CHAPTER

8

Definite Integration

- Definite Integration
- Properties of Definite Integrals
- Definite Integration of Odd and Even Functions
- Definite Integration of Periodic Functions
- Leibnitz's Rule
- Inequalities

)EFINITE INTEGRATION

)efinite Integral as the Limit of a Sum 'Integration by First Principle Rule)

.et f be a continuous function defined on a close interval $[a, b]$. \ssume that all the values taken by the function are non-ıegative, so the graph of the function is a curve above the x-axis.

The definite integral $\int_a^b f(x)\,dx$ is the area bounded by the urve $y=f(x)$, the ordinates $x=a$, $x=b$ and the x-axis. To evaluate ıis area, consider the region $PRSQP$ between the curve, x-axis nd the ordinates $x=a$, $x=b$.

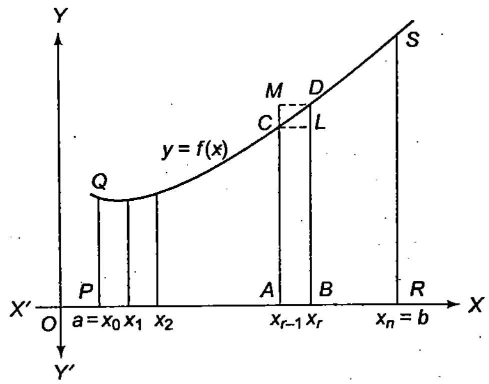

Fig. 8.1

)ivide the interval $[a, b]$ into n equal sub-intervals denoted by $[x_0, x_1], [x_1, x_2] \ldots [x_{r-1}, x_r] \ldots [x_{n-1}, x_n]$, where $x_0 = a, x_1 = a+h$, $x_2 = a+2h, \ldots, x_r = a+rh$ and $x_n = b = a+nh$ or $n = \frac{b-a}{h}$.

Ve note that as $n \to \infty$, $h \to 0$.

The region $PRSQP$ under consideration is the sum of n ub-regions, where each sub-region is defined on sub-intervals $[x_{r-1}, x_r], r = 1, 2, 3, \ldots, n$.

From Fig. 8.1, we have area of the rectangle $ABLC <$ area of the ≥gion $ABDCA <$ area of the rectangle $ABDM$. (1)

Evidently, as $x_r - x_{r-1} \to 0$, i.e., $h \to 0$, all the three areas ıown in Fig. 8.1 become nearly equal to each other.

Now, we form the following sums:

$$s_n = h[f(x_0) + \cdots + f(x_{n-1})] = h\sum_{r=0}^{n-1} f(x_r) \quad (2)$$

nd $$S_n = h[f(x_1) + f(x_2) + \cdots + f(x_n)] = h\sum_{r=1}^{n} f(x) \quad (3)$$

Here, s_n and S_n denote the sum of area of all lower and upper ≥ctangles raised over sub-intervals $[x_{r-1}, x_r]$ for $r = 1, 2, 3, \ldots, n$, ≥spectively.

ı view of the inequality (1) for an arbitrary sub-interval $[x_{r-1}, x_r]$, we ave

$<$ area of the region $PRSQP < S_n$ (4)

As $n \to \infty$ the strips become narrower. It is assumed that the miting values of equations (2) and (3) are same in both the cases ıd the common limiting value is the required area under the ırve. Symbolically, we can write

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n = \text{area of the region } PRSQP = \int_a^b f(x)\,dx \quad (5)$$

$$\Rightarrow \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=0}^{n-1} hf(a+rh) = \lim_{n\to\infty} \sum_{r=1}^{n} hf(a+rh)$$

Some Important Series

a. $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

b. $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

c. $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$

d. *In G.P., sum of n terms,* $S_n = \frac{a(r^n - 1)}{(r-1)}$, *where r is common ratio* ($r \neq 1$) *and a is the first term.*

e. $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha + \overline{n-1}\,\beta)$

$$= \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \sin\left(\frac{2\alpha+(n-1)\beta}{2}\right)$$

f. $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha + \overline{n-1}\,\beta)$

$$= \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \cos\left(\frac{2\alpha+(n-1)\beta}{2}\right)$$

g. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots = \frac{\pi^2}{12}$

h. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots = \frac{\pi^2}{6}$

i. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

j. $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \frac{\pi^2}{24}$

Second Fundamental Theorem of Integral Calculus

We state below an important theorem which enables us to evaluate definite integrals using anti-derivative.

Theorem: Let f be a continuous function defined on a closed interval $[a, b]$ and F be an anti-derivative of f. Then $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$, where a and b are called the limits of integration, a being the lower or inferior limit and b being the upper or superior limit.

Note:

- *If $f(x)$ is not defined at $x = a$ and $x = b$, and defined in the open interval (a, b), then $\int_a^b f(x)dx$ can be evaluated.*
- *If $\int_a^b f(x)dx = 0$, then the equation $f(x) = 0$ has at least one root lying in (a, b), provided f is a continuous function in (a, b).*
- *In $\int_a^b f(x)dx$, the function f needs to be well-defined and continuous in the closed interval $[a, b]$. For instance, the consideration of the definite integral $\int_{-2}^{3} x(x^2-1)^{\frac{1}{2}}dx$ is erroneous since the function f expressed by $f(x) = x(x^2-1)^{1/2}$ is not defined in the portion $-1 < x < 1$ on the closed interval $[-2, 3]$.*

Geometrical Interpretation of the Definite Integral

First, we construct the graph of the integrand $y = f(x)$, then in the case of $f(x) \geq 0$, $\forall\ x \in [a, b]$, the integral $\int_a^b f(x)dx$ is numerically equal to the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$.

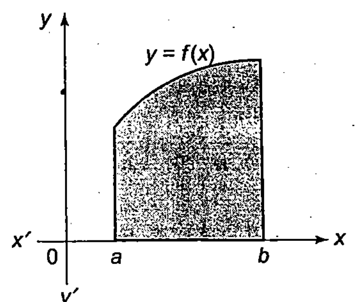

Fig. 8.2

$\int_a^b f(x)dx$ is numerically equal to the area of curvilinear trapezoid bounded by the given curve, the straight lines $x = a$ and $x = b$ and the x-axis.

In general, $\int_a^b f(x)dx$ represents an algebraic sum of areas of the region bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$.

The area above the x-axis are taken positive, while those below the x-axis are taken negative.

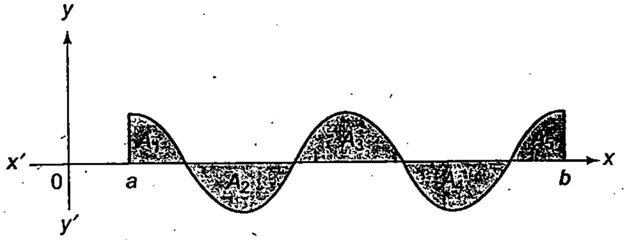

Fig. 8.3

$$\therefore \int_a^b f(x)dx = A_1 - A_2 + A_3 - A_4 + A_5$$

where A_1, A_2, A_3, A_4, A_5 are the areas of the shaded region.

Example 8.1 Evaluate $\int_a^b e^x dx$ using limit of sum.

Sol. We have

$$\int_a^b e^x\ dx = \lim_{n\to\infty} h\left[e^a + e^{a+h} + e^{a+2h} + \cdots + e^{a+(n-1)h}\right]$$

where $b - a = nh$

$$= \lim_{n\to\infty} he^a\left[1 + e^h + e^{2h} + \cdots + e^{(n-1)h}\right]$$

$$= \lim_{n\to\infty} he^a\left[\left\{(e^h)^n - 1\right\}/(e^h - 1)\right]$$

$$= \lim_{n\to\infty} e^a(e^{nh} - 1)\cdot[h/(e^h - 1)]$$

$[\because$ as $n \to \infty, h \to 0$, and $nh = b - a]$

$$= e^a(e^{b-a} - 1)1$$

$$= e^b - e^a$$

Example 8.2 Evaluate $\int_a^b \sin x dx$ using limit of sum.

Sol. We have

$$\int_a^b \sin x\, dx = \lim_{n\to\infty} h\left[\sin a + \sin(a+h) + \sin(a+2h) + \cdots + \sin\{a + (n-1)h\}\right]$$

where $nh = b - a$

$$= \lim_{n\to\infty} h\left[\frac{\sin\left\{a + \frac{1}{2}(n-1)h\right\}\sin\left(\frac{1}{2}nh\right)}{\sin\left(\frac{1}{2}h\right)}\right]$$

$$= \lim_{h\to 0} \frac{2\sin\left\{a + \frac{1}{2}(nh - h)\right\}\sin\left(\frac{1}{2}nh\right)\cdot\left(\frac{1}{2}h\right)}{\sin\left(\frac{1}{2}h\right)}$$

$$= 2\sin\left\{a + \frac{1}{2}(b - a - 0)\right\}\cdot\sin\frac{1}{2}(b - a)\cdot 1$$

$[\because$ as $n \to \infty, h \to 0$; and $nh = b - a]$

$$= 2\sin\left\{\frac{1}{2}(b + a)\right\}\sin\left\{\frac{1}{2}(b - a)\right\}$$

$$= \cos a - \cos b$$

Example 8.3 Evaluate $\int_a^b x^2 dx$ using limit of sum.

Sol. $\int_a^b x^2\ dx = \lim_{n\to\infty} h[a^2 + (a + h)^2 + (a + 2h)^2 + \cdots + \{a + (n - 1)h\}^2]$, where $nh = b - a$

$= \lim_{n\to\infty} h[na^2 + 2ah\{1+2+\cdots+(n-1)\} + h^2\{1^2+2^2 + \cdots + (n-1)^2\}]$

$= \lim_{n\to\infty} \left[nh\cdot a^2 + 2ah^2 \sum_{r=1}^{n-1} r + h^3 \sum_{r=1}^{n-1} r^2 \right]$

$= \lim_{n\to\infty} [nh\cdot a^2 + 2ah^2 \cdot \frac{1}{2}(n-1)n + h^3(1/6)(n-1)n(2n-1)]$

$= \lim_{n\to\infty} [(nh)a^2 + a(nh-h)(nh) + (1/6)(nh-h)(nh)\times (2nh-h)]$

$[\because$ as $n\to\infty, h\to 0$, and $nh = b-a]$

$= (b-a)a^2 + a(b-a-0)(b-a) + (1/6)(b-a-0)(b-a)\times \{2(b-a)-0\}$

$= \frac{1}{3}(b-a)\left[3a^2 + 3a(b-a) + (b-a)^2\right]$

$= \frac{1}{3}(b-a)\left(a^2+ab+b^2\right) = \frac{1}{3}\left(b^3-a^3\right)$

Example 8.4 Evaluate $\int_a^b \frac{dx}{\sqrt{x}}$, where $a, b > 0$.

Sol. $I = \int_a^b \frac{dx}{\sqrt{x}}$ $a>0, b>0$

$$I = h \lim_{n\to\infty} \left[\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{a+h}} + \frac{1}{\sqrt{a+2h}} + \cdots + \frac{1}{\sqrt{a+(n-1)h}} \right]$$

We know that $\sqrt{r} + \sqrt{r-h} < 2\sqrt{r} < \sqrt{r+h} + \sqrt{r}$ (for sufficiently small $h>0$).

$$\Rightarrow \frac{1}{\sqrt{r+h}+\sqrt{r}} < \frac{1}{2\sqrt{r}} < \frac{1}{\sqrt{r-h}+\sqrt{r}}$$

$$\Rightarrow \frac{\sqrt{r+h}-\sqrt{r}}{h} < \frac{1}{2\sqrt{r}} < \frac{\sqrt{r}-\sqrt{r-h}}{h}$$

Let put $r = a, a+h, a+2h, \cdots, a+(n-1)h$

$$\Rightarrow \frac{\sqrt{a+h}-\sqrt{a}}{h} < \frac{1}{2\sqrt{a}} < \frac{\sqrt{a}-\sqrt{a-h}}{h}$$

$$\frac{\sqrt{a+2h}-\sqrt{a+h}}{h} < \frac{1}{2\sqrt{a+h}} < \frac{\sqrt{a+h}-\sqrt{a}}{h}$$

$$\frac{\sqrt{a+3h}-\sqrt{a+2h}}{h} < \frac{1}{2\sqrt{a+2h}} < \frac{\sqrt{a+2h}-\sqrt{a+h}}{h}$$

$$\vdots$$

$$\frac{\sqrt{a+nh}-\sqrt{a+(n-1)h}}{h} < \frac{1}{2\sqrt{a+(n-1)h}} < \frac{\sqrt{a+(n-1)h}-\sqrt{a+(n-2)h}}{h}$$

Adding, we get

$$\frac{\sqrt{a+nh}-\sqrt{a}}{h} < \sum_{r=0}^{n-1} \frac{1}{2\sqrt{a+rh}} < \frac{\sqrt{a+(n-1)h}-\sqrt{a-h}}{h}$$

$$\Rightarrow 2\left(\sqrt{a+b-a}-\sqrt{a}\right) < h\sum_{r=0}^{n-1}\frac{1}{\sqrt{a+rh}} < 2\left(\sqrt{a+b-a-h}-\sqrt{a-h}\right) \text{ (Put } nh = b-a)$$

$$\Rightarrow \lim_{h\to 0} 2\left(\sqrt{a+b-a}-\sqrt{a}\right) < \lim_{h\to 0} h\sum_{r=0}^{n-1}\frac{1}{\sqrt{a+rh}} < \lim_{h\to 0} 2\left(\sqrt{a+b-a-h}-\sqrt{a-h}\right)$$

$$\Rightarrow 2\left(\sqrt{b}-\sqrt{a}\right) < \lim_{h\to 0} h\sum_{r=0}^{n-1}\frac{1}{\sqrt{a+rh}} < 2\left(\sqrt{b}-\sqrt{a}\right)$$

$$\Rightarrow 2\left(\sqrt{b}-\sqrt{a}\right) < \int_a^b \frac{1}{\sqrt{x}}dx < 2\left(\sqrt{b}-\sqrt{a}\right)$$

$$\Rightarrow \int_a^b \frac{1}{\sqrt{x}}dx = 2\left(\sqrt{b}-\sqrt{a}\right)$$

Limits using Definite Integration

We know that $\int_a^b f(x)\,dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh)$.

Now in a special case, let $a = 0$ and $b = 1$, then we have

$\int_0^1 f(x)\,dx = \lim_{h\to 0} \frac{1}{n}\sum_{n=1}^{n} f\left(\frac{r}{n}\right)$.

More generally, $\int_0^k f(x)\,dx = \lim_{h\to 0} \frac{1}{n}\sum_{r=1}^{kn} f\left(\frac{r}{n}\right)$

Example 8.5 Evaluate

$$\lim_{n\to\infty} n\left[\frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+4)} + \cdots + \frac{1}{6n^2}\right].$$

Sol. The given limit is

$$L = \lim_{n\to\infty} \sum_{r=1}^{n} n\cdot\frac{1}{(n+r)(n+2r)}$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n}\frac{1}{(1+r/n)(1+2r/n)}$$

$$= \int_0^1 \frac{dx}{(1+x)(1+2x)}$$

$$= \int_0^1 \left(\frac{-1}{1+x} + \frac{2}{1+2x}\right)dx$$

$$= \left[-\log(1+x) + \log(1+2x)\right]_0^1$$

$$= [(-\log 2 + \log 3) - (-\log 1 + \log 1)]$$

$$= \log(3/2)$$

Example 8.6 Evaluate $\lim_{n\to\infty}\left[\frac{1}{na}+\frac{1}{na+1}+\frac{1}{na+2}+\cdots+\frac{1}{nb}\right]$.

Sol. The given limit is

$$L=\lim_{n\to\infty}\left[\frac{1}{na}+\frac{1}{na+1}+\frac{1}{na+2}+\cdots+\frac{1}{na+n(b-a)}\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{(b-a)n}\frac{1}{na+r}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{(b-a)n}\frac{1}{a+r/n}$$

$$=\int_0^{(b-a)}\frac{dx}{a+x}=\left[\log(a+x)\right]_0^{b-a}$$

$$=\log b-\log a=\log(b/a)$$

Example 8.7 Evaluate $\lim_{n\to\infty}\frac{[(n+1)(n+2)\cdots(n+n)]^{1/n}}{n}$.

Sol. Let $L=\lim_{n\to\infty}\frac{[(n+1)(n+2)\cdots(n+n)]^{1/n}}{n}$

$$=\lim_{n\to\infty}\left[\frac{(n+1)(n+2)\cdots(n+n)}{n^n}\right]^{1/n}$$

$$=\lim_{n\to\infty}\left[\frac{n+1}{n}\cdot\frac{n+2}{n}\cdots\frac{n+n}{n}\right]^{1/n}$$

$$=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{1/n}$$

$$\log L=\lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\cdots+\log\left(1+\frac{n}{n}\right)\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\left(1+\frac{r}{n}\right)=\int_0^1\log(1+x)\,dx$$

$$=\left[x\log(1+x)\right]_0^1-\int_0^1\frac{x}{1+x}\,dx$$

$$=\log 2-\int_0^1\left[1-(1/(1+x))\right]dx=\log 2-[x-\log(1+x)]_0^1$$

$$=\log 2-[(1-\log 2)-(0-\log 1)]$$

$$=2\log 2-1=\log(2^2/e)$$

$$\therefore L=2^2/e=4/e$$

Example 8.8 Evaluate

$$\lim_{n\to\infty}\frac{(1^2+2^2+3^3+\cdots+n^2)(1^3+2^3+3^3+\cdots+n^3)}{1^6+2^6+3^6+\cdots+n^6}.$$

Sol. The given limit is $\lim_{n\to\infty}\frac{\sum_{r=1}^{n}r^2\times\sum_{r=1}^{n}r^3}{\sum_{r=1}^{n}r^6}$

$$=\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^2\times\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^3}{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^6}$$

$$=\frac{\int_0^1x^2dx\int_0^1x^3dx}{\int_0^1x^6dx}=\frac{\left[\frac{x^3}{3}\right]_0^1\left[\frac{x^4}{4}\right]_0^1}{\left[\frac{x^7}{7}\right]_0^1}=\frac{\frac{1}{3}\times\frac{1}{4}}{\frac{1}{7}}=\frac{7}{12}$$

Concept Application Exercise 8.1

1. Evaluate the following integrals using limit of sum.

   a. $\int_a^b\cos x\,dx$ b. $\int_a^b x^3dx$

2. Evaluate the following limits:

   a. $\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-2^2}}+\cdots+\frac{1}{\sqrt{3n^2}}\right)$

   b. $\lim_{n\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\frac{4}{n^2}+\cdots+\frac{1}{n}\sec^2 1\right]$

   c. $\lim_{n\to\infty}\sum_{K=1}^{n}\frac{K}{n^2+K^2}$

   d. $\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\sqrt{r}\sum_{r=1}^{n}\frac{1}{\sqrt{r}}}{\sum_{r=1}^{n}r}$

   e. $\lim_{n\to\infty}\left[\frac{n!}{n^n}\right]^{1/n}$

Example 8.9 Evaluate $\int_{-1}^{0}\frac{dx}{x^2+2x+2}$.

Sol. $\int_{-1}^{0}\frac{dx}{x^2+2x+2}=\int_{-1}^{0}\frac{dx}{(x+1)^2+1}$

$$=|\tan^{-1}(x+1)|_{-1}^{0}$$

$$=\tan^{-1}1-\tan^{-1}0=\frac{\pi}{4}$$

Example 8.10 If $f(x)=\min\left(|x|,1-|x|,\frac{1}{4}\right)$, $\forall x\in R$, then find the value of $\int_{-1}^{1}f(x)\,dx$.

Sol. $f(x)=\min\left(|x|,1-|x|,\frac{1}{4}\right)$

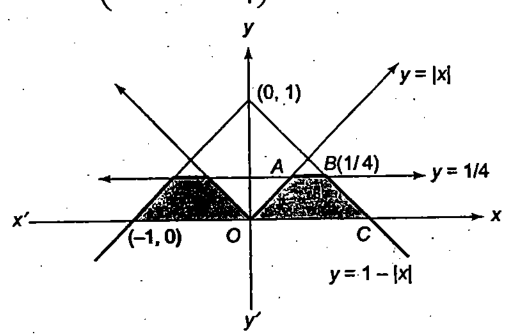

Fig. 8.4

Now, from Fig. 8.4,

$$\int_{-1}^{1} f(x)dx = 2(\text{Area of trapezium } OABC)$$

$$= 2\left(\frac{1}{2}\left(1+\frac{1}{2}\right)\frac{1}{4}\right) = \frac{3}{8}$$

Example 8.11 Find the mistake in the following evaluation of the integral $I = \int_0^{\pi} \frac{dx}{1+2\sin^2 x}$.

$$I = \int_0^{\pi} \frac{dx}{\cos^2 x + 3\sin^2 x}$$

$$= \int_0^{\pi} \frac{\sec^2 x dx}{1+3\tan^2 x} = \frac{1}{\sqrt{3}}[\tan^{-1}(\sqrt{3}\tan x)]_0^{\pi} = 0.$$

Sol. Here, the anti-derivative

$\frac{1}{\sqrt{3}}[\tan^{-1}(\sqrt{3}\tan x)] = F(x)$ is discontinuous at $x = \pi/2$ in the interval $[0, \pi]$.

$$\text{Since } F\left(\frac{\pi}{2}^{+}\right) = \lim_{h\to 0} F\left(\frac{\pi}{2} + h\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{\sqrt{3}}\right)\tan^{-1}\left\{\sqrt{3}\tan\left(\frac{1}{2}\pi + h\right)\right\}$$

$$= \lim_{h\to 0}(1/\sqrt{3})\tan^{-1}\{-\sqrt{3}\cot h\}$$

$$= \left(\frac{1}{\sqrt{3}}\right)\tan^{-1}(-\infty) = -\pi/(2\sqrt{3})$$

$$\text{Also, } F\left(\frac{1}{2}\pi - 0\right) = \pi/(2\sqrt{3}) \neq F\left(\frac{1}{2}\pi + 0\right).$$

Hence, the second fundamental theorem of integral calculus is not applicable.

Example 8.12 Find the value of $\int_{-1}^{1} \frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)dx$.

Sol. We have $\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) = \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

$$\therefore \int_{-1}^{1}\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)dx$$

$$= \int_{-1}^{1} -\frac{dx}{1+x^2}$$

$$= -2\int_0^1 \frac{dx}{1+x^2}\left(\because \text{for even function } \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx\right)$$

$$= -2\left[\tan^{-1}x\right]_0^1 = -2(\pi/4) = -\pi/2.$$

Note that $\int_{-1}^{1}\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)dx$

$$= \left[\tan^{-1}\frac{1}{x}\right]_{-1}^{1}$$

$$= \tan^{-1}1 - \tan^{-1}(-1) = \pi/2$$

is incorrect, because $\tan^{-1}\frac{1}{x}$ is not an anti-derivative (primitive) of $\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)$ on $[-1, 1]$, as $\tan^{-1}\frac{1}{x}$ does not exist for $x \neq 0$.

Example 8.13 Let $\frac{d}{dx}F(x) = \left(\frac{e^{\sin x}}{x}\right), x > 0$.

If $\int_1^4 \frac{3}{x} e^{\sin x^3} dx = F(k) - F(1)$, then find the possible value of k.

Sol. For $\int_1^4 \frac{3}{x} e^{\sin x^3} dx$, putting $x^3 = t$, so that $3x^2 dx = dt$

When $x = 1, t = 1$; when $x = 4, t = 64$

$$\therefore \int_1^4 \frac{3}{x} e^{\sin x^3} dx$$

$$= \int_1^{64} \frac{3}{x} e^{\sin t} \frac{dt}{3x^2}$$

$$= \int_1^{64} \frac{1}{t} e^{\sin t} dt$$

$$= \int_1^{64} \frac{d}{dt} F(t)\, dt = F(64) - F(1)$$

Hence, $k = 64$.

Example 8.14 If $\int_0^1 \frac{e^{-x}\, dx}{1+e^x} = \log_e(1+e) + K$, then find the value of K.

Sol. $I = \int_0^1 \frac{e^{-x}\, dx}{1+e^x} = \int_0^1 \frac{dx}{e^x(1+e^x)}$

Put $e^x = z \therefore e^x dx = dz \Rightarrow dx = \frac{dz}{e^x} = \frac{dz}{z}$

$$\Rightarrow I = \int_1^e \frac{dz}{z^2(1+z)}$$

$$= \int_1^e \left(\frac{1}{1+z} - \frac{z-1}{z^2}\right) dz$$

$$= \left|\log(1+z) - \log z - \frac{1}{z}\right|_1^e$$

$$= \left(\log(1+e) - \log e - \frac{1}{e}\right) - (\log 2 - \log 1 - 1)$$

$$= \log(1+e) - \frac{1}{e} - \log 2$$

$$\therefore K = -\left(\frac{1}{e} + \log 2\right)$$

Example 8.15 Find the value of $\int_0^1 \log x dx$.

Sol. $I = \int_0^1 \log x dx = x\log x|_0^1 - \int_0^1 1 dx$

$$= 1 \times \log 1 - \left(\lim_{x\to 0} x\log x\right) - 1$$

$$= 0 - \lim_{x\to 0} \frac{\log x}{\frac{1}{x}} - 1$$

$$= -\lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} - 1 \text{ (using L'Hopital's Rule)}$$

$$= \lim_{x\to 0} x - 1 = -1$$

Example 8.16 If $f(0)=1, f(2)=3, f'(2)=5$, then find the value of $\int_0^1 x f''(2x)dx$.

Sol. $I_1 = \int_0^1 x f''(2x)dx$, putting $t = 2x$, i.e.,

$dx = \frac{dt}{2}$, we get

$$I_1 = \frac{1}{4}\int_0^2 t f''(t)dt$$

$$= \frac{1}{4}\left[t\, f'(t)\Big|_0^2 - \int_0^2 f'(t)dt\right] \quad \text{(integrating by parts)}$$

$$= \frac{1}{4}\left[t\, f'(t)\Big|_0^2 - f(t)\Big|_0^2\right]$$

$$\Rightarrow I_1 = \frac{1}{4}(2f'(2) - f(2) + f(0)) = \frac{1}{4}(10-3+1) = 2$$

Concept Application Exercise 8.2

1. Consider the integral $I = \int_0^{2\pi} \frac{dx}{5 - 2\cos x}$.

   Making the substitution $\tan \frac{1}{2}x = t$,

   we have $I = \int_0^{2\pi} \frac{dx}{5-2\cos x}$

   $$= \int_0^0 \frac{2dt}{(1+t^2)[5-2(1-t^2)/(1+t^2)]} = 0$$

   The result is obviously wrong, since the integrand is positive and consequently the integral of this function cannot be equal to zero. Find the mistake.

2. $\int_0^{\pi} \frac{dx}{1+\sin x}$

3. $\int_1^{\infty} (e^{x+1} + e^{3-x})^{-1} dx$

4. $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)\sqrt{1-x^2}} dx$

5. $\int_0^1 \frac{2-x^2}{(1+x)\sqrt{1-x^2}} dx$

6. $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$

PROPERTIES OF DEFINITE INTEGRALS

Property I

Changing dummy variable: $\int_a^b f(x)dx = \int_a^b f(t)dt$,

i.e., the value of the definite integral does not change with the change of argument (variable of integration) provided the limits of integration remains the same.

Property II

Interchanging limits: $\int_a^b f(x)dx = -\int_b^a f(x)dx$,

i.e., the sign of the definite integral is changed when the order of the limits changed.

Property III

Splitting limits: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$,

where c may lie inside or outside the interval $[a, b]$.

This property is useful when the function is in the form of piecewise definition for $x \in (a, b)$ or when $f(x)$ is discontinuous or non-differentiable at $x = c$.

Proof: Analytical Method

Let $\int f(x)dx = F(x)$

$$\text{R.H.S.} = \int_a^c f(x)dx + \int_c^b f(x)dx$$
$$= F(x)\Big|_a^c + F(x)\Big|_c^b$$
$$= F(c) - F(a) + F(b) - F(c)$$
$$= F(b) - F(a) \quad (1)$$

$$\text{L.H.S.} = \int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) \quad (2)$$

From equations (1) and (2), we get

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Graphical Method

The proof of the property is more clear from the graph.

Case I: If $a < c < b$

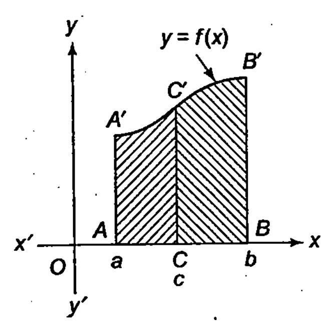

Fig. 8.5

It is clear from the figure,

Area of $ABB'A'A$ = Area of $(ACC'A'A)$ + Area of $(CBB'C'C)$

i.e., $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Case II: If $c < a < b$

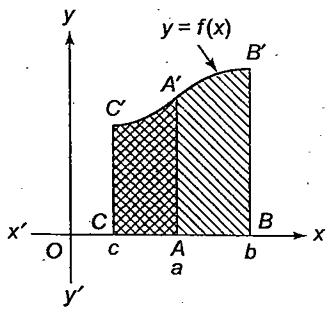

Fig. 8.6

It is clear from the figure,
Area of $(CBB'C'C)$ = Area of $(CAA'C')$ + Area of $(ABB'A')$

$$\text{i.e., } \int_c^b f(x)dx = \int_c^a f(x)dx + \int_a^b f(x)dx$$

$$\Rightarrow \int_a^b f(x)dx = -\int_c^a f(x)dx + \int_c^b f(x)dx$$

$$= \int_a^c f(x)dx + \int_c^b f(x)dx \quad \text{(By Property II)}$$

Case III : If $a < b < c$

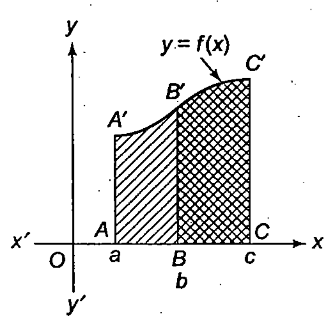

Fig. 8.7

It is clear from the figure,
Area of $(ACC'A'A)$ = Area of $(ABB'A'A)$ + Area of $(BCC'B'B)$

$$\text{i.e., } \int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

$$\Rightarrow \int_a^b f(x)dx = \int_a^c f(x)dx - \int_b^c f(x)dx$$

$$= \int_a^c f(x)dx + \int_c^b f(x)dx \quad \text{(By property II)}$$

Generalization: The above property can be generalized in the following form:

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx \cdots + \int_{c_n}^b f(x)dx$$

Example 8.17 Evaluate $\int_{-1}^{1} |x|\, dx$.

Sol. We know that in the interval $[-1, 1]$,

$$|x| = \begin{cases} -x, & -1 \le x \le 0 \\ x, & 0 \le x \le 1 \end{cases}$$

$$\therefore \int_{-1}^{1} |x|\, dx = \int_{-1}^{0} (-x)dx + \int_0^1 x\, dx$$

$$= \left[-\frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{1}{2}x^2\right]_0^1 = 1$$

Example 8.18 Evaluate $\int_{-\pi/2}^{2\pi} \sin^{-1}(\sin x)\, dx$.

Sol. The graph of $f(x) = \sin^{-1}(\sin x)$ is as shown in Fig. 8.8

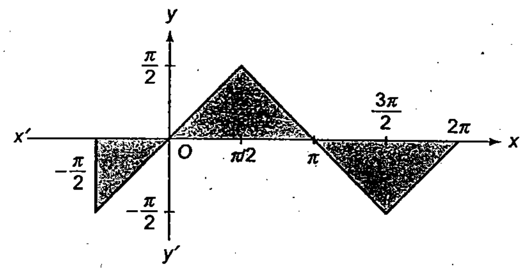

Fig. 8.8

$$\Rightarrow \int_{-\pi/2}^{2\pi} \sin^{-1}(\sin x)\, dx = \int_{-\pi/2}^{0} \sin^{-1}(\sin x)\, dx + \int_0^{\pi} \sin^{-1}(\sin x)\, dx + \int_{\pi}^{2\pi} \sin^{-1}(\sin x)\, dx$$

= Area of shaded region

$$= -\left(\frac{1}{2} \times \frac{\pi}{2} \times \frac{\pi}{2}\right) + \left(\frac{1}{2} \times \pi \times \frac{\pi}{2}\right) - \left(\frac{1}{2} \times \pi \times \frac{\pi}{2}\right)$$

$$= -\frac{\pi^2}{8}$$

Example 8.19 Evaluate $\int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x}\, dx$.

Sol. Given integral

$$I = \int_{-\pi/2}^{\pi/2} \sqrt{\left[\cos x\left(1 - \cos^2 x\right)\right]}\, dx$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{\left(\cos x \sin^2 x\right)}\, dx$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{(\cos x)}\, |\sin x|\, dx \quad (1)$$

$$\text{Now, } |\sin x| = \begin{cases} -\sin x, \text{ if } -\pi/2 \le x < 0 \\ \sin x, \text{ if } 0 < x \le \pi/2 \end{cases}$$

$\therefore$ from equation (1), we have

$$I = \int_{-\pi/2}^{0} \sqrt{(\cos x)}\, (-\sin x)dx + \int_0^{\pi/2} \sqrt{(\cos x)}\, \sin x dx$$

Putting $\cos x = t$, $-\sin x\, dx = dt$

$$\Rightarrow I = \int_0^1 t^{1/2}\, dt - \int_1^0 t^{1/2}\, dt = 2\int_0^1 t^{1/2}\, dt$$

$$= 2\left(\frac{2}{3}\right)\left[t^{3/2}\right]_0^1 = \frac{4}{3}$$

Example 8.20 If $[x]$ denotes the greatest integer less than or equal to x, then find the value of the integral $\int_0^2 x^2\, [x]\, dx$.

Sol. $\int_0^2 x^2[x]\,dx = \int_0^1 x^2[x]\,dx + \int_1^2 x^2[x]dx$

$= \int_0^1 x^2\,(0)dx + \int_1^2 x^2\,(1)dx$

$= 0 + \int_1^2 x^2\,dx = \left|\frac{x^3}{3}\right|_1^2 = \frac{8-1}{3} = \frac{7}{3}$

Example 8.21 Show that $\int_a^b \frac{|x|}{x}dx = |b|-|a|$.

Sol. Case I: If $0 \le a < b$, then $|x|/x = 1$

$\therefore I = \int_a^b 1\,dx = b-a = |b|-|a|$

Case II: If $a < b \le 0$, then $|x| = -x$

$\therefore \text{I} = \int_a^b \frac{-x}{x}dx = \int_a^b (-1)\,dx$

$= [-x]_a^b = -b-(-a) = |b|-|a|$

Case III: If $a < 0 < b$

then $|x| = -x$ when $a < x < 0$

and $|x| = x$ when $0 < x < b$

$I = \int_a^b \frac{|x|}{x}\,dx = \int_a^0 \frac{|x|}{x}dx + \int_0^b \frac{|x|}{x}dx$

$= \int_a^0 \frac{-x}{x}\,dx + \int_0^b \frac{x}{x}dx$

$= \int_a^0 (-1)dx + \int_0^b 1dx$

$= [-x]_a^0 + [x]_0^b = a + b = b-(-a) = |b|-|a|$

Hence, in all the cases, $I = \int_a^b \frac{|x|}{x}\,dx = |b|-|a|$.

Property IV

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Proof:

In R.H.S., put $a+b-x=t$ $\quad \therefore dx = -dt$

When $x = a \Rightarrow t = b$ and $x = b \Rightarrow t = a$

then R.H.S. $= \int_b^a f(t)(-dt) = -\int_b^a f(t)dt = \int_a^b f(t)dt$

$= \int_a^b f(x)dx =$ L.H.S.

Example 8.22 Evaluate $\int_{\pi/6}^{\pi/3} \frac{\sqrt{(\sin x)}\,dx}{\sqrt{(\sin x)}+\sqrt{(\cos x)}}$.

Sol. Given integral $I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{(\sin x)}\,dx}{\sqrt{(\sin x)}+\sqrt{(\cos x)}}$ (1)

$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{(\cos x)}\,dx}{\sqrt{(\cos x)}+\sqrt{(\sin x)}}$ (Replacing x by $\frac{\pi}{2}-x$) (2)

Adding equations (1) and (2), we get

$\Rightarrow 2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{(\sin x)}+\sqrt{(\cos x)}}{\sqrt{(\cos x)}+\sqrt{(\sin x)}}dx$

$= \int_{\pi/6}^{\pi/3} dx = [x]_{\pi/6}^{\pi/3} = \pi/3 - \pi/6 = \pi/6$

Hence, $I = \pi/12$

Example 8.23 Evaluate $\int_{-\pi}^{3\pi} \log(\sec\theta - \tan\theta)d\theta$.

Sol. Let $I = \int_{-\pi}^{3\pi} \log(\sec\theta - \tan\theta)\,d\theta$ (1)

Using the property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, we get

$I = \int_{-\pi}^{3\pi} \log[\sec(2\pi-\theta) - \tan(2\pi-\theta)]\,d\theta$

$= \int_{-\pi}^{3\pi} \log[\sec\theta + \tan\theta]\,d\theta$ (2)

Adding equations (1) and (2), we get

$2I = \int_{-\pi}^{3\pi} \log[(\sec\theta - \tan\theta)(\sec\theta+\tan\theta)]\,d\theta$

$= \int_{-\pi}^{3\pi} \log(1)\,d\theta = \int_{-\pi}^{3\pi} 0.\,d\theta = 0 \Rightarrow I = 0$

Example 8.24 Evaluate $\int_{-\pi}^{\pi} \frac{x\sin x dx}{e^x+1}$.

Sol. Let $I = \int_{-\pi}^{\pi} \frac{x\sin x dx}{e^x+1}$ (1)

Using property IV, we replace x by $0-x$ or $-x$

$\Rightarrow I = \int_{-\pi}^{\pi} \frac{(-x)\sin(-x)dx}{e^{-x}+1} = \int_{-\pi}^{\pi} \frac{e^x\; x\sin x dx}{e^x+1}$ (2)

Adding equations (1) and (2), we get $2I = \int_{-\pi}^{\pi} x\sin x dx$

or $I = \int_0^{\pi} x\sin x dx$

$\Rightarrow I = \int_0^{\pi} (\pi-x)\sin(\pi-x)dx = \int_0^{\pi} \pi\sin x dx - I \Rightarrow I = \pi$

Example 8.25 Evaluate $\int_0^a \frac{dx}{x+\sqrt{(a^2-x^2)}}$ or $\int_0^{\pi/2} \frac{d\theta}{1+\tan\theta}$.

Sol. Putting $x = a\sin\theta$, we get $dx = a\cos\theta\,d\theta$,

when $x = 0 = a\sin\theta$, $\theta = 0$

when $x = a = a\sin\theta$, $\sin\theta = 1$, Therefore, $\theta = \pi/2$.

The given integral

$I = \int_0^{\pi/2} \frac{\cos\theta\;d\theta}{\sin\theta+\cos\theta}$ (1)

Now using Property IV, we get

$I = \int \frac{\cos\left(\frac{1}{2}\pi-\theta\right)d\theta}{\sin\left(\frac{1}{2}\pi-\theta\right)+\cos\left(\frac{1}{2}\pi-\theta\right)}$, or

$I = \int_0^{\pi/2} \frac{\sin\theta\;d\theta}{\cos\theta+\sin\theta}$ (2)

Adding equations (1) and (2), we get

$$2I=\int_0^{\pi/2}\frac{\sin\theta+\cos\theta}{\sin\theta+\cos\theta}\,d\theta=\int_0^{\pi/2}d\theta$$

or $2I=[\theta]_0^{\pi/2}=\pi/2$

$\therefore I=\pi/4$

Example 8.26 Evaluate $\int_0^{\pi/2}\frac{\sin^2 x\,dx}{\sin x+\cos x}$.

Sol. Let $I=\int\frac{\sin^2 x\,dx}{\sin x+\cos x}$ (1)

Using property IV, we have

$$I=\int_0^{\pi/2}\frac{\sin^2\left(\frac{1}{2}\pi-x\right)dx}{\sin\left(\frac{1}{2}\pi-x\right)+\cos\left(\frac{1}{2}\pi-x\right)};\text{ or}$$

$$I=\int_0^{\pi/2}\frac{\cos^2 x\,dx}{\cos x+\sin x} \quad (2)$$

Now adding equations (1) and (2), we get

$$2I=\int_0^{\pi/2}\frac{(\sin^2 x+\cos^2 x)dx}{\sin x+\cos x}$$

or $I=\frac{1}{2}\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}$

$$=\frac{1}{2\sqrt{2}}\int_0^{\pi/2}\frac{dx}{\sin\left(x+\frac{1}{4}\pi\right)}$$

$$=\left(\frac{1}{2\sqrt{2}}\right)\int_0^{\pi/2}\text{cosec}\left(x+\frac{1}{4}\pi\right)dx$$

$$=\left(\frac{1}{2\sqrt{2}}\right)\left[\log\left\{\text{cosec}\left(x+\frac{1}{4}\pi\right)-\cot\left(x+\frac{1}{4}\pi\right)\right\}\right]_0^{\pi/2}$$

$$=(1/2\sqrt{2})\left[\log\left\{\text{cosec}\left(\frac{1}{2}\pi+\frac{1}{4}\pi\right)-\cot\left(\frac{1}{2}\pi+\frac{1}{4}\pi\right)\right\}-\log\left\{\text{cosec}\left(\frac{1}{4}\pi\right)-\cot\left(\frac{1}{4}\pi\right)\right\}\right]$$

$$=(1/2\sqrt{2})\left[\log\left\{\sec\left(\frac{1}{4}\pi\right)+\tan\left(\frac{1}{4}\pi\right)\right\}-\log(\sqrt{2}-1)\right]$$

$$=(1/2\sqrt{2})\left[\log(\sqrt{2}+1)-\log(\sqrt{2}-1)\right]$$

$$=\left(\frac{1}{2\sqrt{2}}\right)\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

Example 8.27 Show that $\int_0^{\pi/2}\sqrt{(\sin 2\theta)}\sin\theta\,d\theta=\pi/4$.

Sol. Let $I=\int_0^{\pi/2}\sqrt{(\sin 2\theta)}\sin\theta\,d\theta$ (1)

Using property IV, we get

$$I=\int_0^{\pi/2}\sqrt{\sin(2(\pi/2)-\theta)}\sin(\pi/2-\theta)d\theta$$

$$=\int_0^{\pi/2}\sqrt{(\sin 2\theta)}\cos\theta\,d\theta \quad (2)$$

Adding equations (1) and (2), we get

$$2I=\int_0^{\pi/2}\sqrt{(\sin 2\theta)}(\sin\theta+\cos\theta)\,d\theta$$

or $I=\frac{1}{2}\int_0^{\pi/2}\sqrt{1-(\sin\theta-\cos\theta)^2}(\sin\theta+\cos\theta)\,d\theta$

$$=\frac{1}{2}\int_{-1}^{1}\sqrt{1-t^2}\,dt \quad [\text{Let }\sin\theta-\cos\theta=t]$$

$$=\frac{1}{2}\left[\frac{1}{2}t\sqrt{(1-t^2)}+\frac{1}{2}\sin^{-1}t\right]_{-1}^{1}=\frac{\pi}{4}$$

Example 8.28 Evaluate $\int_0^1\frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}$.

Sol. Let $I=\int_0^1\frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}$ (1)

Also, $I=\int_0^1\frac{dx}{[5+2(1-x)-2(1-x)^2][1+e^{2-4(1-x)}]}$

$$=\int_0^1\frac{dx}{(5+2x-2x^2)(1+e^{-2+4x})}$$

$$=\int_0^1\frac{e^{2-4x}dx}{(5+2x-2x^2)(e^{2-4x}+1)} \quad (2)$$

Adding equations (1) and (2), we get

$$2I=\int_0^1\frac{(1+e^{2-4x})dx}{(5+2x-2x^2)(1+e^{2-4x})}$$

$$=\int_0^1\frac{dx}{5-2(x^2-x)}=\int_0^1\frac{dx}{\frac{1}{2}+5-2\left(x-\frac{1}{2}\right)^2}$$

$$=\frac{1}{2}\int_0^1\frac{dx}{\frac{11}{4}-\left(x-\frac{1}{2}\right)^2}$$

$$=\frac{1}{4\sqrt{11}/2}\left|\log\frac{\sqrt{11}/2+x-\frac{1}{2}}{\sqrt{11}/2-\left(x-\frac{1}{2}\right)}\right|_0^1$$

$$=\frac{1}{2\sqrt{11}}\left[\log\frac{\frac{\sqrt{11}}{2}+\frac{1}{2}}{\frac{\sqrt{11}}{2}-\frac{1}{2}}-\log\frac{\frac{\sqrt{11}}{2}-\frac{1}{2}}{\frac{\sqrt{11}}{2}+\frac{1}{2}}\right]$$

$$=\frac{1}{2\sqrt{11}}\left[2\log\left(\frac{\sqrt{11}+1}{\sqrt{11}-1}\right)\right]$$

$$=\frac{1}{\sqrt{11}}\log\left(\frac{\sqrt{11}+1}{\sqrt{11}-1}\right)$$

$$= \frac{1}{\sqrt{11}} \log \frac{\sqrt{11}+1}{\sqrt{11}-1} \frac{\sqrt{11}+1}{\sqrt{11}+1}$$

$$= \frac{1}{\sqrt{11}} \log \frac{(\sqrt{11}+1)^2}{10}$$

$$\therefore I = \frac{1}{2\sqrt{11}} \log \frac{(\sqrt{11}+1)^2}{10}$$

Property V

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx \text{ and}$$

$$\int_0^{2a} f(x)\,dx = \begin{cases} 0 \text{ if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx \text{ if } f(2a-x) = f(x) \end{cases}$$

Proof:

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$$

Put $x = 2a - t$ in second integral on R.H.S.

Therefore, $dx = -dt$

When $x = a \Rightarrow t = a$

$x = 2a \Rightarrow t = 0$, then

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^0 f(2a-t)(-dt)$$

$$= \int_0^a f(x)dx + \int_0^a f(2a-t)dt$$

$$= \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

$$= \begin{cases} \int_0^a f(x)dx - \int_0^a f(x)dx = 0, & \text{if } f(2a-x) = -f(x) \\ \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \end{cases}$$

Example 8.29 Evaluate $\int_0^{2\pi} \sin^{100} x \cos^{99} x dx$.

Sol. $I = \int_0^{2\pi} \sin^{100} x \cos^{99} x dx$

Here, $f(x) = \sin^{100} x \cos^{99} x$ for which $f(2\pi - x) = f(x)$

$$\Rightarrow I = 2\int_0^{\pi} \sin^{100} x \cos^{99} x dx$$

$$= 2\int_0^{\pi} \sin^{100}(\pi - x)\cos^{99}(\pi - x)dx \quad \text{(by property IV)}$$

$$= -2\int_0^{\pi} \sin^{100} x \cos^{99} x dx$$

$$= -I \Rightarrow 2I = 0 \Rightarrow I = 0$$

Example 8.30 Evaluate $\int_0^{4\pi} \frac{dx}{\cos^2 x(2+\tan^2 x)}$.

Sol. $\int_0^{4\pi} \frac{\sec^2 x}{(2+\tan^2 x)} dx = 2\int_0^{2\pi} \frac{\sec^2 x\, dx}{2+\tan^2 x}$

$$= 4\int_0^{\pi} \frac{\sec^2 x}{2+\tan^2 x} dx$$

$$= 8\int_0^{\frac{\pi}{2}} \frac{\sec^2 x dx}{2+\tan^2 x}$$

$$= 8\int_0^{\pi/2} \frac{d(\tan x)}{2+\tan^2 x}$$

$$= \frac{8}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)_0^{\pi/2}$$

$$= \frac{8}{\sqrt{2}}\left(\frac{\pi}{2} - 0\right) = 2\sqrt{2}\pi$$

Important Result

$$\int_0^{\pi/2} \log \sin x dx = \int_0^{\pi/2} \log \cos x dx = \frac{1}{2}\pi \log\left(\frac{1}{2}\right)$$

Proof:

Let $I = \int_0^{\pi/2} \log \sin x dx$ (1)

By using property IV, we have

$I = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$ or $I = \int_0^{\pi/2} \log \cos x\, dx$ (2)

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x) dx$$

$$= \int_0^{\pi/2} \log (\sin x \cos x)\, dx$$

$$= \int_0^{\pi/2} \log\{(\sin 2x)/2\}\, dx$$

$$= \int_0^{\pi/2} \log \sin 2x\, dx - \int_0^{\pi/2} (\log 2)\, dx$$

$$= \frac{1}{2}\int_0^{\pi} \log \sin t\, dt - (\log 2)[x]_0^{\pi/2}$$

[Putting $2x = t$, $dx = \frac{1}{2}dt$]

$$= \frac{1}{2} 2\int_0^{\pi/2} \log \sin t\, dt - (\pi/2)\log 2 \quad \text{(by property V)}$$

$$= \int_0^{\pi/2} \log \sin x\, dx - (\pi/2)\log 2 \quad \text{(by property I)}$$

$= I - (\pi/2)\log 2$

or $2I - I = -(\pi/2)\log 2$

Hence, $I = \int_0^{\pi/2} \log \sin x\, dx = -(\pi/2)\log 2$

$$= \frac{1}{2}\pi \log\left(\frac{1}{2}\right)$$

Example 8.31 Evaluate $\int_0^{\pi} x \log \sin x\, dx$.

Sol. Let $I = \int_0^{\pi} x \log \sin x\, dx$ (1)

Now using property IV, we have

$$I = \int_0^{\pi} (\pi - x) \log \sin(\pi - x)\, dx$$

$$\Rightarrow I = \int_0^{\pi} (\pi - x) \log \sin x\, dx \quad (2)$$

Adding equations (1) and (2), we get $2I = \pi \int_0^{\pi} \log \sin x\, dx$

$$\Rightarrow 2I = 2\pi \int_0^{\pi/2} \log \sin x\, dx \quad \text{(by Property V)}$$

$$\Rightarrow I = \pi \int_0^{\pi/2} \log \sin x$$

$$= \pi \left\{\frac{1}{2}\pi \log(1/2)\right\} = \frac{1}{2}\pi^2 \log(1/2)$$

Example 8.32 Evaluate $\int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x) dx$.

Sol. Let $I = \int_{-\pi/4}^{\pi/4} \log\left\{\sqrt{2} \sin\left(x + \frac{\pi}{4}\right)\right\} dx$

Putting $x + \frac{\pi}{4} = \theta$, $dx = d\theta$

$$= \int_0^{\pi/2} \log\left(\sqrt{2} \sin\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \log 2\, d\theta + \int_0^{\pi/2} \log \sin\theta\, d\theta$$

$$= \left(\frac{1}{4}\pi \log 2\right) - \frac{1}{2}\pi \log 2$$

$$= -\frac{1}{4}\pi \log 2$$

Example 8.33 Evaluate $\int_0^{\pi/2} x \cot x dx$.

Sol. Integrating by parts, taking cot x as second function, given integral becomes

$$I = [x \log \sin x]_0^{\pi/2} - \int_0^{\pi/2} \log \sin x\, dx$$

$$= 0 - \lim_{x \to 0} (x \log \sin x) - \int_0^{\pi/2} \log \sin x\, dx = \frac{1}{2}\pi \log 2$$

as $\lim_{x \to 0} x \log \sin x = \lim_{x \to 0} \left(\frac{\log \sin x}{1/x}\right)$

$$= \lim_{x \to 0} \left(\frac{\cot x}{-1/x^2}\right)$$

$$= \lim_{x \to 0} \left(\frac{-x^2}{\tan x}\right)$$

$$= \lim_{x \to 0} \left(-x \times \frac{x}{\tan x}\right) = 0 \times 1 = 0$$

Example 8.34 Evaluate $\int_0^{\infty} \log(x + 1/x) \frac{dx}{1 + x^2}$.

Sol. Putting $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, given integral

$$I = \int_0^{\pi/2} \frac{\log(\tan\theta + \cot\theta)}{1 + \tan^2\theta} \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \log(\sin\theta/\cos\theta + \cos\theta/\sin\theta) d\theta$$

$$= \int_0^{\pi/2} \log\{1/(\sin\theta \cos\theta)\} d\theta$$

$$= -\int_0^{\pi/2} \log \sin\theta d\theta - \int_0^{\pi/2} \log \cos\theta d\theta$$

$$= -2\left(-\frac{1}{2}\pi \log 2\right) = \pi \log 2$$

Property VI

$$\int_0^{2a} f(x) dx = \int_0^{a} \{f(a - x) + f(a + x)\} dx$$

Proof: R.H.S. $= \int_0^a \{f(a - x) + f(a + x)\} dx$

$$= \int_0^a f(a - x) dx + \int_0^a f(a + x) dx$$

$$= \int_0^a f(a - (a - x)) dx + \int_{0+a}^{a+a} f(x) dx$$ [in second integral replace $x + a$ by x]

$$= \int_0^a f(x) dx + \int_a^{2a} f(x) dx$$

$$= \int_0^{2a} f(x) dx = \text{L.H.S.}$$

Property VII

$$\int_a^b f(x) dx = (b - a)\int_0^1 f((b - a)x + a) dx$$

Proof:

R.H.S. $= (b - a) \int_0^1 f((b - a)x + a) dx$

Let $(b - a)x + a = t \Rightarrow dx = \frac{dt}{(b - a)}$. Also when $x = 0$, then $t = a$ and $x = 1$ then $t = b$

Therefore, R.H.S. $= (b - a) \int_a^b f(t) \frac{dt}{(b - a)} = \int_a^b f(t) dt$

$$= \int_a^b f(x) dx = \text{L.H.S.}$$

Concept Application Exercise 8.3

1. If $f(a + b - x) = f(x)$, then prove that
$$\int_a^b x f(x)\, dx = \frac{a + b}{2}\int_a^b f(x)\, dx.$$
2. Find the value of the integral $\int_3^6 \frac{\sqrt{x}}{\sqrt{9 - x} + \sqrt{x}} dx$.
3. Find the value of $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$.

4. Find the value of $\int_0^1 \sqrt[3]{2x^3 - 3x^2 - x + 1}\, dx$.

5. Find the value of $\int_0^1 x(1-x)^n\, dx$.

6. If a continuous function f on $[0, a]$ satisfies $f(x)f(a-x) = 1$, $a > 0$, then find the value of $\int_0^a \frac{dx}{1+f(x)}$.

