

$$\Rightarrow\quad \frac{1}{8} \le P(A \cap B') \le \frac{3}{8}$$

$$\therefore\quad P(A \cap B) = P(B) - P(A' \cap B) \qquad \text{[Using Eq. (1)]}$$

$$\Rightarrow\quad \frac{3}{8} \le P(B) - P(A' \cap B) \le \frac{5}{8}$$

$$\Rightarrow\quad 0 \le P(A' \cap B) \le \frac{1}{4}$$

4. a, c.

Given that A and B are mutually exclusive events.

$$\therefore\quad A \cap B = \phi$$

$$\Rightarrow\quad A \subseteq \bar{B} \text{ and } B \subseteq \bar{A}$$

$$\Rightarrow\quad P(A) \le P(\bar{B}) \text{ and } P(B) \le P(\bar{A})$$

5. a, b, c.

Let 'H' be the event that married man watches the show and 'W' be the probability that married woman watches the show.

$$\therefore\quad P(H) = 0.4,\ P(W) = 0.5,\ P(H/W) = 0.7$$

a. $P(H \cap W) = P(W)\,P(H/W) = 0.5 \times 0.7 = 0.35$

b. $P(W/H) = \dfrac{P(H \cap W)}{P(H)} = \dfrac{0.35}{0.4} = \dfrac{7}{8}$

c. $P(H \cup W) = P(H) + P(W) - P(H \cap W)$

$$= 0.4 + 0.5 - 0.35 = 0.55$$

6. b, c.

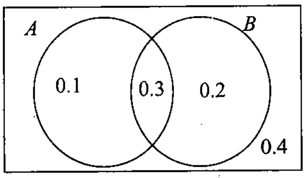

Fig. 9.15

$$P(A) = 0.4,\ P(B) = 0.5,\ P(A \cap B) = 0.3$$

$$\Rightarrow\quad P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6$$

Now,

$$P(E_1) = \frac{P(A \cap B)}{P(A \cup B)} = \frac{0.3}{0.6} = \frac{1}{2}$$

$$P(E_2) = \frac{P(A \cap \bar{B}) + P(\bar{A} \cap B)}{P(A \cup B)} = \frac{0.1 + 0.2}{0.6} = \frac{1}{2}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.5} = \frac{3}{5} = 0.60;$$

$$P(B/A) = \frac{0.3}{0.4} = \frac{3}{4} = 0.75 = 0.75$$

$$P(A/(A \cup B)) = \frac{P(A)}{P(A \cup B)} = \frac{0.4}{0.6} = \frac{2}{3}$$

$$\frac{P(B)}{P(A \cup B)} = \frac{0.5}{0.6} = \frac{5}{6}$$

7. c, d.

$P(E_1) = 1 - P$ (unit's place in both is 1, 2, 3, 4, 6, 7, 8, 9)

$$P(E_1 = 0 \text{ or } 5) = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25}$$

$$P(E_2{:}\ 5) = P(1\ 3\ 5\ 7\ 9) - P(1\ 3\ 7\ 9)$$

$$= \frac{1}{4} - \frac{4}{25}$$

$$= \frac{25 - 16}{100} = \frac{9}{100}$$

$$\frac{P(E_2)}{P(E_1)} = \frac{9}{100} \times \frac{25}{9} = \frac{1}{4}$$

$$P(E_1) = 4P(E_2)$$

$$P(E_2/E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)} = \frac{P(E_2)}{P(E_1)} = \frac{1}{4}$$

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{P(E_2)}{P(E_2)} = 1$$

8. a, c, d.

$$P(E) = \frac{^{2n}C_n}{2^{2n}} = \frac{(2n)!}{n!\,n!\,2^n\,2^n}$$

$$= \frac{1 \times 2 \times 3 \times \cdots \times (2n)}{n!\,n!\,2^n\,2^n}$$

$$= \frac{1 \times 3 \times 5 \cdots \times (2n-1)}{n!\;2^n}$$

Now,

$$\prod_{r=1}^{n}\left(\frac{2r-1}{2r}\right) = \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times (2n)}$$

$$= \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{(1 \times 2 \times 3 \times \cdots \times n)2^n}$$

$$\sum_{r=0}^{n}\left(\frac{^nC_r}{2^n}\right)^2 = \frac{1}{2^n 2^n}\sum_{r=0}^{n}(^nC_r)^2$$

$$= \frac{1}{2^n 2^n}\,^{2n}C_n$$

Also,

$$\frac{\sum_{r=0}^{n}(^nC_r)^2}{\left(\sum_{r=0}^{2n}\,^{2n}C_r\right)} = \frac{^{2n}C_n}{2^{2n}}$$

9. a, b. The probability that both will be alive for 10 years, hence, i.e., the probability that the man and his wife both will be alive 10 years hence is $0.83 \times 0.87 = 0.7221$. The probability that at least one of them will be alive is

$1 - P$ {That none of them remains alive 10 years hence}

$= 1 - (1 - 0.83)(1 - 0.87) = 1 - 0.17 \times 0.13$

$= 0.9779$

10. b, c, d.

a. False.

$$P(TTT \text{ or } HHH) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

b. $$\frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B^c)}{1 - P(B)} = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

$$P(A \cap B)[1 - P(B)] = P(B)P(A) - P(B)P(A \cap B)$$

$$P(A \cap B) = P(A)P(B)$$

Hence, the given statement is true.

c. Let E_1 be the white ball is drawn in first draw; E_2 be the event that black ball is drawn in second draw; E be the event that white ball is drawn in second draw.

$$\therefore\; P(E) = P(E/E_1)P(E_1) + P(E/E_2)P(E_2)$$

$$= \frac{d+w}{w+b+d}\left(\frac{w}{w+b}\right) + \frac{w}{w+b+d}\left(\frac{b}{w+b}\right)$$

$$= \left(\frac{w}{w+b}\right)\left(\frac{d+w}{w+b+d} + \frac{b}{w+b+d}\right)$$

$$= \left(\frac{w}{w+b}\right)$$

which is independent of d.

d. To prove that A, B, C are pairwise independent only. Now,

$$P(A \cap B) = P((A \cap B \cap C) \cup (A \cap B \cap C))$$
$$= P(A \cap B \cap C) + P(A \cap B \cap C)$$
$$= P(A)\,P(B)\,P(\bar{C}) + P(A)\,P(B)\,P(C) \quad \text{(given)}$$
$$= P(A) \times P(B)\,[P(C') + P(\bar{C})]$$
$$= P(A) \times P(B)$$

Similarly, for the other two. Hence, this statement is correct.

11. a, b, c, d.

a. $$P(E_1) = 1 - P(RRR) = 1 - \left[\frac{1}{3} \times \frac{2}{4} \times \frac{3}{5}\right] = 0.9$$

b. $$P(E_2) = 3P(BRR) = 3 \times \frac{2}{3} \times \frac{1}{4} \times \frac{2}{5} = 0.2$$

c. $$P(E_3) = P(RRR/(RRR \cup BBB))$$

$$= \frac{P(RRR)}{P(RRR) + P(BBB)}$$

$$= \frac{0.1}{0.1 + \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5}}$$

$$= \frac{0.1}{0.1 + 0.4} = 0.2$$

d. $$P(E_4) = 1 - P(BBB) = 1 - \frac{2}{5} = 0.6$$

12. a, b, c, d.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \frac{5}{8} = \frac{3}{8} + \frac{4}{8} - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = \frac{2}{8} = \frac{1}{4}$$

Now,

$$P(A^c/B) = \frac{P(A^c \cap B)}{P(B)}$$

$$= \frac{P(B) - P(A \cap B)}{P(B)}$$

$$= 1 - 2\left(\frac{1}{4}\right)$$

$$= \frac{1}{2}$$

$$2P(A/B^c) = \frac{2P(A \cap B^C)}{P(B^C)}$$

$$= \frac{2(P(A) - P(A \cap B))}{1 - P(B)}$$

$$= 4\left(\frac{3}{8} - \frac{2}{8}\right) = \frac{1}{2}$$

Hence option (a) is correct.

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{4} \times \frac{2}{1} = \frac{1}{2} = P(B)$$

Hence (b) is correct. Again,

$$P(A^c/B^c) = \frac{P(A^C \cap B^C)}{P(B^C)}$$

$$= \frac{1 - P(A \cup B)}{1 - P(B)}$$

$$= 2\left(1 - \frac{5}{8}\right) = \frac{3}{4}$$

$$P(B/A^c) = \frac{P(B \cap A^C)}{1 - P(A)}$$

$$= \frac{P(B) - P(A \cap B)}{5/8}$$

$$= \frac{1/2 - 1/4}{5/8}$$

$$= \frac{1}{4} \times \frac{8}{5}$$

$$= \frac{2}{5}$$

Hence,

$$8P(A^c/B^c) = 15P(B/A^c)$$

Hence, (c) is not correct. Again,

$$2P(A/B^c) = \frac{1}{2}$$

$$\Rightarrow P(A/B^c) = \frac{1}{4} = P(A \cap B)$$

Hence (d) is correct.

13. a, c.

Let p_1, p_2 be the chances of happening of the first and second events, respectively. Then according to the given conditions, we have

$$p_1 = p_2^2$$

and

$$\frac{1 - p_1}{p_1} = \left(\frac{1 - p_2}{p_2}\right)^3$$

Hence,

$$\frac{1 - p_2^2}{p_2^2} = \left(\frac{1 - p_2}{p_2}\right)^3 \Rightarrow p_2(1 + p_2) = (1 - p_2)^2$$

$$\Rightarrow 3p_2 = 1 \Rightarrow p_2 = \frac{1}{3}$$

and so

$$p_1 = \frac{1}{9}$$

14. a, b.

Let the number of red and blue balls be r and b, respectively.

Then, the probability of drawing two red balls is

$$p_1 = \frac{{}^rC_2}{{}^{r+b}C_2} = \frac{r(r-1)}{(r+b)(r+b-1)}$$

The probability of drawing two blue balls is

$$p_2 = \frac{{}^bC_2}{{}^{r+b}C_2} = \frac{b(b-1)}{(r+b)(r+b-1)}$$

The probability of drawing one red and one blue ball is

$$p_3 = \frac{{}^rC_1 \times {}^bC_1}{{}^{r+b}C_2} = \frac{2br}{(r+b)(r+b-1)}$$

By hypothesis, $p_1 = 5p_2$ and $p_3 = 6p_2$.

$\therefore$ $r(r-1) = 5b(b-1)$ and $2br = 6b(b-1)$

$\Rightarrow$ $r = 6, b = 3$

15. a, b, d.

We have, the probability that the bomb strikes the target is $p = 1/2$. Let n be the number of bombs which should be dropped to ensure 99% chance or better of completely destroying the target. Then, the probability that out of n bombs at least two bombs strike the target is greater than 0.99. Let X denote the number of bombs striking the target. Then

$$P(X = r) = {}^nC_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{n-r} = {}^nC_r\left(\frac{1}{2}\right)^n, r = 0, 1, 2, \ldots, n$$

We should have

$$P(X \geq 2) \geq 0.99$$

$$\Rightarrow \{1 - P(X < 2)\} \geq 0.99$$

$$\Rightarrow 1 - \{P(X = 0) + P(X = 1)\} \geq 0.99$$

$$\Rightarrow 1 - \left\{(1+n)\frac{1}{2^n}\right\} \geq 0.99$$

$$\Rightarrow 0.001 \geq \frac{1+n}{2^n}$$

$$\Rightarrow 2^n > 100 + 100n \Rightarrow n \geq 11$$

Thus, the minimum number of bombs is 11.

16. a, b, c, d.

We have,

P(exactly one of A, B occurs)

$= P[(A \cap \bar{B}) \cup \bar{A} \cap B)]$

$= P(A \cap \bar{B}) + P(\bar{A} \cap B)$

$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$

$= P(A) + P(B) - 2P(A \cap B)$

$= P(A \cup B) - P(A \cap B)$

Also,

P(exactly one of A, B occurs)

$= [1 - P(\bar{A} \cap \bar{B})] - [1 - P(\bar{A} \cup \bar{B})]$

$= P(\bar{A} \cup \bar{B}) - P(\bar{A} \cap \bar{B})] = P(\bar{A}) + P(\bar{B}) - 2P(\bar{A} \cap \bar{B})$

17. a, b, c.

Option (d) is true if and only if A and B are independent.

18. a, b, c.

$$P(A/B) = P(A) = \frac{1}{2}$$

$$P\{A/(A \cup B)\} = \frac{P[A \cap (A \cup B)]}{P(A \cup B)}$$

$[\because A \cap (A \cup B) = A \cap (A - B - A \cap B)]$

$= A - A \cap B - A \cap B = a\}$

$$\Rightarrow \quad P\left(\frac{A}{A \cup B}\right) = \frac{P(A)}{P(A \cup B)}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2} - \frac{1}{5} - \frac{1}{10}} = \frac{\frac{1}{2}}{\frac{6}{10}} = \frac{5}{6}$$

and similarly,

$$P\left(\frac{A \cap B}{A' \cup B'}\right) = 0$$

19. b, c.

Let $P(A) = x$ and $P(B) = y$. Since A and B are independent events, therefore,

$$P(\bar{A} \cap B) = 2/15 \quad \Rightarrow P(\bar{A})P(B) = 2/15$$

$$\Rightarrow (1 - P(A))\, P(B) = 2/15$$

$$\Rightarrow (1 - x)y = 2/15 \qquad (1)$$

and

$$P(A \cap \bar{B}) = \frac{1}{6} \Rightarrow P(A)\, P(\bar{B}) = \frac{1}{6}$$

$$\Rightarrow x(1 - y) = \frac{1}{6}$$

$$\Rightarrow x - xy = \frac{1}{6} \qquad (2)$$

Subtracting Eq. (1) from Eq. (2), we get

$$x - y = \frac{1}{30} \quad \Rightarrow x = \frac{1}{30} + y$$

Putting this value of x in Eq. (1), we get

$$y - y\left(\frac{1}{30} + y\right) = \frac{2}{15}$$

$\Rightarrow \quad 30y - y - 30y^2 = 2/5$

$\Rightarrow \quad 30y^2 - 29y + 4 = 0$

$\Rightarrow \quad (6y - 1)(5y - 4) = 0$

$\Rightarrow \quad y = 1/6$ or $y = 4/5$

$\Rightarrow \quad P(B) = 1/6$ or $P(B) = 4/5$

20. a, b, c.

a. $P(A) = \frac{1}{5}, P(B) = \frac{7}{25}, P(B/A) = \frac{9}{10}$

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - \left[\frac{1}{5} + \frac{7}{25} - P(A)\, P(B/A)\right]$$

$$= 1 - \left[\frac{1}{5} + \frac{7}{25} - \frac{1}{5} \times \frac{7}{25}\right] = \frac{7}{10}$$

b. $$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(A)\, P(B/A)}{P(B)}$$

$$= \frac{\frac{1}{5} \times \frac{9}{10}}{\frac{7}{25}}$$

$$= \frac{9}{50} \times \frac{25}{7} = \frac{9}{14} = \frac{18}{28}$$

c. $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A)P(B/A)$$

$$= \frac{1}{5} + \frac{7}{25} - \frac{1}{5} \times \frac{9}{10}$$

$$= \frac{10 + 14 - 9}{50}$$

$$= \frac{3}{10}$$

d. $$P(A' \cup B') = 1 - P(A \cap B)$$

$$= 1 - P(A)P(B/A)$$

$$= 1 - \frac{1}{5} \times \frac{9}{10}$$

$$= \frac{41}{50}$$

21. b, c, d.

Roots of $x^2 + px + q = 0$ will be real if $p^2 \geq 4q$.

The possible selections are as follows:

p	q
1	–
2	1
3	1, 2
4	1, 2, 3, 4
5	1, 2, 3, 4, 5, 6
6	1, 2, ..., 9
7	1, 2, ..., 10
8	1, 2, ..., 10
9	1, 2, ..., 10
10	1, 2, ..., 10
Total	62

Therefore, number of favourable ways is 62 and total number of ways is $10^2 = 100$. Hence, the required probability is 62/100 = 31/50.

The probability that the roots are imaginary is $1 - 31/50 = 19/50$.

Roots are equal when $(p, q) \equiv (2, 1), (4, 4), (9, 6)$. The probability that the roots are real and equal is 3/50. Hence, probability that the roots are real and distinct is 3/5.

22. a, b, c.

Here total number of cases is ${}^8C_2 = 28$.

a. Favourable number of case is 13.

For $2 \to 6$ choices
For $1 \to 7$ choices

b. For $6 \to 5$ choices
For $7 \to 6$ choices
For $8 \to 7$ choices

c. For $1 \to 4$ choices (2, 4, 6, 8)
For $2 \to 6$ choices (3, 4, 5, 6, 7, 8)
For $3 \to 3$ choices (4, 6, 8)
For $4 \to 4$ choices (5, 6, 7, 8)
For $5 \to 2$ choices (6, 8)
For $6 \to 2$ choices (7, 8)
For $7 \to 1$ choice (8)

Alternative solution:

a. $$\frac{{}^8C_2 - {}^6C_2}{{}^8C_2} = \frac{13}{28}$$

b. $$\frac{{}^8C_2 - {}^5C_2}{{}^8C_2} = \frac{9}{14}$$

c. $$\frac{{}^8C_2 - {}^4C_2}{{}^8C_2} = \frac{11}{14}$$

23. a, d.

