Chapter 6

Squares and Square Roots

Introduction to Square and Square Roots

Square

The square of a number is that number raised to the power 2.

Or

The square of a number is the product of the number with the number itself. Thus, the square of a = (a × a), denoted by a^2.

If a number is multiplied by itself, then the product is said to be the square of that number, i.e., if m and n are two natural numbers such that $n = m^2$, then n is said to be the square of the number m.

For example,
$2^2 = 2x2 = 4$, so we say that the square of 2 is 4.

$3^2 = 3x3 = 9$, so we say that the square of 3 is 9.

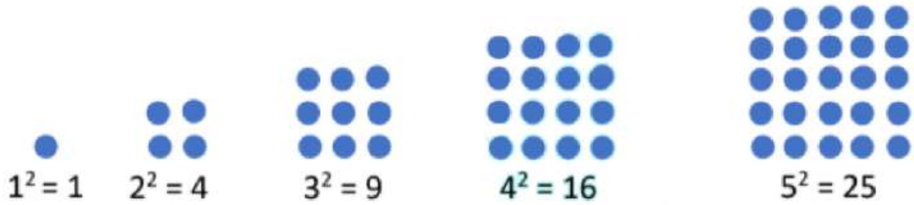
$12^2 = 12x12 = 144$, so we say that the square of 12 is 144.

The following table contains the squares of the first 20 natural numbers.

Number	Square	Number	Square
1	$1^2 = 1 \times 1 = 1$	11	$11^2 = 11 \times 11 = 121$
2	$2^2 = 2 \times 2 = 4$	12	$12^2 = 12 \times 12 = 144$
3	$3^2 = 3 \times 3 = 9$	13	$13^2 = 13 \times 13 = 169$
4	$4^2 = 4 \times 4 = 16$	14	$14^2 = 14 \times 14 = 196$
5	$5^2 = 5 \times 5 = 25$	15	$15^2 = 15 \times 15 = 225$
6	$6^2 = 6 \times 6 = 36$	16	$16^2 = 16 \times 16 = 256$
7	$7^2 = 7 \times 7 = 49$	17	$17^2 = 17 \times 17 = 289$
8	$8^2 = 8 \times 8 = 64$	18	$18^2 = 18 \times 18 = 324$
9	$9^2 = 9 \times 9 = 81$	19	$19^2 = 19 \times 19 = 361$
10	$10^2 = 10 \times 10 = 100$	20	$20^2 = 20 \times 20 = 400$

Perfect Square

A natural number n is called a perfect square or a square number if there exists a natural number m such that $n = m^2$. The perfect squares 1, 4, 9, 16, …… can be represented geometrically by dots forming a square as shown below.



Hence, the name 'perfect squares' or square number is assigned to these numbers.

Checking whether the number is Perfect Square or not

2	144
2	72
2	36
2	18
3	9
3	3
	1

Step 1: Obtain a given number Is 144 a perfect square? If so, find the number whose square is 144.

Step 2: Write the number as a product of prime factors. $144 = 2 \times 2 \times 2 \times 2 \times 3 \times 3$

Step 3: Group the factors in pairs in such a way that both the factors in each pair are equal. $144 = (2 \times 2) \times (2 \times 2) \times (3 \times 3)$

Step 4: See whether some factor is left over not. If nothing is left over after grouping them, then the given number is a perfect square. Otherwise, it is not a perfect square.

Clearly, 144 can be grouped into pairs of equal factors and no factors and no factor is left over. Hence, 144 is a perfect square.

Step 5: To obtain the number whose square is the given number take one factor from each group and multiply them.

$$144 = \underbrace{(2 \times 2)}_{2} \times \underbrace{(2 \times 2)}_{2} \times \underbrace{(3 \times 3)}_{3}$$

$$2 \times 2 \times 3 = 12 \Rightarrow 12 \times 12 = 12^2$$

Checking whether the number is Perfect Square or not

Find the smallest number by which 7203 should be divided so that the result is a perfect square.

