Exercise 6.4

Answer 1E.

The objective is to explain why logarithmic function $y = \ln x$ is used more frequently in calculus.

The natural logarithm is generally written as $y = \ln x$. It is useful to find a solution to differential equations and hyperbolic functions, population growth.

The natural logarithm $y = \ln x$ means $\log_e x$ and the common logarithm is $y = \log_a x$.

The reason why the natural logarithm is used instead of logarithm of base a, when dealing with common logarithmic function there are two variable that can affect the function, that is a and e. But in a natural logarithm the base is e the only factor that effecting the function is x.

In a real world or in nature, things are explained by e which is approximately 2.718.

Therefore, natural logarithm is more used than common logarithm.

The natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_a x$ is the differentiation formula is simplest.

For example, the differentiation is simple using natural logarithms.

In natural logarithm, $\frac{d}{dx}\ln x = \frac{1}{x}$.

In common logarithm, $\frac{d}{dx}\ln_b x = \frac{1}{x\ln b}$, because $\ln_b x = \frac{\ln x}{\ln b}$.

Answer 2E.

Consider $f(x) = x\ln x - x$

Differentiating $f(x) = x\ln x - x$

$$\begin{aligned} f'(x) &= \frac{d}{dx}(f(x)) \\ &= \frac{d}{dx}(x \cdot \ln x - x) \\ &= \frac{d}{dx}(x \cdot \ln x) - \frac{d}{dx}(x) \end{aligned}$$

$$= \left[x\left(\frac{1}{x}\right) + \ln x(1) \right] - 1$$
$$= 1 + \ln x - 1$$
$$= \ln x$$

Therefore $\boxed{f'(x) = \ln x}$

Answer 3E.

Consider the following integral:

$$f(x) = \sin(\ln x)$$

Use chain rule to differentiate the function, $f(x) = \sin(\ln x)$.

$$f(x) = \sin(\ln x)$$

$$f'(x) = \frac{d}{dx}(\sin(\ln x))$$

$$f'(x) = \cos(\ln x) \cdot \frac{d}{dx} \ln x \quad \text{Use chain rule } F(x) = f'(g(x)) \cdot g'(x)$$

$$f'(x) = \cos(\ln x) \cdot \frac{1}{x} \quad \text{Use } \frac{d}{dx} \ln x = \frac{1}{x}$$

$$f'(x) = \frac{\cos(\ln x)}{x}$$

Hence, the derivative of the function $f(x) = \sin(\ln x)$ is $\boxed{f'(x) = \frac{\cos(\ln x)}{x}}$.

Answer 4E.

Given $f(x) = \ln(\sin^2 x)$

Differentiating with respect to x,

$$f'(x) = \frac{d}{dx}(f(x))$$
$$= \frac{d}{dx}\left(\ln(\sin^2 x)\right)$$
$$= \frac{1}{\sin^2 x} \frac{d}{dx}(\sin^2 x) \qquad \left(\text{Since } \frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}\right)$$
$$= \frac{1}{\sin^2 x}(2 \sin x) \cdot \frac{d}{dx}(\sin x) \qquad \left(\frac{d}{dx}(u^n) = nu^{n-1} \cdot \frac{du}{dx}\right)$$
$$= \frac{2}{\sin x} \cdot \cos x$$
$$= 2 \cdot \frac{\cos x}{\sin x}$$
$$= 2 \cot x$$

$$\boxed{f'(x) = 2 \cot x}$$

Answer 5E.

Given $f(x) = \ln\left(\frac{1}{x}\right)$
$$= \ln 1 - \ln x$$
$$= -\ln x$$

Differentiating, $f(x)$

$$f'(x) = \frac{d}{dx}(f(x))$$
$$= \frac{d}{dx}(-\ln x)$$
$$= -\frac{1}{x}$$

Therefore $\boxed{f'(x) = \frac{-1}{x}}$

Answer 6E.

Given $y = \frac{1}{\ln x} = (\ln x)^{-1}$

Differentiating y,

$$y' = \frac{d}{dx}\left((\ln x)^{-1}\right)$$

$$= (-1)(\ln x)^{-2}\frac{d}{dx}(\ln x)$$

$$= \frac{-1}{(\ln x)^2} \times \frac{1}{x}$$

Therefore $\boxed{y' = \frac{-1}{x(\ln x)^2}}$

Answer 7E.