The probability that head appears r times is

$${}^{99}C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{99-r}$$

which is maximum when $r = 49$ or 50.

24. a, c.

Let one probability of choosing one integer k be $P(k) = \lambda/k^4$. (λ is one constant of proportionality). Then,

$$\sum_{k=1}^{2m} \frac{\lambda}{k^4} = 1$$

$$\Rightarrow \lambda \sum_{k=1}^{2m} \frac{1}{k^4} = 1$$

Let x_1 be the probability of choosing the odd number. Then,

$$x_1 = \sum_{k=1}^{m} P(2k-1) = \lambda \sum_{k=1}^{m} \frac{1}{(2k-1)^4}$$

Also,

$$1 - x_1 = \sum_{k=1}^{m} P(2k)$$

$$= \lambda \sum_{k=1}^{m} \frac{1}{(2k)^4}$$

$$< \lambda \sum_{k=1}^{m} \frac{1}{(2k-1)^4}$$

$\Rightarrow 1 - x_1 < x_1$

$\Rightarrow x_1 > 1/2$

$\Rightarrow x_2 < 1/2$

Reasoning Type

1. a. $P(A/B) \geq P(A)$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} \geq P(A)$$

$$\Rightarrow \frac{P(A \cap B)}{P(A)} \geq P(B)$$

$\Rightarrow P(B/A) \geq P(B)$

2. c. Statement 1 is true as there are six equally likely possibilities of which only two are favourable (4 and 6). Hence, probability that the obtained number is composite) is 2/6 = 1/3.

Statement 2 is not true, as the three possibilities are not equally likely.

3. c. The required probability is

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B) - P(A \cap B)]$$
$$= 0.39$$

4. a. $\because P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$$\geq P(A) + P(B) - 1$$

$$\therefore \quad P(A \cap B) \geq \frac{3}{5} + \frac{2}{3} - 1$$

$$\Rightarrow \quad P(A \cap B) \geq \frac{4}{15} \quad (1)$$

$$P(A \cap B) \leq P(A)$$

$$\Rightarrow \quad P(A \cap B) \leq \frac{3}{5} \quad (2)$$

From Eqs. (1) and (2),

$$\frac{4}{15} \leq P A \cap B) \leq \frac{3}{5} \quad (3)$$

From (3),

$$\frac{4}{15P(B)} \leq \frac{P(A \cap B)}{P(B)} \leq \frac{3}{5P(B)}$$

$$\Rightarrow \frac{2}{5} \leq P\left(\frac{A}{B}\right) \leq \frac{9}{10}$$

5. a. $P\{A \cap (B \cap C)\} = P(A \cap B \cap C) = P(A)P(B)P(C)$

$\therefore \quad P[A \cap (B \cup C)]$

$= P[(A \cap B) \cup (A \cap C)]$

$= P[(A \cap B) + (A \cap C) - P[(A \cap B) \cap (A \cap C)]$

$= p(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$

$= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C)$

$= P(A)[P(B) + P(C) - P(B)P(C)]$

$= P(A)\, P(B \cup C)$

Therefore, A and $B \cup C$ are independent events.

6. a. $2n + 1 = 5, n = 2$

$$P(E) = \frac{3n}{4n^2 - 1} = \frac{6}{15} = \frac{2}{5}$$

As a, b, c are in A.P., so

$a + c = 2b$

$\Rightarrow$ $a + c$ is even

Therefore, a and c are both even or both odd. So, the number of ways of choosing a and c is ${}^nC_2 + {}^{n+1}C_2 = n^2$.

$$\therefore \quad P(E) = \frac{n^2}{{}^{2n+1}C_3} = \frac{3n}{4n^2 - 1}$$

7. d. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore$ $1 \geq P(A) + P(B) - P(A \cap B) \geq 3/4$

$\Rightarrow$ $P(A) + P(B) - 1/8 \geq 3/4$

[since minimum value of $P(A \cap B)$ is 1/8)

$\Rightarrow$ $P(A) + P(B) \leq 1/8 + 3/4 = 7/8$

As the maximum value of $P(A \cap B)$ is 3/8, we get

$1 \geq P(A) + P(B) - 3/8$

$\Rightarrow$ $P(A) + P(B) \leq 1 + 3/8 = 11/8$

8. b. Clearly both are correct but statement 2 is not the correct explanation for statement 1.

9. b. $P(A \cup \bar{B}) = 1 - \overline{(A \cup \bar{B})} = 1 - (\bar{A} \cap B) = 1 - P(\bar{A})P(B)$

$\Rightarrow$ $0.9 = 1 - 0.6 \times P(B)$

$\Rightarrow$ $P(B) = \frac{1}{6}$

Clearly, statement 2 is not correct explanation of statement 1.

10. c. According to statement 1, the required probability is

$${}^nC_0\left(\frac{1}{2}\right)^n + {}^nC_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{n-4} + {}^nC_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^{n-8} + \cdots$$

$$= ({}^nC_0 + {}^nC_4 + {}^nC_8 + \cdots)\left(\frac{1}{2}\right)^n$$

Now consider the binomial expansion,

$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \cdots$

Putting $x + i$, where $i = \sqrt{-1}$, we get

$$(1 + i)^n = ({}^nC_0 - {}^nC_2 + {}^nC_4 - \cdots) + i({}^nC_1 - {}^nC_3 + {}^nC_5 - \cdots)$$

$$\Rightarrow \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n = ({}^nC_0 - {}^nC_2 + {}^nC_4 - \cdots) + i({}^nC_1 - {}^nC_3 + {}^nC_5 - \cdots)$$

$$\Rightarrow {}^nC_0 - {}^nC_2 + {}^nC_4 - \cdots = 2^{n/2}\cos\frac{n\pi}{4}$$

Also we know that

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = 2^{n-1}$$

$$\Rightarrow 2({}^nC_0 + {}^nC_4 + {}^nC_8 + \cdots) = 2^{n-1} + 2^{n/2}\cos\frac{n\pi}{4}$$

Hence, the required probability is

$$\frac{1}{4} + \frac{1}{2^{n/2+1}}\cos\left(\frac{n\pi}{4}\right)$$

11. d. $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$ (by definition)

$\Rightarrow$ $P(\bar{B}) = P((A \cup \bar{A}) \cap \bar{B}) = P((A \cap \bar{B}) \cup (\bar{A} \cap \bar{B}))$

Hence, statement 2 is true. Now,

$$P(A/\bar{B}) + P(\bar{A}/\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} + \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})}$$

$$= \frac{P(A \cap \bar{B}) + P(\bar{A} \cap \bar{B})}{P(\bar{B})}$$

$$= \frac{P(\bar{B})}{P(\bar{B})} = 1$$

Therefore, statement 1 is false.

12. a. In binomial theorem, we have proved statement 2. Now, there may be 0, 1, 2, 3, 4 or 5 heads in the last five throws and the same can be for the first 10 throws. The number of cases thus may be given by

$$m = {}^5C_0{}^{10}C_0 + {}^5C_1{}^{10}C_1 + {}^5C_2{}^{10}C_2 + {}^5C_3{}^{10}C_3 + {}^5C_4{}^{10}C_4 + {}^5C_5{}^{10}C_5$$

$$= {}^5C_0{}^{10}C_{10} + {}^5C_1{}^{10}C_9 + {}^5C_2{}^{10}C_8 + {}^5C_3{}^{10}C_7 + {}^5C_4{}^{10}C_6 + {}^5C_5{}^{10}C_5$$

$$= {}^{10+5}C_{10} = {}^{15}C_{10}$$

$$= 3003$$

The total number of ways (N) is $2^{15} = 32768$. Hence, the required probability is $m/N = 3003/32768$.

13. c. $P(A) + P(B) = 1$ is true as A and B are mutually exclusive and exhaustive events, but statement 2 is false as it is not given that the evênts are exhaustive.

14. b. The total number of cases, $n(S) = 4!$. Let E be the event that no letter is mailed in its correct envelope. Then the favourable number of cases is

$$4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9$$

Hence, the required probability is

$$P(E) = \frac{9}{24} = \frac{3}{8}$$

Also, the probability that all the letters are placed in the correct envelope is 1/24.

Hence, the probability that all the letters are not placed in the correct envelope is 23/24.

Hence, statement 2 is correct but does not explain statement 1.

15. a. We have,

$$P(A \cup \bar{B}) = 1 - \overline{(A \cup \bar{B})} = 1 - (\bar{A} \cap B) = 1 - P(\bar{A})\,P(B)$$

$$0.8 = 1 - 0.7 \times P(B)$$

$$\Rightarrow P(B) = \frac{2}{7}$$

Linked Comprehension Type

For Problems 1–3

1. b, 2. c, 3. a.

Sol. Let $P(i)$ be the probability that exactly i students are passing an examination. Now given that

$$P(A_i) = \lambda i^2 \text{ (where } \lambda \text{ is constant)}$$

$$\Rightarrow \sum_{i=1}^{10} P(A_i) = \sum_{i=1}^{10} \lambda i^2 = \lambda \frac{10\times 11\times 21}{6} = \lambda 385 = 1 \Rightarrow \lambda = 1/385$$

Now, $P(5) = 25/385 = 5/77$.

Let A represent the event that selected students have passed the examination.

$$\therefore \quad P(A) = \sum_{i=1}^{10} P(A/A_i)P(A_i)$$

$$= \sum_{i=1}^{10} \frac{i}{10}\frac{i^2}{385}$$

$$= \frac{1}{3850}\sum_{i=1}^{10} i^3$$

$$= \frac{10^2 11^2}{4\times 3850} = \frac{11}{14}$$

Now,

$$P(A_1/A) = \frac{P(A/A_1)P(A_1)}{P(A)}$$

$$= \frac{\frac{1}{385}\frac{1}{10}}{\frac{11}{14}}$$

$$= \frac{1}{11\times 55}\frac{1}{5}$$

$$= \frac{1}{3025}$$

For Problems 4–6

4. c, 5. b, 6. d.

Sol. Let in 8 coupons S, U, R, F appears x_1, x_2, x_3, x_4 times. Then $x_1 + x_2 + x_3 + x_4 = 8$, where $x_1, x_2, x_3, x_4 \geq 0$.

We have to find non-negative integral solutions of the equation. The total number of such solutions is ${}^{8+4-1}C_{4-1} = {}^{11}C_3 = 165$.

If a person gets at least one free packet, then he must get each coupon at least once, which is equal to number of positive integral solutions of the equation. The number of such solutions is ${}^{8-1}C_{4-1} = {}^{7}C_3 = 35$. Then, the probability that he gets exactly one free packet is

$$(35 - 1)/165 = 102/495.$$

The probability that he gets two free packets is $1/{}^{11}C_3 = 1/165$.

For Problems 7–9

7. d, 8. d, 9. b.

Sol. 7. Let p_1 be the probability of being an answer correct from section 1. Then $p_1 = 1/5$. Let p_2 be the probability of being an answer correct from section 2. Then $p_2 = 1/15$. Hence, the required probability is $1/5 \times 1/15 = 1/75$.

8. Scoring 10 marks from four questions can be done in 3 + 3 + 3 + 1 = 10 ways so as to answer 3 questions from section 2 and 1 question from section 1 correctly. Hence the required probability is

$$\frac{{}^{10}C_3\,{}^{10}C_1}{{}^{20}C_4}\frac{1}{5}\left(\frac{1}{15}\right)^3$$

9. To get 40 marks, he has to answer all questions correctly and its probability is $(1/5)^{10}\,(1/15)^{10}$. Hence, probability of getting a score less than 40 is

$$1 - \left(\frac{1}{5}\right)^{10}\left(\frac{1}{15}\right)^{10} = 1 - \left(\frac{1}{75}\right)^{10}$$

For Problems 10–12

10. c., 11. d., 12.c.

Sol. 10. x can be 2, 3, 4, 5, 6. The number of ways in which sum of 2, 3, 4, 5, 6 can occur is given by the coefficients of x^2, x^3, x^4, x^5, x^6 in

$$(3x + 2x^2 + x^3)(x + 2x^2 + 3x^3) = 3x^2 + 8x^3 + 14x^4 + 8x^5 + 3x^6$$

This shows that sum that occurs most often is 4.

11. Sum that occurs for minimum times is 2 or 6.

12. The number of ways in which different sums can occur is (3 + 2 + 1) (1 + 2 + 3) = 36. The probability of 4 is 14/36 = 7/18.

For Problems 13–15

13. c, 14. b, 15. d.

Sol. A: She gets a success

T: She studies 10 h: $P(T) = 0.1$

S: She studies 7 h: $P(S) = 0.2$

F: She studies 4 h: $P(F) = 0.7$

$P(A/T) = 0.8$, $P(A/S) = 0.6$, $P(A/F) = 0.4$

$$P(A) = P(A\cap T) + P(A\cap S) + P(A\cap F)$$

$$= P(T)\,P(A/T) + P(S)\,P(A/S) + P(F)\,P(A/F)$$

$$= (0.1)(0.8) + (0.2)(0.6) + (0.7)(0.4)$$

$$= 0.08 + 0.12 + 0.28 = 0.48$$

$$= P(F/A) = \frac{P(F\cap A)}{P(A)}$$

$$= \frac{(0.7)\,(0.4)}{0.48}$$

$$= \frac{0.28}{0.48} = \frac{7}{12}$$

$$P(F/\bar{A}) = \frac{P(F \cap \bar{A})}{P(\bar{A})}$$

$$= \frac{P(F) - P(F \cap A)}{0.52}$$

$$= \frac{(0.7) - 0.28}{0.52}$$

$$= \frac{0.42}{0.52} = \frac{21}{26}$$

For Problems 16–18

16. b, 17. a, 18. c.

Sol. $P(S/T) = \dfrac{P(S \cap T)}{P(T)}$

$\Rightarrow \quad 0.5 = \dfrac{P(S \cap T)}{0.69}$

$\Rightarrow \quad P(S \cap T) = 0.5 \times 0.69 = P(S)\, P(T)$

Therefore, S and T are independent.

$\therefore \quad P(S \text{ and } T) = P(S)P(T)$

$= 0.69 \times 0.5 = 0.345$

$P(S \text{ or } T) = P(S) + P(T) - P(S \cap T)$

$= 0.5 + 0.69 - 0.345$

$= 0.8450$

For Problems 19–21

19. d, 20. b, 21. a.

Sol. 19. Let E be the event that all the amoeba population dies out;

E_1 be the event that after first second amoeba splits into two;

E_2 be the event that after first second amoeba remains the same. Then,

$P(E) = P(E_1)P(E/E_1) + P(E_2)P(E/E_2)$

$$= \frac{1}{2}\frac{1}{4}\frac{1}{4} + \frac{1}{4}\frac{1}{4}$$

$$= \frac{3}{32}$$

20. After 2s, exactly 4 amoeba are alive, i.e. initially amoeba must split into two and in 2nd second again amoeba must split into two. Hence, the required probability is $(1/2) \times (1/2) \times (1/2) = 1/8$.

21. In first second, mother amoeba splits into two with the probability 1/2.

In 2nd second, two amoeba split into two with probability $(1/2) \times (1/2) = 1/4$.

In 3rd second, four amoeba split into two with probability $(1/2) \times (1/2) \times (1/2) \times (1/2) = 1/16$.

Hence, the probability that the population is maximum after 3s is $(1/2) \times (1/4) \times (1/16) = 1/2^7$.

For Problems 22–24

22. b, 23. b, 24. d.

Sol. The number of cubes having at least one side painted is $9 + 9 + 3 + 3 + 1 + 1 = 26$. The number of cubes having two sides painted is $4 + 4 + 1 + 1 + 1 = 12$.

For Problems 25–27

25. a, 26. d, 27. b.

Sol.

25. For the favourable cases, the points should lie inside the concentric circle of radius $r/2$. So the desired probability is given by

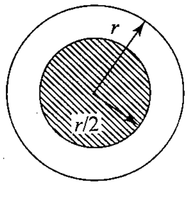

Fig. 9.16

$$\frac{\text{Area of smaller circle}}{\text{Area of larger circle}} = \frac{\pi\left(\frac{r}{2}\right)^2}{\pi r^2} = \frac{1}{4}$$

26.

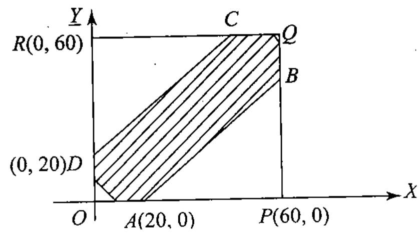

Fig. 9.17

Let A and B arrive at the place of their meeting 'a' minutes and 'b' minutes after 5 pm. Their meeting is possible only if

$|a - b| \le 20$ (1)

Clearly, $0 \le a \le 60$ and $0 \le b \le 60$. Therefore, a and b can be selected as an ordered pair (a, b) from the set $[0, 60] \times [0, 60]$.