3	7203
7	2401
7	343
7	49
7	7
	1

Resolving 7203 into prime factors, we get

$7203 = 3 \times 7 \times 7 \times 7 \times 7$

Now, grouping the factors into pairs of equal factors, we get

$7203 = 3 \times (7 \times 7) \times (7 \times 7)$

We observe that 3 cannot be paired. Thus, if we divide 7203 by 3 the quotient will be a perfect square.

When we divide 7203 by 3, we get 2401 which is the square of 49.

Hence, the smallest number by which 7203 must be divided to make the result a perfect square is 3.

Find the smallest number by which 90 must be multiplied so that the product is a perfect square.

2	90
3	45
3	15
5	5
	1

90 into prime factors, we get

$90 = 2 \times 3 \times 3 \times 5$

Grouping the factors into pairs of equal factors, we get

$90 = 2 \times (3 \times 3) \times 5$

For a number to be a perfect square, it should be possible to pair all its prime factors. In this case, 2 and 5 are without a pair.

Thus, we must multiply the number by 2 and 5 so that the product is a perfect square

Hence, the smallest number by which 90 should be multiplied is 5 and 2 that is 10.

Properties of Square

Properties of Square Numbers

Property 1:

A natural number having 2, 3, 7 and 8 at the unit's place is never a perfect square. In other words, no square numbers end in 2, 3, 7 or 8.

For example,

152, 7693, 88888, 798328 are not the perfect square numbers.

Note:

The phrase "Number ends in a" means that the unit's digits of the number is a.

Property 2:

A natural number having 0, 1, 4, 5, 6 and 9 at the unit's place may or may not be a perfect square number.

Or

It should be noted that a number having unit's digits other than the digits 2, 3, 7 and 8 is not necessarily a perfect square. It may or may not be a perfect square.

For example,

1. 100, 25, 36, 49, 64 and 81 are perfect square numbers.

2. 10, 15, 46, 69, 84 and 91 are not perfect square numbers.

Property 3:

If a number has 0 in the unit's place, then it's square ends in 0.

For example,

Perfect square of 10 is 100, 20 is 400, 60 is 3600 and 80 is 6400.

Property 4:

The number of zero at the end of a perfect square is always even. If a number ending in an odd number of zeros is never a perfect square.

For example,

Number	Number of 0's at unit place	Square	Number of 0's at unit place
10	1	100	2
200	2	40000	4
1000	3	1000000	6
10000	4	100000000	8

Note:

It should be noted that numbers ending in an even number of zeros may or may not be a perfect square.

For example,

2500 is a perfect square but, 1500, 4700, etc. are not perfect square numbers.

Property 5:

If a number has 1 or 9 in the unit's place, then it's square ends in 1.

For example,

$1^2 = 1$, $11^2 = 121$ and $21^2 = 421$ and $9^2 = 81$, $19^2 = 361$ and $29^2 = 841$

Property 6:
If a number has 2 or 8 in the unit's place, then it's square ends in 4.
For example,

$2^2 = 4$, $12^2 = 144$ and $22^2 = 484$ and $8^2 = 64$, $18^2 = 324$ and $28^2 = 2304$

Property 7:

If a number has 3 or 7 in the unit's place, then it's square ends in 9.

For example,

$3^2 = 9$, $13^2 = 169$ and $23^2 = 529$ and $7^2 = 49$, $17^2 = 289$ and $27^2 = 729$

Property 8:

If a number has 4 or 6 in the unit's place, then it's square ends in 6.

For example,

$4^2 = 16$, $14^2 = 169$ and $24^2 = 576$ and $6^2 = 36$, $16^2 = 256$ and $46^2 = 2116$

Property 9:

If a number has 5 in the unit's place, then it's square ends in 5.

For example,

$5^2 = 25$, $15^2 = 225$ and $25^2 = 625$.