Consider the function to find derivative is

$$f(x) = \log_{10}\left(x^3 + 1\right)$$

We know that

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

The given function is converting into in terms of natural logarithm

For any positive number $a(a \neq 1)$, we have

$$\log_a^x = \frac{\ln x}{\ln a}$$

$$f(x) = \log_{10}\left(x^3 + 1\right) = \frac{\ln\left(x^3 + 1\right)}{\ln(10)}$$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(\frac{\ln\left(x^3 + 1\right)}{\ln(10)}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{\ln(10)} \cdot \ln\left(x^3 + 1\right)\right)$$

Recall the constant multiple rule

If c is a constant and g is a differentiable function, then

$$\frac{d}{dx}[cg(x)] = c\frac{d}{dx}g(x)$$

In this case

$$c = \frac{1}{\ln 10}, g(x) = \ln\left(x^3 + 1\right)$$

$$\frac{d}{dx}\left(\frac{1}{\ln(10)} \cdot \ln\left(x^3 + 1\right)\right) = \frac{1}{\ln(10)}\frac{d}{dx}\ln\left(x^3 + 1\right)$$

The function $\ln\left(x^3+1\right)$ is a composition of two functions $\ln x, x^3+1$

Recall the chain rule

If f_2 is differentiable at x and f_1 is differentiable at $f_2(x)$ then $f_1 o f_2$ is differentiable at x and

$$\frac{d}{dx}\left(f_1 o f_2(x)\right)=f_1'\left(f_2(x)\right)f_2'(x)$$

In this case

$$f_1(x)=\ln x, f_2(x)=x^3+1$$
$$(f_1 o f_2)(x)=f_1\left(f_2(x)\right)=f_1\left(x^3+1\right)=\ln\left(x^3+1\right)$$
$$f_1'(x)=\frac{d}{dx}(\ln x)=\frac{1}{x}$$
$$f_1'\left(f_2(x)\right)=\frac{1}{f_2(x)}=\frac{1}{x^3+1}$$
$$f_2'(x)=\frac{d}{dx}\left(x^3+1\right)=3x^{3-1}=3x^2$$
$$\frac{d}{dx}\ln\left(x^3+1\right)=f_1'\left(f_2(x)\right)f_2(x)$$
$$=\frac{1}{x^3+1}\cdot 3x^2$$
$$\frac{1}{\ln(10)}\frac{d}{dx}\ln\left(x^3+1\right)=\frac{1}{\ln(10)}\cdot\frac{1}{x^3+1}\cdot 3x^2$$

Answer 8E.

Consider the following function:

$$f(x)=\log_5\left(xe^x\right)$$

Differentiate the function as shown below:

$$\frac{d}{dx}\left[\log_5\left(xe^x\right)\right]=\frac{d}{dx}\frac{\ln\left(xe^x\right)}{\ln 5}$$ Use the formula $\log_a x=\frac{\ln x}{\ln a}$

$$=\frac{1}{\ln 5}\cdot\frac{d}{dx}\left[\ln\left(xe^x\right)\right]$$

$$=\frac{1}{\ln 5}\cdot\frac{1}{xe^x}\frac{d}{dx}\left(xe^x\right)$$ Use the chain rule $\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(x)\right)g'(x)$

$$=\frac{1}{\ln 5}\cdot\frac{1}{xe^x}\left[e^x+xe^x\right]$$ Use the product rule $\frac{d}{dx}\left[f(x)g(x)\right]=f(x)g'(x)+g(x)f'(x)$

$$=\frac{1}{\ln 5}\cdot\frac{e^x}{xe^x}(x+1)$$ Common out e^x

$$=\frac{1}{\ln 5}\cdot\left(\frac{x+1}{x}\right)$$ Simplify

Therefore, $\boxed{\frac{d}{dx}\left[\log_5\left(xe^x\right)\right]=\frac{x+1}{x\ln 5}}$.

Answer 9E.

Consider the following function:

$$f(x)=\sin x\ln(5x)$$

Differentiate the function as shown below:

$$\frac{d}{dx}[\sin x\ln(5x)]$$

$$=\sin x\frac{d}{dx}\ln(5x)+\ln(5x)\frac{d}{dx}(\sin x)$$

$\left\{\begin{array}{l}\text{Use the product rule}\\ \frac{d}{dx}[f(x)g(x)]=f(x)g'(x)+g(x)f'(x)\end{array}\right.$

$=\sin x\cdot\frac{1}{5x}\cdot 5+\ln(5x)\cos x$ Since $\frac{d}{dx}(\ln x)=\frac{1}{x},\frac{d}{dx}(\sin x)=\cos x$

$=\frac{\sin x}{x}+\cos x\ln(5x)$ Simplify

Therefore, $\boxed{\frac{d}{dx}[\sin x\ln(5x)]=\frac{\sin x}{x}+\cos x\ln(5x)}$.

Answer 10E.

Given $f(u)=\frac{u}{1+\ln u}$

Differentiating $f(u)$,

$$\begin{aligned}f'(u)&=\frac{d}{du}(f(u))\\&=\frac{d}{du}\left(\frac{u}{1+\ln u}\right)\\&=\frac{(1+\ln u)\frac{d}{du}(u)-u\frac{d}{du}(1+\ln u)}{(1+\ln u)^2}\\&=\frac{(1+\ln u)-u\left(\frac{1}{u}\right)}{(1+\ln u)^2}\\&=\frac{\ln u}{(1+\ln u)^2}\end{aligned}$$

Therefore $\boxed{f'(u)=\frac{\ln u}{(1+\ln u)^2}}$

Answer 11E.