Alternatively, it is equivalent to select a point (a, b) from the square $OPQR$, where P is $(60, 0)$ and R is $(0, 60)$ in the Cartesian plane. Now,

$$|a - b| \le 20 \Rightarrow -20 \le a - b \le 20$$

Therefore, points (a, b) satisfy the equation $-20 \le x - y \le 20$.

Hence, favourable condition is equivalent to selecting a point from the region bounded by $y \le x + 20$ and $y \ge x - 20$. Therefore, the required probability is

$$\frac{\text{Area of } OABQCDO}{\text{Area of square } OPQR} = \frac{[\text{Ar}(OPQR) - 2\text{Ar}(\Delta\, APB)]}{\text{Ar}(OPQR)}$$

$$= \frac{60 \times 60 - \frac{2}{2} \times 40 \times 40}{60 \times 60} = \frac{5}{9}$$

27. Picking two points x and y randomly from the intervals [0, 2] and [0, 1] is equivalent to picking a single point (x, y) randomly from the rectangle S shown in the following figure, which has vertices at (0, 0) (2, 0), (2, 1) and (0, 1). So, we take S as our sample space. Now, the condition $y \le x^2$ is satisfied if and only if the point (x, y) lies in the shaded region. It is the portion of the rectangle lying below the parabola $y = x^2$. Therefore, the required probability is given by

$$\frac{\text{Area of the shaded region}}{\text{Area of the rectangle } S}$$

Area of rectangle $S = 2 \times 1 = 2$. Area of shaded region is

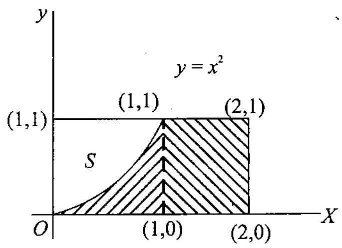

Fig. 9.18

$$\int_0^1 x^2\,dx + 1\times 1 = \frac{1}{3} + \frac{4}{3}$$

Hence, required probability is

$$\frac{\frac{4}{3}}{2} = \frac{2}{3}$$

For Problems 28–30

28. a, 29. a, 30. b.

Sol. 28. The total number of ways of painting first column when colours are not alternating is $2^8 - 2$. The total number of ways when no column has alternating colours is $(2^8 - 2)^8/2^{64}$.

29. The number of ways the square has equal number of red and black squares is ${}^{64}C_{32}$. Hence, the required probability is ${}^{64}C_{32}/2^{64}$.

30. This is possible only when the column are alternating red and black. Hence, the required probability is $2/2^{64} = 1/2^{63}$.

For Problems 31–33

31. a, 32. c, 33. d.

31. $P(A_2) = \frac{18}{36}$

$P(A_3) = \frac{12}{36} = \frac{1}{3}$

$P(A_4) = \frac{9}{36} = \frac{1}{4}$

$P(A_5) = \frac{7}{36}$

$P(A_6) = \frac{6}{36} = \frac{1}{6}$

Hence, A_3 is most probable.

32. $P(A_2) = \frac{1}{2}, P(A_3) = \frac{1}{3}, P(A_6) = \frac{1}{6}$

$\therefore\ P(A_2 \cap A_3) = P(A_2)\,P(A_3)$

$\Rightarrow\ P(A_6) = P(A_2)\,P(A_3)$

$\frac{6}{36} = \frac{1}{2} \times \frac{1}{3}$

Hence, A_2 and A_3 are independent.

33. Note that A_1 is independent with all events $A_1, A_2, A_3, A_4, \ldots, A_{12}$. Now, total number of ordered pairs is 23.

$$\underbrace{(1, 1), (1, 2), (1, 3), \ldots, (1, 11)}_{22} + (1, 12)$$

Also, A_2, A_3 and A_3, A_2 are independent. Hence, there are 25 ordered pairs.

For Problems 34–36

34. a, 35. c, 36. c.

Sol. The scores of n can be reached in the following two mutually exlcusive events:

(i) by throwing a head when the score is $(n - 1)$;

(ii) by throwing a tail when the score is $(n - 2)$

Hence,

$$P_n = P_{n-1} \times \frac{1}{2} + P_{n-2} \times \frac{1}{2} \qquad [\because P(\text{head}) = P(\text{tail}) = 1/2]$$

$$\Rightarrow\ P_n = \frac{1}{2}[P_{n-1} + P_{n-2}] \qquad (1)$$

$$\Rightarrow\ P_0 + \frac{1}{2}P_{n-1} = P_{n-1} + \frac{1}{2}P_{n-2}$$

(adding $(1/2)\,P_{n-1}$ on both sides)

$$= P_{n-2} + \frac{1}{2}P_{n-3}$$

$$\vdots$$

$$= P_2 + \frac{1}{2}P_1 \qquad (2)$$

Now, a score of 1 can be obtained by throwing a head at a single toss.

$$\therefore\ P_1 = \frac{1}{2}$$

And a score of 2 can be obtained by throwing either a tail at a single toss or a head at the first toss as well as second toss.

$$\therefore\ P_2 = \frac{1}{2} + \left(\frac{1}{2} \times \frac{1}{2}\right) = \frac{3}{4}$$

From Eq. (2), we have

$$P_n + \frac{1}{2}P_{n-1} = \frac{3}{4} + \frac{1}{2}\left(\frac{1}{2}\right) = 1$$

$$\Rightarrow\ P_n = 1 - \frac{1}{2}P_{n-1}$$

$$\Rightarrow\ P_n - \frac{2}{3} = 1 - \frac{1}{2}P_{n-1} - \frac{2}{3}$$

$$\Rightarrow\ P_n - \frac{2}{3} = -\frac{1}{2}\left(P_{n-1} - \frac{2}{3}\right)$$

$$= \left(-\frac{1}{2}\right)^2 \left(P_{n-2} - \frac{2}{3}\right)$$

$$= \left(-\frac{1}{2}\right)^3 \left(P_{n-3} - \frac{2}{3}\right)$$

$$= \left(-\frac{1}{2}\right)^{n-1} \left(P_1 - \frac{2}{3}\right)$$

$$= \left(-\frac{1}{2}\right)^{n-1} \left(\frac{1}{2} - \frac{2}{3}\right)$$

$$= \left(-\frac{1}{2}\right)^{n-1} \left(-\frac{1}{6}\right)$$

$$= \left(-\frac{1}{2}\right)^{n} \frac{1}{3}$$

$$\Rightarrow \quad P_n = \frac{2}{3} + \frac{(-1)^n}{2^n}\frac{1}{3} = \frac{1}{3}\left\{2 + \frac{(-1)^n}{2^n}\right\}$$

Now,

$$P_{100} = \frac{2}{3} + \frac{1}{3\times 2^{100}} > \frac{2}{3} \text{ and } P_{101} = \frac{2}{3} - \frac{1}{3\times 2^{101}} < \frac{2}{3}$$

$$\Rightarrow \quad P_{101} < \frac{2}{3} < P_{100}$$

For Problems 37–39

37. b, 38. c, 39. a.

Sol. 37. If a family of n children contains exactly k boys, then, by binomial distribution, its probability is

$${}^nC_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k}$$

Hence, by total probability law, the probability of a family of n children having exactly k boys is given by

$$\alpha p^n \; {}^nC_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \quad \text{(where } n \ge k\text{)}$$

Therefore, the required probability is

$$= \sum_{n=k}^{\infty} \alpha p^n \; {}^nC_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k}$$

$$= \alpha \left(\frac{1}{2}\right)^k p^k \sum_{n=k}^{\infty} {}^nC_k \left(\frac{1}{2}\right)^{n-k} (p^{n-k})$$

$$= \alpha \left(\frac{1}{2}\right)^k p^k \left[1 + {}^{(k+1)}C_1 \left(\frac{p}{2}\right) + {}^{(k+2)}C_2 \left(\frac{p}{2}\right)^2 + \cdots\right]$$

$$= \alpha \left(\frac{1}{2}\right)^k p^k \left(1 - \frac{p}{2}\right)^{-(k+1)}$$

$$= \alpha p^k (2-p)^{-(k+1)}$$

$$= \frac{2\alpha}{2-p}\left(\frac{p}{2-p}\right)^k, \; k \ge 1$$

38. Let A denote the event of a family including at least one boy. Then,

$$P(A) = \frac{2\alpha}{2-p}\sum_{k=1}^{\infty}\left(\frac{p}{2-p}\right)^k$$

$$= \frac{2\alpha}{2-p} \frac{\left(\frac{p}{2-p}\right)}{1 - \left(\frac{p}{2-p}\right)} \quad \text{(sum of infinite terms of G.P.)}$$

$$= \frac{\alpha p}{(2-p)(1-p)} \qquad (1)$$

39. Let B denote the event of a family including at least two or more boys. Then,

$$P(B) = \frac{2\alpha}{2-p}\sum_{k=2}^{\infty}\left(\frac{p}{2-p}\right)^k$$

$$= \frac{\alpha p^2}{(2-p)^2(1-p)} \qquad (2)$$

$$\therefore \quad P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)} \qquad \text{(Since } B \subset A\text{)}$$

$$= \frac{p}{2-p} \qquad \text{[From Eqs. (1) and (2)]}$$

Matrix-Match Type

1. **a → p; b → r; c → q; d → p.**

$$P(A) = \frac{{}^4C_1\,{}^8C_2}{{}^{12}C_3} = \frac{4\,28}{220} = \frac{112}{220} = \frac{28}{55}$$

$$P(B) = \frac{{}^4C_3 + {}^8C_3}{{}^{12}C_3} = \frac{4+56}{220} = \frac{60}{220} = \frac{3}{11}$$

$P(C) = P(WBB$ or BWB or WWB or $BBB)$

$$= \frac{8}{12}\times\frac{4}{11}\times\frac{3}{10}+\frac{4}{12}\times\frac{8}{11}\times\frac{3}{10}+\frac{8}{12}\times\frac{7}{11}\times\frac{4}{10}+\frac{4}{12}\times\frac{3}{11}\times\frac{2}{10}$$

$$= \frac{96+96+224+24}{12\times 110} = \frac{440}{12\times 110} = \frac{1}{3}$$

$A \cap B = \phi \Rightarrow A$ and B are mutually exclusive

$$P(B \cap C) = P(BBB) = \frac{4\times 3\times 2}{12\times 110} = \frac{1}{55}$$

Also,

$$P(B)\,P(C) = \frac{3}{11}\times\frac{1}{3} = \frac{1}{11}$$

Hence, B and C are neither independent nor mutually exclusive.

$$P(C \cap A) = P(WWB)\,\frac{8\times 7\times 4}{12\times 11\times 10} = \frac{28}{3\times 55}$$

$$P(C)\,P(A) = \frac{1}{3}\times\frac{112}{220} = \frac{28}{3\times 55} \Rightarrow C \text{ and } A \text{ are independent}$$

Also, A, B, C are mutually exclusive as A and B are mutually exclusive.

2. a ⟶ q, r, s; b → r; c → p, q; d → p, q, r, s.

a. Suppose the coin is tossed n times. The probability of getting head or tail is 1/2. The probability of not getting any head in n tosses is $(1/2)^n$. The probability of getting at least one head is $1 - (1/2)^n$. Now given that

$$1 - (1/2)^n \geq 0.8$$

$$\Rightarrow \left(\frac{1}{2}\right)^n \leq 0.2$$

$$\Rightarrow 2^n \geq 5$$

Therefore, the least value of n is 3.

b. The total number of mappings is n^n. The number of one-one mappings is $n!$ Hence the probability is

$$\frac{n!}{n^n} = \frac{3}{32} = \frac{6}{64} = \frac{3!}{4^3} = \frac{4!}{4^4}$$

Comparing, we get $n = 4$.

c. Given equation is

$$2x^2 + 2mx + m + 1 = 0$$

$$D = 4m^2 - 8(m + 1) \geq 0$$

$$m^2 - 2m - 2 \geq 0$$

$$(m - 1)^2 - 3 \geq 0$$

$$\Rightarrow m = 3, 4, 5, 6, 7, 8, 9, 10$$

Also, the number of ways of choosing m is 10. Therefore, the required probability is 4/5.

$$\therefore \quad 5k = 4$$

d. $20P^2 - 13P + 2 \leq 0$

$$\Rightarrow (4P - 1)(5P - 2) \leq 0$$

$$\Rightarrow \frac{1}{4} \leq P \leq \frac{2}{5}$$

$$\Rightarrow \frac{1}{4} \leq \frac{1}{5} + \frac{1}{5}\left(\frac{4}{5}\right) + \frac{1}{5}\left(\frac{4}{5}\right)^2 + \cdots + \frac{1}{5}\left(\frac{4}{5}\right)^{n-1} \leq \frac{2}{5}$$

$$\Rightarrow n = 2$$

Hence, maximum as well as minimum value of n is 2.

3. a → r, s; b → p, q, r, s; c → p, q, r, s; d → p.

a. $P(\text{success}) = 1/2$; $P(\text{failure}) = 1/2$

Suppose 'n' bombs are to dropped. Let E be the event that the bridge is destroyed. Then,

$$P(E) = 1 - P(0 \text{ or } 1 \text{ success})$$

$$= 1 - \left(\left(\frac{1}{2}\right)^n + {}^nC_1 \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}\right) = 1 - \left(\frac{1}{2^n} + \frac{n}{2^n}\right) \geq 0.9$$

$$\Rightarrow \frac{1}{10} \geq \frac{n+1}{2^n} \text{ or } \frac{2^n}{10(n+1)} \geq 1$$

b. The bag contains 2 red, 3 white and 5 black balls. Hence $P(S) = 1/5$; $P(F) = 4/5$; Let E be the even of getting a red ball.

$$P(E) = P(S \text{ or } FS \text{ or } FFS \text{ or } \ldots] \geq \frac{1}{2}$$

$$\therefore \quad P(F)^n \leq \frac{1}{2}; \left(\frac{4}{5}\right)^n \leq \frac{1}{2}$$

The value of n consistent is 4.

c. Let there be x red socks and y blue socks and $x > y$. Then

$$\frac{{}^xC_2 + {}^yC_2}{{}^{x+y}C_2} = \frac{1}{2}$$

or

$$\frac{x(x-1) + y(y-1)}{(x+y)(x+y-1)} = \frac{1}{2}$$

Multiplying both sides by $2(x + y)(x + y - 1)$ and expanding, we get

$$2x^2 - 2x + 2y^2 - 2y = x^2 + 2xy + y^2 - x - y$$

Rearranging, we have

$$x^2 - 2xy + y^2 = x + y$$

$$\Rightarrow (x - y)^2 = x + y$$

$$\Rightarrow |x - y| = x + y$$

Now,

$$x + y \leq 17$$

$$x - y \leq \sqrt{17}$$

As $x - y$ must be an integer, so

$$x - y = 4$$

$$\therefore \quad x + y = 16$$

Adding both together and dividing by 2 yields $x \leq 10$.

d. Let the number of green socks be $x > 0$. Let E: be the event that two socks drawn are of the same colour.

$$P(E) = P(RR \text{ or } BB \text{ or } WW \text{ or } GG)$$

$$= \frac{3}{{}^{6+x}C_2} + \frac{{}^xC_2}{{}^{6+x}C_2}$$

$$= \frac{6}{(x+6)(x+5)} + \frac{x(x-1)}{(x+6)(x+5)}$$

$$= \frac{1}{5}$$

$$\Rightarrow 5(x^2 - x + 6) = x^2 + 11x + 30$$

$$\Rightarrow 4x^2 - 16x = 0$$

$$\Rightarrow x = 4$$

4. a → q; b → r; c → s; d → r.

We have,

$$P(A \cap B) = P(A)\,P(B) = \frac{1}{12}$$

a. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{1}{2}$

b. $P\left(\frac{A}{A \cup B}\right) = \frac{P(A)}{P(A \cup B)} = \frac{2}{3}$

c. $(C)\, P\left(\frac{B}{A' \cap B'}\right) = \frac{P(B \cap (A' \cap B'))}{P(A' \cap B')} = \frac{P(\phi)}{P(A' \cap B')} = 0$

d. $P\left(\frac{A'}{B}\right)=\frac{P(A'\cap B)}{P(B)}=P(A')=\frac{2}{3}$

5. a → q; b → p; c → r; d → q.