7. Find the value of $\int_0^{\pi/2} \sin 2x \log \tan x\, dx$.

8. Find the value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\, dx, a > 0$.

9. Find the value of $\int_0^{\pi} \frac{x \sin x dx}{1+\cos^2 x}$.

10. If $I_1 = \int_0^{\pi} x f(\sin^3 x + \cos^2 x) dx$ and $I_2 = \int_0^{\pi/2} f(\sin^3 x + \cos^2 x) dx$, then relate I_1 and I_2.

11. Find the value of the integral $\int_0^{\pi} \log(1+\cos x) dx$.

12. Find the value of: $\int_0^1 \{(\sin^{-1} x)/x\} dx$.

DEFINITE INTEGRATION OF ODD AND EVEN FUNCTIONS

Property I

$$\int_{-a}^{a} f(x)dx = \begin{cases} 0, & \text{if } f(x) \text{ is odd, i.e., } f(-x) = -f(x) \\ 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is even, i.e., } f(-x) = f(x) \end{cases}$$

Proof: Analytical Method

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$$

Put $x = -t$ in 1st term on R.H.S. Therefore, $dx = -dt$

When $x = -a \Rightarrow t = a, x = 0 \Rightarrow t = 0$

$$\Rightarrow \int_{-a}^{a} f(x)dx = \int_a^0 f(-t)(-dt) + \int_0^a f(x)dx$$

$$= \int_0^a f(-t)dt + \int_0^a f(x)dx$$

$$= \int_0^a f(-x)dx + \int_0^a f(x)dx$$

$$= \begin{cases} -\int_0^a f(x)dx + \int_0^a f(x)dx, & \text{if } f(x) \text{ is odd} \\ \int_0^a f(x)dx + \int_0^a f(x)dx, & \text{if } f(x) \text{ is even} \end{cases}$$

$$= \begin{cases} 0, & \text{if } f(x) \text{ is odd} \\ 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is even} \end{cases}$$

Graphical Method

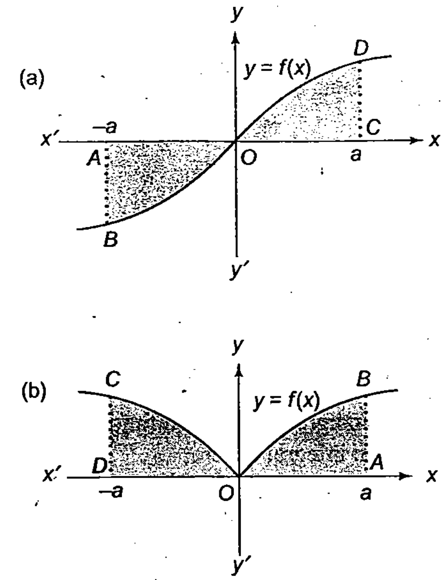

Fig. 8.9

Since, the graph of odd function is symmetrical about origin. It is clear from Fig. 8.9(a).

Area of $OCDO$ = Area of $OABO$

i.e., $\int_0^a f(x)dx = -\int_{-a}^0 f(x)dx$ (1)

($\because$ Left portion below x-axis, $\therefore$ taking -ve sign)

$\therefore \int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$

$= -\int_0^a f(x)dx + \int_0^a f(x)dx = 0$ [from equation (1)]

Also, the graph of even function is symmetrical about y-axis. It is clear from Fig 8.9(b).

Area of $OCDO$ = Area of $OABO$

i.e., $\int_0^a f(x)dx = \int_{-a}^0 f(x)dx$ (2)

$\therefore \int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$

$= \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx$ [from equation (2)]

Property II

If $f(t)$ is an odd function, then $\phi(x) = \int_a^x f(t)\, dt$ is an even function.

Proof:

Since $\phi(x) = \int_a^x f(t)\, dt \Rightarrow \phi(-x) = \int_a^{-x} f(t)\, dt$

Let $t = -y$

Then $\phi(-x) = \int_{-a}^{x} f(-y)(-dy)$

$= \int_{-a}^{x} f(y)dy$ [as given f is an odd function]

$= \int_{-a}^{a} f(y)dy + \int_a^x f(y)dy$

$$= 0 + \int_a^x f(y)dy = \phi(x)$$

Hence, $\phi(x)$ is an even function.

Example 8.35 Evaluate $\int\limits_{-\pi/2}^{\pi/2} \log\left(\frac{a - \sin\theta}{a + \sin\theta}\right) d\theta, a > 0.$

Sol. $f(\theta) = \log\left(\frac{a - \sin\theta}{a + \sin\theta}\right)$

$$\Rightarrow f(-\theta) = \log\left(\frac{a + \sin\theta}{a - \sin\theta}\right)$$

$$= -\log\left(\frac{a - \sin\theta}{a + \sin\theta}\right)$$

$$= -f(\theta)$$

Hence, the integrand is an odd function.

So, the given integral is zero.

Example 8.36 Evaluate

$$\int_{-\pi/2}^{\pi/2} \log\left\{\frac{ax^2 + bx + c}{ax^2 - bx + c}(a + b)\,|\sin x|\right\} dx.$$

Sol. $I = \int_{-\pi/2}^{\pi/2} \log\left\{\frac{ax^2 + bx + c}{ax^2 - bx + c}(a + b)\,|\sin x|\right\} dx$

$$= \int_{-\pi/2}^{\pi/2} \log\left(\frac{ax^2 + bx + c}{ax^2 - bx + c}\right) dx + \int_{-\pi/2}^{\pi/2} \log(a + b)\, dx$$

$$+ \int_{-\pi/2}^{\pi/2} \log|\sin x|\, dx$$

$$= I_1 + I_2 + I_3 \qquad (1)$$

Now let, $f(x) = \log\left(\frac{ax^2 + bx + c}{ax^2 - bx + c}\right)$

$$\Rightarrow f(-x) = \log\left(\frac{ax^2 - bx + c}{ax^2 + bx + c}\right) = -f(x)$$

$$\therefore I_1 = \int_{-\pi/2}^{\pi/2} f(x)\, dx = 0$$

$$I_2 = \log(a + b)\,[x]_{-\pi/2}^{\pi/2}$$

$$= \pi \log(a + b)$$

$$I_3 = \int_{-\pi/2}^{\pi/2} \log|\sin x| dx$$

$$= 2\int_0^{\pi/2} \log|\sin x|\, dx$$

$$= 2\int_0^{\pi/2} \log \sin x\, dx$$

$$= 2\left(-\frac{1}{2}\pi \log 2\right)$$

Hence, from equation (1), we have

$$I = 0 + \pi \log(a + b) - \pi \log 2$$

$$= \pi \log\{(a + b)/2\}$$

Example 8.37 Evaluate $\int\limits_{-\pi/4}^{\pi/4} \frac{x^9 - 3x^5 + 7x^3 - x + 1}{\cos^2 x} dx.$

Sol. $f(x) = \frac{x^9 - 3x^5 + 7x^3 - x}{\cos^2 x} + \sec^2 x$

$$= \sec^2 x\,(x^9 - 3x^5 + 7x^3 - x) + \sec^2 x$$

$$\Rightarrow \int_{-\pi/4}^{\pi/4} f(x)dx = \int_{-\pi/4}^{\pi/4} \sec^2 x dx$$

[$\because \sec^2 x(x^9 - 3x^5 + 7x^3 - x)$ is an odd function]

$$= 2\int_0^{\pi/4} \sec^2 x dx$$

$$= 2 \tan x\Big|_0^{\pi/4} = 2$$

Example 8.38 If f is an odd function, then evaluate

$$I = \int_{-a}^{a} \frac{f(\sin x)}{f(\cos x) + f(\sin^2 x)} dx.$$

Sol. Let $\phi(x) = \frac{f(\sin x)}{f(\cos x) + f(\sin^2 x)}$

$$\Rightarrow \phi(-x) = \frac{f(\sin(-x))}{f(\cos(-x)) + f(\sin^2(-x))}$$

$$= \frac{f(-\sin x)}{f(\cos x) + f(\sin^2 x)} = \frac{-f(\sin x)}{f(\cos x) + f(\sin^2 x)} - \phi(x)$$

$$\Rightarrow I = \int_{-a}^{a} \frac{f(\sin x)}{f(\cos x) + f(\sin^2 x)} dx = 0$$

Example 8.39 Evaluate $\int_{-1/2}^{1/2}\left[\left(\frac{x+1}{x-1}\right)^2 + \left(\frac{x-1}{x+1}\right)^2 - 2\right]^{1/2} dx.$

Sol. Given integral

$$= \int_{-1/2}^{1/2}\left[\left\{\frac{x+1}{x-1} - \frac{x-1}{x+1}\right\}^2\right]^{1/2} dx$$

$$= \int_{-1/2}^{1/2}\left|\frac{x+1}{x-1} - \frac{x-1}{x+1}\right| dx$$

$$= \int_{-1/2}^{1/2}\left|\frac{4x}{x^2 - 1}\right| dx = 2\int_0^{1/2}\left|\frac{4x}{x^2 - 1}\right| dx$$

$$= 2\int_0^{1/2} \frac{4x}{1 - x^2} dx \qquad \because \left|\frac{4x}{x^2 - 1}\right| = -\frac{4x}{x^2 - 1}$$

when $0 \le x \le \frac{1}{2}$

$$= -4\left[\log\left(1 - x^2\right)\right]_0^{1/2}$$

$$= -4 \log(3/4) = 4 \log(4/3)$$

Concept Application Exercise 8.4

Evaluate the following:

1. $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^2 x(\sin x+\cos x)dx$

2. $\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}\, dx$

3. $\int_{-\pi}^{\pi}\left(1-x^2\right) \sin x \cos^2 x\, dx$

4. $\int_{-1}^{1} \frac{\sin x - x^2}{3-|x|}\, dx$

5. $\int_{-\pi/2}^{\pi/2} \sqrt{\cos^{2n-1} x-\cos^{2n+1} x}\, dx$, where $n \in N$.

6. $\int_{-1/2}^{1/2} \cos x \log \frac{1-x}{1+x}\, dx$

7. $\int_{-3\pi/2}^{-\pi/2} [(x+\pi)^3 + \cos^2(x+3\pi)]\, dx$

DEFINITE INTEGRATION OF PERIODIC FUNCTIONS

Property I

$\int_0^{nT} f(x)\, dx = n\int_0^T f(x)\, dx$, where T is the period of the funciton and $n \in I$, (i.e., $f(x+T)=f(x)$).

Proof: Analytical Method

$$\int_0^{nT} f(x)\, dx = \int_0^T f(x)\, dx + \int_T^{2T} f(x)\, dx + \int_{2T}^{3T} f(x)\, dx + \cdots + \int_{(n-1)T}^{nT} f(x)dx$$

$$= \int_0^T f(x)\, dx + \int_0^T f(x+T)\, dx + \int_0^T f(x+2T)\, dx + \cdots + \int_0^T f(x+(n-1)T)\, dx$$

$$= \underbrace{\int_0^T f(x)\, dx + \int_0^T f(x)\, dx + \cdots + \int_0^T f(x)\, dx}_{n \text{ times}}$$

$\{\because f(x)=f(x+T)=f(x+2T)=\cdots=f(x+(n-1)T)\}$

$= n\int_0^T f(x)\, dx$

Hence, $\int_0^{nT} f(x)\, dx = n\int_0^T f(x)\, dx$

Graphical Method

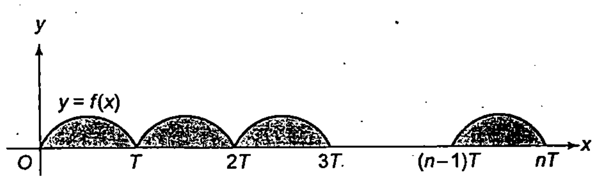

Fig. 8.10

$\therefore$ Figure of $f(x)$ is same from $0 \to T$, $T \to 2T$, $2T \to 3T$, $\cdots$, $(n-1)T \to nT$, then it is clear from the figure that

$$\int_0^{nT} f(x)\, dx = n\int_0^T f(x)\, dx.$$

Property II

$$\int_a^{a+nT} f(x)\, dx = n\int_0^T f(x)\, dx$$

Proof:

$$I = \int_a^{a+nT} f(x)dx$$

$$= \int_a^0 f(x)dx + \int_0^{nT} f(x)dx + \int_{nT}^{a+nT} f(x)dx$$

In the last integral, put $x = y + nT$

$$\Rightarrow \int_{nT}^{a+nT} f(x)dx = \int_o^a f(y+nT)dy = \int_o^a f(y)dy$$

$$\Rightarrow I = \int_a^0 f(x)dx + \int_0^{nT} f(x)dx + \int_0^a f(y)dy$$

$$= n\int_0^T f(x)dx$$

Property III

$\int_{mT}^{nT} f(x)\, dx = (n-m)\int_0^T f(x)\, dx$, where T is the period of the function and $m, n \in I$.

Proof:

$$\text{L.H.S.} = \int_{mT}^{nT} f(x)\, dx$$

$$= \int_0^{(n-m)T} f(x+mT)\, dx$$

$$= \int_0^{(n-m)T} f(x)\, dx \quad [\because f(x) \text{ is periodic}]$$

$$= (n-m)\int_0^T f(x)\, dx$$

Property IV

$\int_{a+nT}^{b+nT} f(x)\, dx = \int_a^b f(x)dx$, where T is the period of the function and $n \in I$.

Proof:

$$\text{L.H.S.} = \int_{a+nT}^{b+nT} f(x)\, dx$$

$$= \int_a^b f(x+nT)\, dx$$

$$= \int_a^b f(x)\, dx \quad [\because f(x+nT)=f(x)]$$

Example 8.40 Evaluate $\int_0^{16\pi/3} |\sin x|\, dx$.

Sol. $\int_0^{16\pi/3} |\sin x|\, dx = \int_0^{5\pi} |\sin x|\, dx + \int_{5\pi}^{5\pi+\pi/3} |\sin x|\, dx$

$$= 5\int_0^{\pi} |\sin x|\, dx + \int_0^{\pi/3} |\sin x|\, dx$$

$[\because |\sin x|$ is periodic with period $\pi]$

$$= 5\int_0^{\pi} \sin x dx + \int_0^{\pi/3} \sin x dx = 5\times 2 + \left(-\frac{1}{2}+1\right) = \frac{21}{2}$$

Example 8.41 Evaluate $\int_0^{100}(x-[x])dx$ (where $[\cdot]$ represents the greatest integer function).

Sol. $x-[x]=\{x\}$ has period 1

$$\Rightarrow \int_0^{100}(x-[x])\,dx = 100\int_0^1 \{x\}\,dx$$
$$= 100\int_0^1 x dx$$
$$= \frac{100}{2}[x^2]_0^1 = 50$$

Example 8.42 Evaluate $\dfrac{\int_0^n [x]dx}{\int_0^n \{x\}dx}$ (where $[x]$ and $\{x\}$ are integral and fractional parts of x and $n \in N$).

Sol. $$I = \frac{\int_0^n [x]dx}{\int_0^n \{x\}dx}$$
$$= \frac{\int_0^n (x-\{x\})dx}{\int_0^n \{x\}dx}$$
$$= \frac{\int_0^n x dx}{\int_0^n \{x\}dx} - 1$$
$$= \frac{\left.\frac{x^2}{2}\right|_0^n}{n\int_0^1 \{x\}dx} - 1$$
$$= \frac{\frac{n^2}{2}}{n\int_0^1 x\,dx} - 1 = \frac{\frac{n^2}{2}}{n\frac{1}{2}} - 1 = n-1$$

Example 8.43 Evaluate $\int_{-\pi/4}^{n\pi-\frac{\pi}{4}} |\sin x + \cos x|\,dx$.

Sol. $$I = \int_{-\pi/4}^{n\pi-\pi/4} |\sin x+\cos x|\,dx$$
$$= \int_{-\pi/4}^{n\pi-\frac{\pi}{4}} \sqrt{2}\,|\sin(x+\pi/4)|\,dx \quad \text{(multiplying and dividing by } \sqrt{2})$$
$$= n\int_0^{\pi}\sqrt{2}\,|\sin(x+\pi/4)|\,dx \quad \text{(as } |\sin(x+\pi/4)| \text{ is periodic with period } \pi)$$
$$= \sqrt{2}n\int_0^{\pi}|\sin(x+\pi/4)|\,dx$$
$$= \sqrt{2}n\left[\int_0^{3\pi/4}\sin(x+\pi/4)dx + \int_{3\pi/4}^{\pi} -\sin(x+\pi/4)dx\right]$$
$$= 2\sqrt{2}n\left[\because \sin\left(x+\frac{\pi}{4}\right) > 0 \text{ for } x\in\left(0,\frac{3\pi}{4}\right)\right]$$

Example 8.44 Let f be a real, valued function satisfying $f(x)+f(x+4)=f(x+2)+f(x+6)$.

Prove that $\int_x^{x+8} f(t)dt$ is a constant function.

Sol. Given that $f(x)+f(x+4)=f(x+2)+f(x+6)$ (1)

Replacing x by $x+2$, we get

$f(x+2)+f(x+6)=f(x+4)+f(x+8)$ (2)

From equations (1) and (2), we get $f(x)=f(x+8)$ (3)

$$\Rightarrow \int_x^{x+8} f(t)dt = \int_0^8 f(t)dt \quad \Rightarrow \quad g \text{ is a constant function.}$$

Example 8.45 A periodic function with period 1 is integrable over any finite interval. Also for two real numbers a, b and for two unequal non-zero positive integers m and n, $\int_a^{a+n} f(x)dx = \int_b^{b+m} f(x)\,dx$. Calculate the value of $\int_m^n f(x)dx$.

Sol. Given $f(1+x)=f(x)$

$\therefore \int_a^{a+n} f(x)\,dx = n\int_0^1 f(x)\,dx$ ($\because f(x)$ is periodic)

Similarly, $\int_b^{b+m} f(x)\,dx = m\int_0^1 f(x)dx$

Given $\int_a^{a+n} f(x)\,dx = \int_b^{b+m} f(x)\,dx$

$\Rightarrow n\int_0^1 f(x)\,dx = m\int_0^1 f(x)dx$

$\Rightarrow (n-m)\int_0^1 f(x)\,dx = 0$

$\Rightarrow \int_0^1 f(x)\,dx = 0$ (1) ($\because n \neq m$)

$\therefore \int_m^n f(x)\,dx = \int_0^{n-m} f(m+x)\,dx = \int_0^{n-m} f(x)\,dx$ ($\because f$ is periodic)

$= (n-m)\int_0^1 f(x)\,dx$ (Assume $n > m$)

$= 0$ [from equation (1)]

Concept Application Exercise 8.5

1. Evaluate $\int_0^{100\pi}\sqrt{(1-\cos 2x)}dx$.
2. If $\int_0^{n\pi} f(\cos^2 x)dx = k\int_0^{\pi} f(\cos^2 x)dx$, then find the value k.
3. Evaluate $\int_0^{n\pi+t}(|\cos x|+|\sin x|)dx$, where $t\in[0,\pi/2]$.
4. Find the value of $\int_0^{10} e^{2x-[2x]}\,d(x-[x])$ (where $[.]$ denotes the greatest integer function).
5. If $f(x)$ is a function satisfying $f(x+a)+f(x)=0$ for all $x\in R$ and positive constant a such that $\int_b^{c+b} f(x)dx$ is independent of b, then find the least positive value of c.

LEIBNITZ'S RULE

If f is a continuous function on $[a, b]$, and $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$, then $\frac{d}{dx}\left\{\int_{u(x)}^{v(x)} f(t)\,dt\right\} = f(v(x))\frac{dv(x)}{dx} - f(u(x))\frac{du(x)}{dx}$.

Proof:

Let $\frac{d}{dx}F(x) = f(x)$

$$\Rightarrow \int_{u(x)}^{v(x)} f(t)\,dt = F(v(x)) - F(u(x))$$

$$\Rightarrow \frac{d}{dx}\left\{\int_{u(x)}^{v(x)} f(t)\,dt\right\} = \frac{d}{dx}(F(v(x)) - F(u(x)))$$

$$\Rightarrow \frac{d}{dx}\left\{\int_{u(x)}^{v(x)} f(t)\,dt\right\} = F'(v(x))\frac{d(v(x))}{dx} - F'(u(x))\frac{d(u(x))}{dx}$$

$$\Rightarrow \frac{d}{dx}\left\{\int_{u(x)}^{v(x)} f(t)\,dt\right\} = f(v(x))\frac{d(v(x))}{dx} - f(u(x))\frac{d(u(x))}{dx}$$

Example 8.46 If $y = \int_{x^2}^{x^3} \frac{1}{\log t}\,dt$ (where $x > 0$), then find $\frac{dy}{dx}$.

Sol. $y = \int_{x^2}^{x^3} \frac{1}{\log t}\,dt$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^3)\frac{1}{\log x^3} - \frac{d}{dx}(x^2)\frac{1}{\log x^2}$$

$$= \frac{3x^2}{3\log x} - \frac{2x}{2\log x} = x(x-1)(\log x)^{-1}$$

Example 8.47 If $\int_{\sin x}^{1} t^2 f(t)dt = 1 - \sin x$, where $x \in \left(0, \frac{\pi}{2}\right)$, then find the value of $f\left(\frac{1}{\sqrt{3}}\right)$.

Sol. $\int_{\sin x}^{1} t^2 f(t)dt = 1 - \sin x$

Differentiating both sides, we get

$1^2 \times f(1)\, 0 - \sin^2 x\, f(\sin x) \cos x = -\cos x$

$\Rightarrow f(\sin x) = \text{cosec}^2 x = \frac{1}{\sin^2 x}$

$\Rightarrow f(z) = \frac{1}{z^2}, \therefore f\left(\frac{1}{\sqrt{3}}\right) = 3$

Example 8.48 Let $f: R \to R$ be a differentiable function having $f(2) = 6, f'(2) = \frac{1}{48}$. Then evaluate

$$\lim_{x\to 2}\int_6^{f(x)} \frac{4t^3}{x-2}\,dt.$$

Sol. $\lim_{x\to 2}\int_6^{f(x)} \frac{4t^3}{x-2}\,dt = \lim_{x\to 2}\frac{\int_6^{f(x)} 4t^3\,dt}{x-2}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 2}\frac{4\left(f(x)\right)^3 f'(x)}{1}$

(applying L'Hopital Rule)

$= 4(f(2))^3 \times f'(2)$

$= 4(6)^3 \times \frac{1}{48}$

$= 18$

Example 8.49 Evaluate $\lim_{x\to\infty}\frac{\left(\int_0^x e^{x^2}dx\right)^2}{\int_0^x e^{2x^2}dx}$.

Sol. Since $e^{x^2} > 0$, $e^{2x^2} > 0$ in $[0, x]$, where $x > 0$,

$\int_0^x e^{x^2}dx$ and $\int_0^x e^{2x^2}dx \to \infty$ as $x \to \infty$

$L = \lim_{x\to\infty}\frac{\left(\int_0^x e^{x^2}\right)^2}{\int_0^x e^{2x^2}}$ is of the form $\frac{\infty}{\infty}$.

Therefore, using L' Hopital's Rule

$$L = \lim_{x\to\infty}\frac{2e^{x^2}\int_0^x e^{x^2}dx}{e^{2x^2}}$$

$$= 2\lim_{x\to\infty}\frac{\int_0^x e^{x^2}dx}{e^{x^2}} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= 2\lim_{x\to\infty}\frac{e^{x^2}}{2xe^{x^2}}$$

$$= \lim_{x\to\infty}\frac{1}{x} = 0$$

Example 8.50 If $\int_0^y \cos t^2 dt = \int_0^{x^2}\frac{\sin t}{t}dt$, then prove that $\frac{dy}{dx} = \frac{2\sin x^2}{x\cos y^2}$.

Sol. Given that $\int_0^y \cos t^2\,dt = \int_0^{x^2}\frac{\sin t}{t}\,dt$

Differentiating w.r.t. x, we get

$$\cos y^2\frac{dy}{dx} = \frac{\sin x^2}{x^2}2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin x^2}{x\cos y^2}$$

Example 8.51 If $x = \int_0^y \frac{dt}{\sqrt{1+9t^2}}$ and $\frac{d^2y}{dx^2} = ay$, then find a.

Sol. $x = \int_0^y \frac{dt}{\sqrt{1+9t^2}}$

Differentiate w.r.t. y, we get

$$\Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{1+9y^2}}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{1+9y^2}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dy}\left(\sqrt{1+9y^2}\right)\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{18y}{2\sqrt{1+9y^2}}\sqrt{1+9y^2} = 9y$$

$$\Rightarrow a = 9$$

Example 8.52 Prove that $y = \int_{1/8}^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_{1/8}^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt$, where $0 \le x \le \pi/2$, is the equation of a straight line parallel to the x-axis. Find its equation.

Sol. Here, we have to prove that y = constant or derivative of y w.r.t. x is zero.

$$y = \int_{\frac{1}{8}}^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_{\frac{1}{8}}^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt \quad (1)$$

$$\frac{dy}{dx} = \sin^{-1}\sqrt{\sin^2 x}\; 2\sin x\cos x + \cos^{-1}\sqrt{\cos^2 x}\,(-2\cos x\sin x)$$

$= 2x\sin x\cos x - 2x\sin x\cos x$

$= 0$ for all x

Therefore, the curve in equation (1) is a straight line parallel to the x-axis.

Now, since y is constant, it is independent of x. So let's select $x = \pi/4$

$$\Rightarrow y = \int_{1/8}^{1/2} \sin^{-1}\sqrt{t}\, dt + \int_{1/8}^{1/2} \cos^{-1}\sqrt{t}\, dt$$

$$= \int_{1/8}^{1/2} (\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t})\, dt$$

$$= \int_{1/8}^{1/2} \pi/2\, dt$$

$$= \frac{\pi}{2}\left[\frac{1}{2} - \frac{1}{8}\right]$$

$$= \frac{3\pi}{16}$$

Therefore, equation of the line is $y = \frac{3\pi}{16}$.

Example 8.53 Let $f: (0, \infty) \to (0, \infty)$ be a differentiable function satisfying, $x\int_0^x (1-t)f(t)dt = \int_0^x tf(t)dt\ \forall\ x \in R^+$ and $f(1) = 1$. Determine $f(x)$.

Sol. We have $x\int_0^x (1-t)f(t)dt = \int_0^x tf(t)dx$

Differentiating both sides w.r.t. x, we get

$$x(1-x)f(x) + \int_0^x (1-t)f(t)dt = xf(x)$$

$$\Rightarrow x^2 f(x) = \int_0^x (1-t)f(t)dt$$

Differentiating both sides w.r.t. x again, we get

$$x^2 f'(x) + 2xf(x) = (1-x)f(x)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \frac{1-3x}{x^2}$$

$$\Rightarrow \int \frac{f'(x)}{f(x)}dx = \int \frac{1-3x}{x^2}dx$$

$$\Rightarrow \log f(x) = -\frac{1}{x} - 3\log x + \log c$$

$$\Rightarrow \log\left[\frac{f(x)}{c}\right] = \frac{-1}{x} - 3\log x$$

Given $f(1) = 1 \Rightarrow \log\left(\frac{1}{c}\right) = -1 \Rightarrow c = e$

$$\Rightarrow \log\left(\frac{f(x)x^3}{e}\right) = -\frac{1}{x}$$

$$\Rightarrow f(x) = \frac{1}{x^3}e^{\left(1-\frac{1}{x}\right)}$$

Example 8.54 If $y = \int_0^x f(t)\sin\{k(x-t)\}\, dt$, then prove that $\frac{d^2y}{dx^2} + k^2y = kf(x)$.

Sol. Since $y = \int_0^x f(t)\sin\{k(x-t)\}\, dt$

$$= \int_0^x f(t)[\sin kx\cos kt - \sin kt\cos kx]\, dt$$

$$= \sin kx \int_0^x f(t)\cos kt dt - \cos kx \int_0^x f(t)\sin kt dt \quad (1)$$

$$\Rightarrow \frac{dy}{dx} = k\cos kx \int_0^x f(t)\cos kt dt + \sin kx\,[f(x)\cos kx] + k\sin kx \int_0^x f(t)\sin kt dt - \cos kx\,[f(x)\sin kx]$$

$$= k\cos kx \int_0^x f(t)\cos kt dt + k\sin kx \int_0^x f(t)\sin kt dt \quad (2)$$

Again differentiating equation (2) w.r.t. x, we get

$$\Rightarrow \frac{d^2y}{dx^2} = -k^2\sin kx \int_0^x f(t)\cos kt dt + k\cos kx\,[f(x)\cos kx] + k^2\cos kx \int_0^x f(t)\sin tdt + k\sin kx\,[f(x)\sin kx]$$

$$= -k^2y + kf(x)$$

$$\Rightarrow \frac{d^2y}{dx^2} + k^2y = kf(x)$$

Concept Application Exercise 8.6

1. Evaluate $\lim_{x\to 4}\int_4^x \frac{(4t-f(t))}{(x-4)}\,dt$.

2. Evaluate $\lim_{x\to 0}\frac{\int_0^x \cos t^2\,dt}{x}$.

3. Find the points of minima for $f(x)=\int_0^x t(t-1)(t-2)\,dt$.

4. If $f(x)=e^{g(x)}$ and $g(x)=\int_2^x \frac{t\,dt}{1+t^4}$, then find the value of $f'(2)$.

5. If $f(x)=\int_{\pi^2/16}^{x^2}\frac{\sin x\sin\sqrt{\theta}}{1+\cos^2\sqrt{\theta}}\,d\theta$, then find the value of $f'\left(\frac{\pi}{2}\right)$.

6. Find the equation of tangent to $y=\int_{x^2}^{x^3}\frac{dt}{\sqrt{1+t^2}}$ at $x=1$.

7. If $\int_{\pi/3}^{x}\sqrt{(3-\sin^2 t)}\,dt+\int_0^y \cos t\,dt=0$, then evaluate $\frac{dy}{dx}$.

INEQUALITIES

Property I

If at every point x of an interval $[a, b]$, the inequalities $g(x)\le f(x)\le h(x)$ are fulfilled, then $\int_a^b g(x)\,dx\le\int_a^b f(x)\,dx\le\int_a^b h(x)\,dx$, where $a<b$.

Proof:

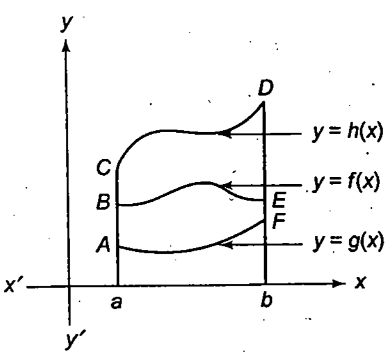

Fig. 8.11

It is clear from the Fig. 8.11,
Area of curvilinear trapezoid $aAFb\le$ Area of curvilinear trapezoid $aBEb\le$ Area of curvilinear trapezoid $aCDb$

i.e., $\int_a^b g(x)\,dx\le\int_a^b f(x)\,dx\le\int_a^b h(x)\,dx$.

Example 8.55 Prove that $0<\int_0^1\frac{x^7\,dx}{\sqrt[3]{(1+x^8)}}<\frac{1}{8}$.

Sol. Since $0<\frac{x^7}{\sqrt[3]{(1+x^8)}}<x^7\ \forall\ 0<x<1$,

then $\int_0^1 0\,dx<\int_0^1\frac{x^7}{\sqrt[3]{(1+x^8)}}\,dx<\int_0^1 x^7\,dx$

Hence, $0<\int_0^1\frac{x^7dx}{\sqrt[3]{(1+x^8)}}<\frac{1}{8}$.

Example 8.56 Prove that $\frac{1}{2}\le\int_0^{1/2}\frac{dx}{\sqrt{1-x^{2n}}}\le\frac{\pi}{6}$ for $n\ge 1$.

Sol. For $n\ge 1$ and $-1\le x\le 1$, we have

$1\ge\sqrt{1-x^{2n}}\ge\sqrt{1-x^2}$

$\Rightarrow\int_0^{1/2}dx\le\int_0^{1/2}\frac{dx}{\sqrt{1-x^{2n}}}\le\int_0^{1/2}\frac{dx}{\sqrt{1-x^2}}=[\sin^{-1}x]_0^{1/2}$

$\Rightarrow\frac{1}{2}\le\int_0^{1/2}\frac{dx}{\sqrt{1-x^{2n}}}\le\frac{\pi}{6}$

Property II

If m is the least value (global minimum) and M is the greatest value (global maximum) of the function $f(x)$ on the interval $[a, b]$ (estimation of an integral), then $m(b-a)\le\int_a^b f(x)\,dx\le M(b-a)$.

Proof: Analytical Method

It is given that $m\le f(x)\le M$, then

$\int_a^b m\,dx\le\int_a^b f(x)\,dx\le\int_a^b M\,dx$

$\Rightarrow m(b-a)\le\int_a^b f(x)\,dx\le M(b-a)$

Graphical Method

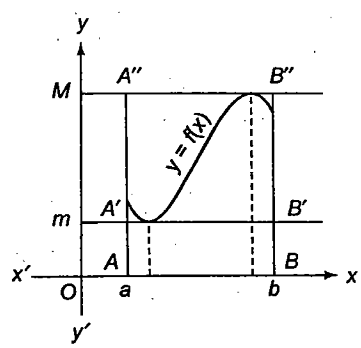

Fig. 8.12

It is clear from the Fig. 8.12

Area of $ABB'A'\le\int_a^b f(x)\,dx\le$ Area of $ABB''A''$

i.e., $m(b-a)\le\int_a^b f(x)\,dx\le M(b-a)$

Example 8.57 Prove that $1 < \int_0^2 \left(\frac{5-x}{9-x^2}\right) dx < \frac{6}{5}$.

Sol. Let $f(x) = \frac{5-x}{(9-x^2)}$.

$\Rightarrow f'(x) = -\frac{(x-9)(x-1)}{(9-x^2)^2}$

For $f'(x) = 0$, we have $x = 1$ as $x \in [0, 2]$.

Now $f(0) = 5/9, f(1) = 1/2, f(2) = 3/5$.

Therefore, the greatest and the least values of the integrand in the interval [0, 2] are, respectively, equal to $f(2) = 3/5$ and $f(1) = 1/2$.

Hence, $(2-0)\frac{1}{2} < \int_0^2 \left(\frac{5-x}{9-x^2}\right) dx < (2-0)\frac{3}{5}$,

or $1 < \int_0^2 \left(\frac{5-x}{9-x^2}\right) dx < \frac{6}{5}$.

Property III

$$\left|\int_a^b f(x)\, dx\right| \le \int_a^b |f(x)|\, dx$$

Proof:

Obviously, $-|f(x)| \le f(x) \le |f(x)|, \forall x \in [a, b]$

$\Rightarrow \int_a^b -|f(x)|\, dx \le \int_a^b f(x)\, dx \le \int_a^b |f(x)|\, dx$

or $-\int_a^b |f(x)|\, dx \le \int_a^b f(x)\, dx \le \int_a^b |f(x)|\, dx$

or $\left|\int_a^b f(x) dx\right| \le \int_a^b |f(x)|\, dx$

Example 8.58 Estimate the absolute value of the integral

$$\int_{10}^{19} \frac{\sin x}{1+x^8}\, dx.$$

Sol. Since $|\sin x| \le 1$ for $x \ge 10$, then $\left|\frac{\sin x}{1+x^8}\right| \le \frac{1}{|1+x^8|}$ (1)

But $10 \le x \le 19 \Rightarrow 1 + x^8 > x^8 \ge 10^8$

$\Rightarrow \frac{1}{1+x^8} < \frac{1}{x^8} \le \frac{1}{10^8}$

$\Rightarrow \frac{1}{|1+x^8|} \le \frac{1}{10^8}$ (2)

From equations (1) and (2), we get $\left|\frac{\sin x}{1+x^8}\right| \le 10^{-8}$

Then $\int_{10}^{19} \frac{\sin x}{1+x^8}\, dx \le (19-10) \times 10^{-8}$

$= 9 \times 10^{-8} = (10-1) \times 10^{-8}$

$= 10^{-7} - 10^{-8} < 10^{-7}$

Hence, $\left|\int_{10}^{19} \frac{\sin x dx}{1+x^8}\right| < 10^{-7}$.

Therefore, the approximate value of the integral $= 10^{-8}$.

Property IV

If $f^2(x)$ and $g^2(x)$ are integrable on the interval $[a, b]$, then

$$\left|\int_a^b f(x)g(x)dx\right| \le \sqrt{\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)}.$$

Proof:

Let $F(x) = \{f(x) - \lambda g(x)\}^2 \ge 0$ where λ is real number.

$\Rightarrow \int_a^b \{f(x) - \lambda g(x)\}^2 dx \ge 0$

$\Rightarrow \int_a^b \{\lambda^2 (g(x))^2 - 2\lambda f(x)g(x) + f^2(x)\} dx \ge 0$

$\Rightarrow \lambda^2 \int_a^b (g(x))^2 dx - 2\lambda \int_a^b f(x)g(x)dx + \int_a^b f^2(x)dx \ge 0$

Therefore, discriminant is non-positive, i.e., $B^2 - 4AC \le 0$.

$\Rightarrow 4\left\{\int_a^b f(x)g(x)dx\right\}^2 \le 4\int_a^b f^2(x)dx \int_a^b g^2(x)dx$

Hence, $\left|\int_a^b f(x)g(x)dx\right| \le \sqrt{\int_a^b f^2(x)dx . \int_a^b g^2(x)dx}$.

Example 8.59 Prove that $\int_0^1 \sqrt{(1+x)(1+x^3)}\, dx$ cannot exceed $\sqrt{15/8}$.

Sol. $\int_0^1 \sqrt{(1+x)(1+x^3)}\, dx \le \sqrt{\left(\int_0^1 (1+x)dx\right)\left(\int_0^1 (1+x^3)dx\right)}$

$= \sqrt{\left(x + \frac{x^2}{2}\right)_0^1 \left(x + \frac{x^4}{4}\right)_0^1}$

$= \sqrt{\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)}$

$= \sqrt{\frac{15}{8}}$

Concept Application Exercise 8.7

1. Prove that $4 \le \int_1^3 \sqrt{3 + x^2} \le 4\sqrt{3}$.

2. If $I_1 = \int_0^1 2^{x^2} dx$, $I_2 = \int_0^1 2^{x^3} dx$, $I_3 = \int_1^2 2^{x^2} dx$, $I_4 = \int_1^2 2^{x^3} dx$, then which of the following is/are true?

   a. $I_1 > I_2$ b. $I_2 > I_1$
   c. $I_3 > I_4$ d. $I_3 < I_4$

3. If $I_1 = \int_0^{\pi/2} \cos(\sin x)dx$, $I_2 = \int_0^{\pi/2} \sin(\cos x)dx$ and $I_3 = \int_0^{\pi/2} \cos x dx$, then find the order in which the values I_1, I_2, I_3 exists.

4. Prove that $\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{\pi}{4\sqrt{2}}$.

MISCELLANEOUS SOLVED PROBLEMS

1. Evaluate $\int_0^{10\pi}[\tan^{-1} x]dx$, where $[x]$ represents greatest integer function.

Sol. Here $y = \tan^{-1}x$ is a monotonic function, so the analytical method is advisable.

We have $\begin{cases} 0 < \tan^{-1} x < 1; & \text{when } 0 < x < \tan 1 \\ 1 < \tan^{-1} x < \dfrac{\pi}{2}; & \text{when } \tan 1 < x < 10\pi \end{cases}$

$$\therefore I = \int_0^{10\pi}[\tan^{-1} x]dx = \int_0^{\tan 1} 0\, dx + \int_{\tan 1}^{10\pi} 1 dx$$
$$= 10\pi - \tan 1$$

2. Evaluate $\int_0^{\infty}[2e^{-x}]dx$, where $[x]$ represents greatest integer function.

Sol. $f(x) = 2e^{-x}$ is decreasing for $x \in [0, \infty)$.

Also, when $x = 0$, $2e^{-x} = 2$,

and when $x \to \infty$, $2e^{-x} \to 0$.

Thus, $[2e^{-x}]$ is discontinuous when $2e^{-x} = 1$ or $x = \log 2$.

Also, for $x > \ln 2$, $[2e^{-x}] = 0$

and for $0 < x < \log 2$, we have $0 < x < 1$.

$$\Rightarrow \int_0^{\infty}[2e^{-x}]dx = \int_0^{\ln 2}[2e^{-x}]dx + \int_{\ln 2}^{\infty}[2e^{-x}]dx$$

$$= \int_0^{\ln 2} 1dx + \int_{\ln 2}^{0} 0dx = (x)_0^{\ln 2} = \ln 2$$

$$\Rightarrow \int_0^{\infty}[2e^{-x}]dx = \ln 2$$

3. Evaluate $\int_0^{5\pi/12} [\tan x]\, dx$, where [.] denotes the greatest integer function.

Sol. Let $I = \int_0^{5\pi/12} [\tan x]\, dx$.

Here, $y = \tan x$ is a monotonically increasing function.

Also, when $x = 0$, $\tan x = 0$ and when $x = \dfrac{5\pi}{12}$, $\tan x = 2 + \sqrt{3}$.

Hence, $[\tan x]$ is discontinuous when $\tan x = 1$, $\tan x = 2$, $\tan x = 3$.

$\Rightarrow x = \tan^{-1} 1, x = \tan^{-1} 2, x = \tan^{-1} 3$

$$\therefore I = \int_0^{\tan^{-1}1}[\tan x]dx + \int_{\tan^{-1}1}^{\tan^{-1}2}[\tan x]dx + \int_{\tan^{-1}2}^{\tan^{-1}3} [\tan x]\, dx + \int_{\tan^{-1}3}^{5\pi/12} [\tan x]\, dx$$

$$= \int_0^{\tan^{-1}1} 0\, dx + \int_{\tan^{-1}1}^{\tan^{-1}2} 1\, dx + \int_{\tan^{-1}2}^{\tan^{-1}3} 2\, dx + \int_{\tan^{-1}3}^{5\pi/12} 3\, dx$$

$$= 0 + (\tan^{-1} 2 - \tan^{-1} 1) + 2(\tan^{-1} 3 - \tan^{-1} 2) + 3\left(\frac{5\pi}{12} - \tan^{-1} 3\right)$$

$$= \frac{5\pi}{4} - \frac{\pi}{4} - \tan^{-1} 3 - \tan^{-1} 2$$

$$= \pi - \left[\pi + \tan^{-1}\left(\frac{3+2}{1-(3)(2)}\right)\right]$$

$$= \pi - \left[\pi + \tan^{-1}(-1)\right] = \pi/4$$

4. Evaluate $\int_0^2\left[x^2 - x + 1\right]dx$, where [.] denotes the greatest integer function.

Sol. Here $f(x) = x^2 - x + 1$ is a non-monotonic function. Such problems should be solved by graphical method. Now, $g(x) = [x^2 - x + 1]$, $\forall x \in [0, 2]$ could be plotted as shown in Fig 8.13.

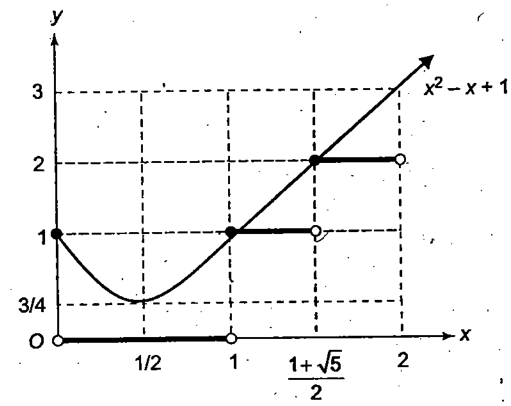

Fig. 8.13

$$\therefore I = \int_0^2\left[x^2 - x + 1\right]dx$$

$$= \int_0^1 0dx + \int_1^{(1+\sqrt{5})/2} 1\, dx + \int_{(1+\sqrt{5})/2}^{2} 2\, dx$$

$$= \left(\frac{1+\sqrt{5}}{2} - 1\right) + 2\left(2 - \frac{1+\sqrt{5}}{2}\right)$$

$$= \frac{5-\sqrt{5}}{2}$$

5. Evaluate $\int_0^{2\pi} [\sin x]dx$, where [.] denotes the greatest integer function.

Sol. $y = \sin x$ is a non-monotonic function in $[0, 2\pi]$. Hence, draw the graph of $f(x) = [\sin x]$.

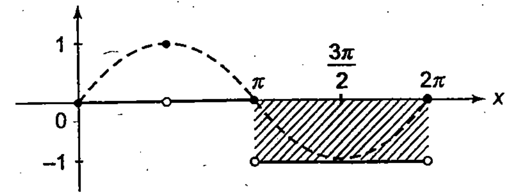

Fig. 8.14

From the graph given in Fig. 8.14,

$$\int_0^{2\pi} [\sin x]\, dx = \text{Algebraic area of the shaded region}$$
$$= (\pi)(-1)$$
$$= -\pi$$

Note:
Students are advised to remember this value. Also, we can prove that $\int_0^{2\pi} [\cos x]\, dx = -\pi$.

6. Evaluate $\int_0^x [\cos t]\, dt$, where $n \in \left(2n\pi, (4n+1)\frac{\pi}{2}\right)$, $n \in N$ and [.] denotes the greatest integer function.

Sol. Let

$$I = \int_0^x [\cos t]\, dt$$
$$= \int_0^{2n\pi} [\cos t] dt + \int_{2n\pi}^x [\cos t] dt$$
$$= n\int_0^{2\pi} [\cos t] dt + \int_{2n\pi}^x [\cos t] dt$$
$$= -n\pi + \int_{2n\pi}^x 0 dt$$
$$= -n\pi$$

7. If $\int_0^1 e^{-x^2} dx = a$, then find the value of $\int_0^1 x^2 e^{-x^2} dx$ in terms of a.

Sol. $I = \int_0^1 x^2 e^{-x^2} dx = \frac{-1}{2}\int_0^1 x(-2x)e^{-x^2} dx$

$$= -\frac{1}{2}\left(x\, e^{-x^2}\Big|_0^1 - \int_0^1 e^{-x^2} dx \right) \quad \text{(Integrating by parts)}$$
$$= -\frac{1}{2e} + \frac{1}{2}a$$

8. If $\int_0^1 \frac{e^t}{1+t} dt = a$, then find the value of $\int_0^1 \frac{e^t}{(1+t)^2} dt$ in terms of a.

Sol. $a = \int_0^1 \frac{e^t}{1+t} dt = \left(\frac{1}{(1+t)} e^t\right)_0^1 + \int_0^1 \frac{e^t}{(1+t)^2} dt$ (integrating by parts)

$$\Rightarrow a = \frac{e}{2} - 1 + \int_0^1 \frac{e^t}{(1+t)^2} dt$$
$$\Rightarrow \int_0^1 \frac{e^t}{(1+t)^2} dt = a + 1 - \frac{e}{2}$$

9. If $I_n = \int_0^{\pi} x^n \sin x dx$, then find the value of $I_5 + 20 I_3$.

Sol. $I_n = \int_0^{\pi} x^n \sin x dx$

$$= \left[-x^n \cos x\right]_0^{\pi} + n\int_0^{\pi} x^{n-1} \cos x dx$$
$$= \pi^n + n[x^{n-1} \sin x]_0^{\pi} - n(n-1)\int_0^{\pi} x^{n-2} \sin x dx$$
$$\Rightarrow I_n = \pi^n + n.0 - n(n-1) I_{n-2}$$

Put $n = 5$,

$$I_5 = \pi^5 - 20 I_3$$
$$I_5 + 20 I_3 = \pi^5$$

10. If $I_n = \int_0^1 x^n (\tan^{-1} x) dx$, then prove that

$$(n+1) I_n + (n-1) I_{n-2} = -\frac{1}{n} + \frac{\pi}{2}.$$

Sol. $I_n = \int_0^1 x^n (\tan^{-1} x) dx = \int_0^1 x^{n-1} (x \tan^{-1} x) dx$

$$\Rightarrow I_n = \left[x^{n-1}\left(\frac{x^2}{2}\tan^{-1} x - \frac{x}{2} + \frac{\tan^{-1} x}{2}\right)\right]_0^1$$
$$-(n-1)\int_0^1 x^{n-2}\left(\frac{x^2}{2}\tan^{-1} x - \frac{x}{2} + \frac{\tan^{-1} x}{2}\right) dx$$
$$\Rightarrow I_n = \frac{\pi}{4} - \frac{1}{2} - \frac{(n-1)}{2} I_n + \frac{(n-1)}{2}\int_0^1 x^{n-1} dx - \frac{1}{2}(n-1) I_{n-2}$$
$$\Rightarrow \frac{(n+1)}{2} I_n = \frac{\pi}{4} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2n} - \frac{1}{2}(n-1) I_{n-2}$$
$$\Rightarrow (n+1) I_n + (n-1) I_{n-2} = -\frac{1}{n} + \frac{\pi}{2}$$

11. If $I_n = \int_0^{\pi/2} \sin^n x\, dx$, then show that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$.

Hence, prove that

$$I_n = \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\left(\frac{1}{2}\right)\frac{\pi}{2} & \text{if } n \text{ is even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\left(\frac{2}{3}\right)1 & \text{if } n \text{ is odd} \end{cases}$$

Sol. $I_n = \int_0^{\pi/2} \sin^n x\, dx$

$$= \int_0^{\pi/2} \sin^{n-1} x \sin x dx$$

$$=\left[-\sin^{n-1}x\cos x\right]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}(n-1)\sin^{n-2}x\cos^2 x\,dx$$

$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x(1-\sin^2 x)\,dx$$

$$=(n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\,dx-(n-1)\int_0^{\frac{\pi}{2}}\sin^n x\,dx$$

$$\Rightarrow I_n+(n-1)I_n=(n-1)I_{n-2}$$

$$\Rightarrow I_n=\left(\frac{n-1}{n}\right)I_{n-2}$$

$$\Rightarrow I_n=\left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\ldots I_0 \text{ or } I_1$$

Accordingly, if n is even or odd,

$$I_0=\frac{\pi}{2},\ I_1=1$$

Hence,

$$I_n=\begin{cases}\left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\left(\frac{1}{2}\right)\frac{\pi}{2} & \text{if } n \text{ is even}\\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\cdots\left(\frac{2}{3}\right)1 & \text{if } n \text{ is odd.}\end{cases}$$

12. Evaluate $\int_0^1\frac{1}{\sqrt{1-x^2}}\sin^{-1}(2x\sqrt{1-x^2})dx$

Sol. $I=\int_0^1\frac{1}{\sqrt{1-x^2}}\sin^{-1}(2x\sqrt{1-x^2})dx$

Putting $x=\sin\theta$, we get

$$\Rightarrow I=\int_0^{\pi/2}\frac{1}{\sqrt{1-\sin^2\theta}}\sin^{-1}(2\sin\theta\cos\theta)\cos\theta\,d\theta$$

$$=\int_0^{\pi/2}\sin^{-1}(\sin 2\theta)d\theta$$

Put $2\theta=t$

$$\Rightarrow I=\frac{1}{2}\int_0^{\pi}\sin^{-1}(\sin t)dt$$

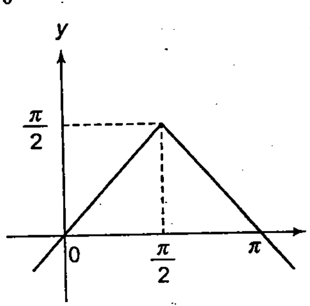

Fig. 8.15

From the graph, $I=\frac{1}{2}$(area of triangle)

$$=\frac{1}{2}\times\frac{1}{2}\frac{\pi}{2}\pi=\frac{\pi^2}{8}$$

13. Prove that $\int_0^x e^{xt}e^{-t^2}dt=e^{x^2/4}\int_0^x e^{-t^2/4}dt.$

Sol. Let $I=\int_0^x e^{xt}e^{-t^2}dt$

$$=e^{x^2/4}\int_0^x e^{-x^2/4}e^{xt}e^{-t^2}dt$$

$$=e^{x^2/4}\int_0^x e^{-(x^2/4-tx+t^2)}dt$$

$$=e^{x^2/4}\int_0^x e^{-(x/2-t)^2}dt$$

The result clearly suggests that we have to substitute $y/2$ for $x/2-t$.

Then $dt=-dy/2$, also when $t=0$, $y=x$ and when $t=x, y=-x$.

$$\Rightarrow I=e^{x^2/4}\int_x^{-x}e^{-y^2/4}(-dy/2)$$

$$=\frac{e^{x^2/4}}{2}\int_{-x}^{x}e^{-y^2/4}dy$$

$$=\frac{e^{x^2/4}}{2}2\int_0^x e^{-y^2/4}dy\quad[\,e^{-y^2/4}\text{ is an even function}]$$

$$=e^{x^2/4}\int_0^x e^{-t^2/4}dt$$

14. Evaluate $\int_{-4}^{-5}e^{(x+5)^2}dx+3\int_{1/3}^{2/3}e^{9\left(x-\frac{2}{3}\right)^2}dx.$

Sol. $I=\int_{-4}^{-5}e^{(x+5)^2}dx+3\int_{1/3}^{2/3}e^{9\left(x-\frac{2}{3}\right)^2}dx$

$$=\int_{-4}^{-5}e^{(x+5)^2}dx+3\int_{1/3}^{2/3}e^{(3x-2)^2}dx$$

$$=I_1+I_2$$

Note that in both I_1 and I_2, function has same format, i.e., e^{t^2}.

Also, e^{t^2} is non-integrable.

Now, in I_1, let $x+5=y$ and in I_2, $3x-2=-t$

$$\Rightarrow I=\int_1^0 e^{y^2}dy+\int_1^0 e^{t^2}(-dt)=0.$$

15. Compute the following integrals.

a. $\int_0^{\infty}f\left(x^n+x^{-n}\right)\log x\frac{dx}{x}$

b. $\int_0^{\infty}f\left(x^n+x^{-n}\right)\log x\frac{dx}{1+x^2}$

c. $\int_{1/e}^{e}\frac{1}{x}\sin\left(x-\frac{1}{x}\right)dx$

Sol. Here limits (reciprocal) and type of functions (reciprocal terms are present, i.e., x and $1/x$) suggest that we must substitute $1/t$ for x.

a. Let $t = 1/x \Rightarrow x = 1/t \Rightarrow dx = -\frac{1}{t^2}dt$.

Also, when $x \to 0, t \to \infty; x \to \infty, t \to 0$

$$\Rightarrow I = \int_0^{\infty} f\left(x^n + x^{-n}\right) \ln x \frac{dx}{x}$$

$$= \int_{\infty}^{0} f\left(t^{-n} + t^n\right) \ln\left(\frac{1}{t}\right) \frac{-\frac{dt}{t^2}}{\frac{1}{t}}$$

$$= -\int_0^{\infty} f\left(t^n + t^{-n}\right) \ln(t) \frac{dt}{t}$$

$$= -I$$

$$\Rightarrow \quad 2I = 0 \Rightarrow I = 0$$

b. Let $I = \int_0^{\infty} f\left(x^n + x^{-n}\right) \ln x \frac{dx}{1+x^2}$

Let $t = 1/x \Rightarrow x = 1/t \Rightarrow dx = -\frac{1}{t^2}dt$.