Consider the function,

$$G(y)=\ln\frac{(2y+1)^5}{\sqrt{y^2+1}}.$$

Then rewrite the function $G(y)$ as,

$$G(y)=\ln(2y+1)^5-\ln\sqrt{y^2+1} \qquad \ln\left(\frac{a}{b}\right)=\ln a-\ln b.$$

$$=\ln(2y+1)^5-\ln\left(y^2+1\right)^{1/2}$$

$$=5\ln(2y+1)-\frac{1}{2}\ln\left(y^2+1\right) \qquad \ln a^n=n\ln a.$$

The objective is to find differentiate the function.

Differentiating $G(y)$ with respect to *y*, to get

$$G'(y)=\frac{d}{dy}\left[5\ln(2y+1)-\frac{1}{2}\ln\left(y^2+1\right)\right]$$

$$=5\cdot\frac{1}{2y+1}\cdot\frac{d}{dy}(2y+1)-\frac{1}{2}\cdot\frac{1}{y^2+1}\cdot\frac{d}{dy}\left(y^2+1\right) \qquad \frac{d}{dx}(\ln x)=\frac{1}{x}.$$

$$=\frac{5}{2y+1}(2)-\frac{1}{2\left(y^2+1\right)}(2y) \qquad \frac{d}{dx}\left(x^n\right)=nx^{n-1}.$$

$$=\frac{10}{2y+1}-\frac{y}{y^2+1}$$

Therefore, $G'(y)=\boxed{\frac{10}{2y+1}-\frac{y}{y^2+1}}$.

Answer 12E.

Consider the following function:

$$h(x)=\ln\left(x+\sqrt{x^2-1}\right)$$

The objective is to differentiate the function.

Differentiate $h(x)$ with respect to *x*, we get

$$h'(x)=\frac{d}{dx}\left[\ln\left(x+\sqrt{x^2-1}\right)\right]$$

Let $g(x)=x+\sqrt{x^2-1}$ then,

$$g'(x)=\frac{d}{dx}\left(x+\sqrt{x^2-1}\right)$$

$$=1+\frac{1}{2\sqrt{x^2-1}}(2x)$$

$$=1+\frac{x}{\sqrt{x^2-1}}$$

Now $h'(x)=\frac{d}{dx}\left[\ln\left(g(x)\right)\right]$

Use the formula, $\frac{d}{dx}\ln\left(g(x)\right)=\frac{g'(x)}{g(x)}$

Then, the derivative becomes,

$$h'(x)=\frac{1}{x+\sqrt{x^2-1}}\left(1+\frac{x}{\sqrt{x^2-1}}\right)$$

$$=\frac{1}{x+\sqrt{x^2-1}}\cdot\left(\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}}+\frac{x}{\sqrt{x^2-1}}\right)$$

$$=\frac{1}{x+\sqrt{x^2-1}}\cdot\left(\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}\right)$$

$$=\boxed{\frac{1}{\sqrt{x^2-1}}}$$

Hence, $h'(x)=\boxed{\frac{1}{\sqrt{x^2-1}}}$.

Answer 13E.

Consider the following function:

$$g(x)=\ln\left(x\sqrt{x^2-1}\right)$$

$$g'(x)=\frac{d}{dx}\left(\ln\left(x\sqrt{x^2-1}\right)\right)$$

$$g'(x)=\frac{1}{x\sqrt{x^2-1}}\frac{d}{dx}\left(x\sqrt{x^2-1}\right)$$ Use $\frac{d}{dx}\ln x=\frac{1}{x}$

$$g'(x)=\frac{1}{x\sqrt{x^2-1}}\left(\sqrt{x^2+1}\frac{d}{dx}x+x\frac{d}{dx}\sqrt{x^2-1}\right)$$

Use $\frac{d}{dx}f(x)\cdot g(x)=g(x)\cdot\frac{d}{dx}f(x)+f(x)\cdot\frac{d}{dx}g(x)$

$$g'(x)=\frac{1}{x\sqrt{x^2-1}}\left(\sqrt{x^2-1}\cdot 1+x\cdot\frac{1}{2}\left(x^2-1\right)^{\frac{1}{2}-1}\cdot\frac{d}{dx}\left[x^2-1\right]\right)$$ Use $\frac{d}{dx}x^n=nx^{n-1}$

$$=\frac{1}{x\sqrt{x^2-1}}\left(\sqrt{x^2-1}+x\cdot\frac{1}{2}\left(x^2-1\right)^{-\frac{1}{2}}\cdot\left(\frac{d}{dx}x^2-\frac{d}{dx}1\right)\right)$$

Simplify further as follows:

$$g'(x)=\frac{1}{x\sqrt{x^2-1}}\left(\sqrt{x^2-1}+\frac{x}{2\sqrt{x^2-1}}\cdot(2x-0)\right)$$

$$=\frac{1}{x\sqrt{x^2-1}}\cdot\left(\sqrt{x^2-1}+\frac{x^2}{\sqrt{x^2-1}}\right)$$

$$=\frac{1}{x\sqrt{x^2-1}}\cdot\left(\frac{\left(\sqrt{x^2-1}\right)\left(\sqrt{x^2-1}\right)+x^2}{\sqrt{x^2-1}}\right)$$

$$=\frac{1}{x\sqrt{x^2-1}}\cdot\left(\frac{x^2-1+x^2}{\sqrt{x^2-1}}\right)$$

$$=\frac{2x^2-1}{x\left(x^2-1\right)}$$

Hence, the derivative of the function $g(x)=\ln\left(x\sqrt{x^2-1}\right)$ is $\boxed{g'(x)=\frac{2x^2-1}{x\left(x^2-1\right)}}$.