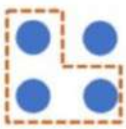
Number having Unit's digit equals to	Square ends with the unit digit
0	0
1 and 9	1
2 and 8	4
3 and 7	9
4 and 6	6
5	5

Some More Interesting Patterns

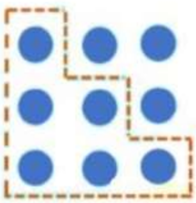
1. The sum of two consecutive triangular numbers is a square number.

Series of triangular numbers is given by,

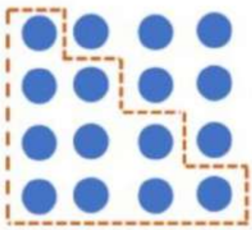
$$X_n = \frac{n(n+1)}{2}$$



$1 + 3 = 4 = 2^2$



$3 + 6 = 9 = 3^2$



$6 + 10 = 16 = 4^2$

2. There are 2n non-perfect square numbers between two consecutive square numbers n^2 and $(n+1)^2$

Consecutive Number	Number of non-perfect square numbers	Non-perfect square numbers between two consecutive square number
1, 2	$2 \times 1 = 2$	2, 3
2, 3	$2 \times 2 = 4$	5, 6, 7, 8
3, 4	$2 \times 3 = 6$	10, 11, 12, 13, 14, 15
:	:	:
n, (n + 1)	$2 \times n = 2n$	$n^2 + 1, n^2 + 2, \ldots, n^2 + 2n$

3. The sum of first n odd natural numbers n^2

n	n^2	Addition of Odd numbers
n = 1	$1^2 = 1$	1 = 1
n = 2	$2^2 = 4$	1 + 3 = 4
n = 3	$3^2 = 9$	1 + 3 + 5 = 9
n = 4	$4^2 = 16$	1 + 3 + 5 + 7 = 16
n = 5	$5^2 = 25$	1 + 3 + 5 + 7 + 9 = 25
n = 6	$6^2 = 36$	1 + 3 + 5 + 7 + 9 + 11 = 36

4. The square of any odd natural number other than 1 can be expressed as the sum of two consecutive numbers which can be given by

$$\left[\frac{n^2-1}{2}, \frac{n^2+1}{2}\right]$$

For example,

$3^2 = 9 = 4+5$

$5^2 = 25 = 12+13$

$7^2 = 49 = 24+25$

$9^2 = 81 = 40+41$

$11^2 = 121 = 60+61$

$13^2 = 169 = 84+85$

5. Product of two consecutive even or odd natural numbers is

$(a + 1)(a - 1) = a^2 - 1$

For example,

a) $11 \times 13 = 143 = 12^2 - 1$

$11 \times 13 = (12 - 1) \times (12 + 1)$

$11 \times 13 = (12 - 1) \times (12 + 1) = 12^2 - 1$

b) $29 \times 31 = 899 = 30^2 - 1$

$29 \times 31 = (30 - 1) \times (30 + 1)$

$29 \times 31 = (30 - 1) \times (30 + 1) = 30^2 - 1$

6. Squares of natural numbers having all digits 1 follow the following patterns.

1^2	1
11^2	1 2 1
111^2	1 2 3 2 1
1111^2	1 2 3 4 3 2 1
11111^2	1 2 3 4 5 4 3 2 1
111111^2	1 2 3 4 5 6 5 4 3 2 1

7. Some more interesting patterns.

7^2	49
67^2	4489
667^2	444889
6667^2	44448889
66667^2	4444488889
666667^2	444444888889

Finding a Square of a Number

We can also find the square of a number without actual multiplication.

For Example:

$43^2 = (40 + 3)^2$

$= (40 + 3) \times (40 + 3)$ $\quad\quad$ $(x + y) \times z = xz + yz$

$= 40(40 + 3) + 3(40 + 3)$

$= 1600 + 120 + 120 + 9$

$= 1600 + 240 + 9$

$= 1849$

Other patterns in Square

Consider a number with unit digit 5 i.e., a5 and here 'a' is a ten's

position of a number.