Also, when $x \to 0, t \to \infty; x \to \infty, t \to 0$

$$\Rightarrow I = \int_0^{\infty} f\left(x^n + x^{-n}\right) \ln x \frac{dx}{1+x^2}$$

$$= \int_{\infty}^{0} f\left(t^{-n} + t^n\right) \ln\left(\frac{1}{t}\right) \frac{-\frac{dt}{t^2}}{1+\frac{1}{t^2}}$$

$$= -\int_0^{\infty} f\left(t^n + t^{-n}\right) \ln(t) \frac{dt}{1+t^2}$$

$$= -I$$

$$\Rightarrow 2I = 0 \Rightarrow I = 0$$

c. $I = \int_{1/e}^{e} \frac{1}{x} \sin\left(x - \frac{1}{x}\right) dx$.

put $x = \frac{1}{t}$; $dx = -\frac{1}{t^2} dt$

$$\Rightarrow I = \int_e^{1/e} t \sin\left(\frac{1}{t} - t\right)\left(-\frac{1}{t^2}\right) dt$$

$$= \int_e^{1/e} \frac{1}{t} \sin\left(t - \frac{1}{t}\right) dt$$

$$= -\int_{1/e}^{e} \frac{1}{t} \sin\left(t - \frac{1}{t}\right) dt$$

$$I = -I \Rightarrow 2I = 0 \Rightarrow I = 0$$

Concept Application Exercise 8.8

1. Find the value of $\int_{-1}^{1} [x^2 + \{x\}] dx$, where [.] and { } denote the greatest function and fractional parts of x.

2. Prove that $\int_0^x [t]\, dt = \frac{[x]([x]-1)}{2} + [x](x - [x])$, where [.] denotes the greatest integer funciton.

3. Prove that $\int_0^{\infty} [ne^{-x}] dx = \ln\left(\frac{n^n}{n!}\right)$, where n is a natural number greater than 1 and [.] denotes the greatest integer function.

4. Evaluate $\int_{\frac{\pi}{2}}^{2\pi} [\cot^{-1} x] dx$, where [.] denotes the greatest integer function.

5. If $\int_0^1 \frac{e^t dt}{t+1} = a$, then evaluate $\int_{b-1}^{b} \frac{e^{-t} dt}{t-b-1}$.

6. If $f(x) = \int_1^x \frac{\log t}{1+t+t^2} dt$, $\forall\ x \ge 1$, then prove that $f(x) = f\left(\frac{1}{x}\right)$.

7. If $f(x)$ is a function satisfying $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$ for all non-zero x, then evaluate $\int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)\, dx$.

8. Evaluate $\int_0^{e-1} \frac{e^{\frac{x^2+2x-1}{2}}}{x+1} dx + \int_1^e x \log x e^{\frac{x^2-2}{2}} dx$.

9. Prove that $I_n = \int_0^{\infty} x^{2n+1} e^{-x^2} dx = \frac{n!}{2}, n \in N$.

10. If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\, dx$, then show that

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,\,n} \qquad (m, n \in N)$$

Hence, prove that

$$I_{m,n} = \begin{cases} \dfrac{(n-1)(n-3)(m-5)\ldots(n-1)(n-3)(n-5)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} \\ \dfrac{\pi}{4} \text{ when both } m \text{ and } n \text{ are even} \\ \dfrac{(m-1)(m-3)(m-5)\ldots(n-1)(n-3)(n-5)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} \\ \text{otherwise} \end{cases}$$

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EXERCISES

Subjective Type

Solutions on page 8.44

1. It is known that $f(x)$ is an odd function in the interval $[-p/2, p/2]$ and has a period p. Prove that $\int_a^x f(t)dt$ is also periodic function with the same period.

2. If $\int_0^{\pi/2} \log \sin\theta\, d\theta = k$, then find the value of $\int_0^{\pi/2} (\theta/\sin\theta)^2\, d\theta$ in terms of k.

3. Let $f(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, then show that
$$f(n) = \int_0^{\pi/2} \cot\left(\frac{\theta}{2}\right)(1-\cos^n\theta)d\theta.$$

4. Evaluate $\int_0^{\pi/4}\left(\tan^{-1}\left(\frac{2\cos^2\theta}{2-\sin 2\theta}\right)\right)\sec^2\theta d\theta.$

5. Evaluate $\int_0^{\sqrt{3}} \frac{1}{1+x^2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx.$

6. If $f(x) = \begin{cases} 1-|x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$, and $g(x) = f(x-1) + f(x+1)$, find the value of $\int_{-3}^{5} g(x)dx$.

7. f, g, h are continuous in $[0, a]$, $f(a-x) = f(x)$, $g(a-x) = -g(x)$, $3h(x) - 4h(a-x) = 5$, then prove that
$$\int_0^a f(x)\, g(x)\, h(x)\, dx = 0.$$

8. Determine a positive integer n such that
$$\int_0^{\pi/2} x^n \sin x dx = \frac{3}{4}\left(\pi^2 - 8\right).$$

9. If $f(x) = \frac{\sin x}{x} \forall x \in (0, \pi]$, prove that
$$\frac{\pi}{2}\int_0^{\pi/2} f(x)\, f\left(\frac{\pi}{2}-x\right)dx = \int_0^{\pi} f(x)dx.$$

10. Let $f(x)$ be a continuous function, $\forall\, x \in R$, except at $x = 0$, such that $\int_0^a f(x)dx$, $a \in R^+$ exists. If $g(x) = \int_x^a \frac{f(t)}{t}dt$, prove that $\int_0^a f(x)dx = \int_0^a g(x)dx.$

11. Let $\int_x^{x+p} f(t)dt$ be independent of x and $I_1 = \int_0^p f(t)dt$, $I_2 = \int_{10}^{p^n+10} f(z)dz$ for some p, where $n \in N$. Then evaluate $\frac{I_2}{I_1}$.

12. If $f(x+f(y)) = f(x) + y\ \forall\ x, y \in R$ and $f(0) = 1$, then prove that $\int_0^2 f(2-x)dx = 2\int_0^1 f(x)dx$.

13. Suppose f is a real-valued differentiable function defined on $[1, \infty)$ with $f(1) = 1$. Moreover, suppose that f satisfies $f'(x) = \frac{1}{x^2 + f^2(x)}$. Show that $f(x) < 1 + \frac{\pi}{4}\ \forall\ x \ge 1$.

14. If $x\int_0^x \sin(f(t))dt = (x+2)\int_0^x t\sin(f(t))dt$, where $x > 0$, then show that $f'(x)\cot f(x) + \frac{3}{1+x} = 0$.

15. Evaluate $\int_0^2 \frac{dx}{(17+8x-4x^2)[e^{6(1-x)}+1]}$.

16. If $\int_0^x [x]dx = \int_0^{[x]} x\, dx$, then prove that either x is purely fractional or x is such that $\{x\} = \frac{1}{2}$ (where [.] and {.} denote the greatest integer and fractional part, respectively).

17. Let f be a continuous function on $[a, b]$.
If $F(x) = \left(\int_a^x f(t)\, dt - \int_x^b f(t)dt\right)(2x-(a+b))$, then prove that there exist some $c \in (a, b)$ such that $\int_a^c f(t)\, dt - \int_c^b f(t)\, dt = f(c)\,(a+b-2c)$.

18. If $\int_a^b |\sin x|\, dx = 8$ and $\int_0^{a+b} |\cos x|\, dx = 9$, then find the value of $\int_a^b x\sin x dx$.

19. $f(x)$ is a continuous and bijective function on R. If $\forall\ t \in R$ area bounded by $y = f(x)$, $x = a - t$, $x = a$ and x-axis is equal to area bounded by $y = f(x)$, $x = a + t$, $x = a$ and x-axis, then prove that $\int_{-\lambda}^{\lambda} f^{-1}(x)dx = 2a\lambda$ (given that $f(a) = 0$).

Objective Type *Solutions on page 8.48*

Each question has four choices a, b, c and d, out of which *only one* is correct.

1. If $S_n = \left[\frac{1}{1+\sqrt{n}} + \frac{1}{2+\sqrt{2n}} + \cdots + \frac{1}{n+\sqrt{n^2}}\right]$, then $\lim_{n\to\infty} S_n$ is equal to
   a. $\log 2$ b. $\log 4$
   c. $\log 8$ d. None of these

2. The value of $\lim_{n\to\infty} \sum_{r=1}^{4n} \frac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r}+4\sqrt{n}\right)^2}$ is equal to
   a. $\frac{1}{35}$ b. $\frac{1}{14}$ c. $\frac{1}{10}$ d. $\frac{1}{5}$

3. Which of the following is incorrect?
   a. $\int_{a+c}^{b+c} f(x)\,dx = \int_a^b f(x+c)dx$
   b. $\int_{ac}^{bc} f(x)dx = c\int_a^b f(cx)dx$
   c. $\int_{-a}^{a} f(x)dx = \frac{1}{2}\int_{-a}^{a} (f(x)+f(-x)\,dx$
   d. None of these

4. The solution for x of the equation $\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{2}$ is
   a. π b. $\frac{\sqrt{3}}{2}$
   c. $2\sqrt{2}$ d. None of these

5. $\int_{-1}^{1/2} \frac{e^x\left(2-x^2\right)dx}{(1-x)\sqrt{1-x^2}}$ is equal to
   a. $\frac{\sqrt{e}}{2}\left(\sqrt{3}+1\right)$ b. $\frac{\sqrt{3e}}{2}$
   c. $\sqrt{3e}$ d. $\sqrt{\frac{e}{3}}$

6. The value of the integral $\int_{-\pi}^{\pi} \sin\, mx \sin nx\, dx$ for $m \neq n$ $(m, n \in I)$ is
   a. 0 b. π
   c. $\pi/2$ d. 2π

7. The value of the integral $\int_0^\infty \frac{x\log x}{(1+x^2)^2}dx$ is
   a. 0 b. $\log 7$
   c. $5\log 1$? d. None of these

8. $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$ is
   a. $\frac{a^2}{4}$ b. $\frac{\pi}{2}$ c. $\frac{\pi}{4}$ d. π

9. $\int_0^{\pi/2} |\sin x - \cos x|\, dx$ is equal to
   a. 0 b. $2\left(\sqrt{2}-1\right)$
   c. $\sqrt{2}-1$ d. $2\left(\sqrt{2}+1\right)$

10. $\int_0^\pi \frac{x\tan x}{\sec x + \cos x}dx$ is
    a. $\frac{\pi^2}{4}$ b. $\frac{\pi^2}{2}$ c. $\frac{3\pi^2}{2}$ d. $\frac{\pi^2}{3}$

11. If $\int_{-1}^{4} f(x)dx = 4$ and $\int_2^4 (3-f(x))dx = 7$, then the value of $\int_2^{-1} f(x)dx$ is
    a. 2 b. –3
    c. –5 d. None of these

12. If $\int_0^1 e^{x^2}(x-\alpha)dx = 0$, then
    a. $1<\alpha<2$ b. $\alpha<0$ c. $0<\alpha<1$ d. $\alpha=0$

13. If $\int_0^\infty \frac{x^2\,dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)\left(x^2+c^2\right)} = \frac{\pi}{2(a+b)(b+c)(c+a)}$, then the value of $\int_0^\infty \frac{dx}{\left(x^2+4\right)\left(x^2+9\right)}$ is
    a. $\frac{\pi}{60}$ b. $\frac{\pi}{20}$ c. $\frac{\pi}{40}$ d. $\frac{\pi}{80}$

14. The value of the integral $\int_0^1 \frac{dx}{x^2+2x\cos\alpha+1}$ is equal to
    a. $\sin\alpha$ b. $\alpha\sin\alpha$ c. $\frac{\alpha}{2\sin\alpha}$ d. $\frac{\alpha}{2}\sin\alpha$

15. The value of $\int_1^e \frac{1+x^2\ln x}{x+x^2\ln x}dx$ is
    a. e b. $\ln(1+e)$
    c. $e+\ln(1+e)$ d. $e-\ln(1+e)$

16. The value of the integral $\int_0^{1/\sqrt{3}} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$ must be
    a. $\frac{\pi}{2\sqrt{2}}$ b. $\frac{\pi}{4\sqrt{2}}$
    c. $\frac{\pi}{8\sqrt{2}}$ d. None of these

17. The value of the integral $\int_0^{\log 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}dx$ is

a. $3+2\pi$
b. $4-\pi$
c. $2+\pi$
d. None of these

18. $\int_0^\infty \frac{xdx}{(1+x)(1+x^2)}$ is equal to

a. $\frac{\pi}{4}$
b. $\frac{\pi}{2}$
c. π
d. None of these

19. $\int_0^\infty \frac{dx}{[x+\sqrt{x^2+1}]^3}$ is equal to

a. $\frac{3}{8}$
b. $\frac{1}{8}$
c. $-\frac{3}{8}$
d. None of these

20. Given $\int_0^{\pi/2} \frac{dx}{1+\sin x+\cos x} = \log 2$, then the value of the definite integral $\int_0^{\pi/2} \frac{\sin x}{1+\sin x+\cos x}dx$ is equal to

a. $\frac{1}{2}\log 2$
b. $\frac{\pi}{2}-\log 2$
c. $\frac{\pi}{4}-\frac{1}{2}\log 2$
d. $\frac{\pi}{2}+\log 2$

21. If $I_1 = \int_{-100}^{101} \frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}$ and $I_2 = \int_{-100}^{101} \frac{dx}{5+2x-2x^2}$, then $\frac{I_1}{I_2}$ is

a. 2
b. $\frac{1}{2}$
c. 1
d. $-\frac{1}{2}$

22. If $f(x)=\frac{e^x}{1+e^x}$, $I_1 = \int_{f(-a)}^{f(a)} xg(x(1-x))\,dx$ and $I_2 = \int_{f(-a)}^{f(a)} g(x(1-x))dx$, then the value of $\frac{I_2}{I_1}$ is

a. -1
b. -2
c. 2
d. 1

23. If $f(y)=e^y, g(y)=y, y>0$ and $F(t)=\int_0^t f(t-y)g(y)\,dt$, then

a. $F(t)=e^t-(1+t)$
b. $F(t)=te^t$
c. $F(t)=te^{-t}$
d. $F(t)=1-e^t(1+t)$

24. The value of the definite integral $\int_0^{\sqrt{\ln\left(\frac{\pi}{2}\right)}} \cos(e^{x^2})\,2xe^{x^2}dx$ is

a. 1
b. $1+(\sin 1)$
c. $1-(\sin 1)$
d. $(\sin 1)-1$

25. The value of $\int_1^{\frac{1+\sqrt{5}}{2}} \frac{x^2+1}{x^4-x^2+1}\log\left(1+x-\frac{1}{x}\right)dx$ is

a. $\frac{\pi}{8}\log_e 2$
b. $\frac{\pi}{2}\log_e 2$
c. $-\frac{\pi}{2}\log_e 2$
d. None of these

26. If $f(x)$ satisfies the condition of Rolle's theorem in [1, 2], then $\int_1^2 f'(x)\,dx$ is equal to

a. 1
b. 3
c. 0
d. None of these

27. The value of the integral $\int_{-1}^3\left(\tan^{-1}\frac{x}{x^2+1}+\tan^{-1}\frac{x^2+1}{x}\right)dx$ is equal to

a. π
b. 2π
c. 4π
d. None of these

28. If $P(x)$ is a polynomial of the least degree that has a maximum equal to 6 at $x=1$, and a minimum equal to 2 at $x=3$, then $\int_0^1 P(x)dx$ equals

a. $\frac{17}{4}$
b. $\frac{13}{4}$
c. $\frac{19}{4}$
d. $\frac{5}{4}$

29. The numbers of possible continuous $f(x)$ defined in [0, 1] for which $I_1 = \int_0^1 f(x)\,dx = 1$, $I_2 = \int_0^1 x\,f(x)dx = a$, $I_3 = \int_0^1 x^2\,f(x)\,dx = a^2$ is/are

a. 1
b. ∞
c. 2
d. 0

30. The value of the definite integral $\int_0^{\pi/2}\sqrt{\tan x}\,dx$ is

a. $\sqrt{2}\pi$
b. $\frac{\pi}{\sqrt{2}}$
c. $2\sqrt{2}\pi$
d. $\frac{\pi}{2\sqrt{2}}$

31. Suppose that $F(x)$ is an anti-derivative of $f(x)=\frac{\sin x}{x}$, where $x>0$, then $\int_1^3 \frac{\sin 2x}{x}dx$ can be expressed as

a. $F(6)-F(2)$
b. $\frac{1}{2}(F(6)-F(2))$
c. $\frac{1}{2}(F(3)-F(1))$
d. $2(F(6)-F(2))$

32. If $\int_0^1 \cot^{-1}(1-x+x^2)dx = \lambda\int_0^1 \tan^{-1}x\,dx$, then λ is equal to

a. 1
b. 2
c. 3
d. 4

33. The value of the integral $\int_{-3\pi/4}^{5\pi/4} \frac{(\sin x+\cos x)}{e^{x-\pi/4}+1}dx$ is

a. 0 b. 1
c. 2 d. None of these

34. $\int_{2-a}^{2+a} f(x)dx$ is equal to (where $f(2-\alpha)=f(2+\alpha)\ \forall\ \alpha\in R$)

a. $2\int_{2}^{2+a} f(x)dx$ b. $2\int_{0}^{a} f(x)dx$

c. $2\int_{2}^{2} f(x)dx$ d. None of these

35. The value of the integral $\int_0^1 e^{x^2}dx$ lies in the interval

a. $(0, 1)$ b. $(-1, 0)$
c. $(1, e)$ d. None of these

36. $I_1=\int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx$, $I_2=\int_{-\pi/4}^{\pi/4} \ln(\sin x+\cos x)\,dx$, then

a. $I_1=2I_2$ b. $I_2=2I_1$
c. $I_1=4I_2$ d. $I_2=4I_1$

37. If $I_1 = \int_0^{\pi/2} \frac{\cos^2 x}{1+\cos^2 x}dx$, $I_2 = \int_0^{\pi/2} \frac{\sin^2 x}{1+\sin^2 x}dx$,

$I_3=\int_0^{\pi/2} \frac{1+2\cos^2 x\sin^2 x}{4+2\cos^2 x\sin^2 x}dx$, then

a. $I_1=I_2>I_3$ b. $I_3>I_1=I_2$
c. $I_1=I_2=I_3$ d. None of these

38. If $f(x)$ is continuous for all real values of x, then $\sum_{r=1}^{n}\int_0^1 f(r-1+x)\,dx$ is equal to

a. $\int_0^n f(x)dx$ b. $\int_0^1 f(x)dx$

c. $n\int_0^1 f(x)dx$ d. $(n-1)\int_0^1 f(x)dx$

39. $I_1=\int_0^{\frac{\pi}{2}} \frac{\sin x-\cos x}{1+\sin x\cos x}dx$, $I_2=\int_0^{2\pi} \cos^6 x\,dx$,

$I_3=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x\,dx$, $I_4=\int_0^1 \ln\left(\frac{1}{x}-1\right)dx$, then

a. $I_2=I_3=I_4=0, I_1\neq 0$
b. $I_1=I_2=I_3=0, I_4\neq 0$
c. $I_1=I_3=I_4=0, I_2\neq 0$
d. $I_1=I_2=I_3=0, I_4\neq 0$

40. If $f(x)$ and $g(x)$ are continuous functions, the $\int_{\ln\lambda}^{\ln 1/\lambda} \frac{f(x^2/4)[f(x)-f(-x)]}{g(x^2/4)[g(x)+g(-x)]}dx$ is

a. dependent on λ b. a non-zero constant
c. zero d. None of these

41. $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}dx$ is equal to

a. π b. π^2
c. 0 d. None of these

42. $f(x)>0\ \forall\ x\in R$ and is bounded. If $\lim_{n\to\infty}\left[\int_0^a \frac{f(x)dx}{f(x)+f(a-x)} + a\int_a^{2a}\frac{f(x)dx}{f(x)+f(3a-x)} + a^2\int_{2a}^{3a}\frac{f(x)dx}{f(x)+f(5a-x)} + \cdots + a^{n-1}\int_{(n-1)a}^{na}\frac{f(x)dx}{f(x)+f[(2n-1)a-x]}\right]=7/5$ (where $a<1$), then a is equal to

a. $\frac{2}{7}$ b. $\frac{1}{7}$ c. $\frac{14}{19}$ d. $\frac{9}{14}$

43. If $f(x)=\int_0^{\pi}\frac{t\sin t\,dt}{\sqrt{1+\tan^2 x\,\sin^2 t}}$ for $0<x<\frac{\pi}{2}$, then

a. $f(0^+)=-\pi$

b. $f\left(\frac{\pi}{4}\right)=\frac{\pi^2}{8}$

c. f is continuous and differentiable in $\left(0,\frac{\pi}{2}\right)$

d. f is continuous but not differentiable in $\left(0,\frac{\pi}{2}\right)$

44. If $\int_{-\pi/4}^{3\pi/4}\frac{e^{\pi/4}dx}{(e^x+e^{\pi/4})(\sin x+\cos x)}=k\int_{-\pi/2}^{\pi/2}\sec x\,dx$, then the value of k is

a. $\frac{1}{2}$ b. $\frac{1}{\sqrt{2}}$ c. $\frac{1}{2\sqrt{2}}$ d. $-\frac{1}{\sqrt{2}}$

45. $\int_{-\pi/3}^{0}\left[\cot^{-1}\left(\frac{2}{2\cos x-1}\right)+\cot^{-1}\left(\cos x-\frac{1}{2}\right)\right]dx$ is equal to

a. $\frac{\pi^2}{6}$ b. $\frac{\pi^2}{3}$ c. $\frac{\pi^2}{8}$ d. $\frac{3\pi^2}{8}$

46. $\int_0^{\infty}\left(\frac{\pi}{1+\pi^2x^2}-\frac{1}{1+x^2}\right)\log x\,dx$ is equal to

a. $-\frac{\pi}{2}\ln\pi$ b. 0

c. $\frac{\pi}{2}\ln 2$ d. None of these

47. If $f(x) = \cos(\tan^{-1}x)$, then the value of the integral $\int_0^1 xf''(x)\,dx$ is

a. $\frac{3-\sqrt{2}}{2}$ b. $\frac{3+\sqrt{2}}{2}$

c. 1 d. $1-\frac{3}{2\sqrt{2}}$

48. The equation of the curve is $y=f(x)$. The tangents at $[1, f(1)]$, $[2, f(2)]$ and $[3, f(3)]$ make angle $\frac{\pi}{6}, \frac{\pi}{3}$ and $\frac{\pi}{4}$, respectively, with the positive direction of x-axis, then the value of $\int_2^3 f'(x) f''(x) dx + \int_1^3 f''(x) dx$ is equal to

a. $-1/\sqrt{3}$ b. $1/\sqrt{3}$
c. 0 d. None of these

49. The value of $\int_1^e \left(\frac{\tan^{-1} x}{x} + \frac{\log x}{1+x^2}\right) dx$ is

a. $\tan e$ b. $\tan^{-1} e$
c. $\tan^{-1}(1/e)$ d. None of these

50. If $f(\pi)=2$ and $\int_0^\pi (f(x)+f''(x)) \sin x\, dx = 5$, then $f(0)$ is equal to (it is given that $f(x)$ is continuous in $[0, \pi]$)

a. 7 b. 3 c. 5 d. 1

51. If $\int_1^2 e^{x^2} dx = a$, then $\int_e^{e^4} \sqrt{\ln x}\, dx$ is equal to

a. $2e^4 - 2e - a$ b. $2e^4 - e - a$
c. $2e^4 - e - 2a$ d. $e^4 - e - a$

52. $\int_{-\pi/2}^{\pi/2} \frac{e^{|\sin x|} \cos x}{(1+e^{\tan x})} dx$ is equal to

a. $e+1$ b. $1-e$
c. $e-1$ d. None of these

53. The value of the expression $\frac{\int_0^a x^4 \sqrt{a^2 - x^2}\, dx}{\int_0^a x^2 \sqrt{a^2 - x^2}\, dx}$ is equal to

a. $\frac{a^2}{6}$ b. $\frac{3a^2}{2}$ c. $\frac{3a^2}{4}$ d. $\frac{a^2}{2}$

54. If $A = \int_0^\pi \frac{\cos x}{(x+2)^2} dx$, then $\int_0^{\pi/2} \frac{\sin 2x}{x+1} dx$ is equal to

a. $\frac{1}{2} + \frac{1}{\pi+2} - A$ b. $\frac{1}{\pi+2} - A$
c. $1 + \frac{1}{\pi+2} - A$ d. $A - \frac{1}{2} - \frac{1}{\pi+2}$

55. $\int_0^4 \frac{(y^2-4y+5)\sin(y-2)dy}{[2y^2-8y+11]}$ is equal to

a. 0 b. 2
c. -2 d. None of these

56. $\int_{\sin\theta}^{\cos\theta} f(x\tan\theta)dx$ $\left(\text{where } \theta \neq \frac{n\pi}{2}, n \in I\right)$ is equal to

a. $-\cos\theta \int_1^{\tan\theta} f(x\sin\theta)dx$

b. $-\tan\theta \int_{\cos\theta}^{\sin\theta} f(x)dx$

c. $\sin\theta \int_1^{\tan\theta} f(x\cos\theta)dx$

d. $\frac{1}{\tan\theta} \int_{\sin\theta}^{\sin\theta\tan\theta} f(x)dx$

57. Let $I_1 = \int_0^1 \frac{e^x dx}{1+x}$ and $I_2 = \int_0^1 \frac{x^2 dx}{e^{x^3}(2-x^3)}$, then $\frac{I_1}{I_2}$ is equal to

a. $3/e$ b. $e/3$ c. $3e$ d. $1/3e$

58. If $\int_0^1 \frac{\sin t}{1+t} dt = \alpha$, then the value of the integral $\int_{4\pi-2}^{4\pi} \frac{\sin \frac{t}{2}}{4\pi+2-t} dt$ is

a. 2α b. -2α c. α d. $-\alpha$

59. $\int_0^1 \frac{\tan^{-1} x}{x} dx$ is equal to

a. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx$ b. $\int_0^{\frac{\pi}{2}} \frac{x}{\sin x} dx$

c. $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx$ d. $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{x}{\sin x} dx$

60. If $I_k = \int_1^e (\ln x)^k dx$ (where $k \in I^+$), then I_4 equals

a. $9e-24$ b. $12-2e$ c. $24-9e$ d. $6e-12$

61. If $I(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$, then $(m, n \in I, m, n \geq 0)$

a. $I(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m-n}} dx$

b. $I(m,n) = \int_0^\infty \frac{x^m}{(1+x)^{m+n}} dx$

c. $I(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$

d. $I(m,n) = \int_0^\infty \frac{x^n}{(1+x)^{m+n}} dx$

62. The value of $\int_0^\pi \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} dx$ is, $n \in I, n \geq 0$

a. $\frac{\pi}{2}$ b. 0 c. π d. 2π

63. The value of the definite integral $\int_0^{\pi/2} \frac{\sin 5x}{\sin x} dx$ is

a. 0 b. $\frac{\pi}{2}$ c. π d. 2π

64. If $I_n=\int_0^{\pi} e^x(\sin x)^n\,dx$, then $\frac{I_3}{I_1}$ is equal to

a. 3/5 b. 1/5 c. 1 d. 2/5

65. Given $I_m = \int_1^e (\log x)^m\, dx$. If $\frac{I_m}{K}+\frac{I_{m-2}}{L}=e$, then the values of K and L are

a. $\frac{1}{1-m}, \frac{1}{m}$ b. $(1-m), \frac{1}{m}$

c. $\frac{1}{1-m}, \frac{m(m-2)}{m-1}$ d. $\frac{m}{m-1}, m-2$

66. Let $f(x)=\min(\{x\},\{-x\})\ \forall\ x\in R$, where $\{\cdot\}$ denotes the fractional part of x, then $\int_{-100}^{100} f(x)\,dx$ is equal to

a. 50 b. 100

c. 200 d. None of these

67. $\int_1^4 \{x-0.4\}dx$ equals (where $\{x\}$ is a fractional part of x)

a. 13 b. 6.3 c. 1.5 d. 7.5

68. The value of $\int_1^a [x]\,f'(x)dx$, where $a>1$, where $[x]$ denotes the greatest integer not exceeding x is

a. $af(a)-\{f(1)+f(2)+\cdots+f([a])\}$
b. $[a]f(a)-\{f(1)+f(2)+\cdots+f([a])\}$
c. $[a]f([a])-\{f(1)+f(2)+\cdots+fA\}$
d. $af([a])-\{f(1)+f(2)+\cdots+fA\}$

69. The value of $\int_0^x [\cos t]dt$, $x\in\left[(4n+1)\frac{\pi}{2},(4n+3)\frac{\pi}{2}\right]$ and $n\in N$ is equal to (where $[\cdot]$ represents greatest integer function

a. $\frac{\pi}{2}(2n-1)-2x$ b. $\frac{\pi}{2}(2n-1)+x$

c. $\frac{\pi}{2}(2n+1)-x$ d. $\frac{\pi}{2}(2n+1)+x$

70. If $f(x)=\int_0^1 \frac{dt}{1+|x-t|}$, then $f'\left(\frac{1}{2}\right)$ is equal to

a. 0 b. $\frac{1}{2}$

c. 1 d. None of these

71. Let $f(x)=\int_2^x \frac{dt}{\sqrt{1+t^4}}$ and g be the inverse of f. Then the value of $g'(0)$ is

a. 1 b. 17

c. $\sqrt{17}$ d. None of these

72. The value of the definite integral $\int_2^4 (x(3-x)(4+x)(6-x)(10-x)+\sin x)\,dx$ equals

a. $\cos 2+\cos 4$ b. $\cos 2-\cos 4$

c. $\sin 2+\sin 4$ d. $\sin 2-\sin 4$

73. If $x=\int_c^{\sin t}\sin^{-1} z\,dz$, $y=\int_k^{\sqrt{t}}\frac{\sin z^2}{z}dz$, then $\frac{dy}{dx}$ is equal to

a. $\frac{\tan t}{2t}$ b. $\frac{\tan t}{t^2}$ c. $\frac{\tan t}{2t^2}$ d. $\frac{\tan t^2}{2t^2}$

74. If $f(x)=\cos x-\int_0^x (x-t)\,f(t)\,dt$, then $f''(x)+f(x)$ is equal to

a. $-\cos x$ b. $-\sin x$

c. $\int_0^x (x-t)\ f(t)\,dt$ d. 0

75. A function f is continuous for all x (and not every where zero) such that $f^2(x)=\int_0^x f(t)\frac{\cos t}{2+\sin t}dt$, then $f(x)$ is

a. $\frac{1}{2}\ln\left(\frac{x+\cos x}{2}\right)$; $x\neq 0$

b. $\frac{1}{2}\ln\left(\frac{3}{2+\cos x}\right)$; $x\neq 0$

c. $\frac{1}{2}\ln\left(\frac{2+\sin x}{2}\right)$; $x\neq n\pi,\ n\in I$

d. $\frac{\cos x+\sin x}{2+\sin x}$; $x\neq n\pi+\frac{3\pi}{4},\ n\in I$

76. $\lim_{x\to 0}\frac{1}{x}\left[\int_y^a e^{\sin^2 t}dt-\int_{x+y}^a e^{\sin^2 t}dt\right]$ is equal to

a. $e^{\sin^2 y}$ b. $\sin 2y e^{\sin^2 y}$

c. 0 d. None of these

77. $f(x)=\int_1^x \frac{e^t}{t}dt$, where $x\in R^+$. Then the complete set of values of x for which $f(x)\le \ln x$ is

a. $(0,1]$ b. $[1,\infty)$

c. $(0,\infty)$ d. None of these

78. If $\int_0^x f(t)dt = x+\int_x^1 tf(t)dt$, then the value of $f(1)$ is

a. 1/2 b. 0 c. 1 d. −1/2

79. If $\int_{\cos x}^1 t^2 f(t)dt = 1-\cos x\ \ \forall\ x\in\left(0,\frac{\pi}{2}\right)$, then the value of $\left[f\left(\frac{\sqrt{3}}{4}\right)\right]$ is ([.] denotes the greatest integer function)

a. 4 b. 5

c. 6 d. −7

80. If $\int_0^{f(x)} t^2\,dt = x\cos\pi x$, then $f'(9)$ is

a. $-\frac{1}{9}$ b. $-\frac{1}{3}$

c. $\frac{1}{3}$ d. non-existent

81. If $f(x) = 1 + \frac{1}{x}\int_1^x f(t)dt$, then the value of $f(e^{-1})$ is

a. 1
b. 0
c. -1
d. None of these

82. If $A = \int_0^1 x^{50}(2-x)^{50}\,dx$; $B = \int_0^1 x^{50}(1-x)^{50}\,dx$, which of the following is true ?

a. $A = 2^{50}B$
b. $A = 2^{-50}B$
c. $A = 2^{100}B$
d. $A = 2^{-100}B$

83. The value of $\int_0^1 \left(\prod_{r=1}^{n}(x+r)\right)\left(\sum_{k=1}^{n}\frac{1}{x+k}\right)dx$ equals

a. n
b. $n!$
c. $(n+1)!$
d. $n \cdot n!$

84. If $I = \int_{-20\pi}^{20\pi} |\sin x|[\sin x]dx$ (where [.] denotes the greatest integer function), then the value of I is

a. -40
b. 40
c. 20
d. -20

85. Given that f satisfies $|f(u) - f(v)| \le |u - v|$ for u and v in $[a, b]$, then $\left|\int_a^b f(x)\,dx - (b-a)f(a)\right| \le$

a. $\frac{(b-a)}{2}$
b. $\frac{(b-a)^2}{2}$
c. $(b-a)^2$
d. None of these

86. $\int_0^\infty \frac{\sin^2 x}{x^2}dx$ must be same as

a. $\int_0^\infty \frac{\sin x}{x}dx$
b. $\left(\int_0^\infty \frac{\sin x}{x}dx\right)^2$
c. $\int_0^\infty \frac{\cos^2 x}{x^2}dx$
d. None of these

87. If $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$, then $\int_0^\infty \frac{\sin^3 x}{x}dx$ is equal to

a. $\pi/2$
b. $\pi/4$
c. $\pi/6$
d. $3\pi/2$

88. $\int_0^x [\sin t]dt$, where $x \in (2n\pi, (2n+1)\pi)$, $n \in N$ and [.] denotes the greatest integer function, is equal to

a. $-n\pi$
b. $-(n+1)\pi$
c. $-2n\pi$
d. $-(2n+1)\pi$

89. $f(x)$ is a continuous function for all real values of x and satisfies $\int_0^x f(t)dt = \int_x^1 t^2 f(t)\,dt + \frac{x^{16}}{8} + \frac{x^6}{3} + a$, then the value of a is equal to

a. $-\frac{1}{24}$
b. $\frac{17}{168}$
c. $\frac{1}{7}$
d. $-\frac{167}{840}$

90. $\int_0^x \frac{2^t}{2^{[t]}}dt$, where [.] denotes the greatest integer function, and $x \in R^+$, is equal to

a. $\frac{1}{\ln 2}([x] + 2^{\{x\}} - 1)$
b. $\frac{1}{\ln 2}([x] + 2^{\{x\}})$
c. $\frac{1}{\ln 2}([x] - 2^{\{x\}})$
d. $\frac{1}{\ln 2}([x] + 2^{\{x\}} + 1)$

91. $f(x)$ is a continuous function for all real values of x and satisfies $\int_n^{n+1} f(x)\,dx = \frac{n^2}{2}\ \forall\ n \in I$, then $\int_{-3}^{5} f(|x|)\,dx$ is equal to

a. 19/2
b. 35/2
c. 17/2
d. None of these

92. The value of $\int_{1/e}^{\tan x}\frac{t\,dt}{1+t^2} + \int_{1/e}^{\cot x}\frac{dt}{t(1+t^2)}$, where $x \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$, is equal to

a. 0
b. 2
c. 1
d. None of these

93. Let $I_1 = \int_{-2}^{2}\frac{x^6 + 3x^5 + 7x^4}{x^4 + 2}dx$ and $I_2 = \int_{-3}^{1}\frac{2(x+1)^2 + 11(x+1) + 14}{(x+1)^4 + 2}dx$, then the value of $I_1 + I_2$ is

a. 8
b. 200/3
c. 100/3
d. None of these

94. For $x \in R$ and a continuous function f, let $I_1 = \int_{\sin^2 t}^{1+\cos^2 t} x f\{x(2-x)\}dx$ and $I_2 = \int_{\sin^2 t}^{1+\cos^2 t} f\{x(2-x)\}dx$. Then $\frac{I_1}{I_2}$ is

a. -1
b. 1
c. 2
d. 3

95. Given a function f: [0, 4] → R is differentiable, then for some $\alpha, \beta \in (0, 2)$, $\int_0^4 f(t)dt$ equals to

a. $f(\alpha^2) + f(\beta^2)$
b. $2\alpha f(\alpha^2) + 2\beta f(\beta^2)$
c. $\alpha f(b^2) + \beta f(\alpha^2)$
d. $f(\alpha)f(\beta)[f(\alpha) + f(\beta)]$

96. $\int_{-3}^{3} x^8\{x^{11}\}\,dx$ is equal to (where {.} is the fractional part of x)

a. 3^8
b. 3^7
c. 3^9
d. None of these

97. If $S = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\frac{1}{3} + \left(\frac{1}{2}\right)^3\frac{1}{4} + \left(\frac{1}{2}\right)^4\frac{1}{5} + \cdots$, then

a. $S = \ln 8 - 2$
b. $S = \ln\frac{4}{e}$
c. $S = \ln 4 + 1$
d. None of these

98. Let $f: R \to R$ be a continuous function and $f(x) = f(2x)$ is true $\forall\ x \in R$. If $f(1) = 3$, then the value of $\int_{-1}^{1} f(f(x))dx$ is equal to
 a. 6 b. 0 c. $3f(3)$ d. $2f(0)$

99. $\int_{-1}^{2}\left[\frac{[x]}{1+x^2}\right]dx$, where $[\cdot]$ denotes the greatest integer function, is equal to
 a. -2 b. -1
 c. zero d. None of these

100. f is an odd function. It is also known that $f(x)$ is continuous for all values of x and is periodic with period 2. If $g(x) = \int_0^x f(t)\,dt$, then
 a. $g(x)$ is odd b. $g(n) = 0, n \in N$
 c. $g(2n) = 0, n \in N$ d. $g(x)$ is non-periodic

101. $\int_0^x |\sin t|dt$, where $x \in (2n\pi, (2n+1)\pi)$, where $n \in N$, is equal to
 a. $4n - \cos x$ b. $4n - \sin x$
 c. $4n + 1 - \cos x$ d. $4n - 1 - \cos x$

102. If $f(x) = \int_{-1}^{x}|t|\,dt$, , then for any $x \ge 0, f(x)$ equals
 a. $\frac{1}{2}(1-x^2)$ b. $\frac{1}{2}x^2$
 c. $\frac{1}{2}(1+x^2)$ d. None of these

103. If $g(x) = \int_0^x (|\sin t| + |\cos t|)\,dt$, then $g\left(x+\frac{\pi n}{2}\right)$ is equal to, where $n \in N$
 a. $g(x) + g(\pi)$ b. $g(x) + g\left(\frac{n\pi}{2}\right)$
 c. $g(x) + g\left(\frac{\pi}{2}\right)$ d. None of these

104. The value of $\int_{-2}^{1}\left[x\left[1+\cos\left(\frac{\pi x}{2}\right)\right]+1\right]dx$, where [.] denotes the greatest integer function, is
 a. 1 b. 1/2
 c. 2 d. None of these

105. If $a > 0$ and $A = \int_0^a \cos^{-1} x\,dx$,
then $\int_{-a}^{a} (\cos^{-1} x - \sin^{-1}\sqrt{1-x^2})dx = \pi a - \lambda A$, then λ is
 a. 0 b. 2
 c. 3 d. None of these

106. The value of $\int_a^b (x-a)^3(b-x)^4\,dx$ is
 a. $\frac{(b-a)^4}{6^4}$ b. $\frac{(b-a)^8}{280}$
 c. $\frac{(b-a)^7}{7^3}$ d. None of these

107. If $\int_0^t \frac{bx\cos 4x - a\sin 4x}{x^2}dx = \frac{a\sin 4t}{t} - 1$, where $0 < t < \frac{\pi}{4}$, then the values of a, b are equal to
 a. $\frac{1}{4}, 1$ b. $-1, 4$ c. $2, 2$ d. $2, 4$

108. If $\lambda = \int_0^1 \frac{e^t}{1+t}$, then $\int_0^1 e^t \log_e(1+t)dt$ is equal to
 a. 2λ b. $e\log_e 2 - \lambda$
 c. λ d. $e\log_e 2 + \lambda$

109. Let f be integrable over $[0, a]$ for any real value of a. If
$I_1 = \int_0^{\pi/2} \cos\theta\, f\left(\sin\theta + \cos^2\theta\right)d\theta$ and
$I_2 = \int_0^{\pi/2} \sin 2\theta\, f(\sin\theta + \cos^2\theta)\,d\theta$, then
 a. $I_1 = -2I_2$ b. $I_1 = I_2$ c. $2I_1 = I_2$ d. $I_1 = -I_2$

110. The range of the function $f(x) = \int_{-1}^{1}\frac{\sin x\,dt}{(1-2t\cos x+t^2)}$ is
 a. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ b. $[0, \pi]$
 c. $\{0, \pi\}$ d. $\left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$

111. The value of $\lim_{n\to\infty}\left[\tan\frac{\pi}{2n}\tan\frac{2\pi}{2n}\cdots\tan\frac{n\pi}{2n}\right]^{1/n}$ is
 a. e b. e^2 c. 1 d. e^3

112. If $f'(x) = f(x) + \int_0^1 f(x)dx$, given $f(0) = 1$, then the value of $f(\log_e 2)$ is
 a. $\frac{1}{3+e}$ b. $\frac{5-e}{3-e}$
 c. $\frac{2+e}{e-2}$ d. None of these

113. If $f(x)$ is monotonic differentiable function on $[a, b]$, then $\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(x)dx =$
 a. $bf(a) - af(b)$ b. $bf(b) - af(a)$
 c. $f(a) + f(b)$ d. cannot be found

114. If $\alpha, \beta\ (\beta > \alpha)$ are the roots of $g(x) = ax^2 + bx + c = 0$ and $f(x)$ is an even function, then $\int_\alpha^\beta \frac{e^{f\left(\frac{g(x)}{x-\alpha}\right)}dx}{e^{f\left(\frac{g(x)}{x-\alpha}\right)} + e^{f\left(\frac{g(x)}{x-\beta}\right)}}$ is equal to
 a. $\left|\frac{b}{2a}\right|$ b. $\frac{\sqrt{b^2-4ac}}{|2a|}$
 c. $\left|\frac{b}{a}\right|$ d. None of these

115. If $y^r = \dfrac{n!\,^{n+r-1}C_{r-1}}{r^n}$, where $n = kr$ (k is constant), then $\lim\limits_{r\to\infty} y$ is equal to

a. $(k-1)\log_e(1+k)-k$ b. $(k+1)\log_e(k-1)+k$
c. $(k+1)\log_e(k-1)-k$ d. $(k-1)\log_e(k-1)+k$

116. $\int\limits_3^{10}[\log[x]]dx$ is equal to (where [.] represents the greatest integer function)

a. 9 b. $16-e$ c. 10 d. $10+e$

117. If the function f: [0, 8] $\to R$ is differentiable, then for $0<a, b<2$, $\int_0^8 f(t)dt$ is equal to

a. $3[\alpha^3 f(\alpha^2)+\beta^2 f(\beta^2)]$ b. $3[\alpha^3 f(\alpha)+\beta^3 f(\beta)]$
c. $3[\alpha^2 f(\alpha^3)+\beta^2 f(\beta^3)]$ d. $3[\alpha^2 f(\alpha^2)+\beta^2 f(\beta^2)]$

118. The function f and g are positive and continuous. If f is increasing and g is decreasing, then

$$\int\limits_0^1 f(x)\,[g(x)-g(1-x)]\,dx$$

a. is always non-positive
b. is always non-negative
c. can take positive and negative values
d. None of these

Multiple Correct Answers Type

Solutions on page 8.62

Each question has four choices a, b, c and d, out of which *one or more* answers are correct.

1. A function $f(x)$ which satisfies the relation

$$f(x) = e^x + \int\limits_0^1 e^x f(t)\,dt, \text{ then}$$

a. $f(0)<0$
b. $f(x)$ is a decreasing function
c. $f(x)$ is an increasing function
d. $\int\limits_0^1 f(x)\,dx > 0$

2. Let $f(x) = \int\limits_1^x \dfrac{3^t}{1+t^2}dt$, where $x>0$, then

a. for $0<\alpha<\beta$, $f(\alpha)<f(\beta)$
b. for $0<\alpha<\beta$, $f(\alpha)>f(\beta)$
c. $f(x)+\pi/4<\tan^{-1}x$, $\forall\ x\ge 1$
d. $f(x)+\pi/4>\tan^{-1}x$, $\forall\ x\ge 1$

3. The values of a for which the integral $\int\limits_0^2 |x-a|dx \ge 1$ is satisfied are

a. $[2,\infty)$ b. $(-\infty, 0]$
c. $(0,2)$ d. None of these

4. If $\int_a^b |\sin x|\,dx = 8$ and $\int_0^{a+b}|\cos x|\,dx = 9$, then

a. $a+b=\dfrac{9\pi}{2}$ b. $|a-b|=4\pi$
c. $\dfrac{a}{b}=15$ d. $\int_a^b \sec^2 x dx = 0$

5. Let $I = \int\limits_1^3 \sqrt{3+x^3}\,dx$, then the values of I will lie in the interval

a. [4,6] b. [1,3]
c. $[4, 2\sqrt{30}]$ d. $[\sqrt{15}, \sqrt{30}]$

6. If $g(x) = \int\limits_0^x 2|t|\,dt$, then

a. $g(x)=x|x|$
b. $g(x)$ is monotonic
c. $g(x)$ is differentiable at $x=0$
d. $g'(x)$ is differentiable at $x=0$

7. Let $f: [1,\infty)\to R$ and $f(x) = x\int\limits_1^x \dfrac{e^t}{t}dt - e^x$, then

a. $f(x)$ is an increasing function
b. $\lim\limits_{x\to\infty} f(x)\to\infty$
c. $f'(x)$ has a maxima at $x=e$
d. $f(x)$ is a decreasing function

8. The value of $\int\limits_0^1 \dfrac{2x^2+3x+3}{(x+1)(x^2+2x+2)}dx$ is

a. $\dfrac{\pi}{4}+2\log 2-\tan^{-1}2$ b. $\dfrac{\pi}{4}+2\log 2-\tan^{-1}\dfrac{1}{3}$
c. $2\log 2-\cot^{-1}3$ d. $-\dfrac{\pi}{4}+\log 4+\cot^{-1}2$

9. If $A_n = \int\limits_0^{\pi/2}\dfrac{\sin(2n-1)x}{\sin x}dx$; $B_n = \int\limits_0^{\pi/2}\left(\dfrac{\sin nx}{\sin x}\right)^2 dx$, for $n\in N$, then

a. $A_{n+1}=A_n$ b. $B_{n+1}=B_n$
c. $A_{n+1}-A_n=B_{n+1}$ d. $B_{n+1}-B_n=A_{n+1}$

10. If $f(x) = \int_a^x [f(x)]^{-1}dx$ and $\int_a^1 [f(x)]^{-1}dx = \sqrt{2}$, then

a. $f(2)=2$ b. $f'(2)=1/2$
c. $f^{-1}(2)=2$ d. $\int_0^1 f(x)dx = \sqrt{2}$

11. The value of $\int\limits_0^\infty \dfrac{dx}{1+x^4}$ is

a. same as that of $\int\limits_0^\infty \dfrac{x^2+1dx}{1+x^4}$

b. $\dfrac{\pi}{2\sqrt{2}}$

c. same as that of $\int_0^{\infty} \frac{x^2 dx}{1+x^4}$

d. $\frac{\pi}{\sqrt{2}}$

12. If $f(x) = \int_0^x |t-1|\, dt$, where $0 \le x \le 2$, then

a. range of $f(x)$ is $[0, 1]$

b. $f(x)$ is differentiable at $x = 1$

c. $f(x) = \cos^{-1} x$ has two real roots

d. $f'(1/2) = 1/2$

13. If $I_n = \int_0^{\pi/4} \tan^n x\, dx$ ($n > 1$ and is an integer), then

a. $I_n + I_{n-2} = \frac{1}{n+1}$

b. $I_n + I_{n-2} = \frac{1}{n-1}$

c. $I_2 + I_4, I_4 + I_6, \ldots$, are in H.P.

d. $\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$

14. If $\int_a^b \frac{f(x)}{f(a)+f(a+b-x)} dx = 10$, then

a. $b = 22, a = 2$ b. $b = 15, a = -5$

c. $b = 10, a = -10$ d. $b = 10, a = -2$

15. If $I_n = \int_0^1 \frac{dx}{(1+x^2)^n}$, where $n \in N$, which of the following statements hold good ?

a. $2n I_{n+1} = 2^{-n} + (2n-1)I_n$ b. $I_2 = \frac{\pi}{8} + \frac{1}{4}$

c. $I_2 = \frac{\pi}{8} - \frac{1}{4}$ d. $I_3 = \frac{3\pi}{32} + \frac{1}{4}$

16. If $f(x)$ is integrable over $[1, 2]$, then $\int_1^2 f(x)\, dx$ is equal to

a. $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)$ b. $\lim_{n\to\infty} \frac{1}{n} \sum_{r=n+1}^{2n} f\left(\frac{r}{n}\right)$

c. $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r+n}{n}\right)$ d. $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} f\left(\frac{r}{n}\right)$

17. If $f(2-x) = f(2+x)$ and $f(4-x) = f(4+x)$ for all x and $f(x)$ is a function for which $\int_0^2 f(x)\, dx = 5$, then $\int_0^{50} f(x)\, dx$ is equal to

a. 125 b. $\int_{-4}^{46} f(x) dx$

c. $\int_1^{51} f(x)\, dx$ d. $\int_2^{52} f(x) dx$

18. $\int_0^x \left\{ \int_0^u f(t)\, dt \right\} du$ is equal to

a. $\int_0^x (x-u) f(u) du$ b. $\int_0^x u f(x-u) du$

c. $x \int_0^x f(u) du$ d. $x \int_0^x u f(u-x) du$

19. Which of the following statement(s) is/are true?

a. If function $y = f(x)$ is continuous at $x = c$ such that $f(c) \ne 0$, then $f(x)f(c) > 0$ $\forall x \in (c-h, c+h)$ where h is sufficiently small positive quantity.

b. $\lim_{n\to\infty} \frac{1}{n} \ln\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right) = 1 + 2\ln 2.$

c. Let f be a continuous and non-negative function defined on $[a, b]$. If $\int_a^b f(x)dx = 0$, then $f(x) = 0$ $\forall x \in [a,b]$.

d. Let f be a continuous function defined on $[a, b]$ such that $\int_a^b f(x)dx = 0$, then there exists at least one $c \in (a, b)$ for which $f(c) = 0$.

20. The value of $\int_0^1 e^{x^2-x} dx$ is

a. < 1 b. > 1 c. $> e^{-\frac{1}{4}}$ d. $< e^{-\frac{1}{4}}$

21. If $f(x) = \int_0^x (\cos(\sin t) + \cos(\cos t)) dt$, then $f(x+\pi)$ is

a. $f(x) + f(\pi)$ b. $f(x) + 2f(\pi)$

c. $f(x) + f\left(\frac{\pi}{2}\right)$ d. $f(x) + 2f\left(\frac{\pi}{2}\right)$

Reasoning Type

Solutions on page 8.66

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. if both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1

b. if both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1

c. if STATEMENT 1 is TRUE and STATEMENT 2 is FALSE

d. if STATEMENT 1 is FALSE and STATEMENT 2 is TRUE

1. Let $f(x)$ is continuous and positive for $x \in [a, b]$, $g(x)$ is continuous for $x \in [a,b]$ and $\int_a^b |g(x)|\, dx > \left|\int_a^b g(x) dx\right|$, then

Statement 1: The value of $\int_a^b f(x)g(x)dx$ can be zero.

Statement 2: Equation $g(x) = 0$ has at least one root for $x \in (a,b)$.

2. **Statement 1:** The value of $\int_{-4}^{-5} \sin(x^2 - 3)dx + \int_{-2}^{-1} \sin(x^2 + 12x + 33)$ is zero.

Statement 2: $\int_{-a}^{a} f(x)dx = 0$ if $f(x)$ is an odd function.

3. **Statement 1:** The value of $\int_0^1 \tan^{-1} \frac{2x-1}{(1+x-x^2)} dx = 0.$

Statement 2: $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx.$

4. **Statement 1:** On the interval $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$, the least value of the function $f(x) = \int_{5\pi/4}^{x} (3\sin t + 4\cos t)\,dt$ is 0.

Statement 2: If $f(x)$ is a decreasing function on the interval $[a,b]$, then the least value of $f(x)$ is $f(b)$.

5. Consider the function $f(x)$ satisfying the relation $f(x+1) + f(x+7) = 0$, $\forall\ x \in R$,

Statement 1: The possible least value of t for which $\int_a^{a+t} f(x)dx$ is independent of a is 12.

Statement 2: $f(x)$ is a periodic function.

6. Consider $I_1 = \int_0^{\pi/4} e^{x^2} dx,\ I_2 = \int_0^{\pi/4} e^x dx,$

$I_3 = \int_0^{\pi/4} e^{x^2} \cos x\, dx,\ I_4 = \int_0^{\pi/4} e^{x^2} \sin x\, dx.$

Statement 1: $I_2 > I_1 > I_3 > I_4.$

Statement 2: for $x \in (0, 1)$, $x > x^2$ and $\sin x > \cos x$.

7. **Statement 1:** Let m be any integer. Then the value of $I_m = \int_0^{\pi} \frac{\sin 2mx}{\sin x} dx$ is zero.

Statement 2: $I_1 = I_2 = I_3 = \cdots = I_m.$

8. **Statement 1:** $\int_0^{\pi} \sqrt{1 - \sin^2 x}dx = 0.$

Statement 2: $\int_0^{\pi} \cos x dx = 0.$

9. **Statement 1:** The value of $\int_0^{2\pi} \cos^{99} x dx$ is 0.