Answer 14E.

Given $g(r)=r^2\ln(2r+1)$

Differentiating of $g(r)$,

$$g'(r)=\frac{d}{dr}(g(r))$$

$$=\frac{d}{dr}\left(r^2\ln(2r+1)\right)$$

$$=\left[\frac{d}{dr}\left(r^2\right)\right]\left(\ln(2r+1)\right)+r^2\cdot\frac{d}{dr}\left(\ln(2r+1)\right)$$

$$=(2r)\left(\ln(2r+1)\right)+r^2\cdot\frac{2}{2r+1}$$

Therefore $\boxed{g'(r)=(2r)\left(\ln(2r+1)\right)+\frac{2r^2}{2r+1}}$

Answer 15E.

We have $f(u)=\dfrac{\ln u}{1+\ln(2u)}$

Which is the from of $f(u)=\dfrac{g(u)}{h(u)}$

Where $g(u)=\ln u$ then $g'(u)=\dfrac{1}{u}$

And $h(u)=1+\ln(2u)$ then $h'(u)=\dfrac{2}{2u}=\dfrac{1}{u}$

We use quotient rule to differentiate $f(u)$ with respect to u

$$f'(u)=\frac{h(u).g'(u)-g(u).h'(u)}{[h(u)]^2}$$

$$=\frac{[1+\ln(2u)]\frac{1}{u}-(\ln u)\frac{1}{u}}{[1+\ln(2u)]^2}$$

$$=\frac{1+\ln 2u-\ln u}{u[1+\ln(2u)]^2}$$

$$=\frac{1+\ln 2+\ln u-\ln u}{u[1+\ln(2u)]^2} \qquad [\ln(mn)=\ln m+\ln n]$$

$$\Rightarrow \boxed{f'(u)=\frac{1+\ln 2}{u[1+\ln(2u)]^2}}$$

Answer 16E.

Given $y=\ln\left|1+t-t^3\right|$

On differentiation,

$$y'=\frac{d}{dt}(y)$$

$$=\frac{d}{dt}\left(\ln\left|1+t-t^3\right|\right)$$

$$=\frac{1}{1+t-t^3}\frac{d}{dt}\left(1+t-t^3\right)$$

$$=\frac{1-3t^2}{1+t-t^3}$$

Therefore $\boxed{y'=\dfrac{1-3t^2}{1+t-t^3}}$

Answer 17E.

Given $f(x)=x^5+5^x$

On differentiation

$$f'(x)=\frac{d}{dx}\left(x^5+5^x\right)$$

$$=\frac{d}{dx}\left(x^5\right)+\frac{d}{dx}\left(5^x\right)$$

$$=5x^4+5^x\ln 5$$

Therefore $\boxed{f'(x)=5x^4+5^x\ln 5}$

Answer 18E.

Given $g(x)=x\sin\left(2^x\right)$

On differentiation,

$$g'(x)=\frac{d}{dx}(g(x))$$

$$=\frac{d}{dx}\left(x\sin\left(2^x\right)\right)$$

$$=\frac{d}{dx}(x)\cdot\sin\left(2^x\right)+x\cdot\frac{d}{dx}\left(\sin\left(2^x\right)\right)$$

$$= \sin(2^x) + x\cdot\cos(2^x)\frac{d}{dx}(2^x)$$
$$= \sin(2^x) + x\cos(2^x)\cdot 2^x \ln 2$$
$$= \sin(2^x) + x2^x\cos(2^x)\ln 2$$

Therefore $\boxed{g'(x) = \sin(2^x) + x2^x\cos(2^x)\ln 2}$

Answer 19E.

Given $y = \tan(\ln(ax+b))$

On differentiation,

$$y' = \frac{d}{dx}(y)$$
$$= \frac{d}{dx}(\tan(\ln(ax+b)))$$

$$= \sec^2(\ln(ax+b))\cdot\frac{d}{dx}(\ln(ax+b))$$
$$= \sec^2(\ln(ax+b))\cdot\frac{1}{ax+b}\frac{d}{dx}(ax+b)$$
$$= \sec^2(\ln(ax+b))\cdot\frac{1}{ax+b}\cdot a$$

Therefore $\boxed{y' = \frac{a}{ax+b}\sec^2(\ln(ax+b))}$

Answer 20E.