Let E_i denote the event that the bag contains i black and $(12-i)$ white balls ($i = 0, 1, 2, \ldots, 12$) and A denote the event that the four balls drawn are all black. Then

$$P(E_i)=\frac{1}{13}\ (i=0,1,2,\ldots,12)$$

$$P\left(\frac{A}{E_i}\right)=0 \text{ for } i=0,1,2,3$$

$$P\left(\frac{A}{E_i}\right)=\frac{{}^iC_4}{{}^{12}C_4} \text{ for } i\geq 4$$

a. $P(A)=\sum_{i=0}^{12}P(E_i)\,P\left(\frac{A}{E_i}\right)$

$$=\frac{1}{13}\times\frac{1}{{}^{12}C_4}\left[{}^4C_4+{}^5C_4+\cdots+{}^{12}C_4\right]$$

$$=\frac{{}^{13}C_5}{13\times{}^{12}C_4}=\frac{1}{5}$$

b. Clearly ,

$$P\left(\frac{A}{E_{10}}\right)=\frac{{}^{10}C_4}{{}^{12}C_4}=\frac{14}{33}$$

c. By Bayer's theorem,

$$P\left(\frac{E_{10}}{A}\right)=\frac{P(E_{10})\left(\frac{A}{E_{10}}\right)}{P(A)}$$

$$=\frac{\frac{1}{13}\times\frac{14}{33}}{\frac{1}{5}}=\frac{70}{429}$$

b. Let B denote the probability of drawing 2 white and 2 black balls. Then

$$P\left(\frac{B}{E_i}\right)=0 \text{ if } i=0,1 \text{ or } 11,12$$

$$P\left(\frac{B}{E_i}\right)=\frac{{}^iC_2\times{}^{12-i}C_2}{{}^{12}C_4} \text{ for } i=2,3,\ldots,10$$

$$\therefore\quad P(B)=\sum_{i=0}^{12}P(E_i)\,P\left(\frac{B}{E_i}\right)$$

$$=\frac{1}{13}\times\frac{1}{{}^{12}C_4}[2\{{}^2C_2\times{}^{10}C_2+{}^3C_2+\cdots+{}^{10}C_2\times{}^2C_2]$$

$$=\frac{1}{13}\times\frac{1}{{}^{12}C_4}[2\{{}^2C_2\ {}^{10}C_2+{}^3C_2\ {}^9C_2+\cdots+{}^5C_2\times{}^7C_2\}$$

$$+{}^6C_2\times{}^6C_2]$$

$$=\frac{1}{13}\times\frac{1}{495}(1287)$$

$$=\frac{1}{5}$$

6. a → q; b → p; c → q; d → s.

a. When no box remain empty, then

$$n(S)=3^6-{}^3C_1 2^6+{}^3C_2=3(243-64+1)$$

$$=540$$

When each box contains equal number of balls, then

$$n(E)=\frac{6!}{2!^3\,3!}\,3!=90$$

Therefore, the required probability is 90/540 = 1/6.

b. The required probability is

$$\frac{3^6-{}^3C_1 2^6+{}^3C_2}{3^6}=\frac{20}{27}$$

c. Let A be the event that A is throwing sum of 9 and B be the event that B throws a number greater than that thrown by A. We have to find $P(B/A)=P(A\cap B)/P(A)=P(B)$ (as A and B are independent). The probability that is throwing dice so that sum is higher than 9 is

$$P(B)=P((4,6)\text{ or }(6,4)\text{ or }(5,5)\text{ or }(6,5)\text{ or }(5,6)\text{ or }(6,6))$$

$$=\frac{6}{36}=\frac{1}{6}$$

d. $P(A\cup\bar{B})=P(A)+P(\bar{B})-P(A\cap\bar{B})$

$$=P(A)+P(\bar{B})-P(A)P(\bar{B})$$

$\Rightarrow\ 0.8=0.3+P(\bar{B})-0.3\,P(\bar{B})$

$\Rightarrow\ 0.5=0.7\,P(\bar{B})$

$\Rightarrow\ P(\bar{B})=\frac{5}{7}$

$\Rightarrow\ P(B)=1-\frac{5}{7}=\frac{2}{7}$

7. a → q, r; b → r; c → p; d → a, b, c, d.

a. $\frac{{}^rC_2}{{}^{r+b}C_2}=\frac{1}{2}\ 2r(r-1)=(r-b)(r+b-1)$

$$=2r(r-1)=(r+b)(r+b-1)$$

$\Rightarrow\ 2r^2-2r=r^2+(2b-1)r+b^2-1$

$\Rightarrow\ r^2-(1+2b)r+1-b^2=0$

$\Rightarrow\ b^2+2br+r-r^2-1=0$

$\Rightarrow\ b=\frac{-2r\pm\sqrt{4r^2-4(r-r^2-1)}}{2}$

$$=-r\pm\sqrt{2r^2-r+1}$$

Since b is integer, possible values of r are 3 and 8.

b. $^4C_2\left(\frac{r}{r+10}\right)^2\left(\frac{10}{r+10}\right)^2=\frac{3}{8}$

c. $\left(\frac{r}{r+10}\right)^2\left(\frac{10}{r+10}\right)^2=\frac{1}{16}$

$\Rightarrow\quad r=10$

d. Probability of getting exactly n red balls in $2n$ draws is always equal to probability of getting exactly n black balls in $2n$ draws for any value of r and b, hence the ratio r/b can be 10, 3, 8, 2.

8. a → r; b → p; c. → q; d → s.

a. The required event will occur if last digit in all the chosen numbers is 1, 3, 7 or 9. Therefore, the required probability is $(4/10)^n$.

b. The required probability is equal to the probability that the last digit is 2, 4, 6, 8 and is given by P (last digit is 1, 2, 3, 4, 6, 7, 8, 9) $-P$ (last digit is 1, 3, 7, 9) $=\frac{8^n-4^n}{10^n}$

c. $P(1, 3, 5, 7, 9)-P(1, 3, 7, 9)=\frac{5^n-4^n}{10^n}$

d. The required probability is

$$P(0, 5)-P(5)=\frac{(10^n-8^n)-(5^n-4^n)}{10^n}$$

$$=\frac{10^n-8^n-5^n+4^n}{10^n}$$

Integer Type

1.(2) Total number of cases $= n(S) = 6!$

Now sum the given digits is $1+2+3+4+5+6=21$, which is divisible by 3.

Now we have to form the number which is divisible by 6, then we have to ensure that the digit in unit place is even.

$\Rightarrow$ Favorable cases $= n(A) = 3\cdot 5!$

Hence, $P(A)=\frac{3\cdot 5!}{6!}=\frac{1}{2}$.

2.(6) Total number of cases $n(S)=6^3=216$

Product is prime only when two outcomes are 1 and the third is prime i.e. 2, 3, 5.

If it is 2, 1, 1, then the number of cases is 3.

Similarly, 3 cases for 3, 1, 1 and 5, 1, 1 each

Hence, favorable cases = 9.

Hence, required probability $p=\frac{1}{24}$

$\Rightarrow\quad \frac{1}{4p}=6$

3.(7) Let the probability of the faces 1, 3, 5 or 6 be p for each face.

Hence, probability of each of the faces 2 or 4 is $3p$

Now according to the question $4p+6p=1$

$\Rightarrow\quad p=\frac{1}{10}$

$\therefore\quad P(1)=P(3)=P(5)=P(6)=\frac{1}{10}$

and $P(2)=P(4)=\frac{3}{10}$

$\Rightarrow$ Required probability

$p = P$(total of 7 with a draw of dice)

$=P(16, 61, 25, 52, 43, 34)$

$=2\left(\frac{1}{10}\cdot\frac{1}{10}\right)+2\left(\frac{3}{10}\cdot\frac{1}{10}\right)+2\left(\frac{3}{10}\cdot\frac{1}{10}\right)$

$=\frac{2+6+6}{100}=\frac{14}{100}=\frac{7}{50}$

4.(1) There are n white balls in the turn.

$\Rightarrow$ Probability of Mr. A to draw two balls of same color is

$\frac{^3C_2+{}^nC_2}{^{n+3}C_2}=\frac{1}{2}$ (given)

$\Rightarrow\quad \frac{6+n(n-1)}{(n+3)(n+2)}=\frac{1}{2}$

$\Rightarrow\quad n^2-7n+6=0$

$\Rightarrow\quad n=1$ or 6 (1)

Also required probability for Mr. B according to the question is

$\frac{3}{n+3}\frac{3}{n+3}+\frac{n}{n+3}\frac{n}{n+3}=\frac{5}{8}$ (given)

Solving, we get $n^2-10n+9=0$; $n=1$ or 9 (2)

From (1) and (2), $n=1$

5.(2) When A and B are mutually exclusive, $P(A\cap B)=0$

$\therefore\quad P(A\cup B)=P(A)+P(B)$ (1)

$\Rightarrow\quad 0.8=0.5+p$

$\Rightarrow\quad p=0.3$ (2)

$P(A\cup B)=P(A)+P(B)$

$=P(A)+P(B)-P(A\cap B)$

$=P(A)+P(B)-P(A)P(B)$

$\Rightarrow\quad 0.8=0.5+q-(0.5)q$

$\Rightarrow\quad 0.3=q/2$

$\Rightarrow\quad q=0.6$ (3)

$\Rightarrow\quad q/p=2$

6.(4) Let the number of green socks be $x>0$.

E: Two socks drawn are of the same color

$\Rightarrow\quad P(E)=P(R\,R$ or $B\,B$ or $W\,W$ or $G\,G)$

$=\frac{3}{^{6+x}C_2}+\frac{^xC_2}{^{6+x}C_2}$

$=\frac{6}{(x+6)(x+5)}+\frac{x(x-1)}{(x+6)(x+5)}=\frac{1}{5}$ (given)

$\Rightarrow\quad 5(x^2-x+6)=x^2+11x+30$

$\Rightarrow\quad 4x^2-16x=0$

$\Rightarrow\quad x=4$

7.(7) Let there be x red socks and y blue socks. Let $x>y$

Then $\frac{^xC_2+{}^yC_2}{^{x+y}C_2}=\frac{1}{2}$

$\Rightarrow\quad \frac{x(x-1)+y(y-1)}{(x+y)(x+y-1)}=\frac{1}{2}$

$\Rightarrow\quad 2x^2-2x+2y^2-2y=x^2+2xy+y^2-x-y.$

$\Rightarrow\quad x^2-2xy+y^2=x+y$

$\Rightarrow\quad (x-y)^2=x+y$

$\Rightarrow \quad |x-y| = (x+y)^{1/2}$

Since $x+y \le 17, x-y \le \sqrt{17}$.

As $x-y$ must be an integer $\Rightarrow x-y=4$

$\therefore \quad x+y=16$

Solving, we get $x=7$.

8.(6) Let the two numbers are 'a' and 'b'

According to the question $a+b=4p$ and $a-b=4q$ where $p, q \in I$

$\Rightarrow \quad 2a = 4(p+q)$ and $2b = 4(p-q)$

$\Rightarrow \quad a = 2I_1$ and $b = 2I_2$

Hence, both a and b must be even.

Also if $(a-b)$ is a multiple of 4 then $(a+b)$ will also be a multiple of 4.

Hence, $n(S) = {}^{11}C_2$

$n(A) = (0, 4), (0, 8), (2, 6), (2, 10), (4, 8), (6, 10) = 6$

$\therefore \quad P(A) = \dfrac{6}{{}^{11}C_2} = \dfrac{6}{55}$

9.(3) For ranked 1 and 2 players to be winners and runners up, respectively, they should not be paired with each other in any round

$$\Rightarrow p = \frac{30}{31} \times \frac{14}{15} \times \frac{6}{7} \times \frac{2}{3} = \frac{16}{31}$$

10.(4) Total ways of distribution $= n(S) = 4^5$

Total ways of distribution so that each child get at least one game

$$n(E) = 4^5 - {}^4C_1\,3^5 + {}^4C_2\,2^5 - {}^4C_3$$
$$= 1024 - 4 \times 243 + 6 \times 32 - 4 = 240$$

Required probability $p = \dfrac{n(E)}{n(S)} = \dfrac{240}{4^5} = \dfrac{15}{64}$

11.(5) $P(n) = Kn^2$

Given $P(1) = K$; $P(2) = 2^2K$; $P(3) = 3^2K$; $P(4) = 4^2K$; $P(5) = 5^2K$; $P(6) = 6^2K$

$\therefore \quad$ Total $= 91K = 1$

$\Rightarrow \quad K = \dfrac{1}{91}$

$\therefore \quad P(1) = \dfrac{1}{91}$; $P(2) = \dfrac{4}{91}$ and so on

Let three events A, B, C are defined as

$A : a < b$

$B : a = b$

$C : a > b$

By symmetry, $P(A) = P(C)$. Also $P(A) + P(B) + P(C) = 1$ (1)

Since $\quad P(B) = \sum_{i=1}^{6} [P(i)]^2$

$$= \left[\frac{1+16+81+256+625+1296}{91 \times 91}\right]$$

$$= \frac{2275}{91 \cdot 91} = \frac{25}{91}$$

Now $2P(A) + P(B) = 1$

$\Rightarrow \quad P(A) = \dfrac{1}{2}[1 - P(B)] = \dfrac{33}{91}$

12.(5) The number of ways of drawing 7 balls (second draws) $= {}^{10}C_7$.

For each set of 7 balls of the second draw, 3 must be common to the set of 5 balls of the first draw, i.e., 2 other balls can be drawn in ${}^3C_2$ ways.

Thus, for each set of 7 balls of the second draw, there are ${}^7C_3 \times {}^3C_2$ ways of making the first draw so that there are 3 balls common.

Hence, the probability of having three balls in common is

$$\frac{{}^7C_3 \times {}^3C_2}{{}^{10}C_7} = \frac{5}{12}$$

13.(6) $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.6}{0.8} = \dfrac{3}{4}$

(Maximum value of $P(A \cap B) = P(A) = 0.6$)

14.(5) Highest number in three throws 4

$\Rightarrow \quad$ At least one of the throws must be equal to 4.

Number of ways when three blocks are filled from $\{1,2,3,4\} = 4^3$

$\therefore \quad$ number of ways when filled from $\{1,2,3\} = 3^3$

$\therefore \quad$ required number of ways $= 4^3 - 3^3$

$\therefore \quad$ Probability $p = \dfrac{4^3 - 3^3}{6^3} = \dfrac{37}{216}$

15.(4) Let event A: Card is of heart but not king (12 cards)

Event B : King but not heart (3 cards)

Event C: Heart and king (1 card)

$\therefore \quad$ required probability

$$p = P(E) = \frac{{}^{12}C_1 \cdot {}^3C_1 + {}^3C_1 \cdot {}^1C_1 + {}^1C_1 \cdot {}^{12}C_1}{{}^{52}C_2} = \frac{2}{52}$$

$\therefore \quad 104p = 4$

Archives

Subjective Type

1. To draw 2 black, 4 white and 3 red balls in order is same as arranging two balls at first 2 places, 4 white balls at next 4 places, (3rd to 6th place) and 3 red balls at next 3 places (7th to 9th place), i.e., $B_1B_2W_1W_2W_3W_4R_1R_2R_3$, which can be done in $2! \times 4! \times 3!$ ways. And total number of ways of arranging all $2+4+3=9$ balls is $9!$ Therefore the required probability is

$$\frac{2! \times 4! \times 3!}{9!} = \frac{1}{1260}$$

2. (i) The number of ways in which all the 6 girls sit together is $6! \times 7!$ (considering all 6 girls as one person). Therefore, the probability of all girls sitting together is $(6! \times 7!)/12! = 720/(12 \times 11 \times 10 \times 9 \times 8) = (1/132)$.

(ii) Starting with a boy, boys can sit in 6! ways leaving one place between every two boys and one at last

B_B_B_B_B_

These places can be occupied by girls in 6! ways. Therefore, if we start with a boy, number of ways of boys and girls sitting alternately is $6! \times 6!$

G_G_G_G_G_G_

Thus total number of ways of alternative sitting arrangements is $6! \times 6! + 6! \times 6! = 2 \times 6! \times 6!$

Therefore, the probability of making alternative sitting arrangement for 6 boys and 6 girls is

$$\frac{2 \times 6! \times 6!}{12!} = \frac{2 \times 720}{12 \times 11 \times 10 \times 9 \times 8 \times 7} = \frac{1}{462}$$

3. a. Let us define the events as follows:

$E_1 \equiv$ First shot hits the target plane,

$E_2 \equiv$ Second shot hits the target plane

$E_3 \equiv$ Third shot hits the target plane

$E_4 \equiv$ Fourth shot hits the target plane

Given

$$P(E_1) = 0.4, P(E_2) = 0.3, P(E_3) = 0.2, P(E_4) = 0.1$$

$$\Rightarrow \quad P(\bar{E}_1) = 0.6, P(\bar{E}_2) = 0.7, P(\bar{E}_3) = 0.8, P(\bar{E}_4) = 0.9$$

Now the gun hits the plane if at least one of the four shots hit the plane. Therefore,

P(at least one shot hits the plane)

$= 1 - P$ (none of the shots hits the plane)

$$= 1 - P(\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3 \cap \bar{E}_4)$$

$$= 1 - P(\bar{E}_1)\, P(\bar{E}_2)\, P(\bar{E}_3)\, P(\bar{E}_4)$$

$$= 1 - 0.6 \times 0.7 \times 0.8 \times 0.9$$

$$= 1 - 0.3024 = 0.6976$$

4. $P(A) = 0.5, P(A \cap B) \le 0.3$

So,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

gives

$$P(B) = P(A \cup B) - P(A) + P(A \cap B)$$

$$\le 1 + 0.3 - 0.5$$

$$= 0.8 \qquad [\because P(A \cup B) \le 1]$$

Hence, $P(B) = 0.9$ is not possible.