$(a5)^2 = (10a + 5)^2$

$= (10a + 5) \times (10a + 5)$

$= 10a\ (10a + 5) + 5(10a + 5)$

$= 100a^2 + 50a + 50a + 25$

$= 100a\ (a + 1) + 25$

$= a\ (a + 1)$ hundred $+ 25$

For example:

$(45)^2 = (40 + 5)^2 = (10 \times 4 + 5)^2$ $\quad\quad$ where $a = 4$

$= 100 \times 4(4 + 1) + 25$

$= 100 \times 4(5) + 25$

$= 4 \times (4 + 1)$ hundred $+ 25$

$= 4 \times 5$ hundred $+ 25$

$= 2000 + 25 = 2025$

Pythagorean Triplet

Three natural number m, n, p is said to form a Pythagorean triplet (m, n, p) if $(m^2 + n^2) = p^2$.

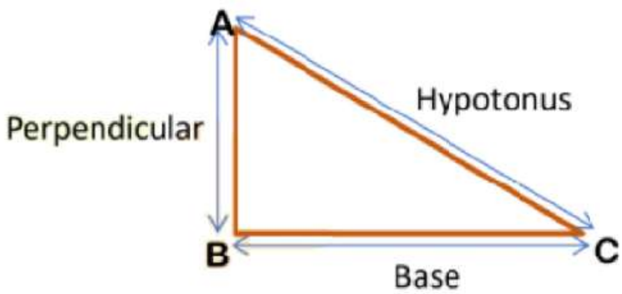
We can also say that collection of three numbers such that the sum of the squares of the two numbers is equal to the square of the third number

For any natural number m>1, we have $(2m)^2 + (m^2 - 1)^2 = (m^2 + 1)^2$

So, 2m, $(m^2 - 1)$ and $(m^2 + 1)$ form a Pythagorean triplet.

$(\text{Hypotonus})^2 = (\text{Perpendicular})^2 + (\text{Base})^2$

$h^2 = p^2 + b^2$



For example,

a) (3, 4, 5), (5, 12, 13), (8, 15, 17) are Pythagorean triplets.

$3^2 + 4^2 = 9 + 16 = 25 = 5^2$

$5^2 + 12^2 = 25 + 144 = 169 = 13^2$

$8^2 + 15^2 = 64 + 225 = 289 = 17^2$

b) What will be Pythagorean triplets whose smallest member is 8?

We can get the Pythagorean triplets by using general form 2m,

$(m^2 - 1)$ and $(m^2 + 1)$.

$m^2 - 1 = 8$

$m^2 = 8 + 1 = 9$

$m = 3$

Therefore, $2m = 2 \times 3 = 6$ and $m^2 + 1 = (3)^2 + 1 = 9 + 1 = 10$

The triplet is thus 6, 8, 10.

Since 8 is not the smallest member of this.

So, let us try

$2m = 8$

Then $m = 4$

We get $m^2 - 1 = (4)^2 - 1 = 16 - 1 = 15$

and $m^2 + 1 = (4)^2 + 1 = 16 + 1 = 17$

Hence, the triplet is 8, 15, 17 with 8 as the smallest member.

Introduction to Square Root

Square Roots

The square root of a number 'a' is that number which when multiplied by itself 'a' as the product.

When a number multiplies itself, the product is the square number. The number is the square root. We get perfect square roots for a

perfect square number. A square root is represented by a $\sqrt{}$(power is $\frac{1}{2}$) sign.

Thus, if b is the square root of a number a, then

$b \times b = a$ or $b^2 = a$

The square root of a number 'a' is denoted by $\sqrt{a}$.

It follows from this that

$$b = \sqrt{a} \Leftrightarrow b = (a)^{1/2} \Leftrightarrow b^2 = a$$

i.e., b is the square root of 'a' if only if a is the square of b.

For example,

a) Square root of 4 is:

$$\sqrt{4} = 2 \qquad \text{because } 2^2 = 2 \times 2 = 4$$

b) Square root of 324 is:

$$\sqrt{324} = 18 \qquad \text{because } 18^2 = 18 \times 18 = 324$$

Finding square roots through repeated subtraction

This method can be used to find the square roots of small natural numbers.