Consider the following function:

$$H(z) = \ln\sqrt{\frac{a^2-z^2}{a^2+z^2}}$$

Differentiate the function.

Rewrite the given function as follows:

$$H(z) = \ln\left(\frac{a^2-z^2}{a^2+z^2}\right)^{\frac{1}{2}}$$

$$H(z) = \frac{1}{2}\ln\left(\frac{a^2-z^2}{a^2+z^2}\right)$$ Use the property $\ln(a^n) = n\ln a$

$$H(z) = \frac{1}{2}\left[\ln(a^2-z^2) - \ln(a^2+z^2)\right]$$ Use the property $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

Differentiate $H(z)$ as follows:

$$\frac{d}{dz}(H(z)) = \frac{d}{dz}\left[\frac{1}{2}\left[\ln(a^2-z^2) - \ln(a^2+z^2)\right]\right]$$ Use the difference rule

$$= \frac{1}{2}\left[\frac{d}{dz}(\ln(a^2-z^2)) - \frac{d}{dz}(\ln(a^2+z^2))\right]$$

$$= \frac{1}{2}\left[\frac{1}{a^2-z^2}\cdot -2z - \frac{1}{a^2+z^2}\cdot 2z\right]$$ Since $\frac{d}{dx}(\ln x) = \frac{1}{x}$

$$=-\frac{z}{a^2-z^2}-\frac{z}{a^2+z^2}$$ Simplify

$$=\frac{-z(a^2+z^2)-z(a^2-z^2)}{(a^2-z^2)(a^2+z^2)}$$ Take the L.C.M

$$=\frac{-a^2z-z^3-a^2z+z^3}{(a^2-z^2)(a^2+z^2)}$$ Multiply

$$=\frac{-2a^2z}{(a^2-z^2)(a^2+z^2)}$$ Simplify

Therefore, $\boxed{\frac{d}{dz}\left[\ln\sqrt{\frac{a^2-z^2}{a^2+z^2}}\right]=\frac{-2a^2z}{(a^2-z^2)(a^2+z^2)}}$.

Answer 21E.

Consider the following function.

$$y=\ln\left(e^{-x}+xe^{-x}\right)$$

Differentiate the given function $y=\ln\left(e^{-x}+xe^{-x}\right)$, with respect to *x*.

Let $g(x)=e^{-x}+xe^{-x}=e^{-x}(1+x)$, then the function y will be as follows:

$$y=\ln(g(x))$$

Use the chain rule $\left[\frac{d}{dx}[\ln(g(x))]=\frac{g'(x)}{g(x)}\right]$, to differentiate the function $y=\ln(g(x))$ with respect to *x*.

$$\frac{dy}{dx}=\frac{d}{dx}[\ln(g(x))]$$

$$=\frac{1}{e^{-x}(1+x)}\left[\frac{d}{dx}\left(e^{-x}(1+x)\right)\right]$$ Since, $g(x)=e^{-x}(1+x)$.

$$=\frac{1}{e^{-x}(1+x)}\left[e^{-x}\cdot\frac{d}{dx}(1+x)+\frac{d}{dx}\left(e^{-x}\right)\cdot(1+x)\right]$$

Using product rule: $\frac{d}{dx}[f(x)\cdot g(x)]=f(x)\cdot g'(x)+f'(x)\cdot g(x)$.

$$=\frac{1}{e^{-x}(1+x)}\left[e^{-x}\cdot(1)+e^{-x}\cdot\frac{d}{dx}(-x)\cdot(1+x)\right]$$ Using chain rule: $\frac{d}{dx}\left[e^{g(x)}\right]=e^{g(x)}\frac{d}{dx}[g(x)]$.

$$=\frac{1}{e^{-x}(1+x)}\left[e^{-x}+e^{-x}(-1)\cdot(1+x)\right]$$ Differentiating $-x$ with respect to *x*

Simplify the result.

$$\frac{dy}{dx}=\frac{1}{e^{-x}(1+x)}\left[e^{-x}-e^{-x}(1+x)\right]$$

$$=\frac{1}{e^{-x}(1+x)}\left[\cancel{e^{-x}}-\cancel{e^{-x}}-xe^{-x}\right]$$

$$=\frac{-xe^{-x}}{e^{-x}(1+x)}$$

$$=\frac{-x\cancel{e^{-x}}}{\cancel{e^{-x}}(1+x)}$$

$$=-\frac{x}{1+x}$$

Therefore, the value of the function $y=\ln\left(e^{-x}+xe^{-x}\right)$, when differentiating with respect to *x* is

$\boxed{\frac{dy}{dx}=-\frac{x}{1+x}}$.

Answer 22E.

Given $y = \ln\left|\cos(\ln x)\right|$

On differentiation,

$$y' = \frac{d}{dx}(y)$$

$$= \frac{d}{dx}\left[\ln\left|\cos(\ln x)\right|\right]$$

$$= \frac{1}{\cos(\ln x)} \cdot \frac{d}{dx}\left(\cos(\ln x)\right)$$

$$= \frac{1}{\cos(\ln x)} \cdot \left(-\sin(\ln x)\right) \times \frac{d}{dx}(\ln x)$$

$$= -\frac{\sin(\ln x)}{\cos(\ln x)} \times \frac{1}{x}$$

$$= -\frac{\tan(\ln x)}{x}$$

Therefore $\boxed{y' = -\frac{\tan(\ln x)}{x}}$

Answer 23E.