Statement 2: $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$, if $f(2a-x) = f(x)$.

10. **Statement 1:** $\int_a^x f(t)dt$ is an even function if $f(x)$ is an odd function.

Statement 2: $\int_a^x f(t)dt$ is an odd function if $f(x)$ is an even function.

11. **Statement 1:** $f(x)$ is symmetrical about $x = 2$, then $\int_{2-a}^{2+a} f(x)dx$ is equal to $2\int_2^{2+a} f(x)dx$.

Statement 2: If $f(x)$ is symmetrical about $x = b$, then $f(b-\alpha) = f(b+\alpha)\ \forall\ (\alpha \in R)$.

12. **Statement 1:** The value of $\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \frac{\pi}{8}\log 2.$

Statement 2: The value of $\int_0^{\pi/2} \log \sin\theta\, d\theta = -\pi \log 2.$

13. **Statement 1:** The value of $\int_0^{\pi} \sin^{100} x \cos^{99} x\, dx$ is zero.

Statement 2: $\int_a^b f(x)\, dx = \int_{a+c}^{b+c} f(x-c)\, dx$ and for odd function $\int_{-a}^{a} f(x)dx = 0$.

14. **Statement 1:** $\int_0^{\pi} x \sin x \cos^2 x\, dx = \frac{\pi}{2}\int_0^{\pi} \sin x \cos^2 x\, dx.$

Statement 2: $\int_a^b x f(x)\, dx = \frac{a+b}{2}\int_a^b f(x)dx.$

15. Let f be a polynomial function of degree n.

Statement 1: There exist a number $x \in [a, b]$ such that $\int_a^x f(t)dt = \int_x^b f(t)\,dt.$

Statement 2: $f(x)$ is a continuous function.

16. **Statement 1:** $\int_0^x |\sin t|\, dt$, for $x \in [0, 2\pi]$ is a non-differentiable function.

Statement 2: $|\sin t|$ is non-differentiable at $x = \pi$.

17. **Statement 1:** If $f(x)$ is continuous on $[a,b]$, then there exists a point $c \in (a, b)$ such that $\int_a^b f(x)dx = f(c)(b-a).$

Statement 2: For $a < b$, if m and M are, respectively, the smallest and greatest values of $f(x)$ on $[a, b]$, then $m(b-a) \le \int_a^b f(x)dx \le (b-a)M.$

Linked Comprehension Type

Solutions on page 8.67

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which *only one* is correct.

For Problems 1–3

$y = f(x)$ satisfies the relation $\int_2^x f(t)dt = \frac{x^2}{2} + \int_x^2 t^2 f(t)dt.$

1. The range of $y = f(x)$ is

a. $[0, \infty)$ b. R

c. $(-\infty, 0]$ d. $\left[-\frac{1}{2}, \frac{1}{2}\right]$

2. The value of $\int_{-2}^{2} f(x)dx$ is

a. 0 b. -2

c. $2\log_e 2$ d. None of these

3. The value of x for which $f(x)$ is increasing is

a. $(-\infty, 1]$ b. $[-1, \infty)$

c. $[-1, 1]$ d. None of these

For Problems 4 – 6

Let $f: R \to R$ be a differentiable function such that

$$f(x) = x^2 + \int_0^x e^{-t} f(x-t)dt.$$

4. $f(x)$ increases for

a. $x > 1$ b. $x < -2$

c. $x > 2$ d. None of these

5. $y = f(x)$ is

a. injective but not surjective

b. surjective but not injective

c. bijective

d. neither injective nor surjective

6. The value of $\int_0^1 f(x)dx$ is

a. $\frac{1}{4}$ b. $-\frac{1}{12}$ c. $\frac{5}{12}$ d. $\frac{12}{7}$

For Problems 7–9

$f(x)$ satisfies the relation $f(x) - \lambda \int_0^{\pi/2} \sin x \cos t\, f(t)dt = \sin x$.

7. If $\lambda > 2$, then $f(x)$ decreases in which of the following interval?

a. $(0, \pi)$ b. $(\pi/2, 3\pi/2)$

c. $(-\pi/2, \pi/2)$ d. None of these

8. If $f(x) = 2$ has at least one real root, then

a. $\lambda \in [1, 4]$ b. $\lambda \in [-1, 2]$

c. $\lambda \in [0, 1]$ d. $\lambda \in [1, 3]$

9. If $\int_0^{\pi/2} f(x)dx = 3$, then the value of λ is

a. 1 b. 3/2

c. 4/3 d. None of these

For Problems 10–13

Let $f(x)$ and $\phi(x)$ are two continuous functions on R satisfying $\phi(x) = \int_a^x f(t)dt$, $a \neq 0$ and another continuous function $g(x)$ satisfying $g(x + \alpha) + g(x) = 0\ \forall\ x \in R, \alpha > 0$ and $\int_b^{2k} g(t)dt$ is independent of b.

10. If $f(x)$ is an odd function, then

a. $\phi(x)$ is also an odd function

b. $\phi(x)$ is an even function

c. $\phi(x)$ is neither as even nor an odd function

d. For $\phi(x)$ to be an even function, it must satisfy $\int_0^a f(x)\, dx = 0$

11. If $f(x)$ is an even function, then

a. $\phi(x)$ is also an even function

b. $\phi(x)$ is an odd function

c. If $f(a-x) = -f(x)$, then $\phi(x)$ is an even function

d. If $f(a-x) = -f(x)$, then $\phi(x)$ is an odd function

12. Least positive value of c if c, k, b are in A.P. is

a. 0 b. 1 c. α d. 2α

13. If m, n are even integers and $p, q \in R$, then $\int_{p+m\alpha}^{q+n\alpha} g(t)dt$ is equal to

a. $\int_p^q g(x)dx$ b. $(n-m)\int_0^\alpha g(x)dx$

c. $\int_p^q g(x)dx + (n-m)\int_0^\alpha g(2x)dx$

d. $\int_p^q g(x)dx + (n-m)\int_0^\alpha g(x)dx$

For Problems 14–17

Evaluating Integrals Dependent on a Parameter

Differentiate I with respect to the parameter within the sign of integrals taking variable of the integrand as constant. Now, evaluate the integral so obtained as a function of the parameter and then integrate the result to get I. Constant of integration can be computed by giving some arbitrary values to the parameter and the corresponding value of I.

14. The value of $\int_0^1 \frac{x^a - 1}{\log x} dx$ is

a. $\log(a-1)$ b. $\log(a+1)$

c. $a\log(a+1)$ d. None of these

15. The value of $\int_0^{\pi/2} \log(\sin^2\theta + k^2\cos^2\theta)d\theta$, where $k \geq 0$, is

a. $\pi\log(1+k) + \pi\log 2$ b. $\pi\log(1+k)$

c. $\pi\log(1+k) - \pi\log 2$ d. $\log(1+k) - \log 2$

16. The value of $\frac{dI}{da}$ when $I = \int_0^{\pi/2} \log\left(\frac{1+a\sin x}{1-a\sin x}\right)\frac{dx}{\sin x}$ (where $|a| < 1$) is

a. $\frac{\pi}{\sqrt{1-a^2}}$ b. $-\pi\sqrt{1-a^2}$

c. $\sqrt{1-a^2}$ d. $\frac{\sqrt{1-a^2}}{\pi}$

17. If $\int_0^\pi \frac{dx}{(a-\cos x)} = \frac{\pi}{\sqrt{a^2-1}}$, then the value of $\int_0^\pi \frac{dx}{(\sqrt{10}-\cos x)^3}$ is

a. $\frac{\pi}{81}$ b. $\frac{7\pi}{162}$

c. $\frac{7\pi}{81}$ d. None of these

For Problems 18–20

$$f(x) = \sin x + \int_{-\pi/2}^{\pi/2} (\sin x + t\cos x) f(t)dt$$

18. The range of $f(x)$ is

a. $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$ b. $\left[-\frac{\sqrt{5}}{3}, \frac{\sqrt{5}}{3}\right]$

c. $\left[-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right]$ d. None of these

19. $f(x)$ is not invertible for

a. $x \in \left[-\frac{\pi}{2} - \tan^{-1}2, \frac{\pi}{2} - \tan^{-1}2\right]$

b. $x \in \left[\tan^{-1}\frac{1}{2}, \pi + \tan^{-1}\frac{1}{2}\right]$

c. $x \in \left[\pi + \cot^{-1}2, 2\pi + \cot^{-1}2\right]$

d. None of these

20. The value of $\int_0^{\pi/2} f(x)dx$ is

a. 1 b. −2 c. −1 d. 2

Matrix-Match Type

Solutions on page 8.70

Each question contains statements given in two columns which have to be matched.
Statements a, b, c, d in column 1 have to be matched with statements p, q, r, s in column 2. If the correct match is a-p, a-s, b-r, c-p, and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	(p)	(q)	(r)	(s)
b	(p)	(q)	(r)	(s)
c	(p)	(q)	(r)	(s)
d	(p)	(q)	(r)	(s)

1. If [.] denotes the greatest integer function, then match the following columns:

Column 1	Column 2
a. $\int_{-1}^{1} [x+[x+[x]]]\, dx$	p. 3
b. $\int_{2}^{5} ([x]+[-x])dx$	q. 5
c. $\int_{-1}^{3} \sin(x-[x])dx$	r. 4
d. $25\int_0^{\pi/4}\left(\tan^6(x-[x]) + \tan^4(x-[x])\right)dx$	s. −3

2.

Column 1	Column 2
a. $\lim_{n\to\infty}\left[\int_0^2 \frac{\left(1+\frac{t}{n+1}\right)^n}{n+1} dt\right]$ is equal to	p. $e - \frac{1}{2}e^2 - \frac{3}{2}$
b. Let $f(x)$ be a function satisfying $f'(x) = f(x)$ with $f(0) = 1$ and g be the function satisfying $f(x) + g(x) = x^2$, then the value of the integral $\int_0^1 f(x)g(x)\,dx$ is	q. e^2
c. $\int_0^1 e^{e^x}(1 + xe^x)dx$ is equal to	r. $e^2 - 1$
d. $\lim_{k\to 0}\frac{1}{k}\int_0^k (1+\sin 2x)^{\frac{1}{x}}\, dx$ is equal to	s. e^e

3.

Column 1	Column 2
a. If $f(x)$ is an integrable function for $x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ and $I_1 = \int_{\pi/6}^{\pi/3} \sec^2\theta f(2\sin 2\theta)\,d\theta$ and $I_2 = \int_{\pi/6}^{\pi/3} \operatorname{cosec}^2\theta f(2\sin 2\theta)d\theta$, then I_1/I_2	p. 3
b. If $f(x+1) = f(3+x)$ for $\forall\ x$, and the value of $\int_a^{a+b} f(x)dx$ is independent of a then the value of b can be	q. 1
c. The value of $\int_1^4 \frac{\tan^{-1}[x^2]}{\tan^{-1}[x^2] + \tan^{-1}[25 + x^2 - 10x]}$ (where [.] denotes the greatest integer function) is	r. 2
d. If $I = \int_0^2 \sqrt{x + \sqrt{x + \sqrt{x + \cdots\infty}}}\ dx$ (where $x > 0$), then $[I]$ is equal to (where [.] denotes the greatest integer function)	s. 4

4.

Column 1	Column 2
a. If $I=\int_{-2}^{2}(\alpha x^3+\beta x+\gamma)dx$, then I is	p. independent of α
b. Let α, β be the distinct positive roots of the equation $\tan x = 2x$, then $\gamma\int_0^1(\sin \alpha x \cdot \sin \beta x)\,dx$ (where $\gamma \neq 0$) is	q. independent of β
c. If $f(x+\alpha)+f(x)=0$, where $\alpha>0$, then $\int_{\beta}^{\beta+2\gamma\alpha} f(x)dx$, where $\gamma \in N$, is	r. independent of γ
d. $\gamma\int_0^{\alpha}[\sin x]dx$ is, where $\gamma\neq 0$, $\alpha\in[(2\beta+1)\pi,(2\beta+2)\pi]$ $n \in N$, and where [.] denotes the greatest integer function	s. depends on α

Integer Type

Solutions on page 8.72

1. Consider the polynomial $f(x) = ax^2 + bx + c$. If $f(0) = 0$, $f(2) = 2$, then the minimum value of $\int_0^2 |f'(x)|\,dx$ is
2. Consider a real valued continuous function f such that $f(x) = \sin x + \int_{-\pi/2}^{\pi/2} (\sin x + t\,f(t))\,dt$. If M and m are maximum and minimum value of the function f, then the value of M/m is
3. A continuous real function f satisfies $f(2x) = 3f(x)\ \forall x \in R$. If $\int_0^1 f(x)\,dx = 1$, then the value of definite integral $\int_1^2 f(x)\,dx$ is
4. Let $f(x) = x^3 - \frac{3x^2}{2} + x + \frac{1}{4}$. Then the value of $\left(\int_{1/4}^{3/4} f(f(x))\,dx\right)^{-1}$ is
5. $\lim\limits_{n\to\infty} \frac{n}{2^n}\int_0^2 x^n\,dx$ equals
6. Let f: $[0, \infty) \to R$ be a continuous strictly increasing function, such that $f^3(x) = \int_0^x t \cdot f^2(t)\,dt$ for every $x \geq 0$, then value of $f(6)$ is
7. If the value of the definite integral $\int_0^1 {}^{207}C_7\, x^{200} \cdot (1-x)^7\,dx$ is equal to $\frac{1}{k}$ where $k \in N$, then the value of $k/26$ is
8. If $I = \int_0^{3\pi/5} ((1+x)\sin x + (1-x)\cos x)\,dx$, then the value of $(\sqrt{2}-1)I$ is
9. If the value of $\lim\limits_{n\to\infty}(n^{-3/2})\cdot\sum_{j=1}^{6n}\sqrt{j}$ is equal to $\sqrt{N}$, then the value of $N/12$ is
10. If f is continuous function and $F(x) = \int_0^x \left((2t+3)\cdot\int_t^2 f(u)\,du\right)dt$, then $|F''(2)/f(2)|$ is equal to
11. If the value of the definite integral $\int_0^1 \frac{\sin^{-1}\sqrt{x}}{x^2 - x + 1}\,dx = \frac{\pi^2}{\sqrt{n}}$ (where $n \in N$), then the value of $n/27$ is
12. Let $f(x) = \int_0^x \frac{dt}{\sqrt{1+t^3}}$ and $g(x)$ be the inverse of $f(x)$, then the value of $4\frac{g''(x)}{(g(x))^2}$ is
13. If $U_n = \int_0^1 x^n(2-x)^n\,dx$ and $V_n = \int_0^1 x^n(1-x)^n\,dx$ $n \in N$, and if $\frac{V_n}{U_n} = 1024$, then the value of n is
14. If $\int_0^{\infty} x^{2n+1} \cdot e^{-x^2}dx = 360$, then the value of n is
15. Let $f(x)$ is a derivable function satisfying $f(x) = \int_0^x e^t \sin(x-t)\,dt$ and $g(x) = f''(x) - f(x)$, then the possible integers in the range of $g(x)$ is
16. If $F(x) = \frac{1}{x^2}\int_4^x [4t^2 - 2F'(t)]dt$, then $(9F'(4))/4$ is
17. If $\int_0^{100} f(x)dx = 7$, then $\sum_{r=1}^{100}\left(\int_0^1 f(r-1+x)dx\right) =$
18. The value of $\int_0^{\frac{3\pi}{2}} \frac{|\tan^{-1}\tan x| - |\sin^{-1}\sin x|}{|\tan^{-1}\tan x| + |\sin^{-1}\sin x|}\,dx$ is equal to
19. If $I_n = \int_0^1 (1-x^5)^n\,dx$, then $\frac{55}{7}\frac{I_{10}}{I_{11}}$ is equal to
20. The value of $2^{2010}\frac{\int_0^1 x^{1004}(1-x)^{1004}dx}{\int_0^1 x^{1004}(1-x^{2010})^{1004}dx}$ is

21. If $f(x)=x+\int_0^1 t(x+t)\,f(t)\,dt$, then the value of $\frac{23}{2}f(0)$ is equal to

22. The value of the definite integral $\int_{\sqrt{2}-1}^{\sqrt{2}+1} \frac{x^4+x^2+2}{(x^2+1)^2}dx$ equals

23. Let $J=\int_{-5}^{-4}(3-x^2)\tan(3-x^2)\,dx$ and $K=\int_{-2}^{-1}(6-6x+x^2)\tan(6x-x^2-6)dx$, then $(J+K)$ equals

24. Let $g(x)$ be differentiable on R and $\int_{\sin t}^{1} x^2 g(x)\,dx = (1-\sin t)$, where $t\in\left(0,\frac{\pi}{2}\right)$. Then the value of $g\left(\frac{1}{\sqrt{2}}\right)$ is

Archives

Solutions on page 8.75

Subjective

1. Show that $\lim\limits_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{6n}\right)=\log 6.$ **(IIT-JEE, 1981)**

2. Evaluate $\int_0^1(tx+1-x)^n\,dx$, where n is a positive integer and t is a parameter independent of x. Hence, show that $\int_0^1 x^k(1-x)^{n-k}\,dx=[{}^nC_k(n+1)]^{-1}$ for $k=0,1,\ldots,n$. **(IIT-JEE, 1981)**

3. Show that $\int_0^\pi xf(\sin x)\,dx=\frac{\pi}{2}\int_0^\pi f(\sin x)\,dx.$ **(IIT-JEE, 1982)**

4. Find the value of $\int_{-1}^{3/2}|x\sin\pi x|\,dx.$ **(IIT-JEE, 1982)**

5. Evaluate $\int_0^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}dx.$ **(IIT-JEE, 1983)**

6. Evaluate the following $\int_0^{1/2}\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx.$ **(IIT-JEE, 1984)**

7. Given a function $f(x)$ such that
   a. it is integrable over every interval on the real line, and
   b. $f(t+x)=f(x)$, for every x and a real t, then show that the integral $\int_a^{a+t}f(x)\,dx$ is independent of a.

8. Evaluate $\int_0^{\pi/2}\frac{x\sin x\cos x}{\cos^4 x+\sin^4 x}dx.$ **(IIT-JEE, 1985)**

9. Evaluate $\int_0^\pi\frac{x\,dx}{1+\cos\alpha\sin x}$, where $0<\alpha<\pi$. **(IIT-JEE, 1986)**

10. If f and g are continuous functions on $[0,a]$ satisfying $f(x)=f(a-x)$ and $g(x)+g(a-x)=2$, then show that $\int_0^a f(x)g(x)\,dx=\int_0^a f(x)\,dx.$ **(IIT-JEE, 1989)**

11. Show that $\int_0^{\pi/2}f(\sin 2x)\sin x\,dx=\sqrt{2}\int_0^{\pi/4}f(\cos 2x)\cos x\,dx.$ **(IIT-JEE, 1990)**

12. Prove that for any positive integer k, $\frac{\sin 2kx}{\sin x}=2[\cos x+\cos 3x+\cdots+\cos(2k-1)x]$. Hence, prove that $\int_0^{\pi/2}\sin 2kx\cot x\,dx=\frac{\pi}{2}.$ **(IIT-JEE, 1990)**

13. If f is a continous function with $\int_0^x f(t)\,dt\to\infty$ as $|x|\to\infty$, then show that every line $y=mx$ intersects the curve $y^2+\int_0^x f(t)\,dt=2.$ **(IIT-JEE, 1991)**

14. Evaluate $\int_0^\pi\frac{x\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx.$ **(IIT-JEE, 1991)**

15. Determine a positive integer $n\le 5$ such that $\int_0^1 e^x(x-1)^n=16-6e.$ **(IIT-JEE, 1992)**

16. Evaluate $\int_2^3\frac{2x^5+x^4-2x^3+2x^2+1}{(x^2+1)(x^4-1)}dx.$ **(IIT-JEE, 1993)**

17. Show that $\int_0^{n\pi+v}|\sin x|\,dx=2n+1-\cos v$, where n is a positive integer and $0\le v<\pi$. **(IIT-JEE, 1994)**

18. If $U_n=\int_0^\pi\frac{1-\cos nx}{1-\cos x}dx$, where n is positive integer or zero, then show that $U_{n+2}+U_n=2U_{n+1}$. Hence, deduce that $\int_0^{\pi/2}\frac{\sin^2 n\theta}{\sin^2\theta}=\frac{1}{2}n\pi.$ **(IIT-JEE, 1995)**

19. Evaluate the definite integral $\int_{-1/\sqrt{3}}^{1/\sqrt{3}}\left(\frac{x^4}{1-x^4}\right)\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx.$ **(IIT-JEE, 1995)**

20. Evaluate $\int_0^{\pi/4} \ln(1+\tan x)dx$.

21. Let $a+b=4$, where $a<2$, and let $g(x)$ be a differentiable function. If $\frac{dg}{dx}>0$ for all x, prove that $\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases as $(b-a)$ increases. **(IIT-JEE, 1997)**

22. Determine the value of $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}dx$. **(IIT-JEE, 1997)**

23. Prove that $\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx = 2\int_0^1 \tan^{-1} x\,dx$.
Hence or otherwise, evaluate the integral $\int_0^1 \tan^{-1}(1-x+x^2)dx$. **(IIT-JEE, 1998)**

24. For $x>0$, let $f(x) = \int_1^x \frac{\log t}{1+t}dt$. Find the function $f(x)+f\left(\frac{1}{x}\right)$ and find the value of $f(e)+f\left(\frac{1}{e}\right)$. **(IIT-JEE, 2000)**

25. If $y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}$, then find $\frac{dy}{dx}$ at $x=\pi$. **(IIT-JEE, 2004)**

26. Find the value of $\int_{-\pi/3}^{\pi/3} \frac{\pi+4x^3}{2-\cos\left(|x|+\frac{\pi}{3}\right)}dx$. **(IIT-JEE, 2004)**

27. Evaluate $\int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\,dx$ **(IIT-JEE, 2005)**.

28. Evaluate $5050\frac{\int_0^1 (1-x^{50})^{100}dx}{\int_0^1 (1-x^{50})^{101}dx}$. **(IIT-JEE, 2006)**

Objective

Fill in the blanks

1. $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec} x^2 \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$.
Then $\int_0^{\pi/2} f(x)dx =$ ______. **(IIT-JEE, 1987)**

2. The integral $\int_0^{1.5} [x^2]\,dx$, where $[\cdot]$ denotes the greatest integer function, equals ______. **(IIT-JEE, 1988)**

3. The value of $\int_{-2}^{2} |1-x^2|\,dx$ is ______. **(IIT-JEE, 1989)**

4. The value of $\int_{\pi/4}^{3\pi/4} \frac{\phi}{1+\sin\phi}d\phi$ is ______. **(IIT-JEE, 1993)**

5. The value of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$ is ______. **(IIT-JEE, 1994)**

6. If for non-zero x, $af(x)+bf\left(\frac{1}{x}\right)=\frac{1}{x}-5$, where $a\neq b$, then $\int_1^2 f(x)\,dx =$ ______. **(IIT-JEE, 1996)**

7. For $n>0$, $\int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x+\cos^{2n} x}dx =$ ______. **(IIT-JEE, 1996)**

8. The value of $\int_1^{e^{37}} \frac{\pi\sin(\pi\ln x)}{x}dx$ is ______. **(IIT-JEE, 1997)**

9. Let $\frac{d}{dx}F(x)=\frac{e^{\sin x}}{x}, x>0$. If $\int_1^4 \frac{2e^{\sin x^2}}{x}dx = F(k)-F(1)$, then one of the possible value of k is ______. **(IIT-JEE, 1997)**

10. Let $f: R\to R$ be a continuous function which satisfies $f(x)=\int_0^x f(t)\,dt$. Then the value of $f(\ln 5)$ is ______ **(IIT-JEE, 2009)**

11. The value of $\frac{\pi^2}{\ln 3}\int_{7/6}^{5/6} \sec(\pi x)\,dx$ is ______ **(IIT-JEE, 2011)**

12. If $\int_a^b (f(x)-3x)\,dx = a^2-b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is ______ **(IIT-JEE, 2011)**

True or false

1. The value of the integral $\int_0^{2a}\left[\frac{f(x)}{\{f(x)+f(2a-x)\}}\right]dx$ is equal to a. **(IIT-JEE, 1988)**

Multiple choice questions with one correct answer

1. The value of the definite integral $\int_0^1\left(1+e^{-x^2}\right)dx$ is
a. -1
b. 2
c. $1+e^{-1}$
d. None of these
(IIT-JEE, 1981)

2. Let a, b, c be non-zero real numbers such that
$\int_0^1 (1+\cos^8 x)(ax^2+bx+c)dx = \int_0^2 (1+\cos^8 x)(ax^2+bx+c)dx$.
Then, the quadratic equation $ax^2+bx+c=0$ has

a. no root in (0, 2)
b. at least one root in (0, 2)
c. a double root in (0, 2)
d. two imaginary roots (IIT-JEE, 1981)

3. The value of the integral $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx$ is
a. $\pi/4$
b. $\pi/2$
c. π
d. None of these
(IIT-JEE, 1983)

4. For any integer n, the integral $\int_0^{\pi} e^{\cos^2 x}\cos^3(2n+1)x\,dx$ has the value
a. π
b. 1
c. 0
d. None of these
(IIT-JEE, 1985)

5. Let $f: R \to R$ and $g: R \to R$ be continuous functions. Then the value of the integral
$\int_{-\pi/2}^{\pi/2}[f(x)+f(-x)]\,[g(x)-g(-x)]\,dx$ is
a. π
b. 1
c. -1
d. 0 (IIT-JEE, 1990)

6. The value of $\int_0^{\pi/2}\frac{dx}{1+\tan^3 x}$ is
a. 0
b. 1
c. $\pi/2$
d. $\pi/4$ (IIT-JEE, 1993)

7. If $f(x)=A\sin\left(\frac{\pi x}{2}\right)+B$, $f'\left(\frac{1}{2}\right)=\sqrt{2}$ and $\int_0^1 f(x)\,dx=\frac{2A}{\pi}$, then constants A and B are
a. $\frac{\pi}{2}$ and $\frac{\pi}{2}$
b. $\frac{2}{\pi}$ and $\frac{3}{\pi}$
c. 0 and $\frac{-4}{\pi}$
d. $\frac{4}{\pi}$ and 0 (IIT-JEE, 1995)

8. The value of $\int_0^{2\pi}[2\sin x]dx$, where [.] represents the greatest integral function, is
a. $\frac{-5\pi}{3}$
b. $-\pi$
c. $\frac{5\pi}{3}$
d. -2π (IIT-JEE, 1995)

9. Let f be a positive function. Let $I_1=\int_{1-k}^{k} xf[x(1-x)]\,dx$, $I_2=\int_{1-k}^{k} f[x(1-x)]dx$, where $2k-1>0$. Then $\frac{I_1}{I_2}$ is
a. 2
b. k
c. $\frac{1}{2}$
d. 1 (IIT-JEE, 1997)

10. If $g(x)=\int_0^x \cos^4 t\,dt$, then $g(x+\pi)$ equals
a. $g(x)+g(\pi)$
b. $g(x)-g(\pi)$
c. $g(x)\,g(\pi)$
d. $\frac{g(x)}{g(\pi)}$ (IIT-JEE, 1997)

11. $\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos x}$ is equal to
a. 2
b. -2
c. 1/2
d. $-1/2$ (IIT-JEE, 1999)

12. If for a real number y, $[y]$ is the greatest integral function less than or equal to y, then the value of the integral $\int_{\pi/2}^{3\pi/2}[2\sin x]dx$ is
a. $-\pi$
b. 0
c. $-\pi/2$
d. $\pi/2$ (IIT-JEE, 1999)

13. Let $g(x)=\int_0^x f(t)dt$, where f is such that $\frac{1}{2}\le f(t)\le 1$, for $t\in[0,1]$ and $0\le f(t)\le\frac{1}{2}$, for $t\in[1,2]$. Then $g(2)$ satisfies the inequality
a. $-\frac{3}{2}\le g(2)<\frac{1}{2}$
b. $\frac{1}{2}\le g(2)\le\frac{3}{2}$
c. $\frac{3}{2}<g(2)\le\frac{5}{2}$
d. $2<g(2)<4$
(IIT-JEE, 2000)

14. If $f(x)=\begin{cases} e^{\cos x}\sin x, & \text{for } |x|\le 2 \\ 2, & \text{otherwise}\end{cases}$, then $\int_{-2}^{3} f(x)\,dx$ is equal to
a. 0
b. 1
c. 2
d. 3 (IIT-JEE, 2000)

15. The value of the integral $\int_{e^{-1}}^{e^2}\left|\frac{\log_e x}{x}\right|dx$ is
a. 3/2
b. 5/2
c. 3
d. 5 (IIT-JEE, 2000)

16. The value of $\int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^x}dx$, where $a>0$, is
a. π
b. $a\pi$
c. $\pi/2$
d. 2π (IIT-JEE, 2001)

17. Let $f(x)=\int_1^x\sqrt{2-t^2}\,dt$. Then the real roots of the equation $x^2-f'(x)=0$ are
a. ± 1
b. $\pm\frac{1}{\sqrt{2}}$
c. $\pm\frac{1}{2}$
d. 0 and 1 (IIT-JEE, 2002)

18. Let $T>0$ be a fixed real number. Suppose f is continuous function such that for all $x\in R$, $f(x+T)=f(x)$. If $I=\int_0^T f(x)\,dx$, then the value of $\int_3^{3+3T} f(2x)dx$ is
a. $3/2I$
b. $2I$
c. $3I$
d. $6I$ (IIT-JEE, 2002)

19. The integral $\int_{-1/2}^{1/2}\left([x]+\ln\left(\frac{1+x}{1-x}\right)\right)dx$, is equal to (where $[\cdot]$ represents the greatest integer function)

a. $-\frac{1}{2}$
b. 0
c. 1
d. $2\ln\left(\frac{1}{2}\right)$ **(IIT-JEE, 2002)**

20. If $L(m, n)=\int_0^1 t^m(1+t)^n\,dt$, then the expression for $L(m, n)$ in terms of $(m+1, n-1)$ is $(m, n \in N)$

a. $\frac{2^n}{m+1}-\frac{n}{m+1}L(m+1,n-1)$

b. $\frac{n}{m+1}L(m+1,n-1)$

c. $\frac{2^n}{m+1}+\frac{n}{m+1}L(m+1,n-1)$

d. $\frac{m}{n+1}L(m+1,n-1)$ **(IIT-JEE, 2003)**

21. If $f(x)=\int_{x^2}^{x^2+1}e^{-t^2}dt$, then $f(x)$ increases in

a. $(0,2)$
b. no value of x
c. $(0,\infty)$
d. $(-\infty,0)$ **(IIT-JEE, 2003)**

22. If $f(x)$ is differentiable and $\int_0^{t^2}x f(x)\,dx=\frac{2}{5}t^5$, then $f\left(\frac{4}{25}\right)$ equals

a. 2/5
b. –5/2
c. 1
d. 5/2 **(IIT-JEE, 2004)**

23. The value of the integral $\int_0^1\sqrt{\frac{1-x}{1+x}}dx$ is

a. $\frac{\pi}{2}+1$
b. $\frac{\pi}{2}-1$
c. -1
d. 1 **(IIT-JEE, 2004)**

24. $\int_{-2}^{0}\{x^3+3x^2+3x+3+(x+1)\cos(x+1)\}\,dx$ is equal to

a. -4
b. 0
c. 4
d. 6 **(IIT-JEE, 2005)**

25. Let f be a non-negative function defined on the interval $[0, 1]$. If $\int_0^x\sqrt{1-(f'(t))^2}\,dt=\int_0^x f(t)\,dt, 0\le x\le 1$, and $f(0)=0$, then

a. $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$

b. $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$

c. $f\left(\frac{1}{2}\right)<\frac{1}{3}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$

d. $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$ **(IIT-JEE, 2009)**

26. The value of $\int_0^1\frac{x^4(1-x)^4}{1+x^2}dx$ is (are)

a. $\frac{22}{7}-\pi$
b. $\frac{2}{105}$
c. 0
d. $\frac{71}{15}-\frac{3\pi}{2}$

(IITJEE 2010)

27. Let f be a real-valued function defined on the interval $(-1, 1)$ such that $e^{-x}f(x)=2+\int_0^x\sqrt{t^4+1}\,dt$, for all $x\in(-1, 1)$ and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to

a. 1
b. 1/3
c. 1/2
d. $1/e$

(IITJEE 2010)

28. The value of $\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}}\frac{x\sin x^2}{\sin x^2+\sin(\ln 6-x^2)}dx$ is

a. $\frac{1}{4}\ln\frac{3}{2}$
b. $\frac{1}{2}\ln\frac{3}{2}$
c. $\ln\frac{3}{2}$
d. $\frac{1}{6}\ln\frac{3}{2}$

(IITJEE 2011)

29. Let f: $[-1, 2]\to[0,\infty)$ be a continuous function such that $f(x)=f(1-x)$ for all $x\in[-1, 2]$. Let $R_1=\int_{-1}^{2}xf(x)dx$, and R_2 be the area of the region bounded by $y=f(x)$, $x=-1$, $x=2$, and the x-axis. Then

a. $R_1=2R_2$
b. $R_1=3R_2$
c. $2R_1=R_2$
d. $3R_1=R_2$

Multiple choice questions with one or more than one correct answer

1. If $\int_0^x f(t)dt=x+\int_x^1 t\,f(t)\,dt$, then the value of $f(1)$ is

a. 1/2
b. 0
c. 1
d. –1/2 **(IIT-JEE, 1998)**

2. Let $f(x)=x-[x]$, for every real number x, where $[x]$ is the integral part of x. Then $\int_{-1}^{1}f(x)\,dx$ is

a. 1
b. 2
c. 0
d. 1/2 **(IIT-JEE, 1998)**

3. Let $S_n=\sum_{k=1}^{n}\frac{n}{n^2+kn+k^2}$ and $T_n=\sum_{k=0}^{n-1}\frac{n}{n^2+kn+k^2}$ for $n=1, 2, 3, \ldots$, then

a. $S_n<\frac{\pi}{3\sqrt{3}}$
b. $S_n>\frac{\pi}{3\sqrt{3}}$
c. $T_n<\frac{\pi}{3\sqrt{3}}$
d. $T_n>\frac{\pi}{3\sqrt{3}}$ **(IIT-JEE, 2008)**

4. Let $f(x)$ be a non-constant twice differentiable function defined on $(-\infty,\infty)$ such that $f(x)=f(1-x)$ and $f'\left(\frac{1}{4}\right)=0$, then

a. $f'(x)$ vanishes at least twice on $[0, 1]$

b. $f'\left(\frac{1}{2}\right)=0$

c. $\int_{-1/2}^{1/2} f\left(x+\frac{1}{2}\right)\sin x\, dx = 0$

d. $\int_{0}^{1/2} f(t)e^{\sin \pi t}\, dt = \int_{1/2}^{1} f(1-t)\, e^{\sin \pi t}\, dt$ **(IIT-JEE, 2008)**

5. If $I_n = \int_{-\pi}^{\pi} \frac{\sin n\pi}{(1+\pi^x)\sin x} dx$, $n = 0, 1, 2, \ldots$, then

a. $I_n = I_{n+2}$

b. $\sum_{m=1}^{10} I_{2m+1} = 10\pi$

c. $\sum_{m=1}^{10} I_{2m} = 0$

d. $I_n = I_{n+1}$

(IIT-JEE, 2009)

6. Let f be a real-values functions defined on the interval $(0, \infty)$ by $f(x) = \ln x + \int_{0}^{x} \sqrt{1+\sin t}\, dt$. Then which of the following statement(s) is (are) true ?

a. $f''(x)$ exists for all $x \in (0, \infty)$

b. $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$ but not differentiable on $(0, \infty)$

c. there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

d. there exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (0, \infty)$ **(IITJEE 2010)**

Match the column type

1.

	Column I		Column II
a.	$\int_{-1}^{1} \frac{dx}{1+x^2}$	p.	$\frac{1}{2}\log\left(\frac{2}{3}\right)$
b.	$\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}$	q.	$2\log\left(\frac{2}{3}\right)$
c.	$\int_{2}^{3} \frac{dx}{1-x^2}$	r.	$\frac{\pi}{3}$
d.	$\int_{1}^{2} \frac{dx}{x\sqrt{x^2-1}}$	s.	$\frac{\pi}{2}$

(IIT-JEE, 2006)

Linked comprehension type

Let the definite integral be defined by the formula $\int_a^b f(x)dx = \frac{b-a}{2}(f(a)+f(b))$. For more accurate result for $c \in (a,b)$, we can use $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = F(c)$ so that for $c = \frac{a+b}{2}$, we get $\int_a^b f(x)dx = \frac{b-a}{4}(f(a)+f(b)+2f(c))$.

1. $\int_0^{\pi/2} \sin x dx$ is equal to

a. $\frac{\pi}{8}(1+\sqrt{2})$

b. $\frac{\pi}{4}(1+\sqrt{2})$

c. $\frac{\pi}{8\sqrt{2}}$

d. $\frac{\pi}{4\sqrt{2}}$

2. If $\lim_{x\to a} \frac{\int_a^x f(x)dx - \left(\frac{x-a}{2}\right)(f(x)+f(a))}{(x-a)^3} = 0$, then $f(x)$ is of maximum degree

a. 4 b. 3 c. 2 d. 1

3. If $f''(x) < 0\ \forall\ x \in (a, b)$ and c is a point such that $a < c < b$, and $(c, f(c))$ is the point lying on the curve for which $F(c)$ is maximum, then $f'(c)$ is equal to

a. $\frac{f(b)-f(a)}{b-a}$

b. $\frac{2(f(b)-f(a))}{b-a}$

c. $\frac{2f(b)-f(a)}{2b-a}$

d. 0

(IIT-JEE, 2008)

Integer type

1. For any real number x, let $[x]$ denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval $[-10, 10]$ by

$$f(x) = \begin{cases} x-[x] & \text{if } [x] \text{ is odd} \\ 1+[x]-x & \text{if } [x] \text{ is even} \end{cases}$$

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\, dx$ is

(IITJEE 2010)

2. Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in R$, where $f'(x)$ denotes $\frac{df(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. then the value of $y(2)$ is

ANSWERS AND SOLUTIONS

Subjective Type

1. Let $F(x)=\int_a^x f(t)\,dt$

$\therefore F(x+p)=\int_a^{x+p} f(t)\,dt=\int_a^x f(t)+\int_x^{x+p} f(t)dt$

$=F(x)+\int_x^{x+p} f(t)\,dt$ (1)

Obviously, now we have to prove that $\int_x^{x+p} f(t)\,dt$ is zero.

Given that $f(x)$ has period p, then $\int_x^{x+p} f(t)\,dt$ is independent of x.

Let $x=-p/2$, then $\int_x^{x+p} f(t)\,dt=\int_{-p/2}^{p/2} f(t)\,dt=0$

[as given $f(x)$ is an odd function].

$\therefore \quad F(x+p)=F(x)$

Thus, $F(x)$ is periodic with period P.

2. $I=\int_0^{\pi/2}\left(\frac{\theta}{\sin\theta}\right)^2 d\theta$

$=\int_0^{\pi/2}\theta^2\operatorname{cosec}^2\theta d\theta$

$=[\theta^2(-\cot\theta)]_0^{\pi/2}-\int_0^{\pi/2}2\theta\cdot(-\cot\theta)d\theta$

(Integrating by parts)

$=[\lim_{\theta\to 0}\theta^2\cdot\cot\theta]+2\int_0^{\pi/2}\theta\cot\theta d\theta$

$=0+2\left[[\theta\log\sin\theta]_0^{\pi/2}-\int_0^{\pi/2}\log\sin\theta d\theta\right]$

(Integrating by parts)

$=2\left[-\lim_{\theta\to 0}\theta\ln\sin\theta-k\right]$

$=-2k$

3. Let $g(x)=1+x+x^2+\cdots+x^{n-1}=\frac{x^n-1}{x-1}$

$\therefore f(n)=\int_0^1 g(x)dx=\int_0^1\frac{x^n-1}{x-1}dx$

Put $x=\cos\theta \Rightarrow dx=-\sin\theta\,d\theta$

$\therefore f(n)=\int_{\pi/2}^{0}\frac{(\cos^n\theta-1)(-\sin\theta)}{(\cos\theta-1)}d\theta$

$=\int_0^{\pi/2}\frac{(1-\cos^n\theta)2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}d\theta$

$=\int_0^{\pi/2}\cot\left(\frac{\theta}{2}\right)(1-\cos^n\theta)d\theta$

4. Given integral is $\int_0^{\pi/4}\tan^{-1}\left(\frac{2\cos^2\theta}{2-\sin 2\theta}\right)\sec^2\theta d\theta$

$=\int_0^{\pi/4}\tan^{-1}\left(\frac{1}{\sec^2\theta-\tan\theta}\right)\sec^2\theta d\theta$

$=\int_0^{\pi/4}\tan^{-1}\left(\frac{1}{1+\tan^2\theta-\tan\theta}\right)\sec^2\theta d\theta$

Put $\tan\theta=t$, then $\sec^2\theta\,d\theta=dt$

The given integral reduces to

$\int_0^1\tan^{-1}\left(\frac{1}{1+t^2-t}\right)dt=\int_0^1\tan^{-1}\left(\frac{t-(t-1)}{1+t(t-1)}\right)dt$

$=\int_0^1\tan^{-1}t\,dt-\int_0^1\tan^{-1}(t-1)dt$

$=\int_0^1\tan^{-1}t\,dt-\int_0^1\tan^{-1}((1-t)-1)dt$

$=2\int_0^1 1\tan^{-1}t\,dt$

$=2\left[t\tan^{-1}t\right]_0^1-2\int_0^1\frac{t}{1+t^2}dt$ (integrating by parts)

$=\frac{\pi}{2}-\left[\ln(1+t^2)\right]_0^1=\frac{\pi}{2}-\ln 2.$

5. For $x\le 1$, $\sin^{-1}\frac{2x}{1+x^2}=2\tan^{-1}x$

For $x\ge 1$, $\sin^{-1}\frac{2x}{1+x^2}=\sin^{-1}\frac{\frac{2}{x}}{1+\frac{1}{x^2}}$

$=2\tan^{-1}(1/x)=2\cot^{-1}x$

Hence, the given integral

$$= \int_0^{\sqrt{3}} \frac{\sin^{-1}\left(\frac{2x}{1+x^2}\right)}{(1+x^2)} dx = \int_0^1 \frac{2}{1+x^2}(\tan^{-1} x)dx$$

$$+ \int_1^{\sqrt{3}} \frac{2\cot^{-1} x}{1+x^2} dx$$

$$= \left[(\tan^{-1} x)^2\right]_0^1 - \left[(\cot^{-1} x)^2\right]_1^{\sqrt{3}} = \frac{\pi^2}{16} - \left(\frac{\pi^2}{36} - \frac{\pi^2}{16}\right)$$

$$= \frac{\pi^2}{8} - \frac{\pi^2}{36} = \frac{7\pi^2}{72}$$

6.

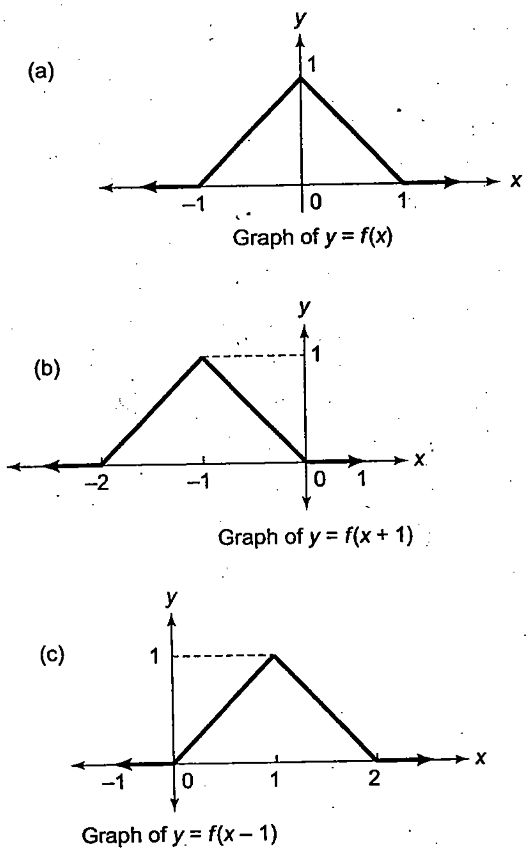

Fig. 8.16

$$\int_{-3}^{5} g(x)dx = \int_{-3}^{5} f(x-1)dx + \int_{-3}^{5} f(x-1)dx$$

= Area of triangle in the graph $y = f(x-1)$

+ Area of triangle in the graph $y = f(x+1)$

$$= 2\frac{1}{2}(2)(1) = 2$$

7. $I = \int_0^a f(x)g(x)h(x)dx$

$$= \int_0^a f(a-x)g(a-x)h(a-x)dx$$

$$= \int_0^a f(x)(-g(x))\left(\frac{3h(x)-5}{4}\right)dx$$

$$= -\frac{3}{4}\int_0^a f(x)g(x)h(x)dx + \frac{5}{4}\int_0^a f(x)g(x)dx$$

$$= -\frac{3}{4}I + \frac{5}{4}\int_0^a f(x)g(x)dx$$

$$\Rightarrow\ I = \frac{5}{7}\int_0^a f(x)g(x)dx$$

$$= \frac{5}{7}\int_0^a f(a-x)g(a-x)dx$$

$$= \frac{5}{7}\int_0^a f(x)(-g(x))dx = -I$$

$$\Rightarrow\ 2I = 0\ \Rightarrow\ I = 0$$

8. Let $I_n = \int_0^{\pi/2} x^n \sin x\, dx$

Integrate by parts and choose $\sin x$ as the second function.

Therefore, $I_n = \left[x^n(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} nx^{n-1}(-\cos x)dx$

$$= 0 + n\int_0^{\pi/2} x^{n-1} \cos x\, dx$$

Again integrating by parts, we get

$$I_n = n\left\{x^{n-1}\sin x\right\}_0^{\pi/2} - n(n-1)\int_0^{\pi/2} x^{n-2}\sin x\, dx$$

$$\Rightarrow\ I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$$

R.H.S. contains π^2. Therefore, put $n = 3$

$$I_3 = 3\left(\frac{\pi}{2}\right)^2 - 3.2I_1 = \frac{3\pi^2}{4} - 6\int_0^{\pi/2} x \sin x\, dx$$

$$= \frac{3\pi^2}{4} - 6\{x(-\cos x) + \sin x\}_0^{\pi/2}$$

$$= \frac{3\pi^2}{4} - 6\{1\}$$

$$= \frac{3}{4}(\pi^2 - 8) \text{ which is true.}$$

Hence, $n = 3$.

9. $\int_0^{\pi} f(x)dx = \int_0^{\pi} \frac{\sin x}{x} dx$

Let $I = \int_0^{\pi/2} f(x) f\left(\frac{\pi}{2} - x\right) dx$

$$= \int_0^{\pi/2} \frac{\sin x}{x} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)} dx$$

$$= \int_0^{\pi/2} \frac{2\sin x \cos x}{x(\pi-2x)} dx$$

$$= \int_0^{\pi/2} \frac{\sin 2x}{x(\pi-2x)} dx$$

$\Rightarrow$ Let $2x = t \Rightarrow dt = 2dx$

$$\Rightarrow I = \int_0^{\pi} \frac{\sin t}{\frac{t}{2}(\pi-t)} \frac{dt}{2} = \int_0^{\pi} \frac{\sin t}{t(\pi-t)} dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin t\left(\frac{1}{t}+\frac{1}{\pi-t}\right) dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\sin t}{t} dt + \frac{1}{\pi}\int_0^{\pi} \frac{\sin t}{(\pi-t)} dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\sin t}{t} dt + \frac{1}{\pi}\int_0^{\pi} \frac{\sin(\pi-t)}{\pi-(\pi-t)} dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\sin t}{t} dt + \frac{1}{\pi}\int_0^{\pi} \frac{\sin t}{t} dt = \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t} dt$$

$$\Rightarrow \frac{\pi}{2}\int_0^{\pi/2} f(x) f\left(\frac{\pi}{2}-x\right) dx = \int_0^{\pi} \frac{\sin x}{x} dx$$

10. We have $g(x) = \int_x^a \frac{f(t)}{t} dt$

Differentiating both sides w.r.t. x, we get

$$g'(x) = -\frac{f(x)}{x} \Rightarrow f(x) = -x g'(x)$$

$$\Rightarrow \int_0^a f(x) dx = -\int_0^a x g'(x) dx = -x\, g(x)\big|_0^a + \int_0^a g(x) dx$$

$$= -a g(a) + \int_0^a g(x) dx = \int_0^a g(x) dx \qquad [\text{as } g(a) = 0]$$

11. Let $g(x) = \int_x^{x+p} f(t) dt$

Since $g(x)$ is independent of x, $g'(x) = 0$.

$\Rightarrow f(x+p) - f(x) = 0$

$\Rightarrow f(x)$ is periodic with period p.

Here, $I_1 = \int_0^p f(t) dt$

$$\text{and } I_2 = \int_{10}^{p^n+10} f(z) dz = \int_{10}^{p^{n-1}p+10} f(z) dz = \int_0^{p^{n-1}p} f(z) dz$$

$$= p^{n-1}\int_0^p f(z) dz \Rightarrow \frac{I_2}{I_1} = p^{n-1}$$

12. It is given that $f(x+f(y)) = f(x) + y$

Putting $y = 0$, we get $f(x+f(0)) = f(x) + 0$

$\Rightarrow f(x+1) = f(x)$

Now, using the property,

$$\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a-x) dx, \text{ we get}$$

$$\int_0^2 f(2-x) dx = \int_0^1 f(2-x) dx + \int_0^1 f(2-(2-x)) dx$$

$$= \int_0^1 f(2-(1-x)) dx + \int_0^1 f(x) dx$$

$$= \int_0^1 f(1+x) dx + \int_0^1 f(x) dx = 2\int_0^1 f(x) dx$$

Alternative method

It is given that $f(x+f(y)) = f(x) + y$

Putting $y = 0$, we get $f(x+f(0)) = f(x) + 0$

$\Rightarrow f(x+1) = f(x)$

$\Rightarrow f(x)$ is periodic with period 1.

Now, $I = \int_0^2 f(2-x) dx$

Putting $2 - x = t$, we get

$$\Rightarrow I = \int_0^2 f(t) dt = 2\int_0^1 f(t) dt = 2\int_0^1 f(x) dx$$

13. Here, $f'(x) = \frac{1}{x^2 + (f(x))^2} > 0\ \forall x \geq 1$

$\Rightarrow f(x)$ is an increasing function $\forall x \geq 1$

Given $f(1) = 1 \Rightarrow f(x) \geq 1\ \forall x \geq 1$

Hence, $f'(x) \leq \frac{1}{1+x^2}\ \forall x \geq 1$

$$\Rightarrow \int_1^x f'(x) dx \leq \int_1^x \frac{1}{1+x^2} dx$$

$\Rightarrow f(x) - f(1) \leq \tan^{-1} x - \tan^{-1} 1$

$\Rightarrow f(x) \leq \tan^{-1} x + 1 - \frac{\pi}{4}$

$\Rightarrow f(x) < \frac{\pi}{2} + 1 - \frac{\pi}{4} \qquad \left(\text{as } \tan^{-1} x < \frac{\pi}{2},\ \forall x \geq 1\right)$

i.e., $f(x) < 1 + \frac{\pi}{4}\ \forall x \geq 1$

14. Given expression is

$$x\int_0^x (1-t)\sin(f(t)) dt = 2\int_0^x t \sin(f(t)) dt$$

Differentiating w.r.t. x, we get

$$\int_0^x (1-t)\sin[f(t)] dt + x(1-x)\sin[f(x)] = 2x\sin[f(x)]$$

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$$\Rightarrow \int_0^x (1-t)\sin[f(t)]dt = x^2\sin[f(x)] + x\sin[f(x)]$$

Again differentiating w.r.t. x, we get

$$(1-x)\sin[f(x)] = 2x\sin[f(x)] + x^2\cos[f(x)]f'(x) + \sin[f(x)] + x\cos[f(x)]f'(x)$$

$$\Rightarrow -3x\sin[f(x)] = (x+x^2)\cos[f(x)]f'(x)$$

$$\Rightarrow \frac{-3x}{x(1+x)} = \cot[f(x)]f'(x)$$

$$\Rightarrow f'(x)\cot f(x) + \frac{3}{1+x} = 0$$

15. Let $I = \int_0^2 \frac{dx}{(17+8x-4x^2)(e^{6(1-x)}+1)}$ (1)

Replace x with $2-x$ (Property IV)

$$\Rightarrow I = \int_0^2 \frac{dx}{(17+8(2-x)-4(2-x)^2)(e^{6(1-(2-x))}+1)}$$

$$= \int_0^2 \frac{dx}{(17+8x-4x^2)(e^{-6(1-x)}+1)} \quad (2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^2 \frac{1}{(17+8x-4x^2)}\left(\frac{1}{(e^{6(1-x)}+1)} + \frac{1}{(e^{-6(1-x)}+1)}\right)dx$$

$$= \int_0^2 \frac{dx}{17+8x-4x^2}$$

$$\Rightarrow I = -\frac{1}{8}\int_0^2 \frac{dx}{x^2-2x-\frac{17}{4}}$$

$$= -\frac{1}{8}\int_0^2 \frac{dx}{(x-1)^2-\frac{21}{4}}$$

$$= -\frac{1}{8}\times\frac{1}{2\times\frac{\sqrt{21}}{2}}\log\left|\frac{x-1-\frac{\sqrt{21}}{2}}{x-1+\frac{\sqrt{21}}{2}}\right|_0^2$$

$$= -\frac{1}{8\sqrt{21}}\log\left|\frac{2x-2-\sqrt{21}}{2x-2+\sqrt{21}}\right|_0^2$$

$$= -\frac{1}{8\sqrt{21}}\left[\log\left|\frac{2-\sqrt{21}}{2+\sqrt{21}}\right| - \log\left(\frac{2+\sqrt{21}}{\sqrt{21}-2}\right)\right]$$

16. Let $x = I + f \Rightarrow [x] = I$ (1)

Now, $\int_0^{[x]} x\,dx = \int_0^I x\,dx = \frac{I^2}{2}$, and

$$\int_0^x [x]\,dx = \int_0^{I+f}[x]\,dx = \int_0^1 0\,dx + \int_1^2 1\,dx + \cdots + \int_{I-1}^I (I-1)\,dx + \int_I^{I+f} I\,dx$$

$= \{1+2+3+\cdots+(I-1)\} + I(I+f-I)$

$$= \frac{I(I-1)}{2} + I(f)$$

$$= \frac{I(I-1)}{2} + I(x-I) \quad \text{[using equation (1)]}$$

Given $\int_0^x [x]dx = \int_0^{[x]} x\,dx$

$$\Rightarrow \frac{I^2}{2} = \frac{I(I-1)}{2} + I(x-I)$$

$\Rightarrow I = 0$ or $2I - 2x + 1 = 0$

i.e., $[x] = 0$ or $x = \frac{2I+1}{2} = I + \frac{1}{2}$

$\Rightarrow 0 \le x < 1$ or $x = [x] + \frac{1}{2}$

$\Rightarrow 0 \le x < 1$ or $\{x\} = \frac{1}{2}$

17. Given $F(x) = \left(\int_a^x f(t)\,dt - \int_x^b f(t)\,dt\right)(2x-(a+b))$ (1)

As f is continuous, hence $F(x)$ is also continuous. Also, put $x = a$.

$$F(a) = \left(-\int_a^b f(t)\,dt\right)(a-b) = (b-a)\int_a^b f(t)\,dt$$

and put $x = b$

$$F(b) = \left(\int_a^b f(t)\,dt\right)(b-a)$$

Hence, $F(a) = F(b)$

Hence, Rolle's Theorem is applicable to $F(x)$.