5. We must have one ace in $n - 1$ attempts and one ace in the n^{th} attempt. The probability of one ace in first $n - 1$ attempts is

$$\frac{{}^4C_1 \times {}^{48}C_{n-2}}{{}^{52}C_{n-1}}$$

and one ace in the n^{th} attempt is

$$\frac{{}^3C_1}{[52 - (n - 1)]} = \frac{3}{53 - n}$$

Hence, the required probability is

$$\frac{4 \times 48!}{(n-2)!\,(50-n)!} \times \frac{(n-1)!\,(53-n)!}{52!} \times \frac{3}{53-n}$$

$$= \frac{(n-1)(52-n)(51-n)}{50 \times 49 \times 17 \times 13}$$

6. Since $P(A \cup B \cup C) \ge 0.75$, therefore,

$$0.75 \le P(A \cup B \cup C) \le 1$$

$$\Rightarrow \quad 0.75 \le P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) \le 1$$

$$\Rightarrow \quad 0.75 \le 0.3 + 0.4 + 0.8 - 0.08 - P(B \cap C) - 0.28 + 0.09 \le 1$$

$$\Rightarrow \quad 0.75 \le 1.23 - P(B \cap C) \le 1$$

$$\Rightarrow \quad -0.48 \le -P(B \cap C) \le -0.23$$

$$\Rightarrow \quad 0.23 \le P(B \cap C) \le 0.48$$

7. Given that A and B are independent events.

$$\therefore \quad P(A \cap B) = P(A)\, P(B) \qquad (1)$$

Also given that

$$P(A \cap B) = \frac{1}{6} \qquad (2)$$

and

$$P(\bar{A} \cap \bar{B}) = \frac{1}{3} \qquad (3)$$

Also,

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$$

$$\Rightarrow \quad P(\bar{A} \cap \bar{B}) = 1 - P(A) - P(B) + P(A \cap B)$$

$$\Rightarrow \quad \frac{1}{3} = 1 - P(A) - P(B) + \frac{1}{6}$$

$$\Rightarrow \quad P(A) + P(B) = \frac{5}{6} \qquad (4)$$

From Eqs. (1) and (2), we get

$$P(A)\, P(B) = \frac{1}{6} \qquad (5)$$

Let $P(A) = x$ and $P(B) = y$. Then Eqs. (4) and (5) become

$$x + y = \frac{5}{6}$$

$$xy = \frac{1}{6}$$

Solving, we get $x = 1/2$ and $y = 1/3$ or $x = 1/3$ and $y = 1/2$. Thus, $P(A) = 1/2$ or $1/3$.

8. Let $P(A)$ and $P(B)$ be the percentage of the population in a city who read newspapers A and B, respectively. Therefore, $P(A) = 25/100 = 1/4$, $P(B) = 20/100 = 1/5$ and $P(A \cap B) = 8/100 = 2/25$.

Therefore, percentage of those who read A but not B is

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

$$= 1/4 - 2/25$$

$$= 17/100 = 17\%$$

Similarly,

$$P(B \cap \bar{A}) = P(B) - P(A \cap B) = 3/25 = 12\%$$

Therefore, percentage of those who read advertisements is

$$30\% \text{ of } P(A \cap \bar{B}) + 40\% \text{ of } P(B \cap \bar{A}) + 50\% \text{ of } P(A \cap B)$$

$$= \frac{30}{100} \times \frac{17}{100} + \frac{40}{100} \times \frac{3}{25} + \frac{50}{100} \times \frac{2}{25} = \frac{139}{1000} = 13.9\%$$

Hence, the percentage of the population who read an advertisement is 13.9%.

9. The total number of ways of ticking one or more alternatives out of 4 is ${}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 15$. Out of these 15 combinations, only one combination is correct. The probability of ticking the alternative correctly at the first trial is 1/15 that at the second trial is (14/15)(1/14) = 1/15 and that at the third trial is (14/15)(13/14)(1/13) = 1/15.

Thus, the probability that the candidate will get marks on the question if he is allowed up to three trials is 1/15 + 1/15 + 1/15 + = 1/5.

10. Let E_i denote the event that the lot contains 2 defcetive articles and E_2 the event that the lot contains 3 defective articles. Suppose that A denotes the event that all the defective articles are found by the twelfth testing. we have,

$$P(E_1) = 0.4, P(E_2) = 0.6$$

Now,

$$P(E_1) + P(E_2) = 0.4 + 0.6 = 1$$

Therefore there can be no other possibility. Now,

$$P(A/E_1) = \frac{({}^2C_1)({}^{18}C_{10})}{({}^{20}C_{11})} = \frac{1}{9} = \frac{11}{190}$$

(Here up to 11th draw, 1 defective and 10 non-defective articles are drawn and the second (i.e. last) defective article is drawn at twelfth draw.)

Also,

$$P(A/E_2) = \frac{({}^3C_2)({}^{17}C_9)}{({}^{20}C_{11})} \cdot \frac{1}{9} = \frac{11}{128}$$

[Here up to 11th draw, 2 defective and 9 non-defective articles are drawn and the third (i.e. last) defective articles is drawn at the twelfth draw.]

We have,

$$P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2)$$

$$= 0.4 \times \frac{11}{190} + 0.6 \times \frac{11}{228} = \frac{99}{1990}$$

11. The man can be one step away from the starting point if (i) either he is one step ahead or (ii) one step behind the starting point. Now if at the end of the 11 steps, the man is one step ahead of the starting point, then out of 11 steps, he must have taken six forward steps and five backward steps. The probability of this event is

$${}^{11}C_6 \times (0.4)^6 \times (0.6)^5 = 462 \times (0.4)^6 \times (0.6)^5$$

Again if at the end of the 11 steps, the man is one step behind the starting point, then out of 11 steps, he must have taken six backward steps and five forward steps. The probability of this event is

$${}^{11}C_6 \times (0.6)^6 \times (0.4)^5 = 462 \times (0.6)^6 \times (0.4)^5$$

Since the events (i) and (ii) are mutually exclusive, therefore the required probability that one of these events happens is

$$[462 \times (0.4)^6 \times (0.6)^5] + [462 \times (0.6)^6 \times (0.4)^5]$$

$$= 462 \times (0.4)^5 \times (0.6)^5 (0.4 + 0.6)$$

$$= 462 \times (0.4 \times 0.6)^5$$

$$= 462(0.24)^5$$

12. For the first two draws, following events may occur.

E_1: Both the balls are white

E_2: First is white and second is black

E_3: First is black and second is white

E_4: Both the balls are black

Let E represent the event that the third ball is black. Then,

$$P(E_1) = \frac{2}{4} \times \frac{1}{2} = \frac{1}{6}$$

$$P(E_2) = \frac{2}{4} \times \frac{2}{3} = \frac{1}{3}$$

$$P(E_3) = \frac{2}{4} \times \frac{2}{5} = \frac{1}{5}$$

$$P(E_4) = \frac{2}{4} \times \frac{3}{5} = \frac{3}{10}$$

The four events E_1, E_2, E_3 and E_4 are mutually exclusive and exhaustive. Using the theorem of total probability,

$$P(E) = P(E_1)\, P(E/E_1) + P(E_2)\, P(E/E_2) + P(E_3)\, P(E/E_3) + P(E_4)\, P(E/E_4)$$

$$= \frac{1}{6} \times \frac{3}{2} + \frac{1}{3} \times \frac{3}{4} + \frac{1}{5} \times \frac{3}{4} + \frac{3}{10} \times \frac{4}{6} = \frac{23}{30}$$

13. Here the total number of coins is $N + 7$. Therefore, the total number of ways of choosing 5 coins out of $N + 7$ is ${}^{n+7}C_5$.

Let E denote the event that the sum of the values of the coins is less than 1 rupee and 50 paise. Then, E' denotes the event that the total values of the five coins is equal to or more than 1 rupee and 50 paise. The number of cases favourable to E' is

$${}^2C_1 \times {}^5C_4 \times {}^NC_0 + {}^2C_2 \times {}^5C_3 \times {}^NC_0 + {}^2C_2 \times {}^5C_2 \times {}^NC_1$$

$$= 2 \times 5 + 10 + 10N$$

$$= 10(N + 2)$$

$$\therefore \quad P(E') = \frac{10(N+2)}{{}^{N+7}C_5}$$

$$\Rightarrow \quad P(E) = 1 - P(E') = 1 - \frac{10(N+2)}{{}^{N+7}C_5}$$

14. The probability p_1 of winning the best of three games is equal to the sum of the probability of winning two games + the probability of wining three games, which is given by

$${}^3C_2\,(0.6)\,(0.4)^2 + {}^3C_3\,(0.4)^3$$

$$= 0.288 + 0.064 = 0.352 \qquad \text{[Using binomial distribution]}$$

Similarly, the probability of winning the best five games is

p_2 = probability of winning three games
+ probability of winning four games
+ probability of winning 5 games

$$= {}^5C_3\,(0.6)^2\,(0.4)^3 + {}^5C_4\,(0.6)\,(0.4)^4 + {}^5C_5\,(0.4)^5$$

$$= 0.2304 + 0.0768 + 0.01024$$

$$= 0.31744$$

As $p_1 > p_2$, therefore A must choose the first offer, i.e., best of three games.

15. Let $A = \{a_1, a_2, \ldots, a_n\}$. Let S be the sample space and E be the event that $P \cap Q = \phi$. The number of subsets of A is 2^n. Each one of P and Q can be selected in 2^n ways. Hence, the total number of ways of selecting P and Q is $2^n\, 2^n = 4^n$.

Now for $P \cap Q = \phi$, element of A either does not belong to any of subsets or it belongs to at most one of them. Therefore, for each element, there are 3 choices, namely

(i) $a_i \notin P, a_i \notin Q$

(ii) $a_i \notin P, a_i \notin Q$

(iii) $a_i \notin P, a_i \notin Q$

Therefore, the total number of ways of selecting P and Q such that

$P \cap Q = \phi$ are $(3)^n$ [3 for each element of A].

$\therefore \quad n(E) = 3^n$

Hence,

$$P(E) = \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n$$

16. Let G denote the event the examinee gueses, C denote the event the examinee copies and K the event the examinee knows. Then given that $P(G) = 1/3$, $P(C) = 1/6$ and $P(K) = 1 - 1/3 - 1/6 = 1/2$. Here, it has been assumed that the events G, C and K are mutually exclusive and totally exhaustive. If R denotes the event that the answer is right, then

$P(R/G) = 1/4$, as out of the four choices only one is correct. $P(R/C) = 1/8$ (given). Also, $P(R/K) = 1$ since the probability of answering correctly when one knows the answer is equal to 1.

Now by Bayes's theorem, the probability that he knows the answer, given that he answered correctly is given by

$$P(K/R) = \frac{P(K)\,P(R/K)}{P(G)\,P(R/G) + P(C)\,P(R/C) + P(K)\,P(R/K)}$$

$$= \frac{(1/2)\times 1}{(1/3)\,(1/4) + (1/6)\,(1/8) + (1/2)\times 1} = \frac{24}{29}$$

17. Let X = defective and Y = non-defective. Then, all possible outcomes are $\{XX, XY, YX, YY\}$. Also,

$$P(XX) = \frac{50}{100} \times \frac{50}{100} = \frac{1}{4}$$

$$P(XY) = \frac{50}{100} \times \frac{50}{100} = \frac{1}{4}$$

$$P(YX) = \frac{50}{100} \times \frac{50}{100} = \frac{1}{4}$$

$$P(YY) = \frac{50}{100} \times \frac{50}{100} = \frac{1}{4}$$

Here,

$$A = (XX) \cup (XY);\ B = (XY) \cup (YY);\ C = (XX) \cup (YY)$$

$$\therefore \quad P(A) = P(XX) + P(XY) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\therefore \quad P(B) = P(XY) + P(YX) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Now,

$$P(A\,B) = P(XY) = \frac{1}{4} = P(A)\,P(B)$$

Therefore, A and B are independent events.

$$P(B\,C) = P(YX) = \frac{1}{4} = P(B)\,P(C)$$

Therefore, B and C are independent events.

$$P(C\,A) = P(X\,Y) = \frac{1}{4} = P(C)\,P(A)$$

Therefore, C and A are independent events.

$$P(A\,B\,C) = 0 \text{ (impossible events)}$$

$$\neq P(A)\,P(B)\,P(C)$$

Therefore, A, B, C are dependent events. Thus, we can conclude that A, B, C are pairwise independent, but A, B, C are dependent.

18. The given numbers are 00, 01, 02, ..., 99. There are total 100 numbers, out of which the numbers, the product of whose digits is 18, are 29, 36, 63 and 92.

$$\therefore \quad p = P(E) = \frac{4}{100} = \frac{1}{25}$$

$$\Rightarrow \quad q = 1 - p = \frac{24}{25}$$

From binomial distribution,

$P(E$ occurring at least 3 times$) = P(E$ occurring 3 times$) + P(E$ occurring 4 times$)$

$$= {}^4C_3\,p^3q + {}^4C_4\,p^4$$

$$= 4 \times \left(\frac{1}{25}\right)^3 \left(\frac{24}{25}\right) + \left(\frac{1}{25}\right)^4$$

$$= \frac{97}{(25)^4}$$

19. $E_1 \equiv$ number noted is 7

$E_2 \equiv$ number noted is 8.

$H \equiv$ getting head on coin

$T \equiv$ getting tail on coin

Then, by total probability theorem,

$$P(E_1) = P(H)\,P(E_1/H) + P(T)\,P(E_1/T)$$

$$P(E_2) = P(H)\,P(E_2/H) + P(T)\,P(E_2/T),$$

where $P(H) = (1/2)$, $P(T) = (1/2)$ and $P(E_1/H)$ is the probability of getting a sum of 7 on two dice.

Here, favourable cases are $\{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4)\,(4, 3)\}$.

$$\therefore \quad P(E_1/H) = \frac{6}{36} = \frac{1}{6}$$

Also, $P(E_1/T)$ is the probability of getting '7' numbered card out of 11 cards

$$\therefore \quad P(E_1/T) = 1/11.$$

$P(E_2/H)$ is the probability of getting a sum of 8 on two dice.

Here, favourable cases are $\{(2, 6)\,(6, 2)\,(4, 4), (5, 3), (3, 5)\}$.

$$\therefore \quad P(E_2/H) = \frac{5}{36}$$

The probability of getting '8' numbered card out of 11 cards is 1/11.

$$\therefore \quad P(E_1) = \frac{1}{2} \times \frac{1}{6} + \frac{1}{2} \times \frac{1}{11} = \frac{1}{12} + \frac{1}{22} = \frac{11+6}{132} = \frac{17}{132}$$

$$P(E_2) = \frac{1}{2} \times \frac{5}{36} + \frac{1}{2} \times \frac{1}{11} = \frac{1}{2}\left[\frac{55+36}{396}\right] = \frac{91}{792}$$

Now, E_1 and E_2 are mutually exclusive events, therefore

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$
$$= \frac{17}{132} + \frac{91}{792}$$
$$= \frac{102+91}{792}$$
$$= \frac{193}{792} = 0.2436$$

20. We have 14 seats in two vans and there are 9 boys and 3 girls. The number of ways of arranging 12 people on 14 seats without restriction is

$$^{14}P_{12} = \frac{14!}{2!} = 7(13!)$$

Now, the number of ways of choosing a van is 2. The number of ways of arranging girls on three adjacent seats is 2(3!). The number of ways of arranging 9 boys on the remaining 11 seats is $^{11}P_9$. Therefore, the required number of ways is

$$2\times(2\times3!)\times{}^{11}P_9 = \frac{4!11!}{2!}12!$$

Hence, the probability of the required events is

$$\frac{12!}{7\times13!} = \frac{1}{91}$$

21. a. The probability of S_1 to be among the eight winners is equal to the probability of S_1 winning in the group, which is given by 1/2.

b. If S_1 and S_2 are in the same pair-then exactly one wins.

If S_1 and S_2 are in two separate pairs, then for exactly one of S_1 and S_2 to be among the eight winners, S_1 wins and S_2 loses or S_1 loses and S_2 wins.

Now the probability of S_1, S_2 being in the same pair and one wins is (Probability of S_1, S_2 being in the same pair) × (Probability of any one winning in the pair). And the probability of S_1, S_2 being in the same pair is

$$\frac{n(E)}{n(S)}$$

The number of ways 16 players are divided into 8 pairs is

$$n(S) = \frac{16!}{(2!)^8\times8!}$$

The number of ways in which 16 persons can be divided in 8 pairs so that S_1 and S_2 are in same pair is

$$n(E) = \frac{14!}{(2!)^7\times7!}$$

Therefore, the probability of S_1 and S_2 being in the same pair is

$$\frac{\dfrac{14!}{(2!)^7\times7!}}{\dfrac{16}{(2!)^8\times8!}} = \frac{2!\times8}{16\times15} = \frac{1}{15}$$

The probability of any one winning in the pair of S_1, S_2 is P(certain event) = 1.

Hence, the prbability tha the pair of S_1, S_2 being in two pairs separately and any one of S_1, S_2 wins is given by the probability of S_1, S_2 being in two pairs separately and S_1 wins, S_2 loses + the probability of S_1, S_2 being in two pairs separately and S_1 loses, S_2 wins. It is given by

$$\left[1-\frac{1}{15}\right]\times\frac{1}{2}\times\frac{1}{2}+\left[1-\frac{1}{15}\right]\times\frac{1}{2}\times\frac{1}{2}$$
$$=\frac{1}{2}\times\frac{14}{15}=\frac{7}{15}$$

Therefore, the required probability is (1/15) + (7/15) + (8/15).