Consider the following function:

$$y = 2x\log_{10}\sqrt{x}$$

Differentiate the function.

$$\frac{d}{dx}\left[2x\log_{10}\sqrt{x}\right]$$

$$= 2x\frac{d}{dx}\log_{10}\sqrt{x} + \log_{10}\sqrt{x}\frac{d}{dx}(2x)$$ Use the product rule $\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$

$$= 2x \cdot \frac{d}{dx}\left(\frac{\ln\sqrt{x}}{\ln 10}\right) + \log_{10}\sqrt{x} \cdot 2$$ Since $\log_a x = \frac{\ln x}{\ln a}, \frac{d}{dx}(2x) = 2$

$$= 2x \cdot \frac{1}{\ln 10}\frac{d}{dx}\left(\ln\sqrt{x}\right) + 2\log_{10}\sqrt{x}$$ Simplify

$$= 2x \cdot \frac{1}{\ln 10} \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} + 2\log_{10} x^{\frac{1}{2}}$$ Since $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}, \frac{d}{dx}(\ln x) = \frac{1}{x}$

$$= 2x \cdot \frac{1}{\ln 10} \cdot \frac{1}{2x} + 2 \cdot \frac{1}{2}\log_{10} x$$ Use the property $\ln\left(a^n\right) = n\ln a$

$$= \frac{1}{\ln 10} + \log_{10} x$$ Simplify

Therefore, $\boxed{\frac{d}{dx}\left[2x\log_{10}\sqrt{x}\right] = \frac{1}{\ln 10} + \log_{10} x}$.

Answer 24E.

Consider the following function:

$$y = \log_2\left(e^{-x}\cos\pi x\right)$$

Differentiate the function.

$$\frac{d}{dx}\left[\log_2\left(e^{-x}\cos\pi x\right)\right]$$

$$\frac{d}{dx}\left[\log_2\left(e^{-x}\cos\pi x\right)\right] = \frac{d}{dx}\frac{\ln\left(e^{-x}\cos\pi x\right)}{\ln 2}$$ $\left[\log_a x = \frac{\ln x}{\ln a}\right]$

$$= \frac{1}{\ln 2} \cdot \frac{d}{dx}\left[\ln\left(e^{-x}\cos\pi x\right)\right]$$

$$= \frac{1}{\ln 2} \cdot \frac{1}{e^{-x}\cos\pi x}\frac{d}{dx}\left(e^{-x}\cos\pi x\right)$$ Use the chain rule $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x)$

Evaluate further.

$$=\frac{1}{\ln 2}\cdot\frac{1}{e^{-x}\cos\pi x}\left[-e^{-x}\cos\pi x-\pi\sin\pi x e^{-x}\right]$$

$\left\{\begin{array}{l}\text{Use the product rule}\\ \frac{d}{dx}\left[f(x)g(x)\right]=f(x)g'(x)+g(x)f'(x)\end{array}\right.$

$$=\frac{1}{\ln 2}\cdot\frac{-e^{-x}\cos\pi x}{e^{-x}\cos\pi x}\left[1+\pi\frac{\sin\pi x}{\cos\pi x}\right] \text{ Simplify}$$

$$=\frac{-1}{\ln 2}\cdot\left[1+\pi\tan\pi x\right] \text{ Simplify}$$

$$=-\frac{(1+\pi\tan\pi x)}{\ln 2} \text{ Simplify}$$

Therefore, $\frac{d}{dx}\left[\log_2\left(e^{-x}\cos\pi x\right)\right]=\boxed{-\frac{(1+\pi\tan\pi x)}{\ln 5}}$

Answer 25E.

Given $f(t)=10^{\sqrt{t}}$

Taking logarithms on both sides

$$\ln\left(f(t)\right)=\ln\left(10^{\sqrt{t}}\right)$$
$$=\sqrt{t}\ln 10$$

On differentiation,

$$\frac{d}{dt}\left(\ln\left(f(t)\right)\right)=\frac{d}{dt}\left(\sqrt{t}\ln 10\right)$$

$$\Rightarrow \quad \frac{1}{f(t)}\cdot f'(t)=(\ln 10)\cdot\frac{1}{2\sqrt{t}}$$

$$\Rightarrow \quad f'(t)=f(t)\frac{\ln 10}{2\sqrt{t}}$$

$$=\frac{10^{\sqrt{t}}\cdot\ln 10}{2\sqrt{t}}$$

Therefore $\boxed{f'(t)=\frac{10^{\sqrt{t}}\ln 10}{2\sqrt{t}}}$

Answer 26E.