$\therefore$ $\exists$ some $c \in (a, b)$ such that $F'(c) = 0$

Now, $F'(x) = = 0$

$\therefore$ $F'(c) = = f(c)[(a+b)-2c$

18. Here $\int_a^b |\sin x|\,dx$ is the area under the curve from $x = a$ to $x = b$.

Also, the area from $x = a$ to $x = a + \pi$ is 2 square units. Hence $b - a = 4\pi$.

Similarly $a + b - 0 = \frac{9\pi}{2}$, i.e., $a + b = \frac{9\pi}{2}$.

$$\Rightarrow a = \frac{\pi}{4},\ b = \frac{17\pi}{4}$$

Hence, $\int_a^b x\sin x\,dx = -x\cos x|_{\pi/4}^{17\pi/4} + \int_{\pi/4}^{17\pi/4}\sin x\,dx$

$$= -\frac{17\pi}{4}\cos\frac{17\pi}{4} + \frac{\pi}{4}\cos\frac{\pi}{4} = -\frac{4\pi}{\sqrt{2}} - 2\sqrt{2}\pi$$

19. Given $\left|\int_{a-t}^a f(x)dx\right| = \left|\int_a^{a+t} f(x)dx\right| \ \forall t \in R$

$$\Rightarrow \int_{a-t}^a f(x)dx = -\int_a^{a+t} f(x)dx$$ (since $f(a) = 0$ and $f(x)$ is monotonic)

$\Rightarrow f(a-t)=-f(a+t)$

$\Rightarrow f(a-t)+f(a+t)=0 \quad (1)$

$f(a+t)=-f(a-t)=x$ (say)

$\Rightarrow t=f^{-1}(x)-a \quad (2)$

and $t=a-f^{-1}(-x) \quad (3)$

From equations (3) and (2), $(a-f^{-1}(x))+(a-f^{-1}(-x))=0$

$$\Rightarrow \int_{-\lambda}^{\lambda} f^{-1}(x)dx=\frac{1}{2}\int_{-\lambda}^{\lambda}(f^{-1}(x)+f^{-1}(-x))\ dx=2a\lambda$$

Objective Type

1.b. $\lim_{n\to\infty} S_n=\lim_{n\to\infty}\left[\frac{1}{1+\sqrt{n}}+\frac{1}{2+\sqrt{2n}}+\cdots+\frac{1}{n+\sqrt{n^2}}\right]$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{\frac{1}{n}+\frac{1}{\sqrt{n}}}+\frac{1}{\frac{2}{n}+\sqrt{\frac{2}{n}}}+\cdots+\frac{1}{\frac{n}{n}+\sqrt{\frac{n}{n}}}\right]$$

$$=\int_0^1\frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)}$$

Put $\sqrt{x}=z$, $\therefore \frac{1}{2\sqrt{x}}dx=dz$

$$\Rightarrow \lim_{n\to\infty} S_n=\int_0^1\frac{2\,dz}{z+1}=2\,|\log(z+1)|_0^1$$

$$=2(\log 2-\log 1)$$

$$=2\log 2=\log 4$$

2.c. $\lim_{n\to\infty}\sum_{r=1}^{4n}\frac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r}+4\sqrt{n}\right)^2}$

$$T_r=\frac{1}{\sqrt{\frac{r}{n}}\,n\left(3\sqrt{\frac{r}{n}}+4\right)^2}$$

$$\Rightarrow S=\lim_{n\to\infty}\frac{1}{n}\sum_{1}^{4n}\frac{1}{\left(3\sqrt{\frac{r}{n}}+4\right)^2\sqrt{\frac{r}{n}}}$$

$$=\int_0^4\frac{dx}{\sqrt{x}(3\sqrt{x}+4)^2}$$

Put $3\sqrt{x}+4=t \Rightarrow \frac{3}{2}\frac{1}{\sqrt{x}}dx=dt$

$$=\frac{2}{3}\int_4^{10}\frac{dt}{t^2}=\frac{2}{3}\left[\frac{1}{t}\right]_{10}^4=\frac{1}{10}$$

3.d. $I=\int_{a+c}^{b+c} f(x)dx$, putting $x=t+c$

$\Rightarrow dx=dt$, we get $I=\int_a^b f(t+c)dt=\int_a^b f(x+c)dx$

$I=\int_{ac}^{bc} f(x)dx$

Putting $x=tc \Rightarrow dx=c\,dt$,

we get $I=c\int_a^b f(ct)\,dt=c\int_a^b f(cx)\,dx$

$$f(x)=\frac{1}{2}(f(x)+f(-x)+f(x)-f(-x))$$

$$\Rightarrow \int_{-a}^{a} f(x)dx$$

$$=\frac{1}{2}\int_{-a}^{a}(f(x)+f(-x)+f(x)-f(-x))dx$$

$$=\frac{1}{2}\int_{-a}^{a}(f(x)+f(-x))dx+\frac{1}{2}\int_{-a}^{a}(f(x)-f(-x))dx$$

$$=\frac{1}{2}\int_{-a}^{a}(f(x)+f(-x)dx$$

as $f(x)+f(-x)$ is even and $f(x)-f(-x)$ is odd.

4.d. $\int_{\sqrt{2}}^{x}\frac{dt}{t\sqrt{t^2-1}}=\frac{\pi}{2}$

$$\Rightarrow [\sec^{-1}t]_{\sqrt{2}}^{x}=\frac{\pi}{2}$$

$$\Rightarrow \sec^{-1}x-\sec^{-1}\sqrt{2}=\frac{\pi}{2}$$

$$\Rightarrow \sec^{-1}x-\frac{\pi}{4}=\frac{\pi}{2}$$

$$\Rightarrow \sec^{-1}x=\frac{\pi}{2}+\frac{\pi}{4}=\frac{3\pi}{4}, \Rightarrow x=-\sqrt{2}$$

5.c. $\int_{-1}^{1/2}\frac{e^x\left(2-x^2\right)dx}{(1-x)\sqrt{1-x^2}}$

$$=\int_{-1}^{1/2}\frac{e^x\left(1-x^2+1\right)}{(1-x)\sqrt{1-x^2}}$$

$$=\int_{-1}^{1/2}e^x\left[\sqrt{\frac{1+x}{1-x}}+\frac{1}{(1-x)\sqrt{1-x^2}}\right]dx$$

$$=e^x\sqrt{\frac{1+x}{1-x}}\Bigg|_{-1}^{1/2}$$

$$=\sqrt{3e}$$

6.a. $\int_{-\pi}^{\pi}\sin nx\sin mx\,dx$

$$=\int_0^{\pi}2\sin mx\sin nx\,dx$$

$$=\int_0^{\pi}[\cos(m-n)x-\cos(m+n)x]\,dx$$

$$=\left|\frac{\sin(m-n)x}{m-n}-\frac{\sin(m+n)x}{m+n}\right|_0^{\pi}=0$$

7.a. $I=\int_0^{\infty}\frac{x\log x\,dx}{(1+x^2)^2}$

Let $x=\frac{1}{t}$

$$\Rightarrow I=\int_{\infty}^{0}\frac{\left(\frac{1}{t}\right)\log\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)dt}{\left(1+\frac{1}{t^2}\right)^2}$$

$$=-\int_0^{\infty}\frac{t\log t}{(1+t^2)^2}dt=-I$$

$$\Rightarrow I=0$$

8.c. Put $x=a\sin\theta \therefore dx=a\cos\theta\, d\theta$

When $x=0,\ \theta=0;\ x=a,\ \theta=\frac{\pi}{2}$

$$\therefore \text{given integral } I=\int_0^{\pi/2}\frac{a\cos\theta\, d\theta}{a\sin\theta+a\cos\theta}$$

$$=\int_0^{\pi/2}\frac{\cos\theta d\theta}{\sin\theta+\cos\theta}$$

$$\text{Also, } I=\int_0^{\pi/2}\frac{\cos\left(\frac{\pi}{2}-\theta\right)d\theta}{\sin\left(\frac{\pi}{2}-\theta\right)+\cos\left(\frac{\pi}{2}-\theta\right)}$$

$$=\int_0^{\pi/2}\frac{\sin\theta\, d\theta}{\cos\theta+\sin\theta}$$

$$\therefore 2I=\int_0^{\pi/2}\frac{\cos\theta+\sin\theta}{\cos\theta+\sin\theta}d\theta=\int_0^{\pi/2}d\theta=\frac{\pi}{2}$$

$$\therefore I=\frac{\pi}{4}$$

9.b. $\int_0^{\pi/2}|\sin x-\cos x|\,dx$

$$=\int_0^{\pi/4}-(\sin x-\cos x)dx+\int_{\pi/4}^{\pi/2}(\sin x-\cos x)dx$$

$$=|\cos x+\sin x|_0^{\pi/4}+|-\cos x-\sin x|_{\pi/4}^{\pi/2}$$

$$=\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-1-0\right)+\left(-0-1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)$$

$$=\frac{4}{\sqrt{2}}-2=2\sqrt{2}-2=2\left(\sqrt{2}-1\right)$$

10.a. Let $I=\int_0^{\pi}\frac{x\tan x}{\sec x+\cos x}dx$ (1)

$$=\int_0^{\pi}\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\cos(\pi-x)}dx$$

$$=\int_0^{\pi}\frac{(\pi-x)\tan x}{\sec x+\cos x}dx \quad (2)$$

Adding equations (1) and (2) gives

$$2I=\pi\int_0^{\pi}\frac{\tan x}{\sec x+\cos x}dx$$

$$=\pi\int_0^{\pi}\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+\cos x}dx=\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$$

Put $\cos x=z$, therefore $-\sin x\,dx=dz$.

When $x=0,\ z=1,\ x=\pi,\ z=-1$

$$\therefore 2I=\pi\int_1^{-1}\frac{-dz}{1+z^2}=\pi\int_{-1}^{1}\frac{dz}{1+z^2}$$

$$=\pi|\tan^{-1}z|_{-1}^{1}$$

$$=\pi[\tan^{-1}1-\tan^{-1}(-1)]$$

$$=\pi\left(\frac{\pi}{4}+\frac{\pi}{4}\right)=\frac{2\pi^2}{4}$$

$$\Rightarrow I=\frac{\pi^2}{4}$$

11.c. We have $\int_2^4(3-f(x))dx=7$

$$\Rightarrow 6-\int_2^4 f(x)dx=7 \Rightarrow \int_2^4 f(x)dx=-1$$

Now,

$$\int_2^{-1}f(x)dx=-\int_{-1}^{2}f(x)dx=-\left[\int_{-1}^{4}f(x)dx+\int_4^2 f(x)dx\right]$$

$$=-\left[\int_{-1}^{4}f(x)dx-\int_2^4 f(x)dx\right]=-[4+1]=-5$$

12.c. We have $\int_0^1 e^{x^2}(x-\alpha)dx=0$

$$\Rightarrow \int_0^1 e^{x^2}x\,dx=\int_0^1 e^{x^2}\alpha\,dx$$

$$\Rightarrow \frac{1}{2}\int_0^1 e^t dt=\alpha\int_0^1 e^{x^2}dx,\ \text{where } t=x^2$$

$$\Rightarrow \frac{1}{2}(e-1)=\alpha\int_0^1 e^{x^2}dx \quad (1)$$

Since, e^{x^2} is an increasing function for $0\le x\le 1$, therefore, $1\le e^{x^2}\le e$ when $0\le x\le 1$.

$$\Rightarrow 1(1-0)\le\int_0^1 e^{x^2}dx\le e(1-0)$$

$$\Rightarrow 1\le\int_0^1 e^{x^2}dx\le e \quad (2)$$

From equations (1) and (2), we find that L.H.S. of equation (1) is positive and $\int_0^1 e^{x^2}dx$ lies between 1 and e. Therefore, α is a positive real number.

Now, from equation (1), $\alpha=\dfrac{\frac{1}{2}(e-1)}{\int_0^1 e^{x^2}dx}$ (3)

The denominator of equation (3) is greater than unity and the numerator lies between 0 and 1. Therefore, $0<\alpha<1$.

13.a. Putting $a=2,\ b=3,\ c=0$, we get

$$\int_0^{\infty}\frac{dx}{\left(x^2+4\right)\left(x^2+9\right)}=\frac{\pi}{2(2+3)(3+0)(0+2)}=\frac{\pi}{60}$$

14.c. Given integral

$$= \int_0^1 \frac{dx}{(x+\cos\alpha)^2 + (1-\cos^2\alpha)}$$

$$= \int_0^1 \frac{dx}{(x+\cos\alpha)^2 + \sin^2\alpha}$$

$$= \frac{1}{\sin\alpha}\left|\tan^{-1}\frac{x+\cos\alpha}{\sin\alpha}\right|_0^1$$

$$= \frac{1}{\sin\alpha}\left[\tan^{-1}\frac{1+\cos\alpha}{\sin\alpha} - \tan^{-1}\frac{\cos\alpha}{\sin\alpha}\right]$$

$$= \frac{1}{\sin\alpha}\left[\tan^{-1}\cot\frac{\alpha}{2} - \tan^{-1}(\cot\alpha)\right]$$

$$= \frac{1}{\sin\alpha}\left[\tan^{-1}\tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) - \tan^{-1}\tan\left(\frac{\pi}{2}-\alpha\right)\right]$$

$$= \frac{1}{\sin\alpha}\left[\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) - \left(\frac{\pi}{2}-\alpha\right)\right] = \frac{\alpha}{2\sin\alpha}$$

5.d. $I = \int_1^e \left(\frac{1}{x}+1\right)dx - \int_1^e \frac{1+\ln x}{1+x\ln x}\,dx$

$= [\ln x + x]_1^e - [\ln(1+x\ln x)]_1^e$

$= e - \ln(1+e)$

6.b. On putting $x = \sin\theta$, we get $dx = \cos\theta\, d\theta$

Integral (without limits) $= \int \frac{\cos\theta\, d\theta}{(1+\sin^2\theta)(\cos\theta)}$

$$= \int \frac{d\theta}{1+\sin^2\theta} = \int \frac{\operatorname{cosec}^2\theta\, d\theta}{2+\cot^2\theta}$$

$$= \int \frac{-dt}{2+t^2} \text{ where } t = \cot\theta$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\tan^{-1}\frac{\cot\theta}{\sqrt{2}}$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$\Rightarrow$ Definite integral $= -\frac{1}{\sqrt{2}}\tan^{-1}1 + \frac{1}{\sqrt{2}}\tan^{-1}\infty$

$$= -\frac{\pi}{4\sqrt{2}} + \frac{\pi}{2\sqrt{2}} = \frac{\pi}{4\sqrt{2}}$$

b. Putting $e^x - 1 = t^2$ in the given integral, we have

$$\int_0^{\log 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}dx = 2\int_0^2 \frac{t^2}{t^2+4}dt = 2\left(\int_0^2 1dt - 4\int_0^2 \frac{dt}{t^2+4}\right)$$

$$= 2\left[\left(t - 2\tan^{-1}\left(\frac{t}{2}\right)\right)_0^2\right]$$

$$= 2[(2 - 2\times\pi/4)] = 4-\pi$$

a. Put $x = \tan\theta \therefore dx = \sec^2\theta\, d\theta$

When $x = \infty$, $\tan\theta = \infty$, $\therefore \theta = \pi/2$

$$\therefore I = \int_0^{\pi/2} \frac{\tan\theta\sec^2\theta}{(1+\tan\theta)(\sec^2\theta)}d\theta \qquad (1)$$

Now changing equation (1) into $\sin\theta$ and $\cos\theta$

$$\therefore I = \int_0^{\pi/2} \frac{\sin\theta\, d\theta}{\cos\theta + \sin\theta} = \frac{\pi}{4}$$

19.a. Putting $x = \tan\theta$, we get

$$\int_0^{\pi/2} \frac{dx}{[x+\sqrt{x^2+1}]^3} = \int_0^{\infty} \frac{\sec^2\theta\, d\theta}{(\tan\theta+\sec\theta)^3}$$

$$= \int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)^3}d\theta$$

$$= \left[-\frac{1}{2(1+\sin\theta)^2}\right]_0^{\pi/2} = -\frac{1}{8}+\frac{1}{2} = \frac{3}{8}$$

20.c. $I = \int_0^{\pi/2} \frac{\sin x\, dx}{1+\sin x+\cos x}$

$$= \int_0^{\pi/2} \frac{\cos x\, dx}{1+\sin x+\cos x}$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{\sin x+\cos x+1-1}{\sin x+\cos x+1}dx$$

$$\Rightarrow 2I = \frac{\pi}{2} - \log 2$$

$$\Rightarrow I = \frac{\pi}{4} - \frac{1}{2}\log 2$$

21.b. $I_1 = \int_{-100}^{101} \frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}$

$$= \int_{-100}^{101} \frac{dx}{(5+2(1-x)-2(1-x)^2)(1+e^{2-4(1-x)})}$$

$$\Rightarrow 2I_1 = \int_{-100}^{101} \frac{dx}{5+2x-2x^2} = I_2$$

$$\Rightarrow \frac{I_1}{I_2} = \frac{1}{2}$$

22.c. $f(x) = \frac{e^x}{1+e^x} \therefore f(a) = \frac{e^a}{1+e^a}$ and $f(-a) = \frac{e^{-a}}{1+e^{-a}}$

$$= \frac{e^{-a}}{1+\frac{1}{e^a}} = \frac{1}{1+e^a}$$

$$\Rightarrow f(a)+f(-a) = \frac{e^a+1}{1+e^a} = 1$$

Let $f(-a) = \alpha \therefore f(a) = 1-a$

Now, $I_1 = \int_\alpha^{1-\alpha} xg(x(1-x))\,dx$

$$= \int_\alpha^{1-\alpha} (1-x)g((1-x)(1-(1-x))\,dx$$

$$= \int_\alpha^{1-\alpha} (1-x)g(x(1-x))\,dx$$

$\therefore 2I_1 = \int_{\alpha}^{1-\alpha} g(x(1-x))\,dx = I_2 \quad \therefore \frac{I_2}{I_1} = 2$

23.a. We have $f(y) = e^y, g(y) = y : y > 0$

$$F(t) = \int_0^t f(t-y)g(y)\,dy$$
$$= \int_0^t e^{t-y} y\,dy$$
$$= e^t \int_0^t e^{-y} y\,dy$$
$$= e^t\left(\left[-ye^{-y}\right]_0^t + \int_0^t e^{-y}\,dy\right)$$
$$= e^t\left(-te^{-t} - \left[e^{-y}\right]_0^t\right)$$
$$= e^t\left(-te^{-t} - e^{-t} + 1\right)$$
$$= e^t - (1+t)$$

24.c. $I = \int_0^{\sqrt{\ln\left(\frac{\pi}{2}\right)}} \cos\left(e^{x^2}\right) 2xe^{x^2}\,dx$

Put $e^{x^2} = t \Rightarrow e^{x^2} 2x\,dx = dt$

$$\Rightarrow I = \int_1^{\pi/2} \cos t\,dt = [\sin t]_1^{\pi/2} = 1 - (\sin 1)$$

25.a. $\int_1^{\frac{1+\sqrt{5}}{2}} \frac{1+\frac{1}{x^2}}{x^2 - 1 + \frac{1}{x^2}} \log\left(1 + x - \frac{1}{x}\right)dx$

$$= \int_1^{\frac{1+\sqrt{5}}{2}} \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2 + 1} \log\left(1 + x - \frac{1}{x}\right)dx$$

Put $x - \frac{1}{x} = t \quad \therefore \left(1 + \frac{1}{x^2}\right)dx = dt$

If $x = 1, t = 0$, and $x = \frac{\sqrt{5}+1}{2}, t = 1$

$$\Rightarrow I = \int_0^1 \frac{\ln(1+t)\,dt}{1+t^2} \text{ Put } t = \tan\theta \quad \therefore dt = \sec^2\theta\,d\theta$$

$$I = \int_0^{\pi/4} \ln(1+\tan\theta)\,d\theta = \frac{\pi}{8}\log_e 2$$

26.c. As $f(x)$ satisfies the conditions of Rolle's theorem in $[1, 2]$, $f(x)$ is continuous in the interval and $f(1) = f(2)$.

Therefore, $\int_1^2 f'(x)dx = [f(x)]_1^2 = f(2) - f(1) = 0$

27.a. $\int_{-1}^3 \left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx$

$$= \int_{-1}^0 \left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx + \int_0^3 \left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx$$

$$= \int_{-1}^0 -\frac{\pi}{2}dx + \int_0^3 \frac{\pi}{2}dx$$
$$= \left[-\frac{\pi}{2}x\right]_{-1}^0 + \left[\frac{\pi}{2}x\right]_0^3$$
$$= \pi$$

28.c. The polynomial function is differentiable everywhere. Therefore, the points of extremum can only be the roots of the derivative. Further, the derivative of a polynomial is a polynomial. The polynomial of the least degree with roots $x = 1$ and $x = 3$ has the form $a(x-1)(x-3)$.

Hence, $P'(x) = a(x-1)(x-3)$.

Since at $x = 1$, we must have $P(1) = 6$, we have

$$P(x) = \int_1^x P'(x)dx + 6 = a\int_1^x (x^2 - 4x + 3)dx + 6$$
$$= a\left(\frac{x^3}{3} - 2x^2 + 3x - \frac{4}{3}\right) + 6$$

Also, $P(3) = 2$ so $a = 3$. Hence, $P(x) = x^3 - 6x^2 + 9x + 2$.

Thus, $\int_0^1 P(x)dx = \frac{1}{4} - 2 + \frac{9}{2} + 2 = \frac{19}{4}$

29.d. Since $a^2 I_1 - 2aI_2 + I_3 = 0$

$$\Rightarrow \int_0^1 (a-x)^2 f(x)dx = 0$$

Hence, no such positive function $f(x)$.

30.b. $I = \int_0^{\pi/2} \sqrt{\tan x}\,dx \quad (1)$

$$\Rightarrow I = \int_0^{\pi/2} \sqrt{\cot x}\,dx \quad (2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} \left(\sqrt{\tan x} + \sqrt{\cot x}\right)dx$$
$$= \sqrt{2}\int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}dx$$
$$= \sqrt{2}\int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}}dx$$
$$= \sqrt{2}\int_{-1}^1 \frac{dt}{\sqrt{1-t^2}} \quad \text{(where } \sin x - \cos x = t\text{)}$$
$$= 2\sqrt{2}\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \sqrt{2}\pi$$
$$\Rightarrow I = \frac{\pi}{\sqrt{2}}$$

31.a. Let $I = \int_1^3 \frac{\sin 2x}{x}dx$

Put $2x = t, \Rightarrow dx = \frac{dt}{2}$

$$\Rightarrow I=\frac{2}{2}\int_2^6 \frac{\sin t}{t}dt=\int_2^6 \frac{\sin t}{t}dt$$

But given $\int \frac{\sin x}{x}dx=F(x)$

$$\Rightarrow \int_2^6 \frac{\sin t}{t}dt=F(6)-F(2)$$

32.b. $\int_0^1 \cot^{-1}(1-x+x^2)\,dx$

$$=\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx$$

$$=\int_0^1 \tan^{-1}\left(\frac{x+(1-x)}{1-x(1-x)}\right)dx$$

$$=\int_0^1 \tan^{-1}x\,dx+\int_0^1 \tan^{-1}(1-x)dx$$

$$=\int_0^1 \tan^{-1}x\,dx+\int_0^1 \tan^{-1}[1-(1-x)]dx$$

$$=2\int_0^1 \tan^{-1}x\,dx \Rightarrow \lambda=2$$

33.a. Let $I=\int_{-3\pi/4}^{5\pi/4}\frac{(\sin x+\cos x)}{e^{x-\pi/4}+1}dx$

$$\Rightarrow I=\int_{-3\pi/4}^{5\pi/4}\frac{\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)}{e^{x-\pi/4}+1}dx$$

Putting $x-\frac{\pi}{4}=t \quad \Rightarrow dx=dt$

$$\Rightarrow I=\int_{-\pi}^{\pi}\frac{\sqrt{2}\cos t}{e^t+1}dt \qquad (1)$$

Replacing t by $\pi+(-\pi)-t$ or $-t$, we get

$$I=\int_{-\pi}^{\pi}\frac{\sqrt{2}\cos(-t)}{e^{-t}+1}dt=\int_{-\pi}^{\pi}\frac{e^t\sqrt{2}\cos t}{e^t+1}dt \qquad (2)$$

Adding equations (1) and (2), we get

$$2I=\sqrt{2}\int_{-\pi}^{\pi}\cos t\,dt \Rightarrow I=0$$

34.a. $f(2-\alpha)=f(2+\alpha)$

$\Rightarrow$ function is symmetric about the line $x=2$.

$$\int_{2-a}^{2+a} f(x)dx=2\int_2^{2+a} f(x)\,dx$$

35.c. Since e^{x^2} is an increasing function on (0, 1), therefore $m=e^0=1$, $M=e^1=e$ (m and M are minimum and maximum values of $f(x)=e^{x^2}$ in the interval (0, 1))

$\Rightarrow 1<e^{x^2}<e$, for all $x\in(0,1)$

$$\Rightarrow 1(1-0)<\int_0^1 e^{x^2}dx<e(1-0)$$

$$\Rightarrow 1<\int_0^1 e^{x^2}dx<e$$

36.a. $I_2=\int_{-\pi/4}^{\pi/4}\ln(\sin x+\cos x)\,dx$

$$=\int_0^{\pi/4}(\ln(\sin x+\cos x)+\ln(\sin(-x)+\cos(-x)))\,dx$$

$$=\int_0^{\pi/4}(\ln(\sin x+\cos x)+\ln(\cos x-\sin x))\,dx$$

$$=\int_0^{\pi/4}\ln(\cos^2 x-\sin^2 x)\,dx$$

$$=\int_0^{\pi/4}\ln(\cos 2x)\,dx$$

Putting $2x=t$, i.e., $\frac{dt}{2}=dx$, we get

$$I_2=\frac{1}{2}\int_0^{\pi/2}\ln(\cos t)\,dt=\frac{1}{2}\int_0^{\pi/2}\ln\left(\cos\left(\frac{\pi}{2}-t\right)\right)dt$$

$$=\frac{1}{2}\int_0^{\pi/2}\ln(\sin t)dt=\frac{1}{2}I_1 \Rightarrow I_1=2I_2$$

37.c. $I_1=\int_0^{\pi/2}\frac{\cos^2 x}{1+\cos^2 x}dx$

$$=\int_0^{\pi/2}\frac{\cos^2(\pi/2-x)}{1+\cos^2(\pi/2-x)}dx$$

$$=\int_0^{\pi/2}\frac{\sin^2 x}{1+\sin^2 x}dx=I_2$$

Also $I_1+I_2=\int_0^{\pi/2}\left(\frac{\sin^2 x}{1+\sin^2 x}+\frac{\cos^2 x}{1+\cos^2 x}\right)dx$

$$=\int_0^{\pi/2}\frac{\sin^2 x+\sin^2 x\cos^2 x+\cos^2 x+\sin^2 x\cos^2 x}{1+\sin^2 x+\cos^2 x+\sin^2 x\cos^2 x}dx$$

$$=\int\frac{1+2\sin^2 x\cos^2 x}{2+\sin^2 x\cos^2 x}dx=2I_3$$

$2I_1=2I_3 \quad \Rightarrow I_1=I_3 \quad \Rightarrow I_1=I_2=I_3$

38.a. $\sum_{r=1}^{n}\int_0^1 f(r-1+x)\,dx$

$$=\int_0^1 f(x)dx+\int_0^1 f(1+x)dx+\int_0^1 f(2+x)dx+\cdots+\int_0^1 f(n-1+x)dx$$

$$=\int_0^1 f(x)dx+\int_1^2 f(x)dx+\int_2^3 f(x)dx+\int_{r-1}^{2} f(x)dx+\cdots+\int_{n-1}^{n} f(x)dx.=\int_0^n f(x)\,dx$$

39.c. $I_1=\int_0^{\pi/2}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$

$$=\int_0^{\pi/2}\frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx$$

$$= \int_0^{\pi/2} \frac{\cos x - \sin x}{1+\sin x \cos x} dx = -I_1$$

$\Rightarrow I_1 = 0$

$I_3 = 0$ as $\sin^3 x$ is odd.

$$I_4 = \int_0^1 \ln\left(\frac{1-x}{x}\right) dx$$

$$= \int_0^1 \ln\left(\frac{1-(1-x)}{1-x}\right) dx$$

$$= \int_0^1 \ln \frac{x}{1-x} dx = -I_4$$

$\Rightarrow I_4 = 0$

$$I_2 = \int_0^{2\pi} \cos^6 x dx = 2\int_0^{\pi} \cos^6 x\, dx \neq 0$$

40.c. $$I = \int_{\log\lambda}^{\log\frac{1}{\lambda}} \frac{f(x^2/4)[f(x)-f(-x)]}{g(x^2/4)[g(x)+g(-x)]} dx$$

$$= \int_{\log\lambda}^{-\log\lambda} \frac{f(x^2/4)[f(x)-f(-x)]}{g(x^2/4)[g(x)+g(-x)]} = 0$$

(as function inside the integration is odd)

41.b. $$I = 0 + 2\int_0^{\pi} \frac{2x \sin x}{1+\cos^2 x} = 4\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = 4\frac{\pi^2}{4} = \pi^2$$

42.c. $$\int_0^a \frac{f(x)}{f(x)+f(a-x)} = \frac{a}{2}$$

$$\Rightarrow \lim_{n\to\infty}\left[\frac{a}{2}+\frac{a^2}{2}+\frac{a^3}{2}+\cdots+\frac{a^n}{2}\right] = \frac{7}{5}$$

$$\Rightarrow \frac{a}{1-a} = \frac{14}{5}$$

$\Rightarrow 5a = 14 - 14a$

$$\Rightarrow a = \frac{14}{19}$$

43.c. $$f(x) = \int_0^{\pi} \frac{t \sin t}{\sqrt{1+\tan^2 x \sin^2 t}} dt \qquad (1)$$

Replacing t by $\pi - t$ and then adding $f(x)$ with equation (1).

$$f(x) = \frac{\pi}{2}\int_0^{\pi} \frac{\sin t}{\sqrt{1+\tan^2 x \sin^2 t}} dt$$

$$= \pi \int_0^{\pi/2} \frac{\sin t}{\sqrt{1+\tan^2 x(1-\cos^2 t)}} dt$$

$$= \pi \int_0^{\pi/2} \frac{\sin t}{\sqrt{\sec^2 x - \tan^2 x \cos^2 t}} dt$$

Let $y = \cos t$

$\therefore\ dy = -\sin t\, dt$

$$\Rightarrow\ f(x) = \pi \int_0^1 \frac{dy}{\sqrt{\sec^2 x - (\tan^2 x) y^2}}$$

$$= \frac{\pi}{\tan x}\int_0^1 \frac{dy}{\sqrt{\operatorname{cosec}^2 x - y^2}}$$

$$= \frac{\pi}{\tan x}\left\{\sin^{-1}\frac{y}{\operatorname{cosec} x}\right\}_0^1$$

$$= \frac{\pi}{\tan x}\sin^{-1}(\sin x) = \frac{\pi x}{\tan x}$$

44.c. $$I = \int_{-\pi/4}^{3\pi/4} \frac{dx}{\sqrt{2}(e^{x-\pi/4}+1)\cos\left(x-\frac{\pi}{4}\right)}$$

Putting $x - \frac{\pi}{4} = t$, we get

$$\Rightarrow I = \frac{1}{\sqrt{2}}\int_{-\pi/2}^{\pi/2} \frac{dt}{(e^t+1)\cos t}$$

$$= \frac{1}{\sqrt{2}}\int_{-\pi/2}^{\pi/2} \frac{e^t dt}{(e^t+1)\cos t}$$

Adding, we get $2I = \frac{1}{\sqrt{2}}\int_{-\pi/2}^{\pi/2} \sec t dt$

$$\therefore I = \frac{1}{2\sqrt{2}}\int_{-\pi/2}^{\pi/2} \sec x dx \quad \therefore\ k = \frac{1}{2\sqrt{2}}$$

45.a. For $x \in \left(-\frac{\pi}{3}, 0\right)$, $2\cos x - 1 > 0$

$$\Rightarrow I = \int_{-\pi/3}^{0} \frac{\pi}{2} dx = \frac{\pi^2}{6}$$

46.a. $$\int_0^{\infty}\left(\frac{\pi}{1+\pi^2 x^2} - \frac{1}{1+x^2}\right)\log x\, dx$$

$$= \int_0^{\infty} \frac{\log\left(\frac{y}{\pi}\right)dy}{1+y^2} - \int_0^{\infty} \frac{\log x}{1+x^2} dx$$

$$= -\int_0^{\infty} \frac{\log \pi}{1+y^2} dy = -\frac{\pi}{2}\ln \pi$$

47.d. $f(x) = \cos(\tan^{-1} x)$

$$\Rightarrow f'(x) = -\frac{\sin(\tan^{-1} x)}{1+x^2}$$

$$\Rightarrow I = \int_0^1 x\, f''(x)\, dx$$

$$= [x\, f'(x)]_0^1 - \int_0^1 f'(x) dx \quad \text{(Integrating by parts)}$$

$$= [f'(1)] - [f(x)]_0^1$$

$$= f'(1) - f(1) + f(0)$$

Now $f(0) = 1$; $f'(1) = -\frac{1}{2\sqrt{2}}$; $f(1) = \frac{1}{\sqrt{2}}$

$$\Rightarrow I = 1 - \frac{3}{2\sqrt{2}}$$

48.a. Given $f'(1)=\tan\pi/6,\quad f'(2)=\tan\pi/3,\quad f'(3)=\tan\pi/4$

Now, $\int_2^3 f'(x)f''(x)dx+\int_1^3 f''(x)\,dx$

$$=\left[\frac{(f'(x))^2}{2}\right]_2^3+[f'(x)]_1^3$$

$$=\frac{(f'(3))^2-(f'(2))^2}{2}+f'(3)-f'(1)$$

$$=\frac{(1)^2-(\sqrt{3})^2}{2}+\left(1-\frac{1}{\sqrt{3}}\right)$$

$$=\frac{1-3}{2}+1-\frac{1}{\sqrt{3}}=\frac{-1}{\sqrt{3}}$$

49.b. $\int_1^e\left(\frac{\tan^{-1}x}{x}+\frac{\log x}{1+x^2}\right)dx$

$$=\int_1^e\frac{\tan^{-1}x}{x}dx+\int_1^e\frac{\log x}{1+x^2}dx$$

$$=\int_1^e\frac{\tan^{-1}x}{x}dx+\left(\log x\tan^{-1}x\right)_1^e-\int_1^e\frac{\tan^{-1}x}{x}dx$$

$$=\tan^{-1}e$$

50.b. $\int_0^\pi[f(x)+f''(x)]\sin x dx$

$$=\int_0^\pi f(x)\sin x dx+\int_0^\pi f''(x)\sin x dx$$

$$=(f(x)(-\cos x))_0^\pi+\int_0^\pi f'(x)\cos x dx$$

$$+\sin x\, f'(x)|_0^\pi-\int_0^\pi\cos x\, f'(x)dx$$

$=f(\pi)+f(0)=5$ (given)

$\Rightarrow f(0)=5-f(\pi)=5-2=3$

51.b. $I_1=\int_e^{e^4}\sqrt{\ln x}dx$, putting $t=\sqrt{\ln x}$, i.e., $dt=\frac{dx}{2x\sqrt{\ln x}}$

$$\Rightarrow dx=2t\,e^{t^2}\,dt$$

$$\Rightarrow\int_e^{e^4}\sqrt{\ln x}dx$$

$$=\int_1^2 2t^2e^{t^2}dt$$

$$=t\cdot e^{t^2}\Big|_1^2-\int_1^2 e^{t^2}dt=2e^4-e-a$$

52.c. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{e^{|\sin x|}\cos x}{(1+e^{\tan x})}dx$

$$=\int_0^{\frac{\pi}{2}}\left(\frac{e^{|\sin x|}\cos x}{1+e^{\tan x}}+\frac{e^{|\sin x|}\cos x}{1+e^{-\tan x}}\right)dx$$

$$=\int_0^{\frac{\pi}{2}}e^{|\sin x|}\cos x dx$$

$$=\int_0^{\frac{\pi}{2}}e^{\sin x}\cos x dx$$

$$=e^{\sin x}\Big|_0^{\frac{\pi}{2}}=e-1$$

53.d. $\int_0^a x^4\sqrt{a^2-x^2}\,dx$

$$=\left[\frac{-x^3\left(a^2-x^2\right)^{3/2}}{3}\right]_0^a+a^2\cdot\frac{3}{6}\int_0^a x^2\sqrt{a^2-x^2}\,dx$$

(Integrating by parts with x^3 as first function and $x\sqrt{a^2-x^2}$ as second function.)

$$=\frac{a^2}{2}\int_0^a x^2\sqrt{a^2-x^2}\,dx$$

$$\Rightarrow\frac{\int_0^a x^4\sqrt{a^2-x^2}\,dx}{\int_0^a x^2\sqrt{a^2-x^2}\,dx}=\frac{a^2}{2}$$

54.a. $I=\int_0^{\pi/2}\frac{\sin 2x}{x+1}dx$. Put $x=y/2$

$$\Rightarrow I=\int_0^\pi\frac{\sin y}{y+2}dy$$

$$=\left(\frac{-\cos y}{y+2}\right)_0^\pi-\int_0^\pi\frac{\cos y}{(y+2)^2}dy\text{ (integrating by parts)}$$

$$\Rightarrow I=\frac{1}{\pi+2}+\frac{1}{2}-A$$

55.a. $I=\int_0^4\frac{(y^2-4y+5)\sin(y-2)}{(2y^2-8y+1)}dy$, put $y-2=z$

$$\Rightarrow I=\int_{-2}^2\frac{z^2+1}{2z^2-7}\sin(z)dz=0$$

56.a. Putting $x\tan\theta=z\sin\theta\Rightarrow dx=\cos\theta\,dz$

$$\Rightarrow I=\cos\theta\int_{\tan\theta}^1 f(z\sin\theta)\,dz$$

$$=-\cos\theta\int_1^{\tan\theta}f(x\sin\theta)\,dx$$

57.c. $I_1=\int_0^1\frac{e^x dx}{1+x}$, $I_2=\int_0^1\frac{x^2dx}{e^{x^3}(2-x^3)}$

In I_2, put $1-x^3=t$

$$\Rightarrow I_2=\frac{1}{3}\int_1^0\frac{-dt}{e^{1-t}(1+t)}$$

$$=\frac{1}{3e}\int_0^1\frac{e^t dt}{1+t}=\frac{1}{3e}I_1$$

$$\Rightarrow\frac{I_1}{I_2}=3e$$

58.d. $I=\int_{4\pi-2}^{4\pi}\frac{\sin\frac{t}{2}}{4\pi+2-t}dt=\frac{1}{2}\int_{4\pi-2}^{4\pi}\frac{\sin\frac{t}{2}}{1+\left(2\pi-\frac{t}{2}\right)}dt$

5

60

61

62.

63.

Put $2\pi - \frac{t}{2} = z$

$\therefore -\frac{1}{2}dt = dz$, i.e., $dt = -2\,dz$

When $t = 4\pi - 2, z = 2\pi - 2\pi + 1 = 1$

When $t = 4\pi, z = 2\pi - 2\pi = 0$

$$\Rightarrow I = \frac{1}{2}\int_1^0 \frac{\sin(2\pi - z)(-2dz)}{1+z}$$

$$= \int_0^1 \frac{-\sin z\,dz}{z+1} = -\int \frac{\sin t}{1+t}dt = -\alpha$$

59.d. $I = \int_0^1 \frac{\tan^{-1}x}{x}dx$

Putting $x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$

$$\Rightarrow I = \int_0^{\pi/4} \frac{\theta}{\tan\theta}\sec^2\theta d\theta$$

$$= \int_0^{\pi/4} \frac{2\theta}{\sin 2\theta}d\theta$$

Putting $2\theta = t$, i.e., $2d\theta = dt$,

we get $I = \frac{1}{2}\int_0^{\pi/2} \frac{t}{\sin t}dt$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{x}{\sin x}dx.$$

60.a. $I_k = \int_1^e (\ln x)^k\,dx = |x(\ln x)^k|_1^e - k\int_1^e (\ln x)^{k-1}dx$

$\Rightarrow I_k = e - kI_{k-1}$

$\Rightarrow I_4 = e - 4I_3$

$= e - 4[e - 3(e - 2I_1)]$

$= 9e - 24 \quad (\because I_1 = 1)$

61.c. Putting $x = \frac{1}{1+y}, dx = -\frac{1}{(1+y)^2}dy$,

we get $I_{(m,n)} = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

$$= \int_\infty^0 \frac{1}{(1+y)^{m-1}}\left(1 - \frac{1}{1+y}\right)^{n-1}\frac{(-1)}{(1+y)^2}dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

Since, $I(m,n) = I(n,m)$

Therefore, $I(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx.$

62.c. We have $I_{n+1} - I_n = 2\int_0^\pi \cos(n+1)x\,dx = 0$

$\therefore I_{n+1} = I_n \Rightarrow I_{n+1} = I_n = \cdots = I_0 \quad \Rightarrow I_n = \pi$ for all $n \geq 0$

63.b. $\sin nx - \sin(n-2)x = 2\cos(n-1)x\sin x$

$$\Rightarrow \int \frac{\sin nx}{\sin x}dx = \int 2\cos(n-1)\,dx + \int \frac{\sin(n-2)x}{\sin x}dx$$

$$\therefore \int_0^{\pi/2} \frac{\sin 5x}{\sin x}dx = \int_0^{\pi/2} 2\cos 4x\,dx + \int_0^{\pi/2} \frac{\sin 3x}{\sin x}dx$$

$$= 0 + \int_0^{\pi/2} \frac{\sin 3x}{\sin x}dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$$

64.a. $I_3 = \int_0^\pi e^x(\sin x)^3\,dx$

$$= e^x(\sin x)^3\Big|_0^\pi - 3\int_0^\pi (\sin x)^2\cos x\,e^x dx$$

$$= 0 - 3(\sin x)^2\cos x\,e^x\Big|_0^\pi + 3\int_0^\pi (2\sin x\cos x\cos x - \sin x\sin^2 x)\,e^x dx$$

$$= 0 + 6\int_0^\pi \sin x\cos^2 x e^x dx - 3\int_0^\pi \sin^3 x e^x dx$$

$$= 6\int_0^\pi \sin x(1-\sin^2 x)e^x dx - 3\int_0^\pi \sin^3 x e^x dx$$

$$= 6\int_0^\pi \sin x e^x dx - 9\int_0^\pi \sin x^3 e^x dx$$

$= 6I_1 - 9I_3$

$\Rightarrow 10I_3 = 6I_1$

$\Rightarrow \frac{I_3}{I_1} = \frac{3}{5}$

65.b. $I_m = \int_1^e (\log x)^m\,dx$

$I_m = \left[x(\log x)^m\right]_1^e - \int_1^e x\frac{m(\log x)^{m-1}}{x}dx$ (integrating by parts)

$\Rightarrow I_m = e - m\int_1^e (\log x)^{m-1}dx = e - mI_{m-1}$ (1)

Replacing m by $m-1$

$I_{m-1} = e - (m-1)I_{m-2}$ (2)

From equations (1) and (2), we have $I_m = e - m[e - (m-1)I_{m-2}]$

$\Rightarrow I_m - m(m-1)I_{m-1} = e(1-m)$

$\Rightarrow \frac{I_m}{1-m} + mI_{m-2} = e$

$\Rightarrow K = 1 - m$ and $L = \frac{1}{m}$

66.a.

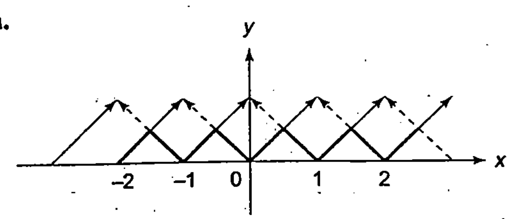

Fig. 8.17

The graph with solid line is the graph of $f(x) = \{x\}$ and the graph with dotted lines is the graph of $f(x) = \{-x\}$. Now the graph of min($\{x\}$, $\{-x\}$) is the graph with dark solid lines.

$\int_{-100}^{100} f(x)\,dx$ = area of 200 triangles shown as solid dark lines in the diagram $= 200\,\frac{1}{2}(1)(\frac{1}{2}) = 50.$

67.c. Put $x - 0.4 = t \Rightarrow \int_{0.6}^{3.6}\{t\}\,dt = \int_{0.6}^{0.6+3}\{t\}\,dt$

$$= 3\int_0^1 (t-[t])\,dt = 3\left(\frac{t^2}{2}\right)_0^1 = \frac{3}{2} = 1.5$$

68.b. Let $I = \int_1^a [x]\,f'(x)\,dx,\ a > 1$

Let $a = k + h$, where $[a] = k$, and $0 \le h < 1$

$$\therefore \int_1^a [x]\,f'(x)\,dx = \int_1^2 1\,f'(x)\,dx + \int_2^3 2\,f'(x)\,dx + \cdots + \int_{k-1}^{k}(k-1)\,f'(x)\,dx + \int_k^{k+h} k\,f'(x)\,dx$$

$$= [f(2)-f(1)] + 2[f(3)-f(2)] + \cdots + (k-1)\,[f(k)-f(k-1)] + k\,[f(k+h)-f(k)]$$

$$= -f(1) - f(2) - f(3) \cdots - f(k) + kf(k+h)$$

$$= [a]f(a) - [f(1) + f(2) + \cdots + f([a])]$$

69.c. $I = \int_0^x [\cos t]dt = \int_0^{2n\pi}[\cos t]dt + \int_{2n\pi}^{x}[\cos t]dt$

$$= n\int_0^{2\pi}[\cos t]\,dt + \int_{2n\pi}^{2n\pi+\pi/2}[\cos t]\,dt + \int_{2n\pi+\frac{\pi}{2}}^{x}[\cos t]\,dt$$

$$= -n\pi + 0 + (x - (2n\pi + \pi/2))\,(-1) = -n\pi + 2n\pi + \pi/2 - x$$

$$= (2n+1)\pi/2 - x$$

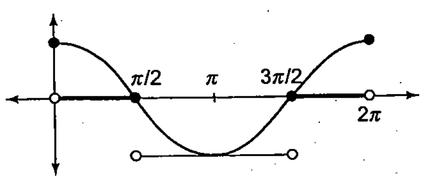

Fig. 8.18

70.a. $f(x) = \int_0^1 \frac{dt}{1+|x-t|} = \int_0^x \frac{dt}{1+x-t} + \int_x^1 \frac{dt}{1-x+t}$

$$\Rightarrow\ f'(x) = \frac{1}{1+x-x} - \frac{1}{1-x+x} = 0$$

71.c. $f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}}$

$$\Rightarrow f'(x) = \frac{1}{\sqrt{1+x^4}} = \frac{dy}{dx}$$

Now $g'(x) = \frac{dx}{dy} = \sqrt{1+x^4}$

when $y = 0$, i.e., $\int_2^x \frac{dt}{\sqrt{1+t^4}} = 0$ then $x = 2$

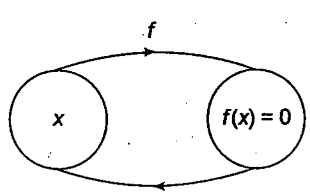

Fig. 8.19

Hence, $g'(0) = \sqrt{1+16} = \sqrt{17}$

72.b. $I = \int_2^4 (x(3-x)(4+x)(6-x)(10-x) + \sin x)\,dx)$ (1)

$$= \int_2^4 ((6-x)(3-(6-x))(4+(6-x))(6-(6-x))\,(10-(6-x)) + \sin(6-x))dx$$

$$= \int_2^4 ((6-x)(x-3)(10-x)x(4+x) + \sin(6-x))dx \quad (2)$$

Adding equations (1) and (2), we get

$$2I = \int_2^4 (\sin x + \sin(6-x))\,dx$$

$$= (-\cos x + \cos(6-x))_2^4$$

$$= -\cos 4 + \cos 2 + \cos 2 - \cos 4$$

$$= 2(\cos 2 - \cos 4)$$

$$\Rightarrow\ I = \cos 2 - \cos 4$$

73.c. $\frac{dx}{dt} = \sin^{-1}(\sin t)\cos t = t\cos t$

and $\frac{dy}{dt} = \frac{\sin t}{\sqrt{t}}\cdot\frac{1}{2\sqrt{t}} = \frac{\sin t}{2t} \Rightarrow \frac{dy}{dx} = \frac{\sin t}{2t.t\cos t} = \frac{\tan t}{2t^2}.$

74.a. $f(x) = \cos x - \int_0^x (x-t)\,f(t)\,dt$

$$\Rightarrow f(x) = \cos x - x\int_0^x f(t)\,dt + \int_0^x tf(t)\,dt$$

$$\Rightarrow f'(x) = -\sin x - xf(x) - \int_0^x f(t)\,dt + xf(x)$$

$$\Rightarrow f'(x) = -\sin x - \int_0^x f(t)\,dt$$

$$\Rightarrow f''(x) = -\cos x - f(x)$$

$$\Rightarrow f''(x) + f(x) = -\cos x$$

75.c. $f^2(x) = \int_0^x f(t)\frac{\cos t}{2+\sin t}\,dt$

$\Rightarrow 2f(x)f'(x) = f(x)\frac{\cos x}{2+\sin x}$ (differentiating w.r.t. x using Leibnitz rule)

$\Rightarrow\ 2f'(x) = \frac{\cos x}{2+\sin x}$ [as $f(x)$ is not zero everywhere]

$$\Rightarrow\ 2\int f'(x)dx = \int\frac{\cos x}{2+\sin x}\,dx$$

$$\Rightarrow 2f(x) = \log_e(2+\sin x) + \log C.$$

Put $x = 0$ we have $2f(0) = \log 2 + \log C$, or $\log C = -\log 2$

$$\Rightarrow f(x) = \frac{1}{2}\ln\left(\frac{2+\sin x}{2}\right);\ x \ne n\pi,\ n \in I$$

76.a. $\lim_{x\to 0}\frac{1}{x}\left[\int_y^a e^{\sin^2 t}dt + \int_a^{x+y} e^{\sin^2 t}dt\right] = \lim_{x\to 0}\frac{1}{x}\int_y^{x+y} e^{\sin^2 t}dt$

$\left(\frac{0}{0}\text{ form}\right)$

Apply L'Hopital Rule

$$= \lim_{x\to 0} \frac{e^{\sin^2(x+y)}\left(1+\frac{dy}{dx}\right) - e^{\sin^2 y}\frac{dy}{dx}}{1}$$

$$= e^{\sin^2 y}\left[1+\frac{dy}{dx}-\frac{dy}{dx}\right] = e^{\sin^2 y}$$

77.a. $f(x) = \int_1^x \frac{e^t}{t}dt \Rightarrow f(1)=0$ and $f'(x) = \frac{e^x}{x}$

Let $g(x) = f(x) - \ln(x), x \in R^+$

$\Rightarrow g'(x) = f'(x) - \frac{1}{x} = \frac{e^x - 1}{x} > 0 \ \forall x \in R^+$

$\Rightarrow g(x)$ is increasing for $x \in R^+$,

$g(1) = f(1) - \ln 1 = 0 - 0 = 0$

$\Rightarrow g(x) > 0 \ \forall x > 1$ and $g(x) \le 0 \ \forall x \in (0, 1]$

$\Rightarrow \ln x \ge f(x) \ \forall x \in (0, 1]$

78.a. $\int_0^x f(t)dt = x + \int_x^1 tf(t)\,dt$

$$\Rightarrow \frac{d}{dx}\left(\int_0^x f(t)dt\right) = \frac{d}{dx}\left(x + \int_x^1 tf(t)dt\right)$$

$\Rightarrow f(x) = 1 + 0 - xf(x)$ [using Leibnitz's Rule]

$\Rightarrow f(x) = 1 - xf(x)$

$\Rightarrow f(x) = \frac{1}{x+1} \Rightarrow f(1) = \frac{1}{2}$

79.b. $\int_{\cos x}^1 t^2 f(t)dt = 1 - \cos x$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}\int_{\cos x}^{1} t^2 f(t)\,dt = \frac{d}{dx}(1-\cos x)$$

$\Rightarrow -\cos^2 x\, f(\cos x)\,(-\sin x) = \sin x$

$\Rightarrow \cos^2 x f(\cos x) \sin x = \sin x$

$\Rightarrow f(\cos x) = \frac{1}{\cos^2 x}$.