Step 1: Obtain the given perfect square whose square root is to be calculated. Let the number be a.

Find the square root of 25 by successive subtractions.

Step 2: Subtract from it successively 1, 3, 5, 7, 9, ... till you get zero.

$$25 - 1 = 24 \leftarrow 1$$

$$24 - 3 = 21 \leftarrow 2$$

$$21 - 5 = 16 \leftarrow 3$$

$$16 - 7 = 9 \leftarrow 4$$

$$9 - 9 = 0 \leftarrow 5$$

Step 3: Count the number of times the subtraction is performed to arrive at zero. Let the number be n.

Clearly, we have performed a subtraction 5 times.

Step 4: Write $\sqrt{a} = n$

Hence, $\sqrt{25} = 5$

For example,

Find the square root of 49 by successive subtractions.

$49 - 1 = 48 \leftarrow 1$

$48 - 3 = 45 \leftarrow 2$

$45 - 5 = 40 \leftarrow 3$

$40 - 7 = 33 \leftarrow 4$

$33 - 9 = 24 \leftarrow 5$

$24 - 11 = 13 \leftarrow 6$

$13 - 13 = 0 \leftarrow 7$

Clearly, we have performed subtraction seven times.

$$\therefore \sqrt{49} = 7$$

This is the simplest method of finding the square root of a perfect square. But it is convenient for small numbers only as it is lengthy and time-consuming for large numbers.

Finding square root through prime factorization

Step 1: Obtain the given number. Find the square root of 3025 by prime factorization.

Step 2: Resolve the given number into prime factors by successive division.

$$3025 = 5 \times 5 \times 11 \times 11$$

Step 3: Make pairs of prime factors such that both the factors in each pair are equal.

$$3025 = (5 \times 5) \times (11 \times 11)$$

5	3025
5	605
11	121
11	11
	1

Step 4: Take one factor from each pair.

5×11

Step 5: Find the product of factors obtained in step 4.

$$5 \times 11 = 55$$

Step 6: The product obtained in step 5 is the required square root.

Hence, $\sqrt{3025} = 55$

For example,

What will be the square root of the 7744 by prime factorization?

Resolving 7744 into prime factors, we get

$$7744 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 11 \times 11$$

Now grouping the factors into pairs of equal factors, we get

$$7744 = (2 \times 2) \times (2 \times 2) \times (2 \times 2) \times (11 \times 11)$$

Now, taking the factors from each pair, we obtain

$$\sqrt{7744} = 2 \times 2 \times 2 \times 11 = 88$$

2	7744
2	3872
2	1936
2	968
2	484
2	242
11	121
11	11
	1

Finding square root through Long Division method

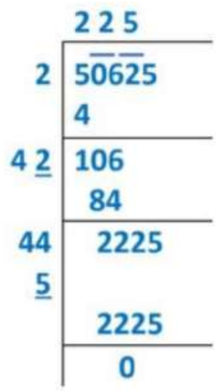
Step 1: Obtain the number whose square root is to be computed. Find the square root of 50625 by long division method.

Step 2: Place the bars over every pair of digits starting with the unit's digit.

Step 3: Also, place a bar on one digit (if any) not forming a pair on the extreme left. Each pair and remaining one digit are called the period.

Step 4: Think of the largest number whose square is less than or equal to the first period. Take this number as the divisor and the quotient

$$2^2 = 4 < 5$$
$$3^2 = 9 > 5$$



Step 5: Put the quotient above the period and write the product of divisor and quotient just below the period.

Step 6: Subtract the product of divisor and quotient from the first period and bring down the next period to the right of the remainder. This becomes the new dividend.

Step 7: Double the quotient as it appears and enter it with a blank on the right for the next digit, as the next possible divisor.

Step 8: Think of a digit, to fill the blank in step 7, such a way that the product of a new divisor and this digit is equal to or just less than the new dividend obtained in step 6 and put the same digit in the quotient.

$$41 \times 1 = 41 < 106$$

$$42 \times 2 = 82 < 106$$

$$43 \times 3 = 129 > 106$$

Step 9: Subtract the product of the digit chosen in step 8 and the new divisor from the dividend obtained in step 6 and bring down the next period to the right of the remainder. This becomes new dividend.