22. The required probability is 1 – (probability of the event that the roots of $x^2 + px + q = 0$ are non-real). The roots of $x^2 + px + q = 0$ will be non-real if and only if

$$p^2 - 4q < 0 \text{ or } p^2 < 4q.$$

We enumerate the possible values of p and q, for which this can happen, in the following table.

q	p	Number of pairs of pq
1	1	1
2	1, 2	2
3	1, 2, 3	3
4	1, 2, 3	3
5	1, 2, 3, 4	4
6	1, 2, 3, 4	4
x7	1, 2, 3, 4, 5	5
8	1, 2, 3, 4, 5	5
9	1, 2, 3, 4, 5	5
10	1, 2, 3, 4, 5, 6	6

Thus, the number if possible pairs is 38. Also, the total number of possible pairs is 10 × 10 = 100. Therefore, the required probability is $1-(38/100) = 1-0.38 = 0.62$.

23. Given that p is the probability that the coin shows a head, then $1 - p$ will be the probability that the coin shows a tail. Now,

$\alpha = P(A$ gets the 1st head in 1st try) + P (A gets the 1st head in 2nd try) + ⋯

$$\Rightarrow \alpha = P(H) + P(T)\,P(T)\,P(T)\,P(H) + P(T)P(T) \times P(T)P(T)P(T)P(T)P(H)$$
$$= p + (1-p)^3 + (1-p)^6 p + \cdots$$
$$= p\,[1 + (1-p)^3 + (1-p)^6 + \cdots]$$
$$= \frac{p}{1-(1-p)^3} \quad (1)$$

Similarly,

$\beta = P(B$ gets the 1st head in 1st try) + $P(B$ gets the 1st head in 2nd try) + ⋯

$$= P(T)P(H) + P(T)\,P(T)\,P(T)\,P(T)\,P(H) + \cdots$$
$$= (1-p)\,p + (1-p)^4 p + \cdots$$
$$= \frac{(1-p)\,p}{1-(1-p)^3} \quad (2)$$

From Eqs. (1) and (2), we get

$\beta = (1-p)\alpha$

Also, Eqs. (1) and (2) give expression for α and β in terms of p.

Also,

$\alpha + \beta + \gamma = 1$ (exhaustive events and mutually exclusive events)

$$\Rightarrow \quad \gamma = 1 - \alpha - \beta$$
$$= 1 - \alpha - (1-p)\alpha$$
$$= 1 - (2-p)\alpha$$
$$= 1 - (2-p)\alpha$$
$$= 1 - (2-p)\frac{p}{1-(1-p)^3}$$
$$= \frac{1-(1-p)^3 - (2p-p^2)}{1-(1-p)^3}$$
$$= \frac{1-1+p^3 + +3p(1-p) - 2p + p^2}{1-(1-p)^3}$$
$$= \frac{p^3 - 2p + p}{1-(1-p)^3}$$
$$= \frac{p(p^2-2p+1)}{1-(1-p)^3}$$
$$= \frac{p(1-p)^2}{1-(1-p)^3}$$

24.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
0	0	0	0	0	0	0	0

Given that if P_i, P_j play with $i < j$, then P_i will win. For the first round, P_4 should be paired with any one from P_5 to P_8. It can be done in ${}^4C_1$ ways. Then P_4 to be the finalist, at least one player from P_5 to P_8 should reach in the second round. Therefore, one pair should be from remaining 3 from P_5 to P_8 in ${}^3C_2$. Then favourable pairings in first round is ${}^4C_1\,{}^3C_2\,{}^3C_2$. Then in the 2nd round, we have four players. Favourable ways is 1. Now total possible pairings is

$$\frac{{}^8C_2 \times {}^6C_2 \times {}^4C_2 \times {}^2C_2}{4!} \times \frac{{}^4C_2 \times {}^2C_2}{2!}$$

Therefore, the probability is

$$\frac{{}^4C_1\,{}^3C_2\,{}^3C_2\,4!\,2!}{{}^8C_2\,{}^6C_2\,{}^4C_2\,{}^2C_2\,{}^4C_2\,{}^2C_2} = \frac{4}{35}$$

25. Given that the probability of showing head by a coin when tossed is p. Therefore the orobability of coin that no two or more consecutive heads occur when tossed n times is given as follows.

The probability of getting one or more or no head, i.e., probability of getting H or T is $P_i = 1$.

Also, the probability of getting one H or number H is

$$p_2 = P(HT) + P(TH) + P(TT)$$
$$= p(1-p) + p(1-p)P + (1-p)(1-p)$$
$$= 1 - p^2$$

For $n \geq 3$:

The probability that (0) two or more consecutive heads occur when tossed n times is

$p_n = p$(last outcome is T) P(two or more consecutive heads in $(n-1)$ throws) + P(last outcome is H) $P((n-1)^{\text{th}}$ throw results in a T) P(number two or more consecutive heads in $(n-2)n$ throws)

$$= (1-p)P_{n-1} + p(1-p)p_{n-2}$$

Hence, proved.

26. Let W_1 (B_1) be the event that a white (a black) ball is drawn in the first draw and let W be the event that white ball is drawn in the second draw. Then,

$$P(W) = P(B_1)P(W/B_1) + P(W_1)P(W/W_1)$$
$$= \frac{n}{m+n}\frac{m}{m+n+k} + \frac{m}{m+n}\frac{m+k}{m+n+k}$$
$$= \frac{m(n+m+k)}{(m+n)(m+n+k)}$$
$$= \frac{m}{m+n}$$

27. The total number of outcomes is 6^n. We can choose three numbers out of 6 in ${}^6C_3$ ways. Now these three numbers must appear at least once in n throws which is equivalent to number of ways of filling 3 boxes with n different objects if no box remains empty which is done in $3^n - {}^3C_1 2^n + {}^3C_2$ ways. Hence, favourable number of cases is ${}^6C_3 \times (3^n - {}^3C_1 2^n + {}^3C_2)$. Hence, the required probability is

$$\frac{{}^6C_3\,[3^n - 3(2^n) + 3]}{6^n}$$

28. Let E_1 be the event that the coin drawn is fair and E_2 be the event that the coin drawn is biased.

$$\therefore \quad P(E_1) = \frac{m}{N} \text{ and } P(E_2) = \frac{N-m}{N}$$

A is the event that on tossing the coin, the head appears first and then appears the tail.

$$\therefore \quad P(A) = P(E_1 \cap A) + P(E_2 \cap A)$$
$$= P(E_1)P(A/E_1) + P(E_2)P(A/E_2)$$
$$= \frac{m}{N}\left(\frac{1}{2}\right)^2 + \left(\frac{N-m}{N}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) \quad (1)$$

We have to find the probability that A has happened because of E_1.

$$\therefore \quad P(E_1/A) = \frac{P(E_1 \cap A)}{P(A)}$$
$$= \frac{\frac{m}{N}\left(\frac{1}{2}\right)^2}{\frac{m}{N}\left(\frac{1}{2}\right)^2 + \frac{N-m}{N}\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)} \quad \text{[Using Eq. (1)]}$$
$$= \frac{m/4}{m/4 + \frac{2(N-m)}{9}} = \frac{9m}{m+8N}$$

29. Let us consider the following events.

E_1: event of passing I exam

E_2: event of passing II exam

E_3: event of passing III exam

Then a student can qualify in anyone of following ways.

1. He passes first and second exam.
2. He passes first, fails in second but passes third exam.
3. He fails in first, passes second and third exam.

Therefore, the required probability is

$$P(E_1)P(E_2/E_1) + P(E_1)\,P(E_2/E_1)\,P(E_3/E_2) + P(E_1)\,P(E_2/E_1)\,P(E_3/E_2)$$

[as an event is dependent on previous one]

$$= p\,p + p\,(1-p)\frac{p}{2} + (1-p)\frac{p}{2}p$$

$$= p^2 + \frac{p^2}{2} - \frac{p^3}{2} + \frac{p^2}{2} - \frac{p^3}{2}$$

$$= 2p^2 - p^3$$

30. Let us consider the following events:

E_1: A hits B; $P(E_1) = 2/3$

E_2: B hits A; $P(E_2) = 1/2$

E_3: C hits A; $P(E_3) = 1/3$

E: A is hit

$$\therefore \quad P(E) = P(E_2 \cup E_3)$$

$$= 1 - P(\bar{E}_2 \cap \bar{E}_3)$$

$$= 1 - P(\bar{E}_2)\cdot P(\bar{E}_3)$$

$$= 1 - \frac{1}{2}\cdot\frac{2}{3}$$

$$= \frac{2}{3}$$

Now,

$$P((E_2 \cap \bar{E}_3)/E) = \frac{P(E_2 \cap \bar{E}_3)}{P(E)}$$

[$\because P(E_2 \cap \bar{E} \cap E) = P(E_2 \cap \bar{E}_3)$, i.e., B hits A and A is hit = B hits A]

$$= \frac{P(E_2)\,P(\bar{E}_3)}{P(E)}$$

$$= \frac{1/2 \times 2/3}{2/3} = \frac{1}{2}$$

31. Given that A and B are two independent events. C is the event in which exactly one of A and B occurs. Let $P(A) = x$, $P(B) = y$. Then,

$$P(C) = P(A \cap \bar{B}) + P(\bar{A} \cap B)$$

$$= P(A)\,P(\bar{B}) + P(\bar{A})\,P(B)$$ [$\because$ if A and B are independent so are 'A and $\bar{B}$' and $\bar{A}$ and B]

$$\Rightarrow \quad P(C) = x(1-y) + y(1-x) \qquad (1)$$

Now consider,

$$P(A \cup B)\,P(\bar{A} \cap \bar{B}) = [P(A) + P(B) - P(A)\,P(B)]\,[P(\bar{A})\,P(\bar{B})]$$

$$= (x + y - xy)(1-x)(1-y)$$

$$= (x+y)(1-x)(1-y) - xy(1-x)(1-y) \cdots$$

$$\times (x+y)(1-x)(1-y) \qquad [\because x, y \in (0, 1)]$$

$$\le x(1-x)(1-y) + y(1-x)(1-y)$$

$$\le x(1-y) + y(1-x) - x^2(1-y) - y^2(1-x) + \cdots$$

$$+ x(1-y) + y(1-x)$$

$$\le P(C) \quad [\text{Using Eq. (1)}]$$

Thus $P(C) \ge P(A \cup B)P(\bar{A} \cap \bar{B})$ is proved.

32. Let us define the following events

A: 4 white balls are drawn in first six draws

B: 5 white balls are drawn in first six draws

C: 6 white balls are drawn in first six draws

E: exactly one white ball is drawn in next two draws (i.e. one white and one red)

Then

$$P(E) = P(E/A)\,P(A) + P(E/B)\,P(B) + P(E/C)\,P(C)$$

But

$P(E/C) = 0$ [as there are only 6 white balls in the bag]

$$\therefore \quad P(E) = P(E/A)\,P(A) + P(E/B)\,P(B)$$

$$= \frac{{}^{10}C_1 \times {}^2C_1}{{}^{12}C_2}\,\frac{{}^{12}C_2 \times {}^6C_4}{{}^{18}C_6} + \frac{{}^{11}C_1 \times {}^1C_1}{{}^{12}C_1}\,\frac{{}^{12}C_1 \times {}^6C_5}{{}^{18}C_6}$$

33. Let us define the following events

C: person goes by car

S: person goes by scooter

B: person goes by bus

T: person goes by train

L: person reaches late

Then, we are given in the question

$$P(C) = \frac{1}{7},\ P(S) = \frac{3}{7},\ P(B) = \frac{2}{7},\ P(T) = \frac{1}{7}$$

$$P(L/C) = \frac{2}{9},\ P(L/S) = \frac{1}{9},\ P(L/B) = \frac{4}{9},\ P(L/T) = \frac{1}{9}$$

We have to find $P(C/\bar{L})$ [Since reaches in time ≡ not late]. Using Bayes's theorem,

$$P(C/\bar{L}) = \frac{P(\bar{L}/C)\,P(C)}{P(\bar{L}/C)P(C) + P(\bar{L}/S)\,P(S) + P(\bar{L}/B) + A(\bar{L}/T)P(T)} \qquad (1)$$

Now,

$$P(\bar{L}/C) = 1 - \frac{2}{9} = \frac{7}{9},\ P(\bar{L}/S) = 1 - \frac{1}{9} = \frac{8}{9}$$

$$P(\bar{L}/B) = 1 - \frac{4}{9} = \frac{5}{9},\ P(\bar{L}/T) = 1 - \frac{1}{9} = \frac{8}{9}$$

Substituting these values in Eq. (1), we get

$$P(C/\bar{L}) = \frac{\frac{7}{9}\times\frac{1}{7}}{\frac{7}{9}\times\frac{1}{7} + \frac{8}{9}\times\frac{3}{7} + \frac{5}{9}\times\frac{2}{7} + \frac{8}{9}\times\frac{1}{7}}$$

$$= \frac{7}{7 + 24 + 10 + 8} = \frac{7}{49} = \frac{1}{7}$$

Objective Type

Fill in the blanks

1. Let E_1 be the event that face 1 has turned up and E_2 be the event that face 1 or 2 has turned up. By the given data,

$$P(E_2) = 0.1 + 0.32 = 0.42,\ P(E_1 \cap E_2) = 0.1$$

Given that E_2 has happened and we have to find then the probability of happening of E_1. Therefore, by conditional probability theorem, we have

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{0.1}{0.42} = \frac{10}{42} = \frac{5}{21}$$

2. Given that

$$P(A \cup B) = P(A \cap B)$$

$\Rightarrow \quad P(A) + P(B) - P(A \cap B) = P(A \cap B)$

$\Rightarrow \quad [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)] = 0$

But

$P(A) - P(A \cap B) \geq 0$ and $P(B) - P(A \cap B) \geq 0$

$[\because P(A \cap B) \leq P(A), P(B)]$

$\Rightarrow \quad P(A) - P(A \cap B) = 0$ and $P(B) - P(A \cap B) = 0$

[since sum of two non-negative numbers can be zero only when these numbers are zeros.]

$\Rightarrow \quad P(A) = P(B) = P(A \cap B)$

which is the required relationship.

3. Let A be the event that maximum number on the two chosen tickets is not more than 10, and B be the event that minimum number on them is 5. We have to find $P(B/A)$. We know that

$$P(B/A) = \frac{P(B \cap A)}{P(A)}$$

The total ways of selecting two tickets out of 100 is ${}^{100}C_2$. The number of ways favourable to A is the number of ways of selecting any 2 numbers from 1 to 10, i.e., ${}^{10}C_2 = 45$. $A \cap B$ contains one number 5 and other greater than 5 and ≤ 10. So, number of ways favourable to $A \cap B$ is ${}^5C_1 = 5$. Therefore,

$$P(A) = \frac{45}{{}^{100}C_2}$$

$$P(B \cap A) = P(B \cap A) = \frac{5}{{}^{100}C_2}$$

Thus,

$$P(B/A) = \frac{5/{}^{100}C_2}{45/{}^{100}C_2} = \frac{5}{45} = \frac{1}{9}$$

4. Let,

$$P(A) = \frac{1+3p}{3},\ P(B) = \frac{1-p}{4},\ P(C) = \frac{1-2p}{2}$$

A, B and C are three mutually exclusive events.

$\therefore \quad P(A) + P(B) + P(C) \leq 1$

$\Rightarrow \quad \frac{1+3p}{3} + \frac{1-p}{4} + \frac{1-2p}{2} \leq 1 \leq 1$

$\Rightarrow \quad 4 + 12p + 3 - 3p + 6 - 12p \leq 12$

$\Rightarrow \quad p \leq 1/3 \qquad (1)$

Also,

$$0 \leq P(A) \leq 1 \Rightarrow 0 \leq \frac{1+3p}{3} \leq 1$$

$$0 \leq 1 + 3p \leq 3$$

$\Rightarrow \quad -\frac{1}{3} \leq p \leq \frac{2}{3} \qquad (2)$

$$0 \leq P(B) \leq 1 \Rightarrow 0 \leq \frac{1-p}{4} \leq 1$$

$\Rightarrow \quad 0 \leq 1 - p \leq 4$

$\Rightarrow \quad -3 \leq p \leq 1 \qquad (3)$

$$0 \leq P(C) \leq 1 \Rightarrow 0 \leq \frac{1-2p}{2} \leq 1$$

$\Rightarrow \quad -\frac{1}{2} \leq p \leq \frac{1}{2} \qquad (4)$

Combining Eqs. (1), (2), (3) and (4), we get

$$\frac{1}{3} \leq p \leq \frac{1}{2}$$

5. First draw is from P, second draw is from Q and third draw is from P. There may be following cases:

Case I:

$R \to R \to R$

The required probability is (6/10) × (5/11) × (6/10) = (18/110).

Case II:

$R \to B \to R$

The required probability is (6/10) × (6/11) × (6/10) = (18/110).

Case III:

$B \to R \to R$

The required probability is (4/10) × (4/11) × (7/10) = (56/550).