Now $f\left(\frac{\sqrt{3}}{4}\right)$ is attained when $\cos x = \frac{\sqrt{3}}{4}$

$f\left(\frac{\sqrt{3}}{4}\right) = \frac{16}{3} = 5.33$

$\left[f\left(\frac{\sqrt{3}}{4}\right)\right] = 5$

80.a. $\int_0^{f(x)} t^2\,dt = x\cos\pi x$ (1)

$\Rightarrow \left.\frac{t^3}{3}\right|_0^{f(x)} = x\cos\pi x$

$\Rightarrow [f(x)]^3 = 3x\cos\pi x$ (2)

$\Rightarrow [f(9)]^3 = -27$

$\Rightarrow f(9) = -3$

Also, differentiating equation (1) w.r.t. x, we get

$[f(x)]^2 f'(x) = \cos\pi x - x\pi\sin\pi x$

$\Rightarrow [f(9)]^2 f'(9) = -1$

$\Rightarrow f'(9) = -\frac{1}{(f(9))^2} = -\frac{1}{9}$

81.b. Given $xf(x) = x + \int_1^x f(t)\,dt$

$f(x) + xf'(x) = 1 + f(x)$

$\Rightarrow f(x) = \log|x| + c$

$f(1) = 1 \quad \Rightarrow \quad f(x) = \log|x| + 1$

$\Rightarrow f(e^{-1}) = 0$

82.c. Given $A = \int_0^1 x^{50}(2-x)^{50}\,dx; B = \int_0^1 x^{50}(1-x)^{50}\,dx$

In A, put $x = 2t \Rightarrow dx = 2dt$

$\Rightarrow A = 2\int_0^{1/2} 2^{50}.t^{50}\,2^{50}(1-t)^{50}\,dt$ (1)

Now, $B = 2\int_0^{1/2} x^{50}(1-x)^{50}\,dx$ (2)

$\left[\text{using } \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(2a-x) = f(x)\right]$

From equations (1) and (2), we get

$A = 2^{100}B$

83.d. The given integrand is a perfect differential coeff. of $\prod_{r=1}^{n}(x+r)$

$$\Rightarrow I = \left[\prod_{r=1}^{n}(x+r)\right]_0^1 = (n+1)! - n! = n\cdot n!$$

84.a. $\int_{-20\pi}^{20\pi} |\sin x|[\sin x]dx$

$= \int_0^{20\pi} |\sin x|([\sin x] + [-\sin x])dx$

$= -20\int_0^{\pi}(\sin x)dx = -20(-\cos x)_0^{\pi} = 20(-2) = -40$

85.b. $\left|\int_a^b f(x)\,dx - (b-a)f(a)\right|$

$= \left|\int_a^b f(x)\,dx - \int_a^b f(a)\,dx\right|$

$= \left|\int_a^b (f(x) - f(a))\,dx\right|$

$$\le \int_a^b |f(x)-f(a)|dx$$

$$\le \int_a^b |x-a|dx = \int_a^b (x-a)dx = \frac{(b-a)^2}{2}$$

86.a. On integrating by parts taking $\sin^2 x$ as first function and $\frac{1}{x^2}$ as second function, we get

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx = \left|\sin^2 x\left(-\frac{1}{x}\right)\right|_0^\infty - \int_0^\infty 2\sin x\cos x\left(-\frac{1}{x}\right)dx$$

Now, $\lim_{x\to\infty} \sin^2 x\left(-\frac{1}{x}\right) = 0$, and

$$\lim_{x\to 0}\frac{\sin^2 x}{x} = \lim_{x\to 0}(\sin x)\frac{\sin x}{x} = 0$$

Thus, $\int_0^\infty \frac{\sin^2 x}{x^2}dx = 0 + \int_0^\infty \frac{\sin 2x}{x}dx$

Now, put $2x = t$, then $dx = dt/2$

$$\int_0^\infty \frac{\sin 2x}{x}dx = \int_0^\infty \frac{\sin t}{t/2}\frac{dt}{2} = \int_0^\infty \frac{\sin t}{t}dt = \int_0^\infty \frac{\sin x}{x}dx$$

87.b. $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

$$\Rightarrow \int_0^\infty \frac{\sin^3 x}{x}dx$$

$$= \frac{3}{4}\int_0^\infty \frac{\sin x}{x}dx - \frac{1}{4}\int_0^\infty \frac{\sin 3x}{x}dx$$

$$= \frac{3}{4}\int_0^\infty \frac{\sin x}{x}dx - \frac{1}{4}\int_0^\infty \frac{\sin u}{u}du \ (u=3x)$$

$$= \frac{3}{4}\frac{\pi}{2} - \frac{1}{4}\frac{\pi}{2} = \frac{\pi}{4}$$

88.a. $I = \int_0^x [\sin t]dt = \int_0^{2n\pi}[\sin t]dt + \int_{2n\pi}^x [\sin t]dt$

$= n\int_0^{2\pi}[\sin t]dt + \int_{2n\pi}^x [\sin t]dt$ (as $[\sin x]$ is periodic with period 2π)

$= -n\pi + 0 = -n\pi$

89.d. $\int_0^x f(t)dt = \int_x^1 t^2 f(t)dt + \frac{x^{16}}{8} + \frac{x^6}{3} + a$ (1)

For $x = 1$, $\int_0^1 f(t)dt = 0 + \frac{1}{8} + \frac{1}{3} + a = \frac{11}{24} + a$

Differentiating both sides of equation (1) w.r.t. x we get,

$f(x) = 0 - x^2 f(x) + 2x^{15} + 2x^5$

$$\Rightarrow f(x) = \frac{2(x^{15}+x^5)}{1+x^2}$$

$$\Rightarrow 2\int_0^1 \frac{x^{15}+x^5}{1+x^2}dx = \frac{11}{24} + a$$

$$\Rightarrow 2\int_0^1 (x^{13} - x^{11} + x^9 - x^7 + x^5)dx = \frac{11}{24} + a$$

$$\Rightarrow 2\left(\frac{1}{14} - \frac{1}{12} + \frac{1}{10} - \frac{1}{8} + \frac{1}{6}\right) = \frac{11}{24} + a$$

$$\Rightarrow a = -\frac{167}{840}$$

90.a. Let $n \le x < n+1$ where $n \in I$

$$I = \int_0^x \frac{2^t}{2^{[t]}}dt = \int_0^n 2^{\{t\}}dt + \int_n^x 2^{\{t\}}dt$$

$$= n\int_0^1 2^{\{t\}}dt + \int_n^x 2^{\{t\}}dt$$

$$= n\int_0^1 2^t dt + \int_n^x 2^{t-n}dt$$

$$= n\left.\frac{2^t}{\ln 2}\right|_0^1 + \frac{1}{2^n}\left.\frac{2^t}{\ln 2}\right|_n^x$$

$$= \frac{n}{\ln 2}(2-1) + \frac{1}{2^n \ln 2}(2^x - 2^n)$$

$$= \frac{n}{\ln 2} + \frac{1}{\ln 2}(2^{x-n} - 1)$$

$$= \frac{[x] + 2^{\{x\}} - 1}{\ln 2}$$

91.b. $\int_{-3}^5 f(|x|)dx = \int_{-3}^3 f(|x|)dx + \int_3^5 f(|x|)dx$

$$= 2\int_0^3 f(x)dx + \int_3^5 f(x)dx$$

$$= 2\left(\int_0^1 f(x)dx + \int_1^2 f(x)dx + \int_2^3 f(x)dx\right) + \left(\int_3^4 f(x)dx + \int_4^5 f(x)dx\right)$$

$$= 2\left(0 + \frac{1}{2} + \frac{2^2}{2}\right) + \left(\frac{9}{2} + \frac{16}{2}\right) = \frac{35}{2}$$

92.c. $f(x) = \int_{\frac{1}{e}}^{\tan x}\frac{t\,dt}{(1+t^2)} + \int_{\frac{1}{e}}^{\cot x}\frac{dt}{t(1+t^2)}$

$$\Rightarrow f'(x) = \frac{\tan x}{1+\tan^2 x}\sec^2 x + \frac{1}{\cot x(1+\cot^2 x)}(-\operatorname{cosec}^2 x)$$

$$= \tan x - \tan x = 0$$

$\Rightarrow f(x)$ is a constant function.

$$f\left(\frac{\pi}{4}\right) = \int_{\frac{1}{e}}^1 \frac{t\,dt}{(1+t^2)} + \int_{\frac{1}{e}}^1 \frac{dt}{t(1+t^2)}$$

$$= \int_{\frac{1}{e}}^1 \frac{1}{t}dt = \ln t\Big|_{1/e}^1 = 1$$

93.c. In I_2, put $x + 1 = t$, then

$$I_2 = \int_{-2}^2 \frac{2t^2+11t+14}{t^4+2}dt = \int_{-2}^2 \frac{2x^2+11x+14}{x^4+2}dx$$

$$\therefore I_1+I_2=\int_{-2}^{2}\frac{x^6+3x^5+7x^4+2x^2+11x+14}{x^4+2}dx$$
$$=\int_{-2}^{2}\frac{(x^2+3x+7)(x^4+2)+5x}{x^4+2}dx$$
$$=\int_{-2}^{2}(x^2+3x+7)dx+5\int_{-2}^{2}\frac{x}{x^4+2}dx$$
$$=2\int_{0}^{2}(x^2+7)dx=\frac{100}{3}$$

(The other integrals are zero, being integrals of odd functions.)

94.b. $I_1=\int_{\sin^2 t}^{1+\cos^2 t} xf(x(2-x))dx$
$$=\int_{\sin^2 t}^{1+\cos^2 t}(2-x)f(x(2-x))dx=2I_2-I_1$$
$$\Rightarrow\quad 2I_1=2I_2\Rightarrow\frac{I_1}{I_2}=1$$

95.b. $I=\int_0^4 f(t)dt$, put $t=x^2$

$\Rightarrow dt=2xdx$, then

$I=2\int_0^2 xf(x^2)\,dx$

From Lagrange's Mean Value Theorem
$$\frac{\int_0^2 2xf(x^2)dx-\int_0^0 2xf(x^2)dx}{2-0}=2yf(y^2)\text{ for some }y\in(0,2)$$
$$\Rightarrow\int_0^2 2xf(x^2)\,dx=2\times 2yf(y^2)$$
$$=2\left\{\frac{2\alpha f(\alpha^2)+2\beta f(\beta^2)}{2}\right\}$$

(where $0<\beta<y<\alpha<2$, and using intermediate Mean Value Theorem.)

96.b. $I=\int_{-3}^{3}x^8\{x^{11}\}dx$ (1)

Replacing x by $-x$, we have $I=\int_{-3}^{3}x^8\{-x^{11}\}dx$ (2)

Adding equations (1) and (2), we get
$$2I=\int_{-3}^{3}x^8\left(\{x^{11}\}+\{-x^{11}\}\right)dx=2\int_0^3 x^8dx=2\left(\frac{x^9}{9}\right)_0^3=2.3^7$$

$\Rightarrow I=3^7$ [as $\{x\}+\{-x\}=1$ for x is not an integer]

97.b. Let $S'=1+x+x^2+x^3+\cdots=\frac{1}{1-x}$

Integrating w.r.t. x, we get $\left|\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right)\right|_0^{1/2}$

$=-|\ln(1-x)|_0^{1/2}$

$\Rightarrow\quad\frac{1}{2}+\frac{1}{2}(S)=\ln 2\Rightarrow S=\ln\frac{4}{e}$

98.a. $f(2x)=f(x)=f\left(\frac{x}{2}\right)=f\left(\frac{x}{2^2}\right)=\cdots=f\left(\frac{x}{2^n}\right)$

So, when $n\to\infty\Rightarrow f(2x)=f(0)$ ($f(x)$ is continuous), i.e., $f(x)$ is a constant function.

$$\Rightarrow f(x)=f(1)=3,\ \int_{-1}^{1}f(f(x))dx=\int_{-1}^{1}3dx=6.$$

99.b. $[x]=0,\forall x\in[0,1)$

For $x\in[1,2)$, $[x]=1$
$$\Rightarrow\frac{[x]}{1+x^2}=\frac{1}{1+x^2}<1,\forall x\in[1,2)\Rightarrow\left[\frac{[x]}{1+x^2}\right]=0$$

For $x\in[-1,0)$, $[x]=-1\Rightarrow\frac{[x]}{1+x^2}=-\frac{1}{1+x^2}$

Clearly, $2\ge 1+x^2>1,\forall x\in[-1,0)$
$$\Rightarrow\frac{1}{2}\le\frac{1}{1+x^2}<1\Rightarrow-\frac{1}{2}\ge-\frac{1}{1+x^2}>-1$$
$$\Rightarrow\left[\frac{[x]}{1+x^2}\right]=-1\ \forall x\in[-1,0)$$

Thus, the given integral $=-\int_{-1}^{0}dx=-1.$

100.c. $g(x)=\int_0^x f(t)dt$
$$g(-x)=\int_0^{-x}f(t)dt=-\int_0^x f(-t)dt=\int_0^x f(t)dt\text{ as }f(-t)=-f(t)$$

$\Rightarrow g(-x)=g(x)$, thus $g(x)$ is even.

Also, $g(x+2)=\int_0^{x+2}f(t)dt$
$$=\int_0^2 f(t)dt+\int_2^{2+x}f(t)dt$$
$$=g(2)+\int_0^x f(t+2)dt$$
$$=g(2)+\int_0^x f(t)dt$$
$$=g(2)+g(x)$$

Now, $g(2)=\int_0^2 f(t)dt=\int_0^1 f(t)dt+\int_1^2 f(t)dt$
$$=\int_0^1 f(t)dt+\int_{-1}^{0}f(t+2)dt$$
$$=\int_0^1 f(t)dt+\int_{-1}^{0}f(t)dt$$
$$=\int_{-1}^{1}f(t)dt=0\text{ as }f(t)\text{ is odd}$$

$\Rightarrow g(2)=0\Rightarrow g(x+2)=g(x)\Rightarrow g(x)$ is periodic with period 2.

$\Rightarrow g(4)=0\Rightarrow f(6)=0, g(2n)=0, n\in N.$

101.c. $\int_0^x|\sin t|\,dt=\int_0^{2n\pi}|\sin t|\,dt+\int_{2n\pi}^{x}|\sin t|\,dt$

$$= 2n\int_0^{\pi}|\sin t|\,dt + \int_{2n\pi}^{x}\sin t\,dt \text{ (as } x \text{ lies in either 1}^{st}\text{ or 2}^{nd}\text{ quadrant)}$$

$$= 2n(-\cos t)_0^{\pi} + (-\cos t)_{2n\pi}^{x} = 4n - \cos x + 1$$

102.c. $$f(x) = \begin{cases} \int_{-1}^{x} -t\,dt & -1 \le x \le 0 \\ \int_{-1}^{0} -t\,dt + \int_0^x t\,dt & x \ge 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}(1-x^2), & -1 \le x \le 0 \\ \frac{1}{2}(1+x^2), & x \ge 0 \end{cases}$$

103.b. $$g\left(x+\frac{\pi n}{2}\right) = \int_0^{x+\frac{n\pi}{2}} (|\sin t| + |\cos t|)\,dt$$

$$= \int_0^{x} (|\sin t| + |\cos t|)\,dt + \int_x^{x+\frac{n\pi}{2}} (|\sin t| + |\cos t|)\,dt$$

$$= g(x) + \int_0^{\frac{n\pi}{2}} (|\sin t| + |\cos t|)\,dt \text{ (as } |\sin t| + |\cos t| \text{ has a period } \pi/2)$$

$$= g(x) + g\left(\frac{n\pi}{2}\right)$$

104.c.

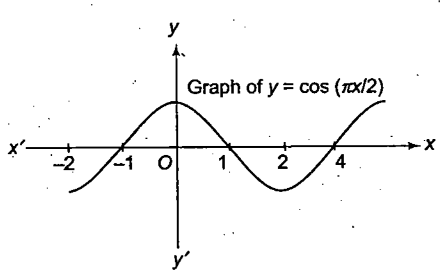

Fig. 8.20

From graph, $$\int_{-2}^{1}\left[x\left[1+\cos\frac{\pi x}{2}\right]+1\right]dx$$

$$= \int_{-2}^{-1}[x[1+(-1)]+1]dx + \int_{-1}^{1}[x[1+0]+1]dx$$

$$= (x)_{-2}^{-1} + \int_{-1}^{1}[x+1]dx = (-1-(-2)) + \int_{-1}^{0} 0\,dx + \int_0^1 1\,dx = 2$$

105.b. $$I = \int_{-a}^{a}\left(\cos^{-1}x - \sin^{-1}\sqrt{1-x^2}\right)dx$$

$$= \int_{-a}^{0}\cos^{-1}x\,dx + A - 2\int_0^a \sin^{-1}\sqrt{1-x^2}\,dx$$

$$= \int_0^a\left(\pi - \cos^{-1}x\right)dx + A - 2A$$

$$= a\pi - 2A \Rightarrow \lambda = 2$$

106.b. Put $x = a\cos^2\theta + b\sin^2\theta, \Rightarrow dx = 2(b-a)\sin\theta\cos\theta\,d\theta$, then

$$\int_a^b (x-a)^3(b-x)^4\,dx$$

$$= 2(b-a)\int_0^{\pi/2}(a\cos^2\theta + b\sin^2\theta - a)^3(b - a\cos^2\theta - b\sin^2\theta)^4\sin\theta\cos\theta\,d\theta$$

$$= 2(b-a)^8\int_0^{\pi/2}\sin^7\theta\cos^9\theta\,d\theta$$

$$= 2(b-a)^8\int_0^{\pi/2}\sin^7\theta(1-\sin^2\theta)^4\cos\theta\,d\theta$$

$$= 2(b-a)^8\int_0^1 x^7(1-x^2)^4\,dx$$

$$= 2(b-a)^8\int_0^1 x^7(1-x^2)^4\,dx$$

$$= 2(b-a)^8\int_0^1 x^7(1-4x^2+6x^4-4x^6+x^8)\,dx$$

$$= 2(b-a)^8\left[\frac{1}{8}-\frac{4}{10}+\frac{6}{12}-\frac{4}{14}+\frac{1}{16}\right] = \frac{(b-a)^8}{280}$$

107.a. $$I = b\int_0^t \frac{1}{x}\cos 4x\,dx - a\int_0^t \frac{1}{x^2}\sin 4x\,dx$$

$$= bI_1 - aI_2$$

$$I_2 = \int_0^t \frac{1}{x^2}\sin 4x\,dx$$

$$= \left\{\left[-\frac{1}{x}\sin 4x\right]_0^t + 4\int_0^t \frac{\cos 4x}{x}dx\right\}$$

$$= \left[-\frac{\sin 4t}{t} + 4 + 4I_1\right], \left\{\lim_{x\to 0}\frac{\sin 4x}{x} = 4\right\}$$

$$\therefore I = bI_1 - a\left\{-\frac{\sin 4t}{t} + 4 + 4I_1\right\}$$

$$= (b-4a)\int_0^t \frac{1}{x}\cos 4x\,dx + \frac{a\sin 4t}{t} - 4a$$

$$= \frac{a\sin 4t}{t} - 1$$

Therefore, $(b-4a)\int_0^t \frac{1}{x}\cos 4x\,dx = 4a - 1$

L.H.S. is a function of t, whereas R.H.S. is a constant. Hence, we must have $b - 4a = 0$ and $4a - 1 = 0$.

$$\therefore a = \frac{1}{4}, b = 1$$

108.b. Given $\lambda = \int_0^1 \frac{e^t}{1+t}dt$

$$\int_0^1 e^t\log_e(1+t)dt = \left[\log_e(1+t)e^t\right]_0^1 - \int_0^1 \frac{e^t}{1+t} = e\log_e 2 - \lambda$$

109.b. $$I_1 - I_2 = \int_0^{\pi/2}(\cos\theta - \sin 2\theta)f(\sin\theta + \cos^2\theta)\,d\theta$$

Put $t = \sin\theta + \cos^2\theta \Rightarrow dt = (\cos\theta - \sin 2\theta)\,d\theta$

$$\Rightarrow I_1 - I_2 = \int_1^1 f(t)dt = 0$$

110.d. We have $f(x)=\int_{-1}^{1}\frac{\sin x\,dt}{\sin^2 x+(t-\cos x)^2}$

$= \frac{\sin x}{\sin x}\tan^{-1}\left(\frac{t-\cos x}{\sin x}\right)\Bigg|_{-1}^{1}$

$= \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)-\tan^{-1}\left(\frac{-1-\cos x}{\sin x}\right)$

$= \tan^{-1}(\tan x/2)+\tan^{-1}(\cot x/2)$

Now, we know that $\tan^{-1}x+\tan^{-1}\frac{1}{x}=\begin{cases}\frac{\pi}{2}, & x>0\\ -\frac{\pi}{2}, & x<0\end{cases}$

$\Rightarrow \tan^{-1}\left(\tan\frac{x}{2}\right)+\tan^{-1}\left(\frac{1}{\tan\frac{x}{2}}\right)=\begin{cases}\frac{\pi}{2}, & \tan\frac{x}{2}>0\\ -\frac{\pi}{2}, & \tan\frac{x}{2}<0\end{cases}$

Hence, range of $f(x)$ is $\left\{-\frac{\pi}{2},\frac{\pi}{2}\right\}$.

111.c. Let $A=\lim_{n\to\infty}\left[\tan\frac{\pi}{2n}\tan\frac{2\pi}{2n}\cdots\tan\frac{n\pi}{2n}\right]^{1/n}$

$\therefore \log A=\lim_{n\to\infty}\frac{1}{n}\left[\log\tan\frac{\pi}{2n}+\log\tan\frac{2\pi}{2n}+\cdots+\log\tan\frac{n\pi}{2n}\right]$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\tan\frac{\pi r}{2n}=\int_0^1\log\tan\left(\frac{\pi}{2}x\right)dx$

$= \frac{2}{\pi}\int_0^{\pi/2}\log\tan y\,dy$ (1)

$\left[\text{Putting } \frac{1}{2}\pi x=y \ \therefore dx=(2/\pi)\,dy\right]$

Now let $I=\int_0^{\pi/2}\log\tan y\,dy$

$I=\int_0^{\pi/2}\log\tan\left(\frac{1}{2}\pi-y\right)dy$ (by Property IV)

$= \int_0^{\pi/2}\log\cot y\,dy$

$= -\int_0^{\pi/2}\log\tan y\,dy=-I$

or $I+I=0$ or $2I=0$ or $I=0$

$\therefore$ from equation (1), $\log A=0$ $\therefore A=e^0=1$

112.b. Differentiating, we get $f''(x)=f'(x)$

$\Rightarrow \int\frac{df'(x)}{f'(x)}=\int dx \Rightarrow \ln f'(x)=x+c \Rightarrow f'(x)=Ae^x$ (1)

$\Rightarrow \int f'(x)dx=\int Ae^x dx \Rightarrow f(x)=Ae^x+B$ (2)

Now, $f(0)=1 \Rightarrow A+B=1$

$\therefore f'(x)=f(x)+\int_0^1(Ae^x+1-A)dx$

$Ae^x=(Ae^x+1-A)+|(Ae^x+(1-A)x)|_0^1$

$\Rightarrow 1-A+(Ae+1-A-A)=0$

$\Rightarrow A(e-3)=-2$

$\Rightarrow A=\frac{2}{3-e}$ and $B=1-\frac{2}{3-e}=\frac{1-e}{3-e}$

$\Rightarrow f(\log_e 2)=\frac{4}{3-e}+\frac{1-e}{3-e}=\frac{5-e}{3-e}$

113.b. $\int_a^b f(x)dx=[xf(x)]_a^b-\int_a^b xf'(x)dx$ (1)

Now, put $f(x)=t$ $\therefore x=f^{-1}(t)$

and $f'(x)\,dx=dt$ and adjust the limits

Therefore, $\int_a^b f(x)\,dx=[bf(b)-af(a)]-\int_{f(a)}^{f(b)}f^{-1}(t)\,dt$ by (1)

$\therefore \int_a^b f(x)+\int_{f(a)}^{f(b)}f^{-1}(x)\,dx=bf(b)-af(a)$

114.b. $2I=\int_\alpha^\beta\frac{e^{f\left(\frac{g(x)}{x-\alpha}\right)}dx}{e^{f\left(\frac{g(x)}{x-\alpha}\right)}+e^{f\left(\frac{g(x)}{x-\beta}\right)}}+\int_\alpha^\beta\frac{e^{f\left(\frac{g(\alpha+\beta-x)}{\beta-x}\right)}dx}{e^{f\left(\frac{g(\alpha+\beta-x)}{\beta-x}\right)}+e^{f\left(\frac{g(\alpha+\beta-x)}{\alpha-x}\right)}}$

$\Rightarrow I=\frac{1}{2}(\beta-\alpha)=\frac{\sqrt{b^2-4ac}}{2a}$

($\because f(x)$ is even function $\Rightarrow \alpha+\beta=0$)

115.a. $y^r=\left(1+\frac{1}{r}\right)\left(1+\frac{2}{r}\right)\left(1+\frac{3}{r}\right)\cdots\left(1+\frac{n-1}{r}\right)$

$\Rightarrow \log y=\frac{1}{r}\sum_{p=1}^{n-1}\log\left(1+\frac{p}{r}\right)$

$\Rightarrow \lim_{n\to\infty}y=\lim_{r\to\infty}y=\int_0^k\log(1+x)dx=(k-1)\log_e(1+k)-k$

116.a. When $e\le[x]\le e^2$ $1<\log[x]<2$

when $e^2\le[x]\le e^3$ $2<\log[x]<3$

$\therefore \int_3^8 1\,dx+\int_8^{10}2\,dx=9$

117.c. Let $g(x)=\int_0^{x^3}f(t)dt$

Now $\int_0^8 f(t)dt=g(2)=\frac{g(2)-g(1)}{2-1}+\frac{g(1)-g(0)}{1-0}$

$=g'(\alpha)+g'(\beta)$

$=3[\alpha^2 f(\alpha^3)+\beta^2 f(\beta^3)]$

118.a. $I=\int_0^1 f(x)\,[g(x)-g(1-x)]\,dx$

$=-\int_0^1 f(1-x)\,[g(x)-g(1-x)]\,dx$

$\Rightarrow 2I=\int_0^1 [f(x)-f(1-x)]\,[g(x)-g(1-x)]\,dx\le 0$

Multiple Correct Answers Type

.a, b.

$$f(x)=e^x+\int_0^1 e^x\ f(t)\ dt=e^x+k\,e^x \text{ where } k=\int_0^1 f(t)\,dt$$

$$\therefore k=\int_0^1\left(e^t+ke^t\right)dt=e+ke-1-k$$

$$\therefore k=\frac{e-1}{2-e},\ \text{thus } f(x)=e^x\left(1+\frac{e-1}{2-e}\right)=\frac{e^x}{2-e}$$

Obviously, $f(0)=\dfrac{1}{2-e}<0$

Also, $f'(x)=\dfrac{e^x}{2-e}<0$ for $\forall x\in R$.

Hence, $f(x)$ is a decreasing function.

Also, $\int_0^1 f(x)\ dx$

$$=\int_0^1\frac{e^x}{2-e}dx$$

$$=\left[\frac{e^x}{2-e}\right]_0^1$$

$$=\frac{e-1}{2-e}<0$$

.a, d.

$$f'(x)=\frac{3^x}{1+x^2}>0\ \forall x>0\Rightarrow f'(x)=\frac{3^x}{1+x^2}>\frac{1}{1+x^2},\forall x\geq 1$$

$$\Rightarrow\int_1^x f'(x)dx>\int_1^x\frac{1}{1+x^2}dx$$

$$\Rightarrow f(x)>\tan^{-1}x-\tan^{-1}1\Rightarrow f(x)+\pi/4>\tan^{-1}x$$

a, b, c.

For $a\leq 0$,

given equation becomes

$$\int_0^2(x-a)dx\geq 1\Rightarrow a\leq\frac{1}{2}\Rightarrow a\leq 0$$

For $0<a<2$,

$$\int_0^2|x-a|\,dx\geq 1\Rightarrow\int_0^a(a-x)dx+\int_a^2(x-a)dx\geq 1$$

$$\Rightarrow\frac{a^2}{2}+2-2a+\frac{a^2}{2}\geq 1\Rightarrow a^2-2a+1\geq 0\Rightarrow(a-1)^2\geq 0$$

For $a\geq 2$,

$$\int_0^2|x-a|\,dx\geq 1\Rightarrow\int_0^2(a-x)dx\geq 1\Rightarrow 2a-2\geq 1\Rightarrow a\geq\frac{3}{2}$$

$$\Rightarrow a\geq 2$$

a, b,

We know $\int_a^b|\sin x|\,dx$ represents the area under the curve from $x=a$ to $x=b$. We also know that area from $x=a$ to $x=a+\pi$ is 2.

$$\therefore\int_a^b|\sin x|\,dx=8\Rightarrow b-a=\frac{8\pi}{2}\quad(1)$$

Similarly, $\int_0^{a+b}|\cos x|\,dx=9\Rightarrow a+b-0=\dfrac{9\pi}{2}$ (2)

From (1) and (2), $a=\dfrac{\pi}{4}$ and $b=\dfrac{17\pi}{4}$

$$\Rightarrow|a+b|=\frac{9\pi}{2},|a-b|=4\pi,\ \frac{a}{b}=17\ \text{and}$$

Obviously $\int_a^b\sec^2 x dx\neq 0$

5. c. Let $f(x)=\sqrt{3+x^3}$

Clearly, $f(x)$ is increasing in $[1,3]$

$\Rightarrow$ The least value of the function, $m=f(1)=\sqrt{3+1^3}=2$

and the greatest value of the function, $M=f(3)=\sqrt{3+3^3}$
$=\sqrt{30}$

Therefore, $(3-1)\ 2\leq\int_1^3\sqrt{3+x^3}dx\leq(3-1)\sqrt{30}$

Here, $4\leq\int_1^3\sqrt{3+x^3}dx\leq 2\sqrt{30}$

6.a, b, c.

$$g(x)=\int_0^x 2|t|\,dt$$

$$=\begin{cases}\int_0^x -2tdt, & x<0\\ \int_0^x 2tdt, & x\geq 0\end{cases}$$

$$=\begin{cases}\left[-t^2\right]_0^x, & x<0\\ \left[t^2\right]_0^x, & x\geq 0\end{cases}$$

$$=\begin{cases}-x^2, & x<0\\ x^2, & x\geq 0\end{cases}$$

$$=x|x|$$

Clearly, continuous and differentiable at $x=0$

Also, $g'(x)=\begin{cases}-2x, & x<0\\ 2x, & x>0\end{cases}$ which is non-differentiable at $x=0$.

7.a, b.

$$f(x)=x\int_1^x\frac{e^t}{t}dt-e^x$$

$$\Rightarrow f'(x)=x\frac{e^x}{x}+\int_1^x\frac{e^t}{t}dt-e^x$$

9.a

10.a

$$\Rightarrow f'(x)=\int_1^x \frac{e^t}{t}dt>0\,[\because x\in[1,\infty)]$$

$\Rightarrow f(x)$ is an increasing function.

8.a, c, d.

$$I=\int_0^1 \frac{2x^2+3x+3}{(x+1)\left(x^2+2x+2\right)}$$

$$=\int_0^1 \frac{2(x^2+2x+2)-(x+1)}{(x+1)\left(x^2+2x+2\right)}dx$$

$$=\int_0^1\left(\frac{2}{x+1}-\frac{1}{x^2+2x+2}\right)dx$$

$$=\left[2\log(x+1)-\tan^{-1}(x+1)\right]_0^1$$

$$=2\log 2-\tan^{-1}2+\tan^{-1}1 \qquad (1)$$

$$=2\log 2-\tan^{-1}2+\frac{\pi}{4}$$

$$=\log 4-\left(\frac{\pi}{2}-\cot^{-1}2\right)+\frac{\pi}{4}$$

$$=-\frac{\pi}{4}+\log 4+\cot^{-1}2$$

From equation (1), $I=2\log 2-\tan^{-1}\left(\frac{2-1}{1+2\times 1}\right)$

$$=2\log 2-\tan^{-1}\frac{1}{3}$$

$$=2\log 2-\cot^{-1}3$$

9.a, d.

$$A_{n+1}-A_n$$

$$=\int_0^{\pi/2}\frac{\sin(2n+1)x-\sin(2n-1)x}{\sin x}$$

$$=\int_0^{\pi/2}2\cos 2nx\,dx=0$$

$$\Rightarrow A_{n+1}=A_n$$

$$B_{n+1}-B_n$$

$$=\int_0^{\pi/2}\frac{\sin^2(n+1)x-\sin^2 nx}{\sin^2 x}dx$$

$$=\int_0^{\pi/2}\frac{\sin(2n+1)x}{\sin x}dx$$

$$=A_{n+1}$$

10.a, b, c.

$$f(x)=\int_a^x\frac{1}{f(x)}dx\Rightarrow f'(x)=\frac{1}{f(x)}\cdot 1-0\Rightarrow f(x)f'(x)=1$$

$$\Rightarrow\int f(x)f'(x)dx=\int 1dx$$

$$\Rightarrow\frac{1}{2}[f(x)]^2=x+c \qquad (1)$$

Now given that $\int_a^1[f(x)]^{-1}dx=\sqrt{2}\Rightarrow f(1)=\sqrt{2}$

$\Rightarrow$ From (1), $\frac{1}{2}[f(1)]^2=1+c\Rightarrow c=0$

$$\Rightarrow f(x)=\pm\sqrt{2x}$$

But $f(1)=\sqrt{2}\Rightarrow f(x)=\sqrt{2x}\Rightarrow f(2)=2$

Also, $f'(x)=\frac{1}{\sqrt{2x}}\Rightarrow f'(2)=1/2$

$$\int_0^1 f(x)dx=\int_0^1\sqrt{2x}dx=\left[\frac{(2x)^{3/2}}{3}\right]_0^1=\frac{(2)^{3/2}}{3}$$

Also, $f^{-1}(x)=\frac{x^2}{2}\Rightarrow f^{-1}(2)=2$

11.b, c.

$$I=\int_0^\infty\frac{dx}{1+x^4} \qquad (1)$$

$$=\int_0^\infty\frac{x^2+1-x^2}{1+x^4}dx$$

$$=\int_0^\infty\frac{x^2}{1+x^4}dx+\int_0^\infty\frac{1-x^2}{1+x^4}dx=I_1+I_2$$

$$I_2=\int_0^\infty\frac{\frac{1}{x^2}-1}{\frac{1}{x^2}+x^2}dx,$$

Put $x+\frac{1}{x}=y$

$$\Rightarrow I_2=\int_\infty^\infty\frac{-1}{y^2-2}dy=0$$

$$\Rightarrow I=\int_0^\infty\frac{dx}{1+x^4}=\int_0^\infty\frac{x^2dx}{1+x^4} \qquad (2)$$

Adding equations (1) and (2), we get

$$\Rightarrow 2I=\int_0^\infty\frac{1+x^2dx}{1+x^4}=\int_0^\infty\frac{\frac{1}{x^2}+1}{\frac{1}{x^2}+x^2}dx,\text{ put } x-\frac{1}{x}=y$$

$$\Rightarrow 2I=\int_{-\infty}^\infty\frac{dy}{y^2+2}=\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{y}{\sqrt{2}}\right]_{-\infty}^\infty=\frac{\pi}{\sqrt{2}}$$

$$\Rightarrow I=\frac{\pi}{2\sqrt{2}}$$

12.a, b, d.

Given that $f(x)=\int_0^x|t-1|\,dt$

$$\Rightarrow f(x)=\int_0^x(1-t)dt,\ 0\le x\le 1$$

$$=x-\frac{x^2}{2}$$

Also $f(x)=\int_0^1(1-t)dt+\int_1^x(t-1)dt$, where $1\le x\le 2$

$= \frac{1}{2} + \frac{x^2}{2} - x + \frac{1}{2} = \frac{x^2}{2} - x + 1$

Thus, $f(x) = \begin{cases} x - \frac{x^2}{2}, & 0 \le x \le 1 \\ \frac{x^2}{2} - x + 1, & 1 < x \le 2 \end{cases}$

$\Rightarrow f'(x) = \begin{cases} 1 - x, & 0 \le x < 1 \\ x - 1, & 1 < x < 2 \end{cases}$

Thus, $f(x)$ is continuous as well as differentiable at $x = 1$.
Also, $f(x) = \cos^{-1} x$ has one real root, draw the graph and verify.

For range of $f(x)$:

$f(x) = \int_0^x |t-1|\,dt$ is the value of area bounded by the curve $y = |t-1|$ and x-axis between the limits $t = 0$ and $t = x$.
Obviously, minimum area is obtained when $t = 0$ and $t = x$ coincide or $x = 0$.
Maximum value of area occurs when $t = 2$,
hence $f(2)$ = area of shaded region = 1.

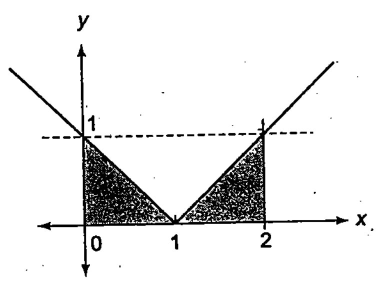

Fig. 8.21

13.b, c, d.

$I_n = \int_0^{\pi/4} \tan^n x\,dx$

$= \int_0^{\pi/4} \tan^{n-2} x \tan^2 x\,dx$

$= \int_0^{\pi/4} \sec^2 x \tan^{n-2} x\,dx - \int_0^{\pi/4} \tan^{n-2} x\,dx$

$= \int_0^1 t^{n-2}\,dt - I_{n-2}$ where $t = \tan x$

$I_n + I_{n-2} = \left(\frac{t^{n-1}}{n-1}\right)_0^1$

$\Rightarrow I_n + I_{n-2} = \frac{1}{n-1}$

$\Rightarrow I_2 + I_4, I_4 + I_6, \ldots$ are in H.P.
For $0 < x < \pi/4$, we have $0 < \tan^n x < \tan^{n-2} x$
So that $0 < I_n < I_{n-2} \Rightarrow I_n + I_{n+2} < 2I_n < I_n + I_{n-2}$

$\Rightarrow \frac{1}{n+1} < 2I_n < \frac{1}{n-1} \Rightarrow \frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$

14.a, b, c.

Let $I = \int_a^b \frac{f(x)}{f(x) + f(a+b-x)}\,dx$ (1)

$= \int_a^b \frac{f(a+b-x)}{f(a+b-x) + f(x)}\,dx$ (2)

Adding equations (1) and (2), we get

$\Rightarrow 2I = \int_a^b 1\,dx = b - a$

$\Rightarrow I = \left(\frac{b-a}{2}\right) = 10$ (given)

$\therefore b - a = 20$

15.a, b, d.

$I_n = \int_0^1 \frac{dx}{(1+x^2)^n} = \int_0^1 (1+x^2)^{-n}\,dx$

$= \frac{x}{(1+x^2)^n}\Bigg|_0^1 - \int_0^1 (-n)(1+x^2)^{-n-1} 2x \times x\,dx$

$= \frac{1}{2^n} + 2n \int_0^1 \frac{x^2\,dx}{(1+x^2)^{n+1}}$

$= \frac{1}{2^n} + 2n \int_0^1 \frac{1+x^2-1}{(1+x^2)^{n+1}}\,dx$

$= \frac{1}{2^n} + 2nI_n - 2nI_{n+1}$

$\Rightarrow 2nI_{n+1} = 2^{-n} + (2n-1)I_n$

$\Rightarrow 2I_2 = \frac{1}{2} + I_1 = \frac{1}{2} + \tan^{-1} x|_0^1$

$\Rightarrow I_2 = \frac{1}{4} + \frac{\pi}{8}$

Also $4I_3 = 2^{-2} + 3I_2$

$= \frac{1}{4} + 3\left(\frac{1}{4} + \frac{\pi}{8}\right) = \frac{1}{4} + \frac{3\pi}{32}$

16.b, c.

$\lim_{n\to\infty} \frac{1}{n} \sum_{r=n+1}^{2n} f\left(\frac{r}{n}\right) = \int_1^2 f(x)\,dx$

$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r+n}{n}\right) = \int_0^1 f(1+x)\,dx$

$= \int_1^2 f(t)\,dt = \int_1^2 f(x)\,dx$

$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$

$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} f\left(\frac{r}{n}\right) = \int_0^2 f(x)\,dx$

17.a, b, d.

$f(2-x) = f(2+x), f(4-x) = f(4+x)$

$\Rightarrow f(4+x) = f(4-x) = f(2+2-x) = f(2-(2-x)) = f(x)$

$\Rightarrow 4$ is a period of $f(x)$

$\int_0^{50} f(x)\,dx = \int_0^{48} f(x)\,dx + \int_{48}^{50} f(x)\,dx$

$$= 12\int_0^4 f(x)dx + \int_0^2 f(x)\,dx$$

(in second integral replacing x by $x+48$ and then using $f(x)=f(x+48)$)

$$= 12\left(\int_0^2 f(x)\,dx + \int_0^2 f(4-x)\,dx\right)+5$$

$$= 12\left(\int_0^2 f(x)\,dx + \int_0^2 f(4+x)dx\right)+5$$

$$= 24\int_0^2 f(x)\,dx + 5 = 125$$

$$\int_{-4}^{46} f(x)dx = \int_{-4}^{-2} f(x)dx + \int_{-2}^{-2+48} f(x)dx$$

$$= \int_0^2 f(x+4)dx + 12\int_0^4 f(x)dx$$

$$= \int_0^2 f(x)dx + 24\int_0^2 f(x)dx$$

$$=125$$

Also $\int_2^{52} f(x)dx = \int_2^4 f(x)dx + \int_4^{4+48} f(x)dx$

$$= \int_0^2 f(4-x)dx + 12\int_0^4 f(x)dx$$

$$= \int_0^2 f(4+x)dx + 24\int_0^2 f(x)dx$$

$$= \int_0^2 f(x)dx + 24\int_0^2 f(x)dx$$

$$=125$$

$$\int_1^{51} f(x)dx = \int_1^3 f(x)dx + \int_3^{3+48} f(x)dx$$

$$= \int_1^3 f(x)dx + 12\int_0^4 f(x)dx$$

$$= \int_0^2 f(x+1)dx + 24\int_0^2 f(x)dx$$

$$\neq 125$$

18.a, b.

L.H.S. $= \int_0^x \left\{\int_0^u f(t)\,dt\right\} du$

Integrating by parts choose '1' as the second funciton

$$= \left\{u\int_0^u f(t)\,dt\right\}_0^x - \int_0^x f(u)\,u\,du$$

$$= x\int_0^x f(t)\,dt - \int_0^x f(u)\,u\,du$$

$$= x\int_0^x f(u)\,du - \int_0^x f(u)\,u\,du = \int_0^x f(u)(x-u)du$$

= R.H.S.

19. a, c, d.

The expression $f(x)f(c)\ \forall\, x \in (c-h, c+h)$ where $h \to 0^+$ is equivalent to $\lim_{x\to 0} f(x)f(c)$ which equals to $(f(c))^2$ because $f(x)$ is continuous.

Therefore, $f(x)f(c) > 0\ \forall\, x \in (c-h, c+h)$ where $h \to 0^+$.

a. We have $I = \lim_{n\to\infty} \frac{1}{n}\ln\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]$

$$= \lim_{n\to\infty}\frac{1}{n}\ln\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)$$

$$= \int_1^2 \ln x\,dx = [x(\ln x - 1)]_1^2 = -1 + 2\ln 2$$

c. Given $f(x) \ge 0 \Rightarrow \int_a^b f(x)dx \ge 0$.

But given $\int_a^b f(x)dx = 0$, so this can be true only when $f(x)=0$.

d. $\int_a^b f(x)dx = 0 \Rightarrow y = f(x)$ cuts x axis at least once.

So, there exists at least one $c \in (a, b)$ for which $f(c) = 0$.

20.a, c.

$$\int_0^1 e^{x^2-x}\,dx$$

For $x \in (0, 1), x^2 - x \in (-1/4, 0)$

$$\Rightarrow e^{-1/4} < e^{x^2-x} < e^0$$

$$\Rightarrow e^{-\frac{1}{4}} < \int_0^1 e^{x^2-x}\,dx < 1$$

21.a, d.

$$f(x+\pi) = \int_0^{x+\pi} (\cos(\sin t) + \cos(\cos t))\,dt$$

$$= \int_0^{\pi} (\cos(\sin t) + \cos(\cos t))\,dt$$

$$+ \int_{\pi}^{x+\pi} (\cos(\sin t) + \cos(\cos t))\,dt$$

$$= f(\pi) + \int_0^x (\cos(\sin t) + \cos(\cos t))\,dt$$

($\because$ for $g(x) = \cos(\sin x) + \cos(\cos x), f(x+\pi) = f(x)$)

$= f(\pi) + f(x)$

$= f(\pi) + 2f\left(\frac{\pi}{2}\right)$ ($\because$ $g(x)$ has period $\pi/2$)

Reasoning Type

1.a. Given that $\int_a^b |g(x)|\,dx > \left|\int_a^b g(x)\,dx\right| \Rightarrow y = g(x)$ cuts the graph at least once, then $y = f(x)\,g(x)$ changes sign at least once in (a, b), hence $\int_a^b f(x)g(x)\,dx$ can be zero.

2.b. $I = \int_{-4}^{-5} \sin(x^2-3)dx + \int_{-2}^{-1} \sin(x^2+12x+33)dx = I_1 + I_2$

$I_2 = \int_{-2}^{-1} \sin(x^2+12x+33)\,dx = \int_{-2}^{-1} \sin((x+6)^2-3)\,dx,$

put $x + 6 = -y$

$\Rightarrow I_2 = -\int_{-4}^{-5} \sin(y^2-3)\,dy = -I_1$

$\Rightarrow I_1 + I_2 = 0 \Rightarrow I = 0$

3.a. $I = \int_0^1 \tan^{-1}\dfrac{2(1-x)-1}{1+(1-x)-(1-x)^2}dx$

$= \int_0^1 \tan^{-1}\dfrac{1-2x}{1+x-x^2}dx$

$= -I$

$\Rightarrow I = 0$

4.d. $f(x) = \int_{5\pi/4}^{x} (3\sin t + 4\cos t)\,dt$

$\Rightarrow f'(x) = 3\sin x + 4\cos x,\ x \in \left[\dfrac{5\pi}{4}, \dfrac{4\pi}{3}\right]$

These values of x are in third quadrant where both $\sin x$ and $\cos x$ are negative.

Then $f'(x) < 0$ for $x \in \left[\dfrac{5\pi}{4}, \dfrac{4\pi}{3}\right]$.

Hence, $f(x)$ is decreasing for these values of x.

Then, the least value of function occurs at $x = \dfrac{4\pi}{3}$.

$\Rightarrow f_{\min} = \int_{5\pi/4}^{4\pi/3} (3\sin t + 4\cos t)\,dt = \dfrac{3}{2} + \dfrac{1}{\sqrt{2}} - 2\sqrt{3}$

5.a. Given $f(x+1) + f(x+7) = 0, \forall x \in R$

Replace x by $x-1$, we have $f(x) + f(x+6) = 0$ (1)

Now, replace x by $x+6$, we have $f(x+6) + f(x+12) = 0$ (2)

From equations (1) and (2), we have $f(x) = f(x+12)$ (3)

Hence, $f(x)$ is periodic with period 12.

$\Rightarrow \int_a^{a+t} f(x)\,dx$ is independent of a if t is positive integral multiple of 12 then possible value of t is 12.

6.c. $x > x^2, \forall x \in \left(0, \dfrac{\pi}{4}\right) \Rightarrow e^x > e^{x^2}\ \forall x \in \left(0, \dfrac{\pi}{4}\right)$

$\cos x > \sin x\ \forall x \in \left(0, \dfrac{\pi}{4}\right)$

$\Rightarrow e^{x^2}\cos x > e^{x^2}\sin x$

$\Rightarrow e^x > e^{x^2} > e^{x^2}\cos x > e^{x^2}\sin x\ \forall x \in \left(0, \dfrac{\pi}{4}\right)$

$\Rightarrow I_2 > I_1 > I_3 > I_4$

7.a. Let $I_m = \int_0^{\pi} \dfrac{\sin 2mx}{\sin x}dx$. Then,

$I_m - I_{m-1} = \int_0^{\pi} \dfrac{\sin 2mx - \sin 2(m-1)x}{\sin x}dx$

$= \int_0^{\pi} 2\cos(2m-1)x\,dx$

$= \dfrac{2}{2m-1}[\sin(2m-1)x]_0^{\pi} = 0$

$I_m = I_{m-1}$ for all $m \in N$

$\Rightarrow I_m = I_{m-1} = I_{m-2} = \cdots = I_1$

But, $I_1 = \int_0^{\pi} \dfrac{\sin 2x}{\sin x}dx = 2\int_0^{\pi} \cos x\,dx = 0$.

$\therefore I_m = 0$ for all $m \in N$

8.d. $\int_0^{\pi} \sqrt{1-\sin^2 x}\,dx$

$= \int_0^{\pi} |\cos x|\,dx$

$= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx$

$= 1 + 1 = 2$

Hence, statement 1 is false.
However, statement 2 is true.

9.b. Let $I = \int_0^{2\pi} \cos^{99} x\,dx$.

Then,

$I = 2\int_0^{\pi} \cos^{99} x\,dx$ $[\because \cos^{99}(2\pi - x) = \cos^{99} x]$

Now, $\int_0^{\pi} \cos^{99} x\,dx = 0$ $[\because \cos^{99}(\pi - x) = -\cos^{99} x]$

$\Rightarrow I = 2 \times 0 = 0$

10.c. Statement 1 is true as it is a fundamental property. (See integration of odd and even function.)

Let $g(x) = \int_a^x f(t)\,dt$

If $f(x)$ is an even function

Then $g(-x) = \int_a^{-x} f(t)\,dt$

$= -\int_{-a}^{x} f(-y)\,dy$

$= -\int_{-a}^{x} f(y)\,dy$

$= -\int_{-a}^{a} f(y)\,dy - \int_a^x f(y)\,dy$

$\neq -g(x)$

Hence, statement 2 is false.

11.a. Statement 2 is a fundamental concept, also we have $f(2-a) = f(2+a)$

$$\int_{2-a}^{2+a} f(x)\,dx = 2\int_{2}^{2+a} f(x)\,dx$$

12.c. Both the statements are true independently, but statement 2 is not a correct explanation of statement 1.

13.a. To prove $\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx$

Put $z = x - c$, then $dz = dx$

When $x = a + c, z = a$ and when $x = b + c, z = b$

$\therefore \int_{a+c}^{b+c} f(x-c)\,dx = \int_a^b f(z)\,dz = \int_a^b f(x)\,dx$

Thus, statement 2 is true

$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx$

Putting $f(x) = \sin^{100} x \cos^{99} x, a = 0, b = \pi$, and $c = -\frac{\pi}{2}$, we get

$\int_0^{\pi} \sin^{100} x \cos^{99} x\,dx$

$= \int_{-\pi/2}^{\pi/2} \sin^{100}\left(x+\frac{\pi}{2}\right)\cos^{99}\left(x+\frac{\pi}{2}\right)dx$

$= -\int_{-\pi/2}^{\pi/2} \cos^{100} x \sin^{99} x\,dx$

$= 0$ $\quad$ [$\because \cos^{100} x \sin^{99} x$ is an odd function]

14.c. $\int_a^b x f(x)\,dx = \int_a^b (a+b-x)\, f(a+b-x)\,dx$

$= (a+b)\int_a^b f(a+b-x)\,dx - \int_a^b x\, f(a+b-x)\,dx$

Therefore, statement 2 is true only when $f(a+b-x) = f(x)$ which holds in statement 1.

Therefore, statement 2 is false and statement 1 is true.

15.a. Let $g(x) = \int_a^x f(t)\,dt - \int_x^b f(t)\,dt$, where $x \in [a,b]$

We have $g(a) = -\int_a^b f(t)\,dt$ and $g(b) = \int_a^b f(t)\,dt$

$\Rightarrow g(a)g(b) = -\left(\int_a^b f(t)\,dt\right)^2 \le 0$

Clearly, $g(x)$ is continuous in $[a, b]$ and $g(a)g(b) \le 0$

It implies that $g(x)$ will becomes zero at least once in $[a, b]$. Hence, $\int_a^x f(t)\,dt = \int_x^b f(t)\,dt$ for at least one value of $x \in [a, b]$.

Hence, both the statements are true and statement 2 is a correct explanation of statement 1.

16.d. Obviously, $|\sin t|$ is non-differentiable at $x = \pi$.

$$\text{But } \int_0^x |\sin t|\,dt = \begin{cases} \int_0^x \sin t, & 0 \le x < \pi \\ \int_0^{\pi} \sin t\,dt + \int_{\pi}^x -\sin t\,dt, & \pi \le x \le 2\pi \end{cases}$$

$$= \begin{cases} -\cos x + 1, & 0 \le x < \pi \\ 3 + \cos x, & \pi \le x \le 2\pi \end{cases}$$

which is continuous as well as differentiable at $x = \pi$.

Hence, statement 1 is false.

17.a. For $a < b$. If m and M are the smallest and greatest values of $f(x)$ on $[a, b]$

then $m(b-a) \le \int_a^b f(x)dx \le (b-a)M$

or $m \le \frac{1}{(b-a)}\int_a^b f(x)dx \le M$

Since $f(x)$ is continuous on $[a, b]$, it takes on all intermediate values between m and M.

Therefore, some values $f(c)$ $(a \le f(c) \le b)$, we will have

$\frac{1}{(b-a)}\int_a^b f(x)dx = f(c)$ or $\int_a^b f(x)dx = f(c)(b-a)$.

Hence, both the statements are true and statement 2 is a correct explanation of statement 1.

Linked Comprehension Type

For Problems 1–3

1. d., 2. a., 3. c.

Sol.

$$\int_2^x f(t)\,dt = \frac{x^2}{2} + \int_x^2 t^2 f(t)\,dt$$

Differentiating w.r.t. x, we get

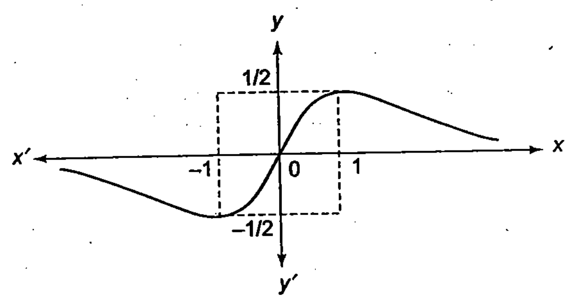

Fig. 8.22

$f(x) = x + (-x^2 f(x))$

$\Rightarrow f(x)[1 + x^2] = x$

$\Rightarrow y = f(x) = \frac{x}{1+x^2}$

$\Rightarrow yx^2 - x + y = 0$

Since x is real, $D \ge 0$

$\Rightarrow 1 - 4y^2 \ge 0$

$\Rightarrow y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Also, $f(x)$ is an odd function, hence $\int_{-2}^{2} f(x)\,dx = 0$

$f'(x) = \frac{1+x^2-2x^2}{1+x^2} = \frac{1-x^2}{1+x^2} \ge 0$

$\Rightarrow x^2 - 1 \le 0$

$\Rightarrow x \in [-1, 1]$

For Problems 4–6

4.b., 5.b., 6.c.