Step 10: Repeat steps 6, 7 and 8 till all periods have been taken up.

$$444 \times 4 = 1776 < 2225$$

$$445 \times 5 = 2225 < 2225$$

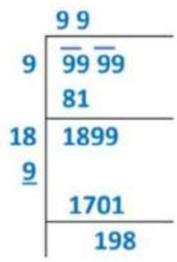
Step 11: Obtain the quotient as the square root of the given number.

For example,

Find the greatest number of four digits which is a perfect square.

The greatest number of four digits = 9999

Let us try to find the square root of 9999



This shows that (99)2

is less than 9999 by 198.

So, the least number to be subtracted is 198

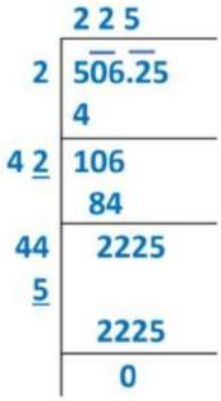
Hence, the required number is (9999 – 189) = 9801

Introduction to Square Root of Decimals

Square Roots of decimals

Step 1: To find the square root of a decimal number, we put bars on the integral part (i.e., 5 and 06) of the number in the usual manner. And place bars on decimal part (i.e., 25) on every pair of digits beginning with the first decimal point.

Let us consider a number 506.25



Step 2: Now proceed in a similar manner. The leftmost bar is on 5 and $22 < 5$ < 32 . Take this number as a divisor and number under the leftmost bar as the

dividend i.e., 5. Divide and get the remainder.

Step 3: The remainder is 1. Write the number under the next bar at the right of this remainder, to get 106.

Step 4: Double the quotient and enter it with a blank on the right.

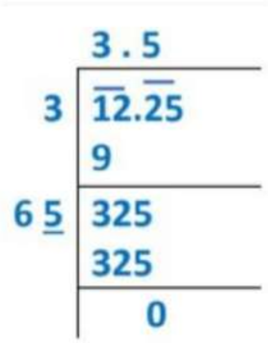
Step 5: We know that $42 \times 2 = 84$.

Step 6: Now, 25 is the decimal part so put a decimal point in the quotient. Repeat the above steps again until we get a remainder equals to zero. We know that $445 \times 5 = 2225$.

Step 7: Since, the remainder is 0 and no bar left, therefore $\sqrt{506.25} = 22.5$

For example,

What will be the square root of 12.25?



Therefore, $\sqrt{12.25} = 3.5$

Square Roots of Product of Two Numbers and Fractional Numbers

For any two positive numbers a and b, we have

For example:

a) $\sqrt{12} = \sqrt{3} \times \sqrt{4}$
b) $\sqrt{16} = \sqrt{8} \times \sqrt{2}$

For example:

a) $\sqrt{\frac{4}{3}} = \frac{\sqrt{4}}{\sqrt{3}}$

Evaluate $\sqrt{\frac{441}{961}}$

We have, $\sqrt{\frac{441}{961}} = \frac{\sqrt{441}}{\sqrt{961}}$

Now, we find the square roots of 441 and 961, as shown below

Thus, $\sqrt{441} = 21$ and $\sqrt{961} = 31$

$$\sqrt{\frac{441}{961}} = \frac{\sqrt{441}}{\sqrt{961}} = \frac{21}{31}$$

Estimating Square Root

Estimate the square root of 250.

We know that $100 < 250 < 400$ and $\sqrt{100} = 10$ and $\sqrt{400} = 20$.

So, $10 < \sqrt{250} < 20$

But still, we are not very close to the square number.

We know that $15^2 = 225$ and $16^2 = 256$

Therefore, $15 < \sqrt{250} < 16$ and 256 is much closer to 250 than 225.

So, $\sqrt{250}$ is approximately 16.