Case IV:

$B \to B \to R$

The required probability is (4/10) × (7/11) × (6/10) = (84/550).

Therefore, the total probability is

$$\frac{18}{110} + \frac{18}{110} + \frac{56}{550} + \frac{84}{550} = \frac{90+90+56+84}{550} = \frac{320}{550} = \frac{32}{55}$$

6. Probability of getting a sum of 5 is 4/36 = 1/9 = $P(A)$ as favourable cases are {(1, 4), (4, 1), (2, 3) (3, 2)}. Similarly, favourable cases of getting a sum of 7 are {(1, 6). (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)}. The total number of such cases is 6. Therefore, the probability of getting a sum of 7 is (6/36) = (1/6). The probability of getting a sum of 5 or 7 is (1/6) + (1/) + (5/18) (As events are mutually exclusive).

Hence, the probability of getting neither a sum of 5 nor a sum of 7 is 1 – (1/18) = (13/18).

Now, we get a sum of 5 before a sum of 7 if either we get a sum of 5, in first chance or we get neither a sum of 5 nor a sum of 7 in first chance and a sum of 5 in second chance and so on. Therefore, the required probability is

$$\frac{1}{9} + \frac{13}{18} \times \frac{1}{9} + \frac{13}{18} \times \frac{13}{18} \times \frac{1}{9} + \cdots \infty$$

$$= \frac{1/9}{1 - 13/18}$$

$$= \frac{1}{9} \times \frac{18}{5} = \frac{2}{5}$$

7. $P(A \cup B) = 0.8$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A)\,P(B)$$

[as A and B are independent events]

$\Rightarrow$ $0.8 = 0.3 + P(B) - 0.3\,P(B)$

$\Rightarrow$ $0.5 = 0.7\,P(B)$

$\Rightarrow$ $P(B) = 5/7$

8. Sample space is $\{Y, Y, Y, R, R, B\}$, where Y stands for yellow colour, R for red and B for blue.

The probability that the colour yellow, red and blue appear in the first, second and third tosses, respectively, is (3/6) × (2/6) × (1/6) = (1/36).

9. Given that $P(A^c) = 0.3$, $P(B) = 0.4$ and $P(A \cap B^c) = 0.5$.

Then,

$$P[B/(A \cup B^c)] = \frac{P[B \cap (A \cup B^C)]}{P(A \cup B^c)}$$

$$= \frac{P((B \cap A) \cup (B \cap B^c))}{P(A \cup B^c)}$$

$$= \frac{P(A \cap B)}{P(A) + (B^c) - P(A \cap B^c)}$$

$$= \frac{P(A) - P(A \cap B^c)}{1 - P(A^c) + 1 - (B) - P(A \cap B^c)}$$

$$= \frac{1 - 0.3 - 0.5}{1 - 0.3 + 1 - 0.4 - 0.5}$$

$$= \frac{0.2}{0.8} = \frac{1}{4}$$

10. Let E_1 be the event of getting minimum number 3, E_2 be the event of getting maximum number 7. Then,

$P(E_1) = P$(getting one number 3 and other 2 from numbers 4 to 10)

$$= \frac{{}^1C_1 \times {}^7C_2}{{}^{10}C_3} = \frac{7}{40}$$

Similarly,

$P(E_2) = P$(getting one number 7 and other 2 from numbers 1 to 6)

$$= \frac{{}^1C_1 \times {}^6C_2}{{}^{10}C_3} = \frac{1}{8}$$

$P(E_1 \cup E_2) = P$ (getting one number 3, second number 7 and third from numbers 4 to 6)

$$= \frac{{}^1C_1 \times {}^1C_1\ {}^3C_1}{{}^{10}C_3} = \frac{1}{40}$$

$\therefore$ $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$$= \frac{7}{40} + \frac{1}{8} - \frac{1}{40}$$

$$= \frac{7 + 5 - 1}{40}$$

$$= \frac{11}{40}$$

True or false

1. Let E be the event 'no two S's occur together'. A, A, I, N can be arranged is 4!/2! = 12 ways.

— A — A — I — N —

In the arrangement shown above there are 5 places for four S.

Out of 5 places, 4 can be selected in ${}^5C_4 = 5$ ways. Therefore, no two S's occur together in 12 × 5 = 60 ways. Hence, the total number of arranging all letters of word ASSASSIN is 8!/(4! 2!) = 840. Therefore, the required probability is (60/840) = (1/14). Now,

$$P(A) + P(B) - P(A)\,P(B) = 0.2 + 0.3 - 0.2 \times 0.3$$

$$= 0.5 - 0.06 = 0.44$$

$$\neq 0.5$$

Hence, the statement is false.

2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Multiple choice question with one correct answer

1. d. The two events can happen simultaneously, e.g., (2, 3). Therefore, they are not mutually exclusive. Also, the two events are not dependent on each other.

2. a. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 0.25 + 0.50 - 0.14$$

$$= 0.61$$

$\therefore$ $P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B)$

$$= 1 - 0.61 = 0.39$$

3. b. $p = 0.4, q = 0.6$

$\therefore$ $P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - {}^3C_0\,(0.4)^0\,(0.6)^3$$

$$= 1 - 0.216 = 0.784$$

4. c. $P(\bar{A}/\bar{B}) = \dfrac{P(\bar{A} \cap \bar{B})}{P(\bar{B})}$

$$= \frac{P(\overline{A \cup B})}{P(\bar{B})}$$

$$= \frac{1 - (A \cup B)}{P(\bar{B})}$$

5. d. Since there are 15 possible cases for selecting a coupon and seven coupons are selected, the total number of cases of selecting seven coupons is 15^7. It is given that the largest number on the selected coupon is 9. Therefore the selection is to be made from the coupons numbered 1 to 9. This can be made in 9^7 ways. Out of these 9^7 cases, 8^7 cases do not contain the number 9. Thus, the favourable number of cases is $9^7 - 8^7$. Hence, the required probability is $(9^7 - 8^7)/(15^7)$.

6. d. Let p be the probability of one coin showing head. Then the probability of one coin showing tail is $1 - p$. According to question, the coin is tossed 100 times and probability of 50 coins showing head is equal to the probability of 51 coins showing head.

Using binomial probability distribution $P(X = r) = {}^nC_r p^r q^{n-r}$, we get

$$^{100}C_5 p^{50}(1-p)^{50} = {}^{100}C_{51} p^{51}(1-p)^{49}$$

$$\Rightarrow \frac{1-p}{p} = \frac{^{100}C_{51}}{^{100}C_{50}} = \frac{50!\,50!}{51!\,49!} = \frac{50}{51}$$

$$\Rightarrow 51 - 51p = 50p$$

$$\Rightarrow 101p = 51 \Rightarrow p = \frac{51}{101}$$

7. b. P(at least 7 points) = P(7 points) + P(8 points)

[∵ at most 8 points can be scored]

Now, 7 points can be scored by scoring 2 points in 3 matches and 1 point in one match. Similarly, 8 points can be scored by scoring 2 points in each of 4 matches. Therefore, the required probability is

$^4C_1 \times [P(2\text{ points})]^3\, P(1\text{ point}) + [P(2\text{ points})]^4 = 4(0.5)^3 \times 0.05 + (0.50)^4$

$= 0.0250 + 0.0625 = 0.0875$

8. a. The minimum face value is not less than 2 and maximum face value is not greater than 5 if we get any of the members 2, 3, 4, 5, while total possible outcomes are 1, 2, 3, 4, 5 and 6. Therefore, in one throw of die, probability of getting any number out of 2, 3, 4 and 5 is $4/6 = 2/3$.

If the die is rolled four times, then all these events being independent, the required probability is $(2/3)^4 = 16/81$.

9. b. Given that

$P(\text{India wins}) = p = 1/2$

$\therefore\ P(\text{India loses}) = p' = 1/2$

Out of 5 matches, India's second win occurs at third test. Hence, India wins third test and simultaneously it has won one match from first two and lost the other. Therefore, the required probability is

$$P(LWW) + P(WLW) = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3$$

$$= \frac{1}{4}$$

10. b. Out of 6 vertices, 3 can be chosen in $^6C_3$ ways. The triangle will be equilateral if it is ΔACE or ΔBDF (2 ways).

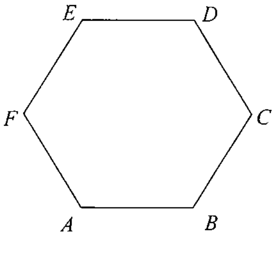

Fig. 9.19

Therefore, the required probability is

$$\frac{2}{^6C_3} = \frac{2}{20} = \frac{1}{10}$$

11. a. We know that

$P(\text{exactly one of } A \text{ or } B \text{ occurs}) = P(A) + P(B) - 2P(A \cap B)$

Therefore,

$$P(A) + P(B) - 2P(A \cap B) = p \quad (1)$$

Similarly,

$$P(B) + P(C) - 2P(B \cap A) = p \quad (2)$$

and

$$P(C) + P(A) - 2P(C \cap A) = p \quad (3)$$

Adding Eqs. (1), (2) and (3) we get

$$2[P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A)] = 3p$$

$$\Rightarrow P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) = 3p/2 \quad (4)$$

It is also given that

$$P(A \cap B \cap C) = p^2 \quad (5)$$

Now,

$P(\text{at least one of } A, B \text{ and } C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$$= \frac{3p}{2} + p^2 \quad \text{[Using Eqs. (4) and (5)]}$$

$$= \frac{3p + 2p^2}{2}$$

12. a. We know that $7^1 = 7$, $7^2 = 49$, $7^3 = 343$, $7^4 = 2401$, $7^5 = 16807$.

Therefore, 7^k (where $k \in Z$) results in a number whose unit's digit is 7 or 9 or 3 or 1.

Now, $7^m + 7^n$ will be divisible by 5 if unit's place digit in the resulting number is 5 or 0. Clearly, it can never be 5. But it can be 0 if we consider values of m and n such that the sum of unit's place digits become 0. And this can be done by choosing

$$\left.\begin{matrix} m = 1, & 5, & 9, \ldots, & 97 \\ n = 3, & 7, & 11, \ldots, & 99 \end{matrix}\right\} \quad \text{(25 options each) } [7 + 3 = 10]$$

or

$$\left.\begin{matrix} m = 2, & 6, & 10, \ldots, & 98 \\ n = 4, & 8, & 12, \ldots, & 100 \end{matrix}\right\} \quad \text{(25 options each) } [9 + 3 = 10]$$

Therefore, the total number of selections of m, n such that $7^m + 7^n$ is divisible by 5 is $(25 \times 25 + 25 \times 25) \times 2$ (since we can interchange values of m and n).

Also the number of total possible selections of m and n out of 100 is 100×100. Therefore, the required probability is

$$\frac{2(25 \times 25 + 25 \times 25)}{100 \times 100} = \frac{1}{4}$$

13. d. The minimum of two numbers will be less than 4 or at least one of the numbers is less than 4.

$\therefore\ P(\text{at least one numbers} < 4) = 1 - P(\text{both the numbers} \geq 4)$

$$= 1 - \frac{3}{6} \times \frac{2}{5}$$

$$= 1 - \frac{6}{30}$$

$$= = 1 - \frac{1}{5} = \frac{4}{5}$$

14. a. Given that $P(B) = 3/4$, $P(A \cap B \cap \bar{C}) = 1/3$

$P(\bar{A} \cap B \cap \bar{C}) = 1/3$. From Venn's diagram, we have

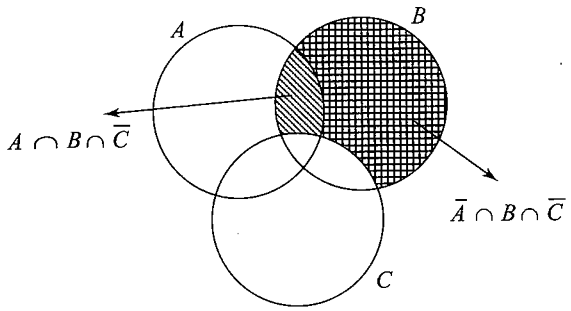

Fig. 9.20

$$P(B \cap C) = P(B) - P(A \cap B \cap \bar{C}) - P(\bar{A} \cap B \cap \bar{C})$$

$$\Rightarrow \quad P(B \cap C) = \frac{3}{4} - \frac{1}{3} - \frac{1}{3} = \frac{9-4-4}{12} = \frac{1}{12}$$

15. d. If a number is to be divisible by both 2 and 3, it should be divisible by their L.C.M. L.C.M. of 2 and 3 is 6. The numbers are 6, 12, 18, ..., 96. The total number is 16. Hence, the probability is

$$\frac{^{16}C_3}{^{100}C_3} = \frac{4}{1155}$$

16. a. In single throw of a dice, probability of getting 1 is 1/6 and probability of not getting 1 is 5/6.

Then, getting 1 in even number of chances is getting 1 in 2nd chance or in 4th chance or in 6th chance and so on. Therefore, the required probability is

$$\frac{5}{6} \times \frac{1}{6} + \left(\frac{5}{6}\right)^3 \times \frac{1}{6} + \left(\frac{5}{6}\right)^5 \times \frac{1}{6} + \cdots + \infty$$

$$= \frac{5}{36}\left[\frac{1}{1-\frac{25}{36}}\right] = \frac{5}{36} \times \frac{36}{11} = \frac{5}{11}$$

17. b. Favourable cases are {(1, 1, 1), (2, 2, 2), ..., (6, 6, 6)}. The number of favourable cases is 6. The total number cases is $6 \times 6 \times 6 = 216$. Therefore, the required probability is $6/216 = 1/36$.

18. c. The probability of getting a white ball in a single draw is $p = 12/24 = 1/2$. The probability of getting a white ball 4th time in the 7th draw is

P(getting 3W in 6th draws and W in 7th draw)

$$= {}^6C_3 \left(\frac{1}{2}\right)^6 \frac{1}{2} = \frac{5}{32}$$

19.(a) $P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$

$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$

$= P(A)\,P(B) + P(A)\,P(C) - P(A)\,P(B)\,P(C)$

$= P(A)\,[P(B) + P(C) - P(B \cap C)] = P(A)\,P(B \cup C)$

Therefore, S_1 is true.

$$P(A \cap (B \cap C)) = P(A)\,P(B)\,P(C) = P(A)\,P(B \cup C)$$

Therefore, S_2 is also true.

20. c. Let E_1 be the event that the Indian man is seated adjacent to his wife and E_2 be the even that each American man is seated adjacent to his wife. Then,

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

Now, $E_1 \cap E_2$ is the evetn that all men are seated adjacent to their wives.

Therefore, we can consider the 5 couples as single–single objects which can be arranged in a circle in 4! ways. But for each couple, husband and wife can interchange their places in 2! ways.

Hence, the umber of ways when all men are seated adjacent to their wives is $4! \times (2!)^5$. Also in all, 10 persons can be seated in a circle in 9! ways.

$$\therefore \quad P(E_1 \cap E_2) = \frac{4! \times (2!)^5}{9!}$$

Similarly, if each American man is seated adjacent to his wife, considering each American couple as single object and Indian woman and man as separate objects, there are 6 different objects which can be arranged in a circle in 5! ways. Also for each American couple, husband and wife can interchange their places in 2! ways.

So, the number of ways in which each American man is seated adjacent to his wife is $5! \times (2!)^4$.

$$\therefore \quad P(E_2) = \frac{5! \times (2!)^4}{9!}$$

Hence,

$$P(E_1/E_2) = \frac{(4! \times (2!)^5)/9!}{(5! \times (2!)^4)/9!} = \frac{2}{5}$$

21. c. We have,

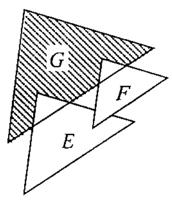

Fig. 9.21

$$E \cap F \cap G = \phi$$

$$P(E^c \cap F^c/G) = \frac{P(E^c \cap F^c \cap G)}{P(G)}$$

$$= \frac{P(G) - P(E \cap G) - P(G \cap F)}{P(G)}$$

[From Venn diagram,
$E^c \cap F^c \cap G = G - E \cap G - F \cap G$]

$$= \frac{P(G) - P(E)P(G) - P(G)P(F)}{P(G)} \quad [\because E, F, G \text{ are pairwise independent}]$$

$$= 1 - P(E) - P(F) = P(E^c) - P(F)$$

22. c.

Even G = original signal is green
E_1 = A receives the signal correct
E_2 = B receives the signal correct
E = Signal received by B is green
P (Signal received by B is green)

$$= P(GE_1E_2) + P(G\bar{E}_1\bar{E}_2) + P(\bar{G}E_1\bar{E}_2) + P(\bar{G}\bar{E}_1E_2)$$

$$P(E) = \frac{46}{5 \times 16}$$

$$P(G/E) = \frac{40/5 \times 16}{46/5 \times 16} = \frac{20}{23}$$

23. c.