Sol.

$$f(x)=x^2+\int_0^x e^{-t} f(x-t)\,dt \qquad (1)$$

$$=x^2+\int_0^x e^{-(x-t)} f(x-(x-t))\,dt$$

$$\left[\text{Using } \int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx\right]$$

$$=x^2+e^{-x}\int_0^x e^{t} f(t)\,dt \qquad (2)$$

Differentiating w.r.t. x, we get

$$\Rightarrow f'(x)=2x-e^{-x}\int_0^x e^{t} f(t)\,dt+e^{-x}e^{x}f(x)$$

$$=2x-e^{-x}\int_0^x e^{t} f(t)\,dt+f(x)$$

$$\Rightarrow f'(x)=2x+x^2 \qquad \text{[using equation (2)]}$$

$$\Rightarrow f(x)=\frac{x^3}{3}+x^2+c$$

Also $f(0)=0$ [from equation (1)]

$$\Rightarrow f(x)=\frac{x^3}{3}+x^2$$

$$\Rightarrow f'(x)=x^2+2x$$

$\Rightarrow f'(x)=0$ has real roots, hence $f(x)$ is non-monotonic.

Hence, $f(x)$ is many-one, but range is R, hence surjective.

$$\int_0^1 f(x)\,dx=\int_0^1\left(\frac{x^3}{3}+x^2\right)dx$$

$$=\left[\frac{x^4}{12}+\frac{x^3}{3}\right]_0^1$$

$$=\frac{1}{12}+\frac{1}{3}=\frac{5}{12}$$

For Problems 7–9

7.c., 8.d., 9.c.

Sol.

$$f(x)-\lambda\int_0^{\pi/2}\sin x\cos t\, f(t)\,dt=\sin x$$

$$\Rightarrow f(x)-\lambda\sin x\int_0^{\pi/2}\cos t\, f(t)\,dt=\sin x$$

$$\Rightarrow f(x)-A\sin x=\sin x \text{ or}$$

$$f(x)=(A+1)\sin x, \text{ where } A=\lambda\int_0^{\pi/2}\cos t\, f(t)\,dt$$

$$\Rightarrow A=\lambda\int_0^{\pi/2}\cos t(A+1)\sin t\,dt$$

$$=\frac{\lambda(A+1)}{2}\int_0^{\pi/2}\sin 2t\,dt$$

$$=\frac{\lambda(A+1)}{2}\left[\frac{-\cos 2t}{2}\right]_0^{\pi/2}$$

$$=\frac{\lambda(A+1)}{2}$$

$$\Rightarrow A=\frac{\lambda}{2-\lambda}$$

$$\Rightarrow f(x)=\left(\frac{\lambda}{2-\lambda}+1\right)\sin x$$

$$\Rightarrow f(x)=\left(\frac{2}{2-\lambda}\right)\sin x$$

$$\left(\frac{2}{2-\lambda}\right)\sin x=2$$

$$\Rightarrow \sin x=(2-\lambda)$$

$$\Rightarrow |2-\lambda|\le 1$$

$$\Rightarrow -1\le\lambda-2\le 1$$

$$\Rightarrow 1\le\lambda\le 3$$

$$\int_0^{\pi/2} f(x)\,dx=3$$

$$\Rightarrow \int_0^{\pi/2}\frac{2}{2-\lambda}\sin x\,dx=3$$

$$\Rightarrow -\left[\frac{2}{2-\lambda}\cos x\right]_0^{\pi/2}=3$$

$$\Rightarrow \frac{2}{2-\lambda}=3$$

$$\Rightarrow \lambda=4/3$$

For Problems 10–13

10.b., 11.d., 12.d., 13.d.

Sol.

10.b. $f(x)$ is an odd function $\Rightarrow f(x)=-f(-x)$

$$\phi(-x)=\int_a^{-x} f(t)\,dt, \text{ put } t=-y$$

$$\Rightarrow \phi(-x)=\int_{-a}^{x} f(-t)(-dt)=\int_{-a}^{x} f(t)\,dt=\int_{-a}^{a} f(t)\,dt$$

$$+\int_a^x f(t)\,dt=0+\int_a^x f(t)\,dt=\phi(x).$$

11.d. If $f(x)$ is an even function, then

$$\phi(-x)=-\int_{-a}^{x} f(t)\,dt$$

$$=-\int_{-a}^{a} f(t)\,dt-\int_a^x f(t)\,dt$$

$$=-2\int_0^a f(t)\,dt-\int_a^x f(t)\,dt \quad \text{(as } f(x) \text{ is an even function)}$$

$$\text{Now, }\int_0^a f(t)\,dt=\int_0^a f(a-t)\,dt$$

$$=\int_0^a -f(t)\,dt \quad [\text{using } f(a-x)=-f(x)]$$

$\Rightarrow \int_0^a f(t)\,dt = 0$

$\Rightarrow \phi(-x) = -\int_a^x f(t)\,dt = -f(x)$

$\Rightarrow \phi(x)$ is an odd function.

12.d. $g(x+\alpha) + g(x) = 0$

$\Rightarrow g(x+2\alpha) + g(x+\alpha) = 0$

$\Rightarrow g(x+2\alpha) = g(x)$

$\Rightarrow g(x)$ is periodic with period 2α

$\Rightarrow \int_b^{2k} g(t)\,dt = \int_b^{b+c} g(x)\,dx \quad (\because \ b, k, c \text{ are in A.P.})$

This is independent of b, then c has least value 2α.

13.d. $\int_{p+m\alpha}^{q+n\alpha} g(t)\,dt = \int_{p+m\alpha}^{p} g(x)\,dx + \int_p^q g(x)\,dx + \int_q^{q+n\alpha} g(x)\,dx$

$= -m\int_0^\alpha g(x)\,dx + \int_p^q g(x)\,dx + n\int_0^\alpha g(x)\,dx$

$= \int_p^q g(x)\,dx + (n-m)\int_0^\alpha g(x)\,dx$

For Problems 14–17

14. b., 15. c., 16. a., 17. c.

Sol.

14.b. Let $I(a) = \int_0^1 \frac{x^a - 1}{\log x}\,dx$ (1)

Differentiating w.r.t. a keeping x as constant

$\therefore \frac{dI(a)}{da} = \int_0^1 \frac{d}{da}\left(\frac{x^a-1}{\log x}\right)dx$

$= \int_0^1 \frac{x^a \log x}{\log x}\,dx$

$= \int_0^1 x^a\,dx$

$= \left.\frac{x^{a+1}}{a+1}\right|_0^1$

$= \frac{1}{(a+1)}$

Integrating both sides w.r.t. a, we get

$I(a) = \log(a+1) + c$

for $a = 0, I(0) = \log 1 + c$ [from equation (1)]

$0 = 0 + c$

$\therefore I = \log(a+1)$

15.c. Let $F(k) = \int_0^{\pi/2} \ln(\sin^2\theta + k^2\cos^2\theta)\,d\theta$

$F'(k) = \int_0^{\pi/2} \frac{1}{\sin^2\theta + k^2\cos^2\theta} 2k\cos^2\theta\,d\theta$

$= 2k\int_0^{\pi/2} \frac{\cos^2\theta}{\sin^2\theta + k^2\cos^2\theta}\,d\theta$

$= 2k\int_0^{\pi/2} \frac{d\theta}{\tan^2\theta + k^2}$

$= 2k\int_0^{\pi/2} \frac{\sec^2\theta - \tan^2\theta}{\tan^2\theta + k^2}\,d\theta$

$= 2k\int_0^{\infty} \frac{dt}{t^2+k^2} - 2k\int_0^{\pi/2} d\theta$

$+2k^3\int_0^{\pi/2} \frac{d\theta}{\tan^2\theta + k^2}$ (Putting $t = \tan\theta$)

$= 2k\left.\frac{1}{k}\tan^{-1}\frac{1}{k}\right|_0^\infty - 2k\frac{\pi}{2} + k^2F'(k)$

$\Rightarrow (1-k^2)F'(k) = \pi - k\pi = \pi(1-k)$

$\Rightarrow F'(k) = \frac{\pi}{1+k}$

$\Rightarrow F(k) = \pi\log(1+k) + c$

For $k = 1, F(1) = 0 \Rightarrow c = -\pi\log 2$

$\Rightarrow F(k) = \pi\log(1+k) - \pi\log 2$

16.a. Let $I(a) = \int_0^{\pi/2} \log\left(\frac{1+a\sin x}{1-a\sin x}\right)\frac{dx}{\sin x}$

$\frac{dI}{da} = \int_0^{\pi/2} \frac{2\sin x}{1-a^2\sin^2 x}\frac{dx}{\sin x}$

$= \int_0^{\pi/2} \frac{2\sec^2 x\,dx}{1+\tan^2 x - a^2\tan^2 x}$

$= \int_0^{\pi/2} \frac{2\sec^2 x\,dx}{1+(1-a^2)\tan^2 x}$

$= \int_0^{\infty} \frac{2dt}{1+(1-a^2)t^2}$ (put $\tan x = t$)

$= \frac{2}{\sqrt{1-a^2}}\left[\tan^{-1}\left(t\sqrt{1-a^2}\right)\right]_0^\infty$

$= \frac{\pi}{\sqrt{1-a^2}}$

$\Rightarrow I = \pi\sin^{-1}a$ [as $I(0) = 0$]

17.c. $\int_0^\pi \frac{dx}{(a-\cos x)} = \frac{\pi}{\sqrt{a^2-1}}$

Differentiating both sides with respect to a, we get

$-\int_0^\pi \frac{dx}{(a-\cos x)^2} = \frac{-\pi a}{(a^2-1)^{3/2}}$

Again differentiating with respect to a, we get

$$2\int_0^{\pi}\frac{dx}{(a-\cos x)^3}=\frac{\pi(1+2a^2)}{(a^2-1)^{5/2}}$$

Put $a=\sqrt{10}$, we get $\int_0^{\pi}\frac{dx}{(\sqrt{10}-\cos x)^3}=\frac{7\pi}{81}$

For Problems 18 – 20

18. b., 19. d., 20. c.

Sol.

$$f(x)=\sin x+\sin x\int_{-\pi/2}^{\pi/2}f(t)\,dt+\cos x\int_{-\pi/2}^{\pi/2}tf(t)\,dt$$

$$=\sin x\left(1+\int_{-\pi/2}^{\pi/2}f(t)\,dt\right)+\cos x\int_{-\pi/2}^{\pi/2}tf(t)\,dt$$

$$=A\sin x+B\cos x$$

Thus, $A=1+\int_{-\pi/2}^{\pi/2}f(t)\,dt$

$$=1+\int_{-\pi/2}^{\pi/2}(A\sin t+B\cos t)\,dt$$

$$=1+2B\int_0^{\pi/2}\cos t\,dt$$

$\Rightarrow\quad A=1+2B \qquad (1)$

$$B=\int_{-\pi/2}^{\pi/2}tf(t)\,dt$$

$$=\int_{-\pi/2}^{\pi/2}t(A\sin t+B\cos t)\,dt$$

$$=2A\int_0^{\pi/2}t\sin t\,dt$$

$$=2A[-t\cos t+\sin t]_0^{\pi/2}$$

$\Rightarrow\quad B=2A \qquad (2)$

From equations (1) and (2), we get

$A=-1/3, B=-2/3$

$\Rightarrow f(x)=-\frac{1}{3}(\sin x+2\cos x)$

Thus, the range of $f(x)$ is $\left[-\frac{\sqrt{5}}{3},\frac{\sqrt{5}}{3}\right]$

$$f(x)=-\frac{1}{3}(\sin x+2\cos x)$$

$$=-\frac{\sqrt{5}}{3}\sin\left(x+\tan^{-1}2\right)$$

$$=-\frac{\sqrt{5}}{3}\cos\left(x-\tan^{-1}\frac{1}{2}\right)$$

$f(x)$ is invertible if $-\frac{\pi}{2}\le x+\tan^{-1}2\le\frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2}-\tan^{-1}2\le x\le\frac{\pi}{2}-\tan^{-1}2$

or $0\le x-\tan^{-1}\frac{1}{2}\le\pi$

$\Rightarrow \tan^{-1}\frac{1}{2}\le x\le\pi+\tan^{-1}\frac{1}{2}$

or $\pi\le x-\tan^{-1}\frac{1}{2}\le 2\pi$

$\Rightarrow x\in\left[\pi+\cot^{-1}2,\ 2\pi+\cot^{-1}2\right]$

$$\int_0^{\pi/2}f(x)\,dx=-\frac{1}{3}\int_0^{\pi/2}(\sin x+2\cos x)\,dx$$

$$=-\frac{1}{3}[-\cos x+2\sin x]_0^{\pi/2}$$

$$=-1$$

Matrix-Match Type

1. a → s. b → s. c → r. d → q.

a. $\int_{-1}^{1}[x+[x+[x]]]\,dx$

(use property $[x+n]=[x]+n$ if n is integer)

$$=\int_{-1}^{1}3[x]\,dx=3\int_{-1}^{1}[x]\,dx=3\int_0^1([x]+[-x])\,dx$$

$=-3$ (as $[x]+[-x]=-1$)

b. $\int_2^5([x]+[-x])\,dx=\int_2^5-1\,dx=-3$

c. $\operatorname{sgn}(x-[x])=\begin{cases}1, & \text{if } x \text{ is not an integer}\\ 0, & \text{if } x \text{ is an integer}\end{cases}$

Hence, $\int_{-1}^{3}\operatorname{sgn}(x-[x])\,dx=4(1-0)=4.$

d. Let $I=25\int_0^{\pi/4}\left(\tan^6(x-[x])+\tan^4(x-[x])\right)dx$

$\left\{\because 0<x\le\frac{\pi}{4}\Rightarrow[x]=0\right\}$

$$\therefore I=25\int_0^{\pi/4}\left(\tan^6 x+\tan^4 x\right)dx$$

$$=25\int_0^{\pi/4}\tan^4 x\left(\tan^2 x+1\right)dx$$

$$=25\int_0^{\pi/4}\tan^4 x\sec^2 x\,dx$$

$$=25\left(\frac{\tan^5 x}{5}\right)_0^{\pi/4}$$

$$=25\times\frac{1}{5}=5$$

2. a → r. b → p. c → s. d → q.

a. $\lim_{n\to\infty}\left[\frac{\int_0^2\left(1+\frac{t}{n+1}\right)^n dt}{n+1}\right]$

d

3

a

$$= \lim_{n\to\infty}\left[\left(1+\frac{t}{n+1}\right)^{n+1}\right]_0^2$$

$$= \lim_{n\to\infty}\left(1+\frac{2}{n+1}\right)^{n+1}-1$$

$$= e^2-1$$

b. $f'(x)=f(x) \Rightarrow f(x)=C\,e^x$ and since $f(0)=1$

$\therefore 1=f(0)=C$

$\therefore f(x)=e^x$ and hence $g(x)=x^2-e^x$.

Thus, $\int_0^1 f(x)g(x)\,dx$

$$= \int_0^1 (x^2e^x-e^{2x})\,dx = x^2e^x\Big|_0^1 - 2\int_0^1 xe^x\,dx - \frac{e^{2x}}{2}\Big|_0^1$$

$$= (e-0)-2\,xe^x\Big|_0^1 + 2\,e^x\Big|_0^1 - \frac{1}{2}(e^2-1)$$

$$= (e-0)-2\,e+2\,e-2-\frac{1}{2}(e^2-1)$$

$$= e-\frac{1}{2}e^2-\frac{3}{2}$$

c. $I=\int_0^1 e^{e^x}(1+x\,e^x)\,dx$

Let $e^x=t$

$$\Rightarrow \int_1^e e^t(1+t\log t)\frac{dt}{t}$$

$$= \int_1^e e^t\left(\frac{1}{t}+\log t\right)dt$$

$$= \left[e^t\log t\right]_1^e$$

$$= e^e$$

d. $L=\lim_{k\to 0}\dfrac{\int_0^k (1+\sin 2x)^{\frac{1}{x}}\,dx}{k}$ $\left(\text{form } \dfrac{0}{0}\right)$

$$\Rightarrow L=\lim_{k\to 0}(1+\sin 2k)^{\frac{1}{k}}$$

$$= e^{\lim_{k\to 0}\frac{1}{k}(\sin 2k)} = e^2$$

3. **a → q., b → r, s., c → p., d → p.**

a. $I_1=\int_{\pi/6}^{\pi/3}\sec^2\theta\, f(2\sin 2\theta)\,d\theta$

Applying property $\int_a^b f(a+b-x)\,dx=\int_a^b f(x)\,dx$

$$I_1=\int_{\pi/6}^{\pi/3}\sec^2\left(\frac{\pi}{2}-\theta\right)f\left(2\sin 2\left(\frac{\pi}{2}-\theta\right)\right)d\theta$$

$$I_1=\int_{\pi/6}^{\pi/3}\operatorname{cosec}^2\theta\, f(2\sin 2\theta)\,d\theta=I_2.$$

b. $f(x+1)=f(x+3) \Rightarrow f(x)=f(x+2)$

$\Rightarrow f(x)$ is periodic with period 2.

Then $\int_a^{a+b} f(x)\,dx$ is independent of a, for which b is multiple of 2.

$\Rightarrow b=2, 4, 6\ldots$

c. Let $I=\int_1^4 \dfrac{\tan^{-1}[x^2]}{\tan^{-1}[x^2]+\tan^{-1}[25+x^2-10x]}$ (1)

Applying $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$, we get

$$I=\int_1^4 \frac{\tan[(5-x)^2]}{\tan^{-1}[(5-x)^2]+\tan^{-1}[x^2]}\,dx \quad (2)$$

Adding equations (1) and (2), we get

$$2I=\int_1^4 dx \Rightarrow 2I=3 \Rightarrow I=3/2$$

d. Let $y=\sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}=\sqrt{x+y}$

$\Rightarrow\ y^2-y-x=0$

$$\Rightarrow\ y=\frac{1\pm\sqrt{1+4x}}{2.1}$$

$$\Rightarrow\ y=\frac{1+\sqrt{1+4x}}{2} \qquad (\because y>1)$$

$$\Rightarrow\ I=\int_0^2 \frac{1+\sqrt{1+4x}}{2}\,dx=\left[\frac{x}{2}+\frac{(1+4x)^{3/2}}{\frac{3}{2}.2.4}\right]_0^2$$

$$= \left[\left(1+\frac{27}{12}\right)-\left(0+\frac{1}{12}\right)\right]=1+\frac{26}{12}=\frac{19}{6}$$

$\Rightarrow [I]=3$

4. **a → p, q. b → p, q, r. c → q, s. d → s.**

a. $I=\int_{-2}^{2}(\alpha x^3+\beta x+\gamma)\,dx$

$\alpha x^3+\beta x$ is an odd function

$I=0+2\int_0^2 \gamma\,dx=2.2\gamma=4\gamma$

b. $I=\frac{1}{2}\int_0^1 2\sin\alpha x\sin\beta x\,dx$

$$= \frac{1}{2}\int_0^1(\cos(\alpha-\beta)x-\cos(\alpha+\beta)x)\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(\alpha-\beta)x}{\alpha-\beta}-\frac{\sin(\alpha+\beta)x}{\alpha+\beta}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{\sin(\alpha-\beta)}{\alpha-\beta} - \frac{\sin(\alpha+\beta)}{\alpha+\beta}\right] \quad (1)$$

Also, $2\alpha = \tan\alpha$ and $2\beta = \tan\beta$

$\Rightarrow 2(\alpha-\beta) = \tan\alpha - \tan\beta$ and $2(\alpha+\beta) = \tan\alpha + \tan\beta$

$$2(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} \text{ and } 2(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$$

Substituting these values, we get

$I = (\cos\alpha\cos\beta) - (\cos\alpha\cos\beta) = 0.$

c. $f(x+\alpha) + f(x) = 0$

$\Rightarrow f(x+2\alpha) + f(x+\alpha) = 0$

$\Rightarrow f(x+2\alpha) = f(x)$

$\Rightarrow f(x)$ is periodic with period 2α

$$\Rightarrow \int_{\beta}^{\beta+2\gamma\alpha} (\alpha x^3 + \beta x + \gamma)\,dx = \gamma\int_0^{2\alpha} f(x)\,dx.$$

d. Let $I = \int_0^{\alpha} [\sin x]dx$, $\alpha \in [(2\beta+1)\pi, (2\beta+2)\pi]$, $\beta \in N$,

[where $[\cdot]$ denotes the greatest integer function.]

$$I = \int_0^{2\beta\pi} [\sin x]dx + \int_{2\beta\pi}^{(2\beta+1)\pi} [\sin x]dx + \int_{(2\beta+1)\pi}^{\alpha} [\sin x]dx$$

$$= \beta\int_0^{2\pi} [\sin x]dx + 0 + \int_{(2\beta+1)\pi}^{\alpha} (-1)\,dx$$

$= -\beta\pi + (2\beta+1)\pi - \alpha$

$= (\beta+1)\pi - \alpha$

$\Rightarrow \gamma\int_0^{\alpha} [\sin x]dx$ depends on α, β and γ

Integer Type

1. (2). $\int_0^2 |f'(x)|dx \ge \left|\int_0^2 f'(x)dx\right|$

$\Rightarrow \int_0^2 |f'(x)|dx \ge |f(2)| = 2$

2. (3) We have $f(x) = \sin x + \int_{-\pi/2}^{\pi/2} (\sin x + t f(t))\,dt = \sin x + \pi\sin x$

$$+ \int_{-\pi/2}^{\pi/2} t f(t)\,dt$$

$\therefore f(x) = (\pi+1)\sin x + A \quad (1)$

Now, $A = \int_{-\pi/2}^{\pi/2} t((\pi+1)\sin t + A)dt = 2(\pi+1)\left(\int_0^{\pi/2} \underset{\text{(I)}}{t}\,\underset{\text{(II)}}{\sin t}\,dt\right)$ (By part)

$\Rightarrow A = 2(\pi+1)$

Hence, $f(x) = (\pi+1)\sin x + 2(\pi+1)$.

Therefore, $f_{\max} = 3(\pi+1) = M$

and $f_{\min} = (\pi+1) = m$.

$\Rightarrow \frac{M}{m} = 3$

3. (5) We have $f(2x) = 3f(x) \quad (1)$

and $\int_0^1 f(x)\,dx = 1 \quad (2)$

From equations (1) and (2), $\frac{1}{3}\int_0^1 f(2x)\,dx = 1$

Put $2x = t$, $\frac{1}{6}\int_0^2 f(t)\,dt = 1$

$\Rightarrow \int_0^2 f(t)\,dt = 6$

$\Rightarrow \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = 6$

Hence, $\int_1^2 f(t)\,dt = 6 - \int_0^1 f(t)\,dt = 6 - 1 = 5.$

4. (4) Given $f(x) = x^3 - \frac{3x^2}{2} + x + \frac{1}{4} = \frac{1}{4}(4x^3 - 6x^2 + 4x + 1)$

$= \frac{1}{4}(4x^3 - 6x^2 + 4x - 1 + 2)$

$f(x) = \frac{1}{4}[x^4 - (1-x)^4] + \frac{2}{4}$

$\therefore f(1-x) = \frac{1}{4}[(1-x)^4 - x^4] + \frac{2}{4}$

$\therefore f(x) + f(1-x) = \frac{2}{4} + \frac{2}{4} = 1 \quad (1)$

Replacing x by $f(x)$ we have

$f[f(x)] + f[1-f(x)] = 1 \quad (2)$

Now $I = \int_{1/4}^{3/4} f(f(x))\,dx \quad (3)$

Also, $I = \int_{1/4}^{3/4} f(f(1-x))\,dx = \int_{1/4}^{3/4} f(1-f(x))\,dx \quad (4)$

{using (1)}

adding (3) and (4),

$$2I = \int_{1/4}^{3/4} [f(f(x)) + f(1-f(x))]dx = \int_{1/4}^{3/4} dx$$

$\Rightarrow 2I = \frac{1}{2} \Rightarrow I = \frac{1}{4}$

$\therefore\ I=\dfrac{1}{4}$

$\therefore\ I^{-1}=4$

(2) $\displaystyle \lim_{n\to\infty}\frac{n}{2^n}\cdot\left.\frac{x^{n+1}}{n+1}\right]_0^2$

$\displaystyle =\lim_{n\to\infty}\frac{n}{2^n}\cdot\frac{2^{n+1}}{n+1}$

$\displaystyle =\lim_{n\to 0}\frac{2}{1+(1/n)}=2$

(6) Given $f^3(x)=\displaystyle\int_0^x t\cdot f^2(t)\,dt$

differentiating, $3f^2(x)f'(x)=xf^2(x)$

$f(x)\neq 0\ \therefore\ f'(x)=\dfrac{x}{3};\ \therefore\ f(x)=\dfrac{x^2}{6}+C$

but $f(0)=0\Rightarrow C=0$
$f(6)=6$

(8) Let $I=\displaystyle\int_0^1 {}^{207}C_7\cdot\underbrace{x^{200}}_{II}\cdot\underbrace{(1-x)^7}_{I}\,dx$

$$I={}^{207}C_7\left[\underbrace{\left.(1-x)^7\cdot\frac{x^{201}}{201}\right|_0^1}_{zero}+\frac{7}{201}\int_0^1(1-x)^6\cdot x^{201}\,dx\right]$$

$$={}^{207}C_7\cdot\frac{7}{201}\int_0^1(1-x)^6\cdot x^{201}\,dx$$

Integrating by parts again 6 more times

$$={}^{207}C_7\cdot\frac{7!}{201.202.203.204.205.206.207}\int_0^1 x^{207}\,dx$$

$$=\frac{(207)!}{7!(200)!}\cdot\frac{7!}{201.202\cdots 207}\cdot\frac{1}{208}$$

$$=\frac{(207)!}{(207)!7!}\cdot\frac{7!}{208}=\frac{1}{208}=\frac{1}{k}\quad\Rightarrow\quad k=208$$

2) $I=\displaystyle\int_0^{3\pi/4}(\sin x+\cos x)\,dx+\int_0^{3\pi/4}\underset{I}{x}\underbrace{(\sin x-\cos x)}_{II}\,dx$

$$=\int_0^{3\pi/4}(\sin x+\cos x)\,dx+\underbrace{x(-\cos x-\sin x)|_0^{3\pi/4}}_{zero}$$

$$+\int_0^{3\pi/4}(\sin x+\cos x)\,dx$$

$$=2\int_0^{3\pi/4}(\sin x+\cos x)\,dx=2(\sqrt{2}+1)$$

8) $I=\displaystyle\lim_{n\to\infty}\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{6n}}{n\sqrt{n}}$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{6n}\sqrt{\frac{r}{n}}=\int_0^6\sqrt{x}\,dx=\left[\frac{2}{3}x^{3/2}\right]_0^6=\frac{2}{3}\cdot 6\sqrt{6}=\sqrt{96}$$

10. (7) $F(x)=(2x+3)\displaystyle\int_x^2 f(u)\,du$

$\therefore\ F'(x)=-(2x+3)f(x)+\left(\displaystyle\int_x^2 f(u)\,du\right)\cdot 2$

$F'(2)=-7f(2)+0$

11. (4) $I=\displaystyle\int_0^1\frac{\sin^{-1}\sqrt{x}}{x^2-x+1}\,dx$ (1)

$I=\displaystyle\int_0^1\frac{\sin^{-1}\sqrt{1-x}}{x^2-x+1}\,dx=\int_0^1\frac{\cos^{-1}\sqrt{x}}{x^2-x+1}\,dx$ (2)

On adding equations (1) and (2), we get

$$2I=\int_0^1\frac{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}{x^2-x+1}\,dx$$

$$=\frac{\pi}{2}\int_0^1\frac{dx}{x^2-x+1}dx$$

$$=\frac{\pi}{2}\int_0^1\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$2I=\frac{\pi}{2}\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\left[\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^1=\frac{\pi^2}{3\sqrt{3}}$$

Hence, $I=\dfrac{\pi^2}{6\sqrt{3}}=\dfrac{\pi^2}{\sqrt{108}}\equiv\dfrac{\pi^2}{\sqrt{n}}$.

12. (6) $y=f(x)\ \Rightarrow\ x=f^{-1}(y)\ \Rightarrow\ x=g(y)$

Given $y=f(x)=\displaystyle\int_0^x\frac{dt}{\sqrt{1+t^3}}$

$\dfrac{dy}{dx}=\dfrac{1}{\sqrt{1+x^3}}\ \Rightarrow\ \dfrac{dx}{dy}=\sqrt{1+x^3}$

$g'(y)=\sqrt{1+g^3(y)}$

$g''(y)=\dfrac{3g^2(y)g'(y)}{2\sqrt{1+g^3(y)}}$

$$\Rightarrow\ 2g''(y)=3g^2(y)\frac{g'(y)}{\sqrt{1+g^3(y)}}=3g^2(y)\frac{\sqrt{1+g^3(y)}}{\sqrt{1+g^3(y)}}=3g^2(y)$$

$\Rightarrow\ 2g''(y)=3g^2(y)$

13. (5) Given $U_n=\int_0^1 x^n\cdot(2-x)^n\,dx;\ V_n=\int_0^1 x^n\cdot(1-x)^n\,dx$

In U_n put $x=2t \Rightarrow dx=2dt$

$\therefore\ U_n=2\int_0^{1/2} 2^n\cdot t^n\,2^n\,(1-t)^n\,dt$ (1)

Now $V_n=2\int_0^{1/2} x^n\,(1-x)^n\,dx$ (2)

From equations (1) and (2) we get $U_n=2^{2n}\cdot V_n$.

14. (6) $I=\int_0^\infty (x^2)^n\cdot x\,e^{-x^2}\,dx$

put $x^2=t \Rightarrow x\,dx=dt/2$

$\Rightarrow I=\frac{1}{2}\int_0^\infty t^n\,e^{-t}\,dt$

$=\frac{1}{2}\left[\left.-t^n\,e^{-t}\right]_0^\infty+n\int_0^\infty t^{n-1}\,e^{-t}\,dt\right]$

$=\frac{1}{2}\left[0+n\int_0^\infty t^{n-1}\,e^{-t}\,dt\right]$

$\Rightarrow I=\frac{n!}{2}=360$

$\Rightarrow n=6$

15. (3) $f(x)\int_0^x e^t\sin(x-t)dt$

$=\int_0^x e^{x-t}\sin(x-(x-t))dt$

$=e^x\int_0^x e^{-t}\sin t\,dt$

$\Rightarrow f'(x)=e^x e^{-x}\sin x+e^x\int_0^x e^{-t}\sin t\,dt$

$=\sin x+e^x\int_0^x e^{-t}\sin t\,dt$

$\Rightarrow f''(x)=\cos x+e^x e^{-x}\sin x+e^x\int_0^x e^{-t}\sin t\,dt$

$=\cos x+\sin x+f(x)$

$\Rightarrow f''(x)-f(x)=\cos x+\sin x$

Range of $g(x)=f''(x)-f(x)$ is $\left[-\sqrt{2},\sqrt{2}\right]$.

Number of integers in the range is 3.

16. (8) $\frac{d}{dx}\int_4^x[4t^2-2F'(t)]dt=[4x^2-2F'(x)]\cdot 1-0$

$\Rightarrow F'(x)=\frac{1}{x^2}[4x^2-2F'(x)]+\frac{-2}{x^3}\int_4^x[4t^2-2F'(t)]dt$

$\Rightarrow F'(4)=\frac{1}{16}[64-2F'(4)]-\frac{1}{32}\int_4^4 g(x)dx$

$\Rightarrow \left(1+\frac{1}{8}\right)F'(4)=4$

$\Rightarrow F'(4)=\frac{32}{9}$

17. (7) $\sum_{r=1}^{100}\left(\int_0^1 f(r-1+x)dx\right)$

$=\int_0^1 f(x)dx+\int_0^1 f(1+x)dx+\int_0^1 f(2+x)dx+\cdots+\int_0^1 f(99+x)dx$

$=\int_0^1 f(x)dx+\int_1^2 f(x)dx+\int_2^3 f(x)dx+\cdots+\int_{99}^{100} f(x)dx$

$=\int_0^{100} f(x)dx=7$

18. (0) $\because$ Integrand is discontinuous at $\frac{\pi}{2}$, then $\int_0^{\pi/2}0\cdot dx+\int_{\pi/2}^{3\pi/2}0\cdot dx=0$

$\because 0<x<\frac{\pi}{2}$, $|\tan^{-1}\tan x|=|\sin^{-1}\sin x|$ and $\frac{\pi}{2}<x<\frac{3\pi}{2}$, $|\tan^{-1}\tan x|=|\sin^{-1}\sin x|$

19. (8) $I_{11}=\int_0^1 \underbrace{(1-x^5)^{11}}_{\text{I}}\cdot\underbrace{1}_{\text{II}}\,dx$

$=\left.(1-x^5)^{11}\cdot x\right]_0^1+11\int_0^1(1-x^5)^{10}5x^4\cdot x\,dx$

$=0-55\int_0^1(1-x^5)^{10}(1-x^5-1)\,dx$

$=-55\int_0^1(1-x^5)^{11}\,dx+55I_{10}$

$\Rightarrow 56I_{11}=55I_{10}$

$\Rightarrow \frac{I_{10}}{I_{11}}=\frac{56}{55}$

20. (4) $I_1=\int_0^1 x^{1004}(1-x)^{1004}dx$

$=2\int_0^{1/2} x^{1004}(1-x)^{1004}dx$ (1)

And $I_2=\int_0^1 x^{1004}(1-x^{2010})^{1004}dx$

Put $x^{1005}=t \Rightarrow 1005\,x^{1004}\,dx=dt$

$\Rightarrow I_2=\frac{1}{1005}\int_0^1(1-t^2)^{1004}dt$

$=\frac{1}{1005}\int_0^1\left(t(2-t)\right)^{1004}dt$

$=\frac{1}{1005}\int_0^1 t^{1004}(2-t)^{2004}dt$

Now put $t=2y \Rightarrow dt=2dy$

$$\Rightarrow I_2=\frac{1}{1005}\int_0^{1/2}(2y)^{1004}(2-2y)^{1004}dt$$

$$=\frac{1}{1005}2\cdot 2^{1004}\cdot 2^{1004}\int_0^{1/2}y^{1004}(1-y)^{1004}dy$$

$$=\frac{1}{1005}2^{2009}\int_0^{1/2}y^{1004}(1-y)^{1004}dy$$

$$=\frac{1}{1005}2^{2008}I_1$$

$$\Rightarrow \frac{I_1}{I_2}=\frac{1005}{2^{2008}}$$

$$\Rightarrow \frac{2^{2010}}{1005}\frac{I_1}{I_2}=4$$

21. (9) $f(x)=x+x\int_0^1 t\,f(t)\,dt+\int_0^1 t^2 f(t)\,dt$

$\therefore\ f(x)=x(1+A)+B$; where $A=\int_0^1 t\,f(t)\,dt$ and $B=\int_0^1 t^2 f(t)\,dt$

Now, $A=\int_0^1 t[t(1+A)+B]dt=\frac{t^3}{3}(1+A)\Big|_0^1+\frac{B}{2}t^2\Big|_0^1$

$\Rightarrow A=\frac{1+A}{3}+\frac{B}{2}$

$\Rightarrow 4A-3B=2$ (1)

Again $B=\int_0^1 t^2[t(1+A)+B]dt=\left[\frac{t^4(1+A)}{4}+\frac{Bt^3}{3}\right]_0^1$

$=\frac{1+A}{4}+\frac{B}{3}$

$\Rightarrow 8B-3A=3$ (2)

Solving equations (1) and (2) we have $B=\frac{18}{23}=f(0)$

22. (2) $I=\int_{\sqrt{2}-1}^{\sqrt{2}+1}\frac{(x^2+1)^2-(x^2-1)}{(x^2+1)^2}dx=\int_{\sqrt{2}-1}^{\sqrt{2}+1}\left(1-\frac{(x^2-1)}{(x^2+1)^2}\right)dx$

$$=2-\underbrace{\int_{\sqrt{2}-1}^{\sqrt{2}+1}\frac{(x^2-1)}{(x^2+1)^2}dx}_{I_1}$$

$I_1=\int_{1/a}^{a}\frac{(x^2-1)}{(x^2+1)^2}dx$ where $(a=\sqrt{2}+1)$;

put $x=\frac{1}{t}\Rightarrow dx=-\frac{1}{t^2}dt$

$$=\int_a^{1/a}\frac{\frac{1}{t^2}-1}{\left(\frac{1}{t^2}+1\right)^2}\cdot\left(-\frac{1}{t^2}\right)dt=-\int_a^{1/a}\frac{(1-t^2)t^4}{t^4(1+t^2)^2}dt$$

$$=-\int_a^{1/a}\frac{(1-t^2)}{(1+t^2)^2}dt=\int_a^{1/a}\frac{t^2-1}{(t^2+1)^2}dt$$

$$=-\int_{1/a}^{a}\frac{t^2-1}{(t^2+1)^2}dt=-I_1$$

$\Rightarrow 2I_1=0$

$\Rightarrow I_1=0$

$\Rightarrow I=2$

23. (0) We have $J=\int_{-5}^{-4}(3-x^2)\tan(3-x^2)dx$.

Put $(x+5)=t$, we get

$$J=\int_0^1(3-(t-5)^2)\tan(3-(t-5)^2)\,dt$$

$$=\int_0^1(-22+10t-t^2)\tan(-22+10t-t^2)dt.$$

Now, $K=\int_{-2}^{-1}(6-6x+x^2)\tan(6x-x^2-6)\,dx.$

Put $(x+2)=z$, we get

$$K=\int_0^1(6-6(z-2)+(z-2)^2)\tan(6(z-2)-(z-2)^2-6)\,dz$$

$$=\int_0^1(22-10z+z^2)\tan(-22+10z-z^2)dz$$

Hence, $(J+K)=0$.

24. (2) We have $\int_{\sin t}^{1}x^2 g(x)\,dx=(1-\sin t)$ (1)

Differentiating both the sides of (1) with respect to 't', we get

$0-(\sin^2 t)\,g(\sin t)\,(\cos t)=-\cos t$

$\Rightarrow g(\sin t)=\frac{1}{\sin^2 t}$ (2)

Putting $t=\frac{\pi}{4}$ in (2),

we get $g\left(\frac{1}{\sqrt{2}}\right)=2$.

Archives

Subjective

1. $L=\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{6n}\right)$

$$=\lim_{n\to\infty}\sum_{r=1}^{5n}\frac{1}{n+r}=\lim_{n\to\infty}\sum_{r=1}^{5n}\left(\frac{1/n}{1+r/n}\right)$$

Now, Lower limit $=\lim_{n\to\infty}(r/n)_{r=1}=\lim_{n\to\infty}(1/n)=0$

Upper limit $=\lim_{n\to\infty}(r/n)_{r=5n}=\lim_{n\to\infty}(5n/5)=5$

Then $L=\int_0^5 \frac{dx}{1+x}=[\log(1+x)]_0^5=\log 6$

2. $\int_0^1 (tx+1-x)^n\, dx$

$=\int_0^1 [(t-1)x+1]^n\, dx$

$=\left[\frac{[(t-1)x+1]^{n+1}}{(t-1)(n+1)}\right]_0^1$

$=\frac{1}{n+1}\left[\frac{t^{n+1}}{t-1}-\frac{1}{t-1}\right]$

$\Rightarrow \int_0^1 (tx+1-x)^n\, dx=\frac{t^{n+1}-1}{(t-1)(n+1)}$ (1)

For $\int_0^1 x^k(1-x)^{n-k}\, dx=\left[{}^nC_k(n+1)\right]^{-1}$

$k=0,1,2,\cdots,n$

Now $[tx+(1-x)]^n$

$=\sum_{k=0}^{n} {}^nC_k\,(tx)^k(1-x)^{n-k}$ [Using binomial theorem]

$=\sum_{k=0}^{n}\left[{}^nC_k\, x^k(1-x)^{n-k}\right]t^k$

Integrating both sides from 0 to 1 w.r.t. x, we get

$\Rightarrow \int_0^1 [tx+(1-x)]^n\, dx=\sum_{k=0}^{n} t^k\, {}^nC_k \int_0^1 x^k(1-x)^{n-k}\, dx$

$\Rightarrow \frac{t^{n+1}-1}{(t-1)(n+1)}=\sum_{k=0}^{n} {}^nC_k t^k\left\{\int_0^1 x^k(1-x)^{n-k}\, dx\right\}$

[Using equation (1)]

$\Rightarrow \sum_{k=0}^{n} {}^nC_k\, t^k\left\{\int_0^1 x^k(1-x)^{n-k}\, dx\right\}$

$=\frac{1}{n+1}[1+t+t^2+t^3+\cdots+t^n]$ [Using sum of G.P.]

Equating the coefficients of t^k on both the sides, we get

${}^nC_k\int_0^1 x^k(1-x)^{n-k}\, dx=\frac{1}{n+1}$

$\Rightarrow \int_0^1 x^k(1-x)^{n-k}\, dx=\frac{1}{{}^nC_k\,(n+1)}$

3. Let $I=\int_0^\pi xf(\sin x)\,dx$ (1)

Now using property IV; we get

$I=\int_0^\pi(\pi-x)f\{\sin(\pi-x)\}\,dx$, or

$I=\int_0^\pi(\pi-x)f(\sin x)\,dx$ (2)

$\therefore$ adding equations (1) and (2), we get $2I=\pi\int_0^\pi f(\sin x)\,dx$

or $I=\frac{\pi}{2}\int_0^\pi f(\sin x)\,dx.$

4. $\because \sin\theta$ is – ve if $-\pi\le\theta\le 0$
is + ve if $0<\theta<\pi$
is – ve if $\pi<\theta\le 3\pi/2$

$\therefore |x\sin\pi x|=\begin{cases}(-x)(-\sin\pi x) & \text{If } -1\le x<0\\ x\sin\pi x & \text{If } 0<x\le 1\\ x(-\sin\pi x) & \text{If } 1<x\le 3/2\end{cases}$

$\therefore \int_{-1}^{3/2}|x\sin\pi x|\,dx$

$=\int_{-1}^{0}x\sin\pi x\,dx+\int_{-1}^{1}x\sin\pi x\,dx+\int_1^{3/2}(-x\sin\pi x)\,dx$

$=\int_{-1}^{1}x\sin\pi x\,dx-\int_1^{3/2}x\sin\pi x\,dx$

$=2\int_0^1 x\sin\pi x\,dx-\int_1^{3/2}x\sin\pi x\,dx$

$=2\left[\left\{x\left(\frac{-1}{\pi}\right)\cos\pi x\right\}_0^1-\int_0^1 1\left(\frac{-1}{\pi}\right)\cos\pi x\,dx\right]$

$-\left\{x\left(\frac{-1}{\pi}\right)\cos\pi x\right\}_1^{3/2}+\int_1^{3/2}1\left(\frac{-1}{\pi}\right)\cos\pi x\,dx$

$=\left(\frac{2}{\pi}\right)+\left(\frac{2}{\pi^2}\right)[\sin\pi x]_0^1+\left\{\frac{3}{(2\pi)}\right\}\cos\frac{3}{2}\pi+\left(\frac{1}{\pi}\right)$

$-\left(\frac{1}{\pi^2}\right)[\sin\pi x]_1^{3/2}$

$=(2/\pi)+0+0+(1/\pi)+(1/\pi^2)$

$=(3\pi+1)/\pi^2$

5. $I=\int_0^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}\,dx$

we know that $(\sin x-\cos x)^2=1-\sin 2x$

$\Rightarrow \sin 2x=1-(\sin x-\cos x)^2$

$\Rightarrow I=\int_0^{\pi/4}\frac{\sin x+\cos x}{9+16(1-(\sin x-\cos x)^2)}\,dx$

$=\int_0^{\pi/4}\frac{\sin x+\cos x}{25-16(\sin x-\cos x)^2}\,dx$

Let $\sin x-\cos x=t$

$\Rightarrow I=\int_{-1}^{0}\frac{dt}{25-16t^2}$

$=\frac{1}{16}\int_{-1}^{0}\frac{dt}{\left(\frac{5}{4}\right)^2-t^2}$

$=\frac{1}{16}\cdot\frac{1}{2.\frac{5}{4}}\log\left[\left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right|\right]_{-1}^{0}$

$=\frac{1}{40}\left[\log 1-\log\frac{1}{9}\right]$

$=\frac{\log 9}{40}=\frac{1}{20}\log 3$

6. Let $I=\int_0^{1/2}\frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Put $x=\sin\theta \Rightarrow dx=\cos\theta\, d\theta$

Also when $x=0$, $\theta=0$ and when $x=1/2$, $\theta=\pi/6$

Thus, $I=\int_0^{\pi/6}\frac{\sin\theta\sin^{-1}(\sin\theta)}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta$

$\Rightarrow I=\int_0^{\pi/6}\theta\sin\theta\, d\theta$

Integrating by parts, we get

$I=\left[\theta(-\cos\theta)\right]_0^{\pi/6}+\int_0^{\pi/6}1\cos\theta\, d\theta$

$=[-\theta\cos\theta+\sin\theta]_0^{\pi/6}$

$=\frac{-\pi}{6}\frac{\sqrt{3}}{2}+\frac{1}{2}=\frac{6-\pi\sqrt{3}}{12}$

7. Given that $f(x)$ is integrable over any interval on real line and $f(t+x)=f(x)$ (1)
for all real x and a real t

Now, $\int_a^{a+t}f(x)\,dx$

$=\int_a^0 f(x)\,dx+\int_0^t f(x)dx+\int_t^{a+t}f(x)\,dx$

In the last integral, put $x=t+y$ so that $dx=dy$

Then $\int_t^{a+t}f(x)\,dx$

$=\int_0^a f(t+y)\,dy=\int_0^a f(y)\,dy$ [Using equation (1)]

$=\int_0^a f(x)\,dx.$

Hence, $\int_t^{a+t}f(x)\,dx$

$=-\int_0^a f(x)\,dx+\int_0^t f(x)\,dx+\int_0^a f(x)\,dx$

$=\int_0^t f(x)\,dx$ which is independent of a.

8. Let $I=\int_0^{\pi/2}\frac{x\sin x\cos x}{\cos^4x+\sin^4x}\,dx$ (1)

$\Rightarrow I=\int_0^{\pi/2}\frac{(\pi/2-x)\sin(\pi/2-x)\cos(\pi/2-x)}{\cos^4(\pi/2-x)+\sin^4(\pi/2-x)}\,dx$

[Using $\int_0^a f(x)dx=\int_0^a f(a-x)dx$]

$\Rightarrow I=\int_0^{\pi/2}\frac{(\pi/2-x)\sin x\cos x}{\sin^4x+\cos^4x}dx$ (2)

Adding equations (1) and (2), we get

$2I=\frac{\pi}{2}\int_0^{\pi/2}\frac{\sin x\cos x}{\sin^4x+\cos^4x}\,dx$

$\Rightarrow I=\frac{\pi}{4}\int_0^{\pi/2}\frac{\sin x\cos x}{\sin^4x+\cos^4x}dx$

$=\frac{\pi}{4}\int_0^{\pi/2}\frac{\sec^2x\tan x}{\tan^4x+1}dx$ (Dividing numerator and denominator by $\cos^4 x$)

$=\frac{\pi}{2\times4}\int_0^{\pi/2}\frac{2\tan x\sec^2x\,dx}{1+\left(\tan^2x\right)^2}$

Put $\tan^2x=t\Rightarrow 2\tan x\sec^2x\,dx=dt$

Also, as $x\to0$, $t\to0$; as $x\to\pi/2$, $t\to\infty$

$\Rightarrow I=\frac{\pi}{8}\int_0^{\infty}\frac{dt}{1+t^2}$

$=\frac{\pi}{8}\left[\tan^{-1}t\right]_0^{\infty}=\frac{\pi}{8}[\pi/2-0]=\pi^2/16.$

9. Let $I=\int_0^{\pi}\frac{x\,dx}{1+\cos\alpha\sin x}$ (1)

$I=\int_0^{\pi}\frac{(\pi-x)\,dx}{1+\cos\alpha\,(\sin(\pi-x))}$

[using $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$]

$\therefore I=\int_0^{\pi}\frac{(\pi-x)\,dx}{1+\cos\alpha\sin x}$ (2)

Adding equations (1) and (2), we get

$2I=\int_0^{\pi}\frac{x+\pi-x}{1+\cos\alpha\sin x}dx$

$=\int_0^{\pi}\frac{\pi}{1+\cos\alpha\sin x}\,dx$

$\therefore I=\frac{\pi}{2}\int_0^{\pi}\frac{1}{1+\cos\alpha\sin x}\,dx$

$=\frac{\pi}{2}\times2\int_0^{\pi/2}\frac{1}{1+\cos\alpha\sin x}dx$

$=\pi\int_0^{\pi/2}\frac{1}{1+\cos\alpha\times\frac{2\tan x/2}{1+\tan^2x/2}}dx$

$=\pi\int_0^{\pi/2}\frac{\sec^2x/2}{1+\tan^2x/2+2\cos\alpha\tan x/2}dx$

Put $\tan x/2=t\Rightarrow\frac{1}{2}\sec^2\frac{x}{2}dx=dt$

Also when $x\to0$, $t\to0$

as $x\to\pi/2$, $t\to1$

$\therefore I=\pi\int_0^1\frac{2dt}{t^2+(2\cos\alpha)t+1}$

$=2\pi\int_0^1\frac{dt}{(t+\cos\alpha)^2+1-\cos^2\alpha}$

$=2\pi\int_0^1\frac{dt}{(t+\cos\alpha)^2+\sin^2\alpha}$

$=2\pi\cdot\frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{t+\cos\alpha}{\sin\alpha}\right)\right]_0^1$

$=\frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{1+\cos\alpha}{\sin\alpha}\right)-\tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right)\right]$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{\frac{1+\cos\alpha}{\sin\alpha}-\frac{\cos\alpha}{\sin\alpha}}{1+\left(\frac{1+\cos\alpha}{\sin\alpha}\frac{\cos\alpha}{\sin\alpha}\right)}\right)\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{\sin\alpha}{1+\cos\alpha}\right)\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\tan\frac{\alpha}{2}\right)\right]$$

$$= \frac{\pi\alpha}{\sin\alpha}$$

10. $\int_0^a f(x)g(x)dx$

$= \int_0^a f(a-x).g(a-x)dx$

$= \int_0^a f(x)\cdot\{2-g(x)\}dx$

$=2\int_0^a f(x)\,dx - \int_0^a f(x)g(x)dx$

$\Rightarrow 2\int_0^a f(x)f(x)\,dx = 2\int_0^a f(x)\,dx$

$\Rightarrow \int_0^a f(x)\,g(x)\,dx = \int_0^a f(x)\,dx$

11. Let $I=\int_0^{\pi/2} f(\sin 2x)\sin x\,dx$ (1)

$= \int_0^{\pi/2} f\left\{\sin 2\left(\frac{1}{2}\pi - x\right)\right\}\sin\left(\frac{1}{2}\pi - x\right)dx$

$= \int_0^{\pi/2} f(\sin 2x)\cos x\,dx$ (2)

Then adding equations (1) and (2), we have

$2I = \int_0^{\pi/2} f(\sin 2x)(\sin x + \cos x)\,dx$

$= \sqrt{2}\int_0^{\pi/2} f(\sin 2x)\sin\left(x+\frac{\pi}{4}\right)dx$

Now, from the result which we have to prove, it is clear that we have to substitute $\frac{\pi}{2}-2\theta = 2x$.

$\Rightarrow dx = -d\theta$ and also when

$x=0$, $\theta=\pi/4$ and when $x=\pi/2$, $\theta=-\pi/4$

$\Rightarrow 2I = \sqrt{2}\int_{-\pi/4}^{\pi/4} f(\cos 2\theta)\cos\theta\,d\theta$

$= 2\sqrt{2}\int_0^{\pi/4} f(\cos 2\theta)\cos\theta\,d\theta$

[as $g(\theta) = f(\cos 2\theta)\cos\theta$ is an even function]

$\therefore\quad I = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$ (3)

Equations (1), (2), (3) give the required result.

12. $2\sin x\cos x + 2\sin x\cos 3x + \cdots + 2\sin x\cos(2x-1)x$

$= \sin 2x + (\sin 4x - \sin 2x) + (\sin 6x - \sin 4x) + \cdots + (\sin 2kx - \sin(2k-2)x)$

$= \sin 2kx =$ R.H.S.

$\Rightarrow \sin 2kx\cot x = \frac{\sin 2kx\cos x}{\sin x}$

$= \cos x\cdot 2[\cos x + \cos 3x + \cdots + \cos(2k-1)x]$

$\therefore\ 1+\cos 2x + \cos 4x + \cos 2x + \cos 6x + \cos 4x + \cdots + \cos 2kx + \cos(2k-2)x$

$\Rightarrow \int_0^{\pi/2} \sin 2kx\cdot\cot x\,dx$

$= \int_0^{\pi/2} dx + \int_0^{\pi/2}(2\cos 2x + 2\cos 4x + \cdots + 2\cos(2k-2)x)\,dx + \int_0^{\pi/2}\cos 2kx = \frac{\pi}{2}$

(as integrals other than 1st one are zero)

13. We are given that f is a continuous function and

$\int_0^x f(t)\,dt \to \infty$ as $|x| \to \infty$

To show that every line $y = mx$ intersects the curve

$y^2 + \int_0^x f(t)\,dt = 2$

If possible, let $y = mx$ intersects the given curve then substituting $y = mx$ in the curves, we get

$m^2x^2 + \int_0^x f(t)\,dt = 2$ (1)

Consider $F(x) = m^2x^2 + \int_0^x f(t)\,dt - 2$

Then $F(x)$ is a continuous function as $f(x)$ is given to be continuous.

Also $F(x) \to \infty$ as $|x| \to \infty$

But $F(0) = -2$

Thus, $F(0) =$ –ve and $F(b) =$ +ve where b is some value of x, and $F(x)$ is continuous.

Therefore, $F(x) = 0$ for some value of $x \in (0, b)$ or equation (1) is solvable for x.

Hence $y = mx$ intersects the given curves.