Given $F(t)=3^{\cos 2t}$

Taking logarithms on both sides.

$$\ln\left(F(t)\right)=\ln\left(3^{\cos 2t}\right)$$
$$=\cos 2t\ln 3$$

On differentiation, we have

$$\frac{d}{dt}\left(\ln\left(F(t)\right)\right)=\frac{d}{dt}\left(\ln 3\cdot\cos 2t\right)$$

$$\Rightarrow \quad \frac{1}{F(t)}F'(t)=\ln 3(-\sin 2t)\times(2)$$

$$\Rightarrow \quad F'(t)=F(t)\ln 3(-\sin 2t)2$$

$$\Rightarrow \quad \boxed{F'(t)=-2(\sin 2t)(\ln 3).3^{\cos 2t}}$$

Answer 27E.

Consider the following function:

$y = x^2 \ln x$

Use logarithms on both sides of the equation and use the properties of logarithms to simplify as shown below:

$$\ln y = \ln\left[x^2 \ln(2x)\right]$$
$$= \ln x^2 + \ln\left[\ln(2x)\right]$$
$$= 2\ln x + \ln\left[\ln(2x)\right]$$

Differentiate implicitly with respect to x.

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{\ln(2x)}\cdot\frac{d}{dx}\left[\ln(2x)\right] \qquad \text{Since } \frac{d}{dx}\ln x = \frac{1}{x}$$
$$= \frac{2}{x} + \frac{1}{\ln(2x)}\cdot\frac{1}{2x}\cdot\frac{d}{dx}(2x) \qquad \text{Use chain rule}$$
$$= \frac{2}{x} + \frac{1}{\ln(2x)}\cdot\frac{1}{2x}\cdot 2 \qquad \text{Use chain rule}$$
$$y' = y\left[\frac{2}{x} + \frac{1}{\ln(2x)}\cdot\frac{1}{x}\right]$$

Substitute, $y = x^2 \ln x$ in the above result.

$$y' = x^2 \ln(2x)\left[\frac{2}{x} + \frac{1}{\ln(2x)}\cdot\frac{1}{x}\right]$$
$$= \boxed{2x\cdot\ln(2x) + x}$$

From the previous step, $y' = x + 2x\ln(2x)$.

Differentiate this equation again with respect to x.

$$y'' = 1 + 2x\cdot\frac{d}{dx}\ln(2x) + \ln(2x)\cdot\frac{d}{dx}2x \qquad \text{Product rule}$$
$$= 1 + 2x\left(\frac{1}{2x}\right)\frac{d}{dx}2x + \ln(2x)\cdot 2$$
$$= 1 + 2 + 2\ln(2x)$$
$$= 3 + 2\ln(2x)$$

Therefore, $\boxed{y'' = 3 + 2\ln(2x)}$.

Answer 28E.

Consider the function,

$$y=\frac{\ln x}{x^2}$$

Differentiate above function using quotient rule.

If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

So,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\ln x}{x^2}\right)$$

$$=\frac{x^2\frac{d}{dx}(\ln x)-\ln x\frac{d}{dx}x^2}{x^4}$$

$$=\frac{x^2\cdot\frac{1}{x}-\ln x(2x)}{x^4}$$

$$=\frac{x-2x(\ln x)}{x^4}$$

$$=\frac{1}{x^3}-2\frac{\ln x}{x^3}$$

Therefore,

$$\boxed{y'=\frac{1}{x^3}-2\frac{\ln x}{x^3}}.$$

Now,

$$y'=\frac{1}{x^3}-2\frac{\ln x}{x^3}$$

Differentiate again with respect to x.

$$\frac{d}{dx}y'=\frac{d}{dx}\left[\frac{1}{x^3}-2\frac{\ln x}{x^3}\right]$$

$$=\frac{d}{dx}\frac{1}{x^3}-2\left[\frac{x^3\cdot\frac{d}{dx}\ln(x)-\ln(x)\cdot\frac{d}{dx}x^3}{x^6}\right] \quad \text{Quotient rule}$$

$$=-3x^{-3-1}-2\left[\frac{x^3\cdot\frac{1}{x}-\ln(x).3x^2}{x^6}\right]$$

$$=-3x^{-4}-2\left[\frac{x^2-3x^2\ln(x)}{x^6}\right]$$

$$=-\frac{3}{x^4}-2\left[\frac{1-3\ln(x)}{x^4}\right]$$

$$=\frac{-5+6\ln(x)}{x^4}$$

Therefore,

$$\boxed{y''=\frac{-5+6\ln(x)}{x^4}}.$$

Answer 29E.