$r_1, r_2, r_3 \in \{1,2,3,4,5,6\}$

r_1, r_2, r_3 are of the form $3k$, $3k+1$, $3k+2$

$$\text{Required probability} = \frac{3! \times {}^2C_1 \times {}^2C_1 \times {}^2C_1}{6 \times 6 \times 6} = \frac{6 \times 8}{216} = \frac{2}{9}$$

Multiple choice questions with one or more than one correct answer

1. a, c, d.

a. $P(M) + P(N) - 2\,P(M \cap N)$
$= [P(M) + P(N) - P(M \cap N)] - P(M \cap N)$
$= P(M \cup N) - P(M \cap N)$
= Probability that exactly one of M and N occurs

b. $P(M) + P(N) - P(M \cap N)$
$= P(M \cup N)$
= Probability that at least of M and N occurs

c. $P(M^c) + P(N^c) - 2\,P(M^c \cap N^c)$
$= 1 - P(M) + 1 - P(N) - 2P(M \cup N)^c$
$= 2 - P(M) - P(N) - 2[1 - P(M \cup N)]$
$= P(M \cup N) + P(M \cup N) - P(M) - P(N)$
$= P(M \cup N) - P(M \cap N)$
= Probability that exactly one of M and N occurs

d. $P(M \cap N^c) + P(M^c \cap N)$
= Probability that M occurs but N does not or probability that M does not occurs but N occurs
= Probability that exactly one of M and N occurs

Thus, (a), (c) and (d) are the correct options.

2. c. Let A, B, C be the events that the student passes tests I, II, III, respectively. Then, according to question, $P(A) = p$, $P(B) = q$, $P(C) = 1/2$.

Now the student is successful if A and B happen or A and C happen or A, B and C happen.

$$\therefore \quad P(AB\bar{C}) + P(AC\bar{B}) + P(ABC) = \frac{1}{2}$$

$$\Rightarrow \quad pq\left(1 - \frac{1}{2}\right) + p\frac{1}{2}(1-q) + pq\frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{2}pq + \frac{1}{2}p - \frac{1}{2}pq + \frac{1}{2}pq = \frac{1}{2}$$

$$\Rightarrow \quad p + pq = 1$$

$$\Rightarrow \quad p(1+q) = 1$$

which holds for $p = 1$ and $q = 0$.

3. c. Given that

$P(A \cup B) = 0.6$; $P(A \cap B) = 0.2$

$$\begin{aligned} \therefore \quad P(\bar{A}) + P(\bar{B}) &= 1 - P(A) + 1 - P(B) \\ &= 2 - (P(A) + P(B)) \\ &= 2 - [P(A \cup B) + P(A \cap B)] \\ &= 2 - [0.6 + 0.2] \\ &= 2 - 0.8 \\ &= 1.2 \end{aligned}$$

4. a, b, c.

We know that

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) \quad (1)$$

Also,

$$P(A \cup B) \le 1$$

$$\Rightarrow \quad -P(A \cup B) \ge -1 \quad (2)$$

$$\therefore \quad P(A \cap B) \ge P(A) + P(B) - 1 \quad \text{[Using Eqs. (1) and (2)]}$$

Therefore, option (a) is correct. Again,

$$P(A \cup B) \ge 0$$

$$\Rightarrow \quad -P(A \cup B) \le 0 \quad (3)$$

$$\Rightarrow \quad P(A \cap B) \le P(A) + P(B) \quad \text{[Using Eqs. (1) and (3)]}$$

Therefore, option (b) is also correct.

From Eq. (1), option (c) is correct and (d) is not correct.

5. b, c, d.

$P(E \cap F) = P(E)\,P(F)$

Now,

$$P(E \cap F) = P(E) - P(E \cap F) = P(E)\,[1 - P(F)] = P(E)P(F^c)$$

and

$$P(E^c \cap F^c) = 1 - P(E \cup F) = 1 - [P(E) + P(F) - P(E \cap F) = [1 - P(E)]\,[1 - P(F)] = P(E^c)\,P(F^c)$$

Also,

$P(E/F) = P(E)$ and $P(E^c/F^c) = P(E^c)$

$$\Rightarrow \quad P(E/F) + P(E^c/F^c) = 1$$

6. a, c.

a. For any two events A and B,

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Now we know that

$$P(A \cup B) \le 1$$

$$\Rightarrow \quad P(A) + P(B) - P(A \cap B) \le 1$$

$$\Rightarrow \quad P(A \cap B) \ge P(A) + P(B) - 1$$

$$\Rightarrow \quad \frac{P(A \cap B)}{P(B)} \ge \frac{P(A) + P(B) - 1}{P(B)} \quad [\text{as } P(B) \ne 0 \therefore P(B) > 0]$$

$$\Rightarrow \quad P(A/B) \ge \frac{P(A) + P(B) - 1}{P(B)}$$

Therefore, option (a) is correct.

b. From Venn's diagram, we can clearly conclude that

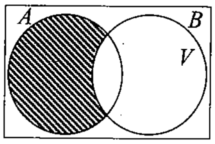

Fig. 9.22

$P(A \cup \bar{B}) = P(A) - P(A \cap B)$

Therefore, option (b) is incorrect

c. $P(A \cap B) = P(A) + P(B) - P(A \cap B)$

$= 1 - P(\bar{A}) + 1 - P(\bar{B}) - P(A)\,P(B)$ [$\because$ A and B are independent events]

$= 2 - P(\bar{A}) - P(\bar{B}) - [1 - P(\bar{A})][1 - P(\bar{B})]$

$= 2 - P(\bar{A}) - P(\bar{B}) - 1 + P(\bar{A}) + P(\bar{B}) - P(\bar{A})P(\bar{B})$

$= 1 - P(\bar{A} \cap \bar{B})$ [$\because$ if A and B are independent, $\bar{A}$ and $\bar{B}$ are also independent]

Therefore, option (c) is the correct statement.

d. For disjoint events,

$P(A \cup B) = P(A) + P(B)$

Therefore, option (d) is he incorrect.

7. a, b.

Let $P(E) = x$ and $P(F) = y$. According to the question,

$$P(E \cap F) = \frac{1}{12}$$

As E and F are independent events, we have

$$P(E \cap F) = P(E)\,P(F)$$

$$\Rightarrow \frac{1}{12} = xy \quad (1)$$

Also,

$$P(\bar{E} \cap \bar{F}) = P(\overline{E \cup F})$$

$$= 1 - P(E \cup F)$$

$$\Rightarrow \frac{1}{2} = 1 - [P(E) + P(F) - P(E)\,P(F)]$$

$$\Rightarrow x + y = \frac{7}{12} \quad (2)$$

Solving Eqs. (1) and (2), we get either $x = 1/3$ and $y = 1/4$ or $x = 1/4$ and $y = 1/3$.

Therefore, options (a) and (b) are correct.

8. a. $P(2 \text{ white and } 1 \text{ black}) = P(W_1\, W_2\, B_3 \text{ or } W_1\, B_2\, W_3 \text{ or } B_1\, W_2\, W_3)$

$= P(W_1\, W_2\, B_3) + P(W_1\, B_2\, W_3) + P(B_1\, W_2\, W_3)$

$= P(W_1)\,P(W_2)\,P(B_3) + P(W_1)\,P(B_2)\,P(W_3) + P(B_1)\,(W_2)\,(W_3)$

$= \frac{3}{4}\times\frac{2}{4}\times\frac{3}{4} + \frac{3}{4}\times\frac{2}{4}\times\frac{1}{4} + \frac{1}{4}\times\frac{2}{4}\times\frac{1}{4}$

$= \frac{1}{32}(9 + 3 + 1)$

$= \frac{13}{32}$

9. a, d.

We have,

$$P(E/F) + P(\bar{E}/F) = \frac{P(E \cap F)}{P(F)} + \frac{P(\bar{E} \cup F)}{P(F)}$$

$$= \frac{P(E \cap F) + P(\bar{E} \cap F)}{P(F)}$$

$$= \frac{P(F)}{P(F)} = 1$$

Therefore, option (a) holds. Also,

$$P(E/F) + P(\bar{E}/F) = \frac{P(E \cap F) + P(\bar{F} \cap E)}{P(F)}$$

$$= \frac{P(E)}{P(F)} \neq 1$$

Therefore option (b) does not hold.

Similarly, we can show that option (c) does not hold but option (d) holds.

10. a. The probability that only two tests are needed is (probability that the first machine tested is faulty) × (probability that the second machine tested is faulty given the first machine tested is faulty), which is given by (2/4) × (1/3) = 1/6.

11. d. Given that $P(E) \le P(F)$ and $P(E \cap F) > 0$. It does not necessarily mean that E is the subset of F. Therefore, the choices (a), (b), (c) do not hold in general. Hence, option (d) is the right choice here.

12. a. The event that the fifth toss result in a head is independent of the event that the first four tosses result in tails. Therefore, the probability of the required event is 1/2.

13. b, c.

According to the problem,

$$m + p + c - mp - mc - pc + mpc = 3/4 \quad (1)$$

$$mp(1 - c) + mc(1 - p) + pc(1 - m) = 2/5 \quad (2)$$

Also,

$$mp + pc + mc - 2mpc = 1/2 \quad (3)$$

From Eqs. (2) and (3),

$$mpc = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$$

$$\therefore \quad mp + mc + pc = \frac{2}{5} + \frac{3}{10} = \frac{7}{10}$$

$$\therefore \quad m + p + c = \frac{3}{4} + \frac{7}{10} - \frac{1}{10} = \frac{15 + 14 - 2}{20} = \frac{27}{20}$$

14. b. The number of ways of arranging 10 balls without any restriction is 10!. As for no two black balls are placed adjacently, first arrange 7 white balls in 7! ways.

$- W - W - W - W - W - W - W -$

Now white balls must be placed in three of eight gaps created in ${}^8C_3 3!$ ways. Hence, number of favorable ways is ${}^8C_3 3!\, 7!$.

Therefore, the required probability is

$$\frac{{}^8C_3 3! 7!}{10!} = \frac{7}{15}$$

15. c., d.

$$P(A \cup B)' = 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$= 1 - P(A) - P(B) + P(A)\,P(B)$

$= [1 - P(A)]\,[1 - P(B)]$

$= P(A')\,P(B')$

Also,

$P(A \cup B) = P(A) + P(B) - P(A)\,P(B)$

$\Rightarrow \quad P(A \cap B) = P(A)\,P(B)$

$\therefore \quad P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)\ \ P(B)}{P(B)} = P(A)$

Also,

$P(B/A) = P(B)$

$P(A' - B') = P(A') - P(A' \cap B')$

$= P(A') - P(A')\,P(B')$

$= P(A')(1 - P(B'))$

$= P(A')P(B)$

16. a, d.

Let $P(E) = e$ and $P(F) = f$

$P(E \cup F) - P(E \cap F) = \dfrac{11}{25}$

$\Rightarrow \quad e + f - 2ef = \dfrac{11}{25}$ (1)

$P(\bar{E} \cap \bar{F}) = \dfrac{2}{25}$

$\Rightarrow \quad (1 - e)(1 - f) = \dfrac{2}{25}$

$\Rightarrow \quad 1 - e - f + ef = \dfrac{2}{25}$ (2)

From (1) and (2)

$ef = \dfrac{12}{25}$ and $e + f = \dfrac{7}{5}$

Solving, we get

$e = \dfrac{4}{5}, f = \dfrac{3}{5}$ or $e = \dfrac{3}{5}, f = \dfrac{4}{5}$

Comprehension

1. b. $P(u_i) \propto i \Rightarrow P(u_i) = K_i,$

But

$\Sigma P(u_i) = 1$

$\Rightarrow \quad \Sigma k_i = 1 \Rightarrow k\,\Sigma i = 1 \Rightarrow K = \dfrac{2}{n(n+1)}$

$\Rightarrow \quad P(u_i) = \dfrac{2i}{n(n+1)}$

By the total probability theorem,

$$P(w) = \sum_{i=1}^{n} P(u_i)\,P(w/u_i)$$

$$= \sum_{i=1}^{n} \frac{2i}{n(n+1)} \times \frac{i}{n+1}$$

$$= \frac{2}{n(n+1)^2}\,\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{2n+1}{3n+3}$$

$$\therefore \quad \lim_{n\to\infty} P(w) = \lim_{n\to\infty} \frac{2n+1}{3n+3} = \lim_{n\to\infty} \frac{2 + 1/n}{3 + 3/n} = \frac{2}{3}$$

2. c. $P(u_i) = c$

Using Bayes's theorem,

$$P(u_n/w) = P(u_m/w)\,\frac{P(w/u_n)\,P(u_n)}{\sum_{i=1}^{n} P(w/u_i)\,P(u_i)}$$

$$= \frac{c \times \dfrac{n}{n+1}}{c\left[\dfrac{1}{n+1} + \dfrac{2}{n+1} + \cdots + \dfrac{n}{n+1}\right]}$$

$$= \frac{n}{n+1} \times \frac{n+1}{\dfrac{n(n+1)}{2}} = \frac{2}{n+1}$$

3. b. $P(w/E) = \dfrac{P(w \cap E)}{P(E)}$

$$= \frac{\dfrac{1}{n} \times \dfrac{2}{n+1} + \dfrac{1}{n} \times \dfrac{4}{n+1} + \dfrac{1}{n} \times \dfrac{6}{n+1} + \cdots + \dfrac{1}{n} \times \dfrac{n}{n+1}}{\dfrac{1}{n} + \dfrac{1}{n} + \cdots + \dfrac{1}{n}\left(\dfrac{n}{2}\text{ times}\right)}$$

$$= \frac{\dfrac{2}{n+1}\left[1 + 2 + 3 + \cdots + \dfrac{n}{2}\right]}{\dfrac{1}{n} \times \dfrac{n}{2}} \quad (n \text{ being even})$$

$$= \frac{4}{n+1}\left[\frac{\dfrac{n}{2}\left(\dfrac{n}{2} + 1\right)}{2}\right]$$

4. a. $P(X = 3) = \left(\dfrac{5}{6}\right)\left(\dfrac{5}{6}\right)\dfrac{1}{6} = \dfrac{25}{216}$

5. b. Given that

$$P(X \le 2) = \frac{1}{6} + \frac{5}{6} \times \frac{1}{6} = \frac{11}{36}$$

Hence, the required probability is $1 - (11/36) = 25/36$.

6. d. For $X \ge 6$, the probability is

$$\frac{5^5}{6^6} + \frac{5^5}{6^7} + \cdots \infty = \frac{5^5}{6^6}\left(\frac{1}{1 - 5/6}\right) = \left(\frac{5}{6}\right)^5$$

For $X > 3$,

$$\frac{5^3}{6^4} + \frac{5^4}{6^5} + \frac{5^5}{6^6} + \cdots \infty = \left(\frac{5}{6}\right)^3$$

Hence, the conditional probability is

$$\frac{(5/6)^6}{(5/6)^3} = \frac{25}{36}$$

7. b. $H \to 1$ ball form U_1 to U_2

$T \to 2$ ball form U_1 to U_2

E : 1 ball drawn from U_2

$P(W \text{ from } U_2)$

$$= \frac{1}{2}\times\left(\frac{3}{5}\times 1\right)+\frac{1}{2}\times\left(\frac{2}{5}\times\frac{1}{2}\right)+\frac{1}{2}\times\left(\frac{{}^3C_2}{{}^5C_2}\times\frac{1}{3}\right)$$

$$+\frac{1}{2}\times\left(\frac{{}^3C_1\,.\,{}^2C_1}{{}^5C_2}\times\frac{2}{3}\right)=\frac{23}{30}$$

8. d. $$P\left(\frac{H}{W}\right) = \frac{P\,(W/H)\times P(H)}{P(W/T)\,.P(T)+(W/H)\,.\,P(H)}$$

$$= \frac{\frac{1}{2}\left(\frac{3}{2}\times 1+\frac{2}{5}\times\frac{1}{2}\right)}{23/30}=\frac{12}{23}$$

Assertion and reason

1. d. We know that

$$P(H_i/E) = \frac{P(H_i \cap E)}{P(E)}$$

$$= \frac{P(E/H_i)\,P(H_i)}{P(E)}$$

$$\Rightarrow \quad P(H_i/E)\,P(E) = P(E/H_i)\,P(H_i)$$

$$\Rightarrow \quad P(E) = \frac{P(E/H_i)\,P(H_i)}{P(H_i/E)}$$

Now given that

$$0 < P(E) < 1$$

$$\Rightarrow \quad 0 < \frac{P(E/H_i)P(H_i)}{P(H_i/E)} < 1$$

$$\Rightarrow \quad P(E/H_i)\,P(H_i) < P(H_i/E)$$

But if $P(H_i \cap E) = 0$, then $P(H_i/E) = P(E/H_i) = 0$. Then $P(E/H_i)$ $(PH_i) < P(H_i/E)$ is not true. Hence, statement 1 is not always true.

Also as $H_1, H_2, \ldots, H_n$ are mutually exclusive and exhaustive events, therefore

$$\sum_{i=1}^{n} P(H_i) = 1$$

Hence, statement 2 is true.

2. b. For unique solution,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

where $a, b, c, d \in \{0, 1\}$.

The total number of cases is 16. Favourable number of cases is 6 (either $ad = 1$, $bc = 0$ or $ad = 0$, $bc = 1$). The probability that system of equations has unique solution is $6/16 = 3/8$. Since $x = 0$ satisfies both the equations, the system has at least one solution.