14. Let $I = \int_0^\pi \frac{x\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$ (1)

Then

$I = \int_0^\pi \frac{(\pi-x)\sin(2\pi-2x)\sin\left(\frac{\pi}{2}\cos(\pi-x)\right)}{2(\pi-x)-\pi}dx$, or

$I = \int_0^\pi \frac{(\pi-x)(-\sin 2x)\sin\left(-\frac{\pi}{2}\cos x\right)}{\pi-2x}$, or

$I = \int_0^\pi \frac{(x-\pi)\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$ (2)

Adding equations (1) and (2), we get

$2I = \int_0^\pi \frac{(2x-\pi)\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}\text{dx}$

$= \int_0^\pi \sin 2x\sin\left(\frac{\pi}{2}\cos x\right)dx$

$= \int_0^\pi 2\sin x\cos x\sin\left(\frac{\pi}{2}\cos x\right)\text{dx}$

$\Rightarrow I = \int_0^{\pi} \sin x \cos x \sin\left(\frac{\pi}{2}\cos x\right) dx$

Put $z = \frac{\pi}{2}\cos x$, then $dz = -\frac{\pi}{2}\sin x\, dx$

When $x = 0, z = \frac{\pi}{2}$ and when $x = -\pi, z = -\frac{\pi}{2}$

$\Rightarrow I = -\frac{2}{\pi}\int_{\pi/2}^{-\pi/2} \frac{2z}{\pi} \sin z\, dz = \frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} z \sin z\, dz = \frac{8}{\pi^2}$

15. Given $\int_0^1 e^x (x-1)^n\, dx = 16 - 6e$

where $n \in N$ and $n \le 5$.

To find the value of n.

Let $I_n = \int_0^1 e^x (x-1)^n\, dx$

$= [(x-1)^n e^x]_0^1 - \int_0^1 n(x-1)^{n-1} e^x\, dx$

$= -(-1)^n - \int_0^1 n(x-1)^{n-1} e^x\, dx$

$\Rightarrow \quad I_n = (-1)^{n+1} - nI_{n-1}.$ (1)

Also $I_1 = \int_0^1 e^x (x-1)\, dx$

$= \left[e^x (x-1)\right]_0^1 - \int_0^1 e^x\, dx$

$= -(-1) - (e^x)_0^1$

$= 1 - (e-1) = 2 - e$

Using equation (1), we get

$I_2 = (-1)^3 - 2I_1 = -1 - 2(2-e) = 2e - 5$

Similarly, $I_3 = (-1)^4 - 3I_2 = 1 - 3(2e-5)$

$= 16 - 6e$

$n = 3$

16. $I = \int_2^3 \frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2+1)(x^4-1)} dx$

$= \int_2^3 \frac{2x^3(x^2-1) + (x^2+1)^2}{(x^2+1)^2(x^2-1)} dx$

$= \int_2^3 \frac{2x^3 dx}{(x^2+1)^2} + \int_2^3 \frac{1}{x^2-1} dx$

$= \int_2^3 \frac{x^2\, 2x dx}{(x^2+1)^2} + \left[\frac{1}{2}\log\frac{x-1}{x+1}\right]_2^3$

$= \int_5^{10} \frac{t-1}{t^2} dt + \left[\frac{1}{2}\left(\log\frac{2}{3} - \log\frac{1}{3}\right)\right]$

Put $x^2 + 1 = t \Rightarrow 2xdx = dt$ when $x \to 2, t \to 5$ and $x \to 3$, $t \to 10$

$= \int_5^{10}\left(\frac{1}{t} - \frac{1}{t^2}\right) dt + \frac{1}{2}\log 2$

$= \left(\log|t| + \frac{1}{t}\right)_5^{10} + \frac{1}{2}\log 2$

$= \log 10 - \log 5 + \frac{1}{10} - \frac{1}{5} + \frac{1}{2}\log 2$

$= \log 2 + \left(-\frac{1}{10}\right) + \frac{1}{2}\log 2$

$= \frac{3}{2}\log 2 - \frac{1}{10}$

17. Let $I = \int_0^{n\pi + v} |\sin x| dx$

$= \int_0^v |\sin x|\, dx + \int_v^{n\pi + v} |\sin x|\, dx$

$= \int_0^v \sin x\, dx + n\int_0^{\pi} |\sin x| dx$ [$\because$ $|\sin x|$ has period π]

$= (-\cos x)_0^v + n(-\cos x)_0^{\pi}$

$= 2n + 1 - \cos v = $ R.H.S.

18. $U_{n+2} - U_{n+1} = \int_0^{\pi} \frac{(1-\cos(n+2)x) - (1-\cos(n+1)x)}{1-\cos x} dx$

$= \int_0^{\pi} \frac{\cos(n+1)x - \cos(n+2)x}{1-\cos x}$

$= \int_0^{\pi} \frac{2\sin\left(n+\frac{3}{2}\right)x \sin\frac{x}{2}}{2\sin^2 x/2} dx$

$\Rightarrow U_{n+2} - U_{n+1} = \int_0^{\pi} \frac{\sin\left(n+\frac{3}{2}\right)x}{\sin\frac{x}{2}} dx$ (1)

$\Rightarrow U_{n+1} - U_n = \int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}} dx$ (2)

From equations (1) and (2), we get

$(U_{n+2} - U_{n+1}) - (U_{n+1} - U_n)$

$= \int \frac{\sin\left(n+\frac{3}{2}\right)x - \sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}} dx$

$\Rightarrow \quad U_{n+2} + U_n - 2U_{n+1} = \int \frac{2\cos(n+1)x \sin x/2}{\sin x/2} dx$

$= 2\int_0^{\pi} \cos(n+1)x\, dx = 2\left(\frac{\sin(n+1)x}{n+1}\right)_0^{\pi} = 0$

$\Rightarrow U_{n+2} + U_n = 2U_{n+1}$

$U_0 = \int_0^{\pi} \frac{1-1}{1-\cos x} dx = 0$, $U_1 = \int_0^{\pi} \frac{1-\cos x}{1-\cos x} dx = \pi$

$U_1 - U_0 = \pi$ (common difference)

$\Rightarrow \quad U_n = U_0 + n\pi = n\pi$

$\Rightarrow \quad U_n = n\pi$

Now, $I_n = \int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2\theta} d\theta = \int_0^{\pi/2} \frac{1-\cos 2n\theta}{1-\cos 2\theta} d\theta$

$$= \frac{1}{2}\int_0^{\pi}\frac{1-\cos nx}{1-\cos x}\,dx \Rightarrow I_n = \frac{1}{2}\,n\pi$$

19. Let $I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Put $x = -y$, so that $dx = -dy$

and $I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{y^4}{1-y^4}\cos^{-1}\left(\frac{-2y}{1+y^2}\right)dy$

But $\cos^{-1}(-x) = \pi - \cos^{-1}x$ for $-1 \le x \le 1$,

$$\Rightarrow I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{y^4}{1-y^4}\left[\pi - \cos^{-1}\left(\frac{2y}{1+y^2}\right)\right]dy$$

$$= \pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\,dx - \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

$$\Rightarrow I = \pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\,dx - I$$

$$\Rightarrow I = \pi\int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4}\,dx$$

$$= \pi\int_0^{1/\sqrt{3}}\left[-1+\frac{1}{1-x^4}\right]dx$$

$$= -\pi\int_0^{1/\sqrt{3}} dx + \pi\int_0^{1/\sqrt{3}}\frac{dx}{1-x^4}$$

$$= -\frac{\pi}{\sqrt{3}} + \frac{\pi}{2}\int_0^{1/\sqrt{3}}\left[\frac{1}{1-x^2}+\frac{1}{1+x^2}\right]dx$$

$$= -\frac{\pi}{\sqrt{3}} + \frac{\pi}{2}\left[\left(-\frac{1}{2}\log_e\left|\frac{1-x}{1+x}\right| + \tan^{-1}x\right)\right]_0^{1/\sqrt{3}}$$

$$= -\frac{\pi}{\sqrt{3}} - \frac{\pi}{4}\log_e\left|\frac{\sqrt{3}+1}{\sqrt{3}-1}\right| + \frac{\pi^2}{12}$$

20. Let $I = \int_0^{\pi/4} \ln(1+\tan x)\,dx$ (1)

$$= \int_0^{\pi/4} \ln(1+\tan(\pi/4-x))\,dx \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/4} \ln\left[1+\frac{1-\tan x}{1+\tan x}\right]dx$$

$$= \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan x}\right)dx$$

$I = \int_0^{\pi/4} [\ln 2 - \ln(1+\tan x)]\,dx$ (2)

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/4} \ln 2\,dx$$

$$= \ln 2\,[x]_0^{\pi/4} = \ln 2\left[\frac{\pi}{4}\right]$$

$$\Rightarrow I = \frac{\pi}{8}\ln 2$$

21. $a + b = 4 \Rightarrow b = 4 - a$

Let $f(a) = \int_0^a g(x)\,dx + \int_0^b g(x)\,dx$

$$= \int_0^a g(x)\,dx + \int_0^{4-a} g(x)\,dx$$

$$\Rightarrow \frac{df(a)}{da} = g(a) - g(4-a)$$

$$\Rightarrow \frac{df(a)}{d(b-a)} = \frac{df(a)}{d(4-2a)} = \frac{df(a)}{-2da} = (g(4-a)-g(a))/2$$

Now given $a < 2$

$\Rightarrow 2a < 4$

$\Rightarrow 4 - a > a$

$\Rightarrow g(4-a) > g(a)$ ($\because g(x)$ is an increasing function)

$$\Rightarrow \frac{df(a)}{d(b-a)} > 0$$

$\Rightarrow f(a) = \int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases as $(b-a)$ increases.

22. $I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$

$$= \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\,dx + 2\int_{-\pi}^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx \quad (1)$$

$$= 0 + 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$$

$$= 4\int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}$$

$$= 4\int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$$

$$= 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx - 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$$

$$\Rightarrow 2I = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$\Rightarrow I = 2\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

Put $\cos x = t$ so that $-\sin x\,dx = dt$

When $x = 0$, $t = 1$; when $x = \pi$, $t = -1$

$$\Rightarrow I = 2\pi\int_1^{-1} \frac{-dt}{1+t^2}$$

$$= 4\pi\left[\tan^{-1}t\right]_0^1$$

$$= 4\pi\,\frac{\pi}{4} = \pi^2$$

23. $\int_0^1 \tan^{-1}\frac{1}{1-x+x^2}\,dx = \int_0^1 \tan^{-1}\frac{x+(1-x)}{1-x(1-x)}\,dx$

$$= \int_0^1 [\tan^{-1}x + \tan^{-1}(1-x)]\,dx$$

$$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}(1-x)\,dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1} [1-(1-x)]\, dx$$

$$= 2\int_0^1 \tan^{-1} x\, dx \qquad (1)$$

Now

$$I = \int_0^1 \tan^{-1}(1-x+x^2)\, dx$$

$$= \int_0^1 \cot^{-1}\left(\frac{1}{1-x+x^2}\right) dx$$

$$= \int_0^1 \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{1-x+x^2}\right)\right] dx$$

$$= \frac{\pi}{2} - 2\int_0^1 \tan^{-1} x\, dx \qquad \text{[from equation (1)]}$$

$$= \frac{\pi}{2} - 2\left\{x \tan^{-1} x - \frac{1}{2}\log(1+x^2)\right\}_0^1$$

$$= \log_e 2$$

24. Let $F(x) = f(x) + f\left(\frac{1}{x}\right)$

$$= \int_1^x \frac{\log t}{1+t} dt + \int_1^{1/x} \frac{\log t}{1+t} dt$$

In 2nd integral, let $t = 1/y \Rightarrow dt = -\frac{1}{y^2} dy$

$$\Rightarrow F(x) = \int_1^x \frac{\log t}{1+t} dt + \int_1^x \frac{-\log y}{1+\frac{1}{y}}\left(-\frac{dy}{y^2}\right)$$

$$= \int_1^x \frac{\log t}{1+t} dt + \int_1^x \frac{\log y}{y(1+y)} dy$$

$$= \int_1^x \frac{\log t}{1+t} dt + \int_1^x \frac{\log t}{t(1+t)} dt$$

$$= \int_1^x \frac{\log t}{t}\, dt = \frac{1}{2}(\log x)^2$$

$$\therefore F(e) = \frac{1}{2}$$

25. We have $y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x \cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta$

$$= \cos x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta$$

$$\Rightarrow \frac{dy}{dx} = -\sin x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta + \cos x \frac{d}{dx}\left[\int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta\right]$$

$$= -\sin x \int_{\pi^2/16}^{x^2} \frac{\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}} d\theta + \cos x\left[\frac{\cos x}{1+\sin^2 x}\, 2x - 0\right]$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\pi} = 0 + \frac{\cos^2 \pi}{1+\sin^2 \pi}\cdot 2\pi = 2\pi$$

26. Let $I = \int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2-\cos\left(|x| + \frac{\pi}{3}\right)} dx$

$$= \int_{-\pi/3}^{\pi/3} \frac{\pi}{2-\cos\left(|x|+\frac{\pi}{3}\right)} dx + \int_{-\pi/3}^{\pi/3} \frac{4x^3}{2-\cos\left(|x|+\frac{\pi}{3}\right)} dx$$

The second integral becomes zero as integrand being an odd function of x.

$$\Rightarrow I = 2\pi \int_0^{\pi/3} \frac{dx}{2-\cos\left(x+\frac{\pi}{3}\right)}$$

Let $x + \pi/3 = y \Rightarrow dx = dy$

Also, as $x \to 0, y \to \pi/3$ as $x \to \pi/3, y \to 2\pi/3$

$$\Rightarrow I = 2\pi \int_{\pi/3}^{2\pi/3} \frac{dy}{2-\cos y}$$

$$= 2\pi \int_{\pi/3}^{2\pi/3} \frac{dy}{2 - \frac{1-\tan^2 y/2}{1+\tan^2 y/2}}$$

$$= 2\pi \int_{\pi/3}^{2\pi/3} \frac{\sec^2 y/2}{3\tan^2 y/2 + 1} dy$$

$$= \frac{4\pi}{3} \int_{\pi/3}^{2\pi/3} \frac{\frac{1}{2}\sec^2 y/2}{\tan^2 y/2 + (1/\sqrt{3})^2} dy$$

$$= \frac{4\pi\sqrt{3}}{3}\left[\tan^{-1}\left(\sqrt{3}\tan y/2\right)\right]_{\pi/3}^{2\pi/3}$$

$$= \frac{4\pi}{\sqrt{3}}[\tan^{-1} 3 - \tan^{-1} 1]$$

$$= \frac{4\pi}{\sqrt{3}}[\tan^{-1} 3 - \pi/4]$$

27. $I = \int_0^{\pi} e^{|\cos x|}\left(2\sin\left(\frac{1}{2}\cos x\right) + 3\cos\left(\frac{1}{2}\cos x\right)\right)\sin x\, dx$

$$= \int_0^{\pi} e^{|\cos x|}\, 2\sin\left(\frac{1}{2}\cos x\right)\sin x\, dx + \int_0^{\pi} e^{|\cos x|}\, 3\cos\left(\frac{1}{2}\cos x\right)\sin x\, dx$$

$$= I_1 + I_2$$

Now using the property that

$$\int_0^{2a} f(x)dx = \begin{cases} 0 \text{ if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx \text{ if } f(2a-x) = f(x) \end{cases}$$

we get $I_1 = 0$ and

$$I_2 = 2\int_0^{\pi/2} e^{|\cos x|}\, 3\cos\left(\frac{1}{2}\cos x\right)\sin x\, dx$$

$$= 6\int_0^{\pi/2} e^{\cos x}\cos\left(\frac{1}{2}\cos x\right)\sin x\, dx$$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$\Rightarrow I_2 = 6\int_0^1 e^t \cos t/2\, dt$$

$$= 6\left[\frac{e^t}{1+\frac{1}{4}}\left(\frac{1}{2}\sin\frac{x}{2} + \cos\frac{x}{2}\right)\right]_0^1$$

$$\left[\text{Using } \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx)\right]$$

$$\Rightarrow I_2 = \frac{24}{5}\left[e\cos\left(\frac{1}{2}\right) + \frac{1}{2}e\sin\left(\frac{1}{2}\right) - 1\right]$$

28. $$\frac{5050\int_0^1 (1-x^{50})^{100}\, dx}{\int_0^1 (1-x^{50})^{101}\, dx} = 5050\frac{I_{100}}{I_{101}}$$

$$I_{101} = \int_0^1 (1-x^{50})(1-x^{50})^{100}\, dx$$

$$= I_{100} - \int_0^1 x.x^{49}(1-x^{50})^{100}\, dx$$

$$= I_{100} - \left[\frac{-x(1-x^{50})^{101}}{101}\right]_0^1 - \int_0^1 \frac{(1-x^{50})^{101}}{5050}$$

$$\Rightarrow I_{101} = I_{100} - \frac{I_{101}}{5050} \Rightarrow 5050\frac{I_{100}}{I_{101}} = 5051$$

Objective

Fill in the blanks

1. Given that

$$f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

Operating $R_1 \to R_1 - \sec x \cdot R_3$, we get

$$= \begin{vmatrix} 0 & 0 & \sec^2 x + \cot x \operatorname{cosec} x - \cos x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

Expanding along R_1, we get

$$= (\sec^2 x + \cot x \operatorname{cosec} x - \cos x)(\cos^4 x - \cos^2 x)$$

$$= \left(\frac{1}{\cos^2 x} + \frac{\cos x}{\sin^2 x} - \cos x\right)\cos^2 x(\cos^2 x - 1)$$

$$= -\left[\frac{\sin^2 x + \cos^3 x - \cos^3 x \sin^2 x}{\cos^2 x \sin^2 x}\right]\cos^2 x \sin^2 x$$

$$= -\sin^2 x - \cos^3 x(1 - \sin^2 x)$$

$$= -\sin^2 x - \cos^5 x$$

$$\therefore \int_0^{\pi/2} f(x)\, dx = -\int_0^{\pi/2}(\sin^2 x + \cos^5 x)\, dx$$

$$= -\int_0^{\pi/2}\left[\frac{1-\cos 2x}{2} + \cos x(1-\sin^2 x)^2\right]dx$$

$$= -\left[\frac{x + \frac{\sin 2x}{2}}{2}\right]_0^{\frac{\pi}{2}} - \left(t - \frac{2t^3}{3} + \frac{t^5}{5}\right)_1^0, \text{ where } t = \sin x$$

$$= -\frac{\pi}{4} + \left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$= -\left(\frac{15\pi - 32}{60}\right)$$

2. When $x = 0, x^2 = 0$ and $x = 1.5, x^2 = 2.25$

$\Rightarrow [x^2]$ is discontinuous when $x^2 = 1$ and $x^2 = 2$ or $x = 1$ and $x = \sqrt{2}$

$$\Rightarrow \int_0^{1.5}[x^2]\, dx = \int_0^1 [x^2]\, dx + \int_1^{\sqrt{2}}[x^2]\, dx + \int_{\sqrt{2}}^{1.5}[x^2]dx$$

$$= 0 + \int_1^{\sqrt{2}} 1\, dx + \int_{\sqrt{2}}^{1.5} 2\, dx$$

$$= 1(\sqrt{2} - 1) + 2(1.5 - \sqrt{2}) = (2 - \sqrt{2})$$

3. Let $I = \int_{-2}^{2}|1 - x^2|\, dx = 2\int_0^2 |1 - x^2|\, dx$

$$= 2\int_0^1 (1 - x^2)dx + 2\int_1^2 (x^2 - 1)dx$$

$$= 2\left[x - \frac{x^3}{3}\right]_0^1 + 2\left[\frac{x^3}{3} - x\right]_1^2$$

$$= 2\left[1 - \frac{1}{3}\right] + 2\left[\frac{8}{3} - 2 - \frac{1}{3} + 1\right]$$

$$= \frac{4}{3} + \frac{8}{3} = 4$$

4. $$I = \int_{\pi/4}^{3\pi/4}\frac{\phi}{1+\sin\phi}\, d\phi \quad (1)$$

$$\Rightarrow I = \int_{\pi/4}^{3\pi/4}\frac{\pi - \phi}{1 + \sin(\pi - \phi)}d\phi$$

$$\left[\text{Using } \int_a^b f(x)dx = \int_a^b f(a+b-x)\, dx\right]$$

$$\Rightarrow I = \int_{\pi/4}^{3\pi/4}\frac{\pi - \phi}{1 + \sin\phi}d\phi \quad (2)$$

Adding equations (1) and (2), we get

$$2I = \int_{\pi/4}^{3\pi/4}\frac{\pi}{1+\sin\phi}d\phi$$

$= \pi \int_{\pi/4}^{3\pi/4} \frac{1-\sin\phi}{1-\sin^2\phi} d\phi$

$= \pi \int_{\pi/4}^{3\pi/4} \frac{1-\sin\phi}{\cos^2\phi} d\phi$

$= \pi \int_{\pi/4}^{3\pi/4} (\sec^2\phi - \sec\phi\tan\phi) d\phi$

$= \pi[\tan\phi - \sec\phi]_{\pi/4}^{3\pi/4}$

$= \pi[\tan 3\pi/4 - \sec 3\pi/4 - \tan\pi/4 + \sec\pi/4]$

$= \pi[-1+\sqrt{2}-1+\sqrt{2}]$

$= 2\pi(\sqrt{2}-1)$

$\Rightarrow I = \pi(\sqrt{2}-1)$

5. Let $I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}} dx$ (1)

$= \int_2^3 \frac{\sqrt{5-x}}{\sqrt{5-(5-x)}+\sqrt{5-x}} dx$

$\left[\text{Using } \int_a^b f(x)\,dx = \int_a^b f(a+b-x)dx\right]$

$\Rightarrow I = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}} dx$ (2)

Adding equations (1) and (2), we get

$2I = \int_2^3 \frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}} dx$

$\Rightarrow I = \frac{1}{2}\int_2^3 1\,dx = \frac{1}{2}(3-2) = \frac{1}{2}$

6. Let's first find the functions satisfying

$af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5.$ (1)

Replacing x by $\frac{1}{x}$, we have $af\left(\frac{1}{x}\right) + bf(x) = x - 5$. (2)

Eliminating $f\left(\frac{1}{x}\right)$ from equations (1) and (2), we get

$\Rightarrow \int_1^2 f(x)\,dx = \int_1^2 \frac{\frac{a}{x}-5a-bx+5b}{a^2-b^2} dx$

$= \frac{1}{a^2-b^2}\left[a\log x - b\frac{x^2}{2} + 5(b-a)x\right]_1^2$

$= \frac{1}{a^2-b^2}\left[a\log 2 - 2b + 10(b-a) + \frac{b}{2} - 5(b-a)\right]$

$= \frac{1}{a^2-b^2}\left[a\log 2 - 5a + \frac{7b}{2}\right]$

7. $\Rightarrow I = \int_0^{2\pi} \frac{x\cos^{2n}x}{\cos^{2n}x+\sin^{2n}x} dx$ (1)

$= 2\int_0^{2\pi} \frac{(2\pi-x)\cos^{2n}x}{\cos^{2n}x+\sin^{2n}x} dx$ (2)

$\left[\text{Using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

Adding equations (1) and (2), we get

$2I = \int_0^{2\pi} \frac{2\pi\cos^{2n}x}{\cos^{2n}x+\sin^{2n}x} dx = 4\pi\int_0^{\pi} \frac{\cos^{2n}x}{\cos^{2n}x+\sin^{2n}x} dx$

$\left[\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(2a-x) = f(x)\right]$

$= 8\pi\int_0^{\pi/2} \frac{\cos^{2n}x}{\cos^{2n}x+\sin^{2n}x} dx$ (3)

[Using the above property again]

$= 8\pi\int_0^{\pi/2} \frac{\cos^{2n}\left(\frac{\pi}{2}-x\right)}{\cos^{2n}\left(\frac{\pi}{2}-x\right)+\sin^{2n}\left(\frac{\pi}{2}-x\right)} dx$

$\left[\text{Using } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= 8\pi\int_0^{\pi/2} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x} dx$ (4)

Adding equations (3) and (4), we have

$4I = 8\pi\int_0^{\pi/2} 1dx$

$\Rightarrow I = \pi^2$

8. Let $I = \int_1^{e^{37}} \frac{\pi\sin(\pi\ln x)}{x} dx$

Let $\pi\ln x = t$

$\Rightarrow \frac{\pi}{x} dx = dt$

$\Rightarrow I = \int_0^{37\pi} \sin t\,dt = [-\cos t]_0^{37\pi} = -\cos 37\pi + 1$

$= -(-1) + 1 = 2$

9. $\int_1^4 \frac{2e^{\sin x^2}}{x} dx = F(k) - F(1) = [F(x)]_1^{16}$

Put $x^2 = t \therefore 2x\,dx = dt$

$\therefore I = \int_1^{16} \frac{e^{\sin t}}{t} dt = F[(t)]_1^{16}$

$\therefore I = F(16) - F(1)$

10. $f(x) = \int_0^x f(t)\,dt \Rightarrow f(0) = 0$

also, $f'(x) = f(x), x > 0$

$\Rightarrow f(x) = ke^x, x > 0$

$\because f(0) = 0$ and $f(x)$ is continuous $\Rightarrow f(x) = 0\ \forall x > 0$

$\therefore f(\ln 5) = 0$

11. $\frac{\pi^2}{\ln 3}\frac{1}{\pi}(\ln(|\sec\pi x + \tan\pi x|))_{7/6}^{5/6}$

$= \frac{\pi}{\ln 3}\left(\ln\left|\sec\frac{5\pi}{6} + \tan\frac{5\pi}{6}\right| - \ln\left|\sec\frac{7\pi}{6} + \tan\frac{7\pi}{6}\right|\right)$

$= \pi$

12. $\int_a^b (f(x) - 3x)\, dx = a^2 - b^2$

$\Rightarrow \int_a^b f(x) dx = \frac{3}{2}(b^2 - a^2) + a^2 - b^2 = \left(\frac{b^2 - a^2}{2}\right)$

$\Rightarrow f(x) = x$

$\Rightarrow f\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$

ue or false

1. Let $I = \int_0^{2a} \frac{f(x)}{f(x) + f(2a-x)} dx$ (1)

$= \int_0^{2a} \frac{f(2a-x)}{f(2a-x) + f(x)}$ (2)

[Using $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$]

Adding equations (1) and (2), we get

$2I = \int_0^{2a} \frac{f(x) + f(2a-x)}{f(x) + f(2a-x)} dx$

$= \int_0^{2a} 1 dx$

$= [x]_0^{2a} = 2a \Rightarrow I = a$

Therefore, the given statement is true.

ltiple choice questions with one correct answer

.d. $\int_0^1 \left(1 + e^{-x^2}\right) dx$

$= \int_0^1 \left(1 + 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \infty\right) dx$

$= \left[2x - \frac{x^3}{3.1!} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \cdots \infty\right]_0^1$

$= \left[2 - \frac{1}{3.1!} + \frac{1}{5.2!} - \frac{1}{7.3!} + \cdots \infty\right]$

Clearly 'd' is the correct alternative.

b. Let $f(x) = \int (1 + \cos^8 x)(ax^2 + bx + c)\, dx$

$\therefore f'(x) = (1 + \cos^8 x)(ax^2 + bx + c)$ (1)

From the given conditions

$f(1) - f(0) = 0 \quad \Rightarrow f(0) = f(1)$ (2)

and $f(2) - f(0) = 0 \quad \Rightarrow f(0) = f(2)$ (3)

From equations (2) and (3), we get $f(0) = f(1) = f(2)$

By Rolle's theorem for $f(x)$ in [0, 1]: $f'(\alpha) = 0$, at least one α such that $0 < \alpha < 1$.

By Rolle's theorem for $f(x)$ in [1, 2]: $f'(\beta) = 0$, at least one β such that $1 < \beta < 2$.

Now, from equation (1), $f'(\alpha) = 0$

$\Rightarrow (1 + \cos^8 \alpha)(a\alpha^2 + b\alpha + c) = 0 \qquad (\because 1 + \cos^8 \alpha \neq 0)$

$\Rightarrow a\alpha^2 + b\alpha + c = 0$

i.e., α is a root of the equation $ax^2 + bx + c = 0$.

Similarly, β is a root of the equation $ax^2 + bx + c = 0$.

But equation $ax^2 + bx + c = 0$ being a quadratic equation cannot have more than two roots.

Hence, equation $ax^2 + bx + c = 0$ has one root α between 0 and 1, and other root β between 1 and 2.

3.a. $I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx$ (1)

$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx$ (2)

[Using $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$]

Adding equations (1) and (2), we get $2I = \int_0^{\pi/2} 1 dx$

$\Rightarrow I = \pi/4$

4.c. $I = \int_0^{\pi} e^{\cos^2 x} \cos^3 (2n+1)x\, dx, n \in Z$ (1)

$= \int_0^{\pi} e^{\cos^2(\pi - x)} \cos^3 [(2n+1)(\pi - x)]\, dx$

[Using $\int_0^a f(x) dx = \int_0^a f(a-x)\, dx$]

$= \int_0^{\pi} e^{\cos^2 x} \cos^3 [(2n+1)\pi - (2n+1)x]\, dx$

$= -\int_0^{\pi} -e^{\cos^2 x} \cos^3 (2n+1)x\, dx$

$= -I$

$\Rightarrow I = 0$

5.d. Since $h(x) = (f(x) + f(-x))(g(x) - g(-x))$

$\Rightarrow h(-x) = (f(-x) + f(x))(g(-x) - g(x))$

$\Rightarrow h(-x) = -h(x)$

$\therefore h(x)$ is odd function,

$\Rightarrow \int_{-\pi/2}^{\pi/2} (f(x) + f(-x))(g(x) - g(-x)) dx = 0$

6.d. Let $I = \int_0^{\pi/2} \frac{dx}{1 + \tan^3 x}$

$= \int_0^{\pi/2} \frac{\cos^3 x}{\sin^3 x + \cos^3 x} dx$ (1)

$= \int_0^{\pi/2} \frac{\cos^3\left(\frac{\pi}{2} - x\right)}{\sin^3\left(\frac{\pi}{2} - x\right) + \cos^3\left(\frac{\pi}{2} - x\right)} dx$

$= \int_0^{\pi/2} \frac{\sin^3 x}{\cos^3 x + \sin^3 x} dx$ (2)

Adding equations (1) and (2), we get

$2I = \int_0^{\pi/2} 1 dx$

$\Rightarrow I = \frac{\pi}{4}$

8.

9.c.

d. $f(x)=A\sin(\pi x/2)+B$

$\Rightarrow f'(x)=\frac{A\pi}{2}\cos\left(\frac{\pi x}{2}\right)$

$\Rightarrow f'\left(\frac{1}{2}\right)=\frac{A\pi}{2}\cos\frac{\pi}{4}=\sqrt{2}$ (given)

$\Rightarrow A=4/\pi$

Also, given $\int_0^1 f(x)\,dx=\frac{2A}{\pi}$

$\Rightarrow \int_0^1\left[A\sin\left(\frac{\pi x}{2}\right)+B\right]dx=\frac{2A}{\pi}$

$\Rightarrow \left|-\frac{2A}{\pi}\cos\left(\frac{\pi x}{2}\right)+Bx\right|_0^1=\frac{2A}{\pi}$

$\Rightarrow B+\frac{2A}{\pi}=\frac{2A}{\pi} \quad \Rightarrow B=0$

8.b. $I=\int_\pi^{2\pi}[2\sin x]\,dx$

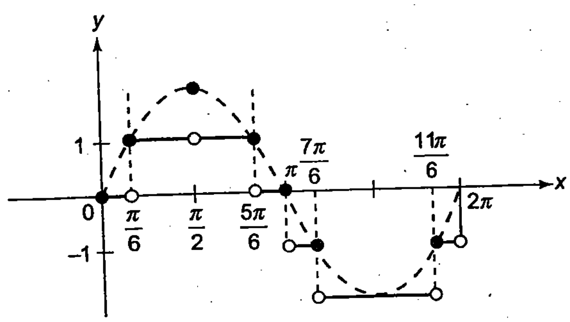

Fig. 8.23

From the graph in Fig. 8.23

$\therefore I=\int_{\pi/6}^{5\pi/6}1dx+\int_\pi^{7\pi/6}-1\,dx+\int_{7\pi/6}^{11\pi/6}-2\,dx+\int_{11\pi/6}^{2\pi}-1dx$

$=\left(\frac{5\pi}{6}-\frac{\pi}{6}\right)+\left(-\frac{7\pi}{6}+\pi\right)+2\left(-\frac{11\pi}{6}+\frac{7\pi}{6}\right)+\left(-2\pi+\frac{11\pi}{6}\right)$

$=\frac{2\pi}{3}-\frac{\pi}{6}-\frac{8\pi}{6}-\frac{\pi}{6}=-\pi$

9.c. Given f is a positive function, and

$I_1=\int_{1-k}^k xf(x(1-x))\,dx$

$I_2=\int_{1-k}^k f[x(1-x)]dx$

Now, $I_1=\int_{1-k}^k f[x(1-x)]dx$ (1)

$=\int_{1-k}^k (1-x)f[(1-x)x]dx$ (2)

$\left[\text{Using the property }\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right]$

Adding equations (1) and (2), we get

$2I_1=\int_{1-k}^k f[x(1-x)]dx=I_2\Rightarrow\frac{I_1}{I_2}=\frac{1}{2}$

10.a. $g(x)=\int_0^x\cos^4 t\,dt$

$\Rightarrow g(x+\pi)=\int_0^{x+\pi}\cos^4 t\,dt$

$=\int_0^x\cos^4 t\,dt+\int_x^{x+\pi}\cos^4 t\,dt$

$=g(x)+\int_0^\pi\cos^4 t\,dt$ [$\because$ period of $\cos^4 t$ is π]

$=g(x)+g(\pi)$

11.a. $I=\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos x}$ (1)

$=\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos(\pi-x)}$

$\left[\text{Using the property }\int_a^b f(x)\,dx=\int_a^b(f(a+b-x))dx\right]$

$=\int_{\pi/4}^{3\pi/4}\frac{dx}{1-\cos x}$ (2)

Adding (1) and (2), we get

$2I=\int_{\pi/4}^{3\pi/4}\left(\frac{1}{1+\cos x}+\frac{1}{1-\cos x}\right)dx$

$=\int_{\pi/4}^{3\pi/4}2\,\text{cosec}^2x\,dx$

$=2(-\cot x)_{\pi/4}^{3\pi/4}$

$=-2[\cot 3\pi/4-\cot\pi/4]$

$=-2(-1-1)=4$

$\Rightarrow I=2$

12.c. Refer to the graph of the question 8 (Fig. 8.23), we have

$\int_{\pi/2}^{3\pi/2}[2\sin x]dx$

$=\int_{\pi/2}^{5\pi/6}1dx+\int_\pi^{7\pi/6}-1dx+\int_{7\pi/6}^{3\pi/2}-2dx$

$=\left[\frac{5\pi}{6}-\frac{\pi}{2}\right]-\left[\frac{7\pi}{6}-\pi\right]-2\left[\frac{3\pi}{2}-\frac{7\pi}{6}\right]$

$=\frac{-\pi}{2}$

13.b. $g(x)=\int_0^x f(t)dt,$

$\Rightarrow f(2)=\int_0^2 f(t)=\int_0^1 f(t)\,dt+\int_1^2 f(t)\,dt$

Now, $\frac{1}{2}\le f(t)\le 1$ for $t\in[0,1]$

$\Rightarrow\int_0^1\frac{1}{2}dt\le\int_0^1 f(t)\,dt\le\int_0^1 1\,dt$

$\Rightarrow\frac{1}{2}\le\int_0^1 f(t)\,dt\le 1$ (1)

Again, $0\le f(t)\le\frac{1}{2}$ for $t\in[1,2]$

$$\Rightarrow \int_1^2 0\,dt \le \int_1^2 f(t) \le \int_1^2 \frac{1}{2}dt$$

$$\Rightarrow 0 \le \int_1^2 f(t)\,dt \le \frac{1}{2} \qquad (2)$$

From equations (1) and (2), we get

$$\frac{1}{2} \le \int_0^1 f(t) + \int_1^2 f(t)\,dt \le \frac{3}{2}$$

$$\Rightarrow \frac{1}{2} \le g(2) \le \frac{3}{2}$$

14.c. If $f(x) = \begin{cases} e^{\cos x}\sin x \text{ for } |x| \le 2 \\ 2 \quad\quad \text{otherwise} \end{cases}$

$$\Rightarrow \int_{-2}^3 f(x)\,dx = \int_{-2}^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

$$= \int_{-2}^2 e^{\cos x}\sin x\,dx + \int_2^3 2\,dx = 0 + 2[x]_2^3 = 2$$

$[\because e^{\cos x}\sin x$ is an odd function$]$

15.b. Let $I = \int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right| dx$

For $\frac{1}{e} < x < 1, \log_e x < 0$, hence $\frac{\log_e x}{x} < 0$

For $1 < x < e^2, \log x > 0$, hence $\frac{\log_e x}{x} > 0$

$$\therefore I = \int_{1/e}^1 -\frac{\log_e x}{x}\,dx + \int_1^2 \frac{\log_e x}{x}\,dx$$

$$= -\frac{1}{2}\left[(\log_e x)^2\right]_{1/e}^1 + \frac{1}{2}\left[(\log_e x)^2\right]_1^{e^2}$$

$$= -\frac{1}{2}\left[0-(-1)^2\right] + \frac{1}{2}\left[(2)^2 - 0\right]$$

$$= \frac{1}{2} + 2 = \frac{5}{2}$$

16.c. $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx \qquad (1)$

$$= \int_{-\pi}^{\pi} \frac{\cos^2(0-x)}{1+a^{(0-x)}}\,dx$$

$\left[\text{Using the property } \int_a^b f(x)\,dx = \int_a^b \left(f(a+b-x)\right)dx\right]$

$$\Rightarrow I = \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{1+a^x}\,dx \qquad (2)$$

Adding equations (1) and (2), we get

$$2I = \int_{-\pi}^{\pi} \cos^2 x\,dx \qquad (3)$$

$$= 2\int_0^{\pi} \cos^2 x\,dx$$

$$= 4\int_0^{\pi/2} \cos^2 x\,dx$$

$\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(2a-x) = f(x)\right]$

$$= 4\int_0^{\pi/2} \sin^2 x\,dx \qquad (4)$$

Adding equations (3) and (4), we get

$$4I = 4\int_0^{\pi/2} 1\,dx$$

$$\Rightarrow I = \pi/2$$

17.a. Here $f(x) = \int_1^x \sqrt{2-t^2}\,dt$

$$\Rightarrow f'(x) = \sqrt{2-x^2}$$

Now the given equation $x^2 - f'(x) = 0$ becomes

$$x^2 - \sqrt{2-x^2} = 0$$

$$\Rightarrow x^2 = \sqrt{2-x^2}$$

$$\Rightarrow x = \pm 1$$

18.c. Let $I_1 = \int_3^{3+3T} f(2x)\,dx$.

Put $2x = y$, so that $I_1 = \frac{1}{2}\int_6^{6+6T} f(y)\,dy$

$$= \frac{1}{2} 6\int_0^T f(y)\,dy \qquad (\because f(x) \text{ has period } T)$$

$$= 3I$$

19.a. $I = \int_{-1/2}^{1/2} \left([x] + \ln\left(\frac{1+x}{1-x}\right)\right) dx$

$$= \int_{-1/2}^{1/2} [x]\,dx + \int_{-1/2}^{1/2} \ln\left(\frac{1+x}{1-x}\right) dx$$

$$= \int_{-1/2}^{0} -1\,dx + \int_0^{1/2} 0\,dx + 0$$

$\left[\because \log\left(\frac{1+x}{1-x}\right) \text{ is an odd function}\right]$

$$= [-x]_{-1/2}^0 = 0 - \left(\frac{1}{2}\right) = -1/2$$

20.a. Given $L(m,n) = \int_0^1 t^m (1+t)^n\,dt$

Integrating by parts considering $(1+t)^n$ as first function, we get

$$L(m,n) = \left[\frac{t^{m+1}}{m+1}(1+t)^n\right]_0^1 - \frac{n}{m+1}\int_0^1 t^{m+1}(1+t)^{n-1}\,dt$$

$$L(m,n) = \frac{2^n}{m+1} - \frac{n}{m+1} L(m+1, n-1)$$

21.d. We have $f(x) = \int_{x^2}^{x^2+1} e^{-t^2}\,dt$

$$\Rightarrow f'(x) = e^{-(x^2+1)^2}.\,2x - e^{-x^4}\cdot 2x$$

$$= 2x\left[e^{-(x^2+1)^2} - e^{-x^4}\right]$$

$$\because (x^2+1)^2 > x^4$$

$$\Rightarrow e^{+(x^2+1)^2} > e^{x^4} \Rightarrow e^{-(x^2+1)^2} < e^{-x^4}$$

$$\Rightarrow e^{-(x^2+1)^2} - e^{-x^4} < 0$$

$\therefore f'(x) \geq 0, \forall x \leq 0$

Therefore, $f(x)$ increases when $x \leq 0$.

2.a. $\int_0^{t^2} xf(x)\, dx = \frac{2}{5}t^5$ (Here, $t > 0$)

Differentiating both sides w.r.t. t, we get

$\Rightarrow t^2 f(t^2) \times 2t = \frac{2}{5} \times 5t^4$

$\Rightarrow f(t^2) = t$

Put $t = \frac{2}{5} \Rightarrow f\left(\frac{4}{25}\right) = \frac{2}{5}$

3.b. $I = \int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$

$= \int_0^1 \frac{1-x}{\sqrt{1-x^2}}\, dx$

$= \sin^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx$

$= \frac{\pi}{2} + \left[\sqrt{1-x^2}\right]_0^1$

$= \frac{\pi}{2} + (0-1) = \frac{\pi}{2} - 1$

4.c. $I = \int_{-2}^{0} [x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)]dx$

$= \int_{-2}^{0} [(x+1)^3 + 2 + (x+1)\cos(x+1)]dx$

$= \int_{-2}^{0} [(-2-x+1)^3 + 2 + (-2-x+1)\cos(-2-x+1)]dx$

$= \int_{-2}^{0} [-(1+x)^3 + 2 - (1+x)\cos(1+x)]dx$

$\Rightarrow \quad 2I = 2\int_{-2}^{0} 2 \quad \Rightarrow \quad I = 4$

5.c. $f' = \pm\sqrt{1-f^2}$

$\Rightarrow f(x) = \sin x$ or $f'(x) = -\sin x$ (not possible)

$\Rightarrow f(x) = \sin x$

Also, $x > \sin, \forall x > 0$.

26.a $\int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\right)dx$

$= \left[\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x\right]_0^1 - \pi$

$= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - \pi = \frac{22}{7} - \pi$

27.b $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\, dt$ (1)

$f(f^{-1}(x)) = x$

$\Rightarrow f'(f^{-1}(x))(f^{-1}(x))' = 1$

$\Rightarrow (f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))}$

$f(0) = 2 \Rightarrow f^{-1}(2) = 0$

$\Rightarrow (f^{-1})'(2) = \frac{1}{f'(0)}$

$e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\, dt$

$\Rightarrow e^{-x}(f'(x) - f(x)) = \sqrt{x^4+1}$

Put $x = 0$

$\Rightarrow f'(0) - 2 = 1$

$\Rightarrow f'(0) = 3$

$(f^{-1})'(2) = 1/3$

28.a Put $x^2 = t \Rightarrow 2x\, dx = dt$

$I = \frac{1}{2}\int_{\ln 2}^{\ln 3} \frac{\sin t}{\sin t + \sin(\ln 6 - t)} dt$

$\Rightarrow I = \frac{1}{2}\int_{\ln 2}^{\ln 3} \frac{\sin(\ln 6 - t)}{\sin(\ln 6 - t) + \sin t} dt$

$\Rightarrow 2I = \frac{1}{2}\int_{\ln 2}^{\ln 3} 1\, dt \Rightarrow I = \frac{1}{4}\ln\frac{3}{2}.$

29.c $R_1 = \int_{-1}^{2} xf(x)\, dx = \int_{-1}^{2} (2-1-x) f(2-1-x)\, dx$

$= \int_{-1}^{2} (1-x) f(1-x)\, dx = \int_{-1}^{2} (1-x) f(x)\, dx$

Hence $2R_1 = \int_{-1}^{2} f(x)\, dx = R_2$.

Multiple choice questions with one or more than one correct answer

1. a. $\int_0^x f(t)\, dt = x + \int_x^1 t f(t)\, dt$

Differentiating both sides w.r.t. x, we get

$f(x) = 1 + 0 - xf(x)$

$\Rightarrow (x+1)f(x) = 1$

$\Rightarrow f(x) = \frac{1}{x+1}$

$\Rightarrow f(1) = \frac{1}{2}$

2. a. $\int_{-1}^{1} f(x)\, dx = \int_{-1}^{1} (x - [x])\, dx$

$= \int_{-1}^{1} x\, dx - \int_{-1}^{1} [x]\, dx$

$= 0 - \int_{-1}^{1} [x] dx$ (1)

[$\because x$ is an odd function]

$$= -\int_{-1}^{0}(-1)\,dx - \int_{0}^{1}0\,dx$$
$$= 1$$

3. a, d.

$$S_n < \lim_{n\to\infty} S_n = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\,\frac{1}{1+k/n+(k/n)^2}$$

$$= \int_0^1 \frac{dx}{1+x+x^2}$$

$$= \int_0^1 \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^1 = \frac{\pi}{3\sqrt{3}}$$

Now, $T_n > \dfrac{\pi}{3\sqrt{3}}$ as

$$h\sum_{k=0}^{n-1} f(k/n) > \int_0^1 f(x)\,dx > h\sum_{k=1}^{n} f(k/n)$$

4. a, b, c, d.

$f(x) = f(1-x)$

Replace x by $\frac{1}{2}+x$, we get

$$\Rightarrow f\left(\frac{1}{2}+x\right) = f\left(\frac{1}{2}-x\right) \qquad (1)$$

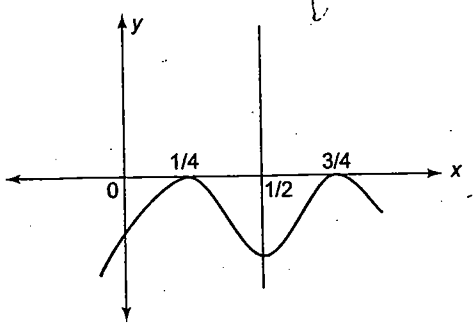

Fig. 8.24

Hence, $f(x+1/2)$ is an even function or $f(x+1/2)\sin x$ is an odd function.

Also, $f'(x) = -f'(1-x)$ (2)

and for $x = 1/2$, we have $f'(1/2) = 0$.

Also $\displaystyle\int_{1/2}^{1} f(1-t)e^{\sin\pi t}\,dt = -\int_{1/2}^{0} f(y)\,e^{\sin\pi y}\,dy$

(by putting, $1-t=y$)

Since, $f'(1/4) = 0 \Rightarrow f'(3/4) = 0$ [from equation (2)]

Also, $f'(1/2) = 0$ [from equation (2)]

$\Rightarrow f'(x) = 0$ at least twice in [0, 1] (Rolle's Theorem).

5. a, b, c.

$$I_n = \int_{-\pi}^{\pi}\frac{\sin nx}{(1+\pi^x)\sin x}\,dx$$

$$= \int_0^{\pi}\left(\frac{\sin nx}{(1+\pi^x)\sin x} + \frac{\pi^x \sin nx}{(1+\pi^x)\sin x}\right)dx = \int_0^{\pi}\frac{\sin nx}{\sin x}$$

Now, $\displaystyle I_{n+2} - I_n = \int_0^{\pi}\frac{\sin(n+2)x - \sin nx}{\sin x}$

$$= \int_0^{\pi}\frac{2\cos(n+1)x\,\sin x}{\sin x}\,dx = 0$$

$$\Rightarrow \quad I_1 = \pi, I_2 = \int_0^{\pi} 2\cos x\,dx = 0$$

1. b, c. $f'(x) = \dfrac{1}{x} + \sqrt{1+\sin x}$

$f'(x)$ is not differentiable at $\sin x = -1$ or $x = 2n\pi - \dfrac{\pi}{2}, n \in N$

In $x \in (1,\infty)$ $f(x) > 0$, $f'(x) > 0$

consider $f(x) - f'(x)$

$$= \ln x + \int_0^x \sqrt{1+\sin t}\,dt - \frac{1}{x} - \sqrt{1+\sin x}$$

$$= \left(\int_0^x \sqrt{1+\sin t}\,dt - \sqrt{1+\sin x}\right) + \ln x - \frac{1}{x}$$

Consider $g(x) = \displaystyle\int_0^x \sqrt{1+\sin t}\;dt - \sqrt{1+\sin x}$

It can be proved that $g(x) \geq 2\sqrt{2} - \sqrt{10}\ \forall x \in (0,\infty)$

Now there exists some $\alpha > 1$ such that $\dfrac{1}{x} - \ln x \leq 2\sqrt{2} - \sqrt{10}$ for all $x \in (\alpha, \infty)$ as $\dfrac{1}{x} - \ln x$ is strictly decreasing function.

$$\Rightarrow \quad g(x) \geq \frac{1}{x} - \ln x$$

Match the column type

1. a-s, b-s, c-p, d-r.

a. $\displaystyle\int_{-1}^{1}\frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_{-1}^{1} = \tan^{-1}(1) - \tan^{-1}(-1)$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{2\pi}{4} = \frac{\pi}{2}$$

b. $\displaystyle\int_0^1\frac{dx}{\sqrt{1-x^2}} = \left(\sin^{-1}x\right)_0^1 = \sin^{-1}(1) - \sin^{-1}(0)$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

c. $\displaystyle\int_2^3\frac{dx}{1-x^2} = \left[\frac{1}{2}\log\left|\frac{1+x}{1-x}\right|\right]_2^3 = \frac{1}{2}\ [\log 2 - \log 3]$

$$= \frac{1}{2}\log 2/3$$

d. $\int_1^2 \frac{dx}{x\sqrt{x^2-1}} = \left[\sec^{-1} x\right]_1^2 = \sec^{-1} 2 - \sec^{-1} 1 = \frac{\pi}{3}$

Linked comprehension type

1.a. $\int_0^{\pi/2} \sin x\, dx = \frac{\left(\frac{\pi}{2}-0\right)}{4}\left(\sin 0 + \sin\frac{\pi}{2} + 2\sin\frac{\pi}{4}\right)$

$$= \frac{\pi}{8}\left(1+\sqrt{2}\right)$$

2.d. $\lim_{x\to a} \frac{\int_a^x f(x)\,dx - \left(\frac{x-a}{2}\right)(f(x)+f(a))}{(x-a)^3} = 0$

$$\lim_{h\to 0} \frac{\int_a^{a+h} f(x)\,dx - \frac{h}{2}(f(a+h)+f(a))}{h^3} = 0$$

$$\Rightarrow \lim_{h\to 0} \frac{f(a+h) - \frac{1}{2}[f(a)+f(a+h)] - \frac{h}{2}(f'(a+h))}{3h^2} = 0$$

[Using L' Hopital's Rule]

$$\Rightarrow \lim_{h\to 0} \frac{\frac{1}{2}f(a+h) - \frac{1}{2}f(a) - \frac{h}{2}f'(a+h)}{3h^2} = 0$$

$$\Rightarrow \lim_{h\to 0} \frac{\frac{1}{2}f'(a+h) - \frac{1}{2}f'(a+h) - \frac{h}{2}f''(a+h)}{6h} = 0$$

[Using L' Hopital's Rule]

$$\Rightarrow \lim_{h\to 0} \frac{-f''(a+h)}{12} = 0$$

$\Rightarrow f''(a) = 0, \forall a \in R$

$\Rightarrow f(x)$ must be of maximum degree 1.

3.b. $f''(x) < 0, \forall x \in (a, b)$, for $c \in (a, b)$

$$F(c) = \frac{c-a}{2}(f(a)+f(c)) + \frac{b-c}{2}(f(b)+f(c))$$

$$= \frac{b-a}{2}f(c) + \frac{c-a}{2}f(a) + \frac{b-c}{2}f(b)$$

$$\Rightarrow F'(c) = \frac{b-a}{2}f'(c) + \frac{1}{2}f(a) - \frac{1}{2}f(b)$$

$$= \frac{1}{2}[(b-a)f'(c) + f(a) - f(b)]$$

$$F''(c) = \frac{1}{2}(b-a)f''(c) < 0$$

$[\because f''(x) < 0, \forall x \in (a, b)$ and $b > a]$

Therefore, $F(c)$ is maximum at the point $(c, f(c))$ where

$$F'(c) = 0 \Rightarrow f'(c) = 2\left(\frac{f(b)-f(a)}{b-a}\right).$$

Integer type

1. (4) $f(x) = \begin{cases} x-1, & 1 \le x < 2 \\ 1-x, & 0 \le x < 1 \end{cases}$

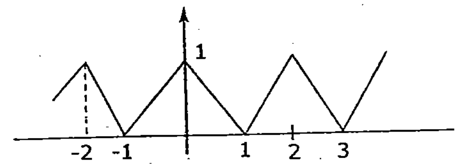

Fig. 8.25

$f(x)$ is periodic with period 2

$$\therefore \quad I = \int_{-10}^{10} f(x)\cos \pi x\, dx$$

$$= 2\int_0^{10} f(x)\cos \pi x\, dx$$

$$= 2\times 5\int_0^{2} f(x)\cos \pi x\, dx$$

$$= 10\left[\int_0^1 (1-x)\cos \pi x\, dx + \int_1^2 (x-1)\cos \pi x\, dx\right] = 10(I_1 + I_2)$$

$I_2 = \int_1^2 (x-1)\cos \pi x\, dx \qquad (\text{put } x-1=t)$

$$I_2 = -\int_0^1 t\cos \pi t\, dt$$

$$I_1 = \int_0^1 (1-x)\cos \pi x\, dx = -\int_0^1 x\cos(\pi x)\,dx$$

$$\therefore \quad I = 10\left[-2\int_0^1 x\cos \pi x\, dx\right]$$

$$= -20\left[x\frac{\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}\right]_0^1$$

$$= -20\left[-\frac{1}{\pi^2} - \frac{1}{\pi^2}\right] = \frac{40}{\pi^2}$$

$$\therefore \quad \frac{\pi^2}{10} I = 4$$

57. (0) $y'(x) + y(x)g'(x) = g(x)g'(x)$

$\Rightarrow e^{g(x)}y'(x) + e^{g(x)}g'(x)y(x) = e^{g(x)}g(x)g'(x)$

$\Rightarrow \frac{d}{dx}(y(x)e^{g(x)}) = e^{g(x)}g(x)g'(x)$

$\therefore\ y(x)e^{g(x)} = \int e^{g(x)}g(x)g'(x)dx$

$= \int e^t\, t\, dt$, where $g(x) = t$

$= (t-1)e^t + c$

$\therefore\ y(x)e^{g(x)} = (g(x)-1)e^{g(x)} + c$

Put $x = 0 \Rightarrow 0 = (0-1).1 + c \Rightarrow c = 1$

Put $x = 2 \Rightarrow y(2)\cdot 1 = (0-1)\cdot(1) + 1$

$y(2) = 0.$