Consider the function,

$$y = \ln\left(x + \sqrt{1+x^2}\right)$$

Differentiate the above equation with respect to x on both sides.

$$\frac{d}{dx}y = \frac{d}{dx}\left[\ln\left(x+\sqrt{1+x^2}\right)\right]$$

$$y' = \frac{1}{x+\sqrt{1+x^2}} \cdot \frac{d}{dx}\left(x+\sqrt{1+x^2}\right)$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{1}{2\sqrt{1+x^2}} \cdot \frac{d}{dx}\left(1+x^2\right)\right) \qquad \text{Since } \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{1}{2\sqrt{1+x^2}} \cdot 2x\right)$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{\sqrt{1+x^2}}$$

Therefore,

$$\boxed{y' = \frac{1}{\sqrt{1+x^2}}}$$

Differentiate again the above equation on both sides

$$\frac{d}{dx}y' = \frac{d}{dx}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

$$y'' = \frac{d}{dx}\left(1+x^2\right)^{-\frac{1}{2}}$$

$$= -\frac{1}{2}\left(1+x^2\right)^{-\frac{1}{2}-1} \cdot \frac{d}{dx}\left(1+x^2\right)$$

$$= -\frac{1}{2}\left(1+x^2\right)^{-\frac{3}{2}} \cdot 2x$$

$$= -\frac{x}{\left(1+x^2\right)^{\frac{3}{2}}}$$

Therefore,

$$\boxed{y'' = -\frac{x}{\left(1+x^2\right)^{\frac{3}{2}}}}.$$

Answer 30E.

Given $y = \ln(\sec x + \tan x)$

Differentiating with respect to x, we get,

$$y' = \frac{dy}{dx} = \frac{d}{dx}\ln(\sec x + \tan x)$$

$$= \frac{1}{(\sec x + \tan x)}\frac{d}{dx}(\sec x + \tan x) \qquad \text{(Chain rule)}$$

$$= \frac{1}{(\sec x + \tan x)}\left(\frac{d}{dx}\sec x + \frac{d}{dx}\tan x\right)$$

$$= \frac{1}{(\sec x + \tan x)}\left[\sec x \tan x + \sec^2 x\right]$$

$$= \frac{\sec x[\tan x + \sec x]}{(\sec x + \tan x)}$$

$$= \sec x$$

Again differentiating with respect to x, we get,

$$y'' = \frac{d}{dx}y' = \frac{d}{dx}\sec x$$
$$= \sec x \tan x$$

Hence

$$\boxed{y' = \sec x \quad \text{and} \quad y'' = \sec x \tan x}$$

Answer 31E.

The given function is

$$f(x) = \frac{x}{1-\ln(x-1)}$$

Differentiating with respect to x, we get

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}\frac{x}{[1-\ln(x-1)]}$$

$$= \frac{[1-\ln(x-1)]\frac{d}{dx}x - x\frac{d}{dx}[1-\ln(x-1)]}{[1-\ln(x-1)]^2} \quad \text{[Using quotient rule]}$$

$$= \frac{\left[1-\ln(x-1).1 - x\left(\frac{d}{dx}1 - \frac{d}{dx}\ln(x-1)\right)\right]}{[1-\ln(x-1)]^2}$$

$$= \frac{\left[1-\ln(x-1) - x\left(0 - \frac{1}{(x-1)}\frac{d}{dx}(x-1)\right)\right]}{[1-\ln(x-1)]^2} \quad \text{[Chain rule]}$$

$$= \frac{[1-\ln(x-1)] - x\left[-\frac{1}{x-1}(1-0)\right]}{[1-\ln(x-1)]^2}$$

$$= \frac{[1-\ln(x-1)] + \frac{x}{x-1}}{[1-\ln(x-1)]^2}$$

$$= \frac{x-1-(x-1)\ln(x-1)+x}{(x-1)[1-\ln(x-1)]^2}$$

$$= \frac{2x-1-(x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}$$

Hence

$$\boxed{f'(x) = \frac{2x-1-(x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}}$$

We have to find the domain of $f(x) = \frac{x}{1-\ln(x-1)}$

The domain of f is the set of those values of x for which $f(x)$ is defined.

Now $f(x)$ is defined

If $x-1>0$ and $1-\ln(x-1) \neq 0$

If $x-1>0 \quad \Rightarrow \quad x>1$

If $1-\ln(x-1) \neq 0$

$\Rightarrow \quad \ln(x-1) \neq 1$

$\Rightarrow \quad e^{\ln(x-1)} \neq e^1$ Applying exponential function to both sides.

$\Rightarrow \quad x-1 \neq e$ Since $e^{\ln x} = x$

$\Rightarrow \quad x \neq 1+e$.

Thus we get

$x > 1$ and $x \neq 1+e$

i.e. the domain of f will be the set of all values of x greater than 1 except 1+e.

Therefore, domain of $f = (1, 1+e) \cup (1+e, \infty)$

Hence

$$\boxed{\text{Domain of } f = (1, 1+e) \cup (1+e, \infty)}$$

Answer 32E.

Given $f(x) = \sqrt{2 + \ln x}$

On differentiation,

$$f'(x) = \frac{d}{dx}\left(\sqrt{2+\ln x}\right)$$

$$= \frac{1}{2\sqrt{2+\ln x}} \frac{d}{dx}(2+\ln x)$$

$$= \frac{1}{2\sqrt{2+\ln x}}\left(\frac{1}{x}\right)$$

$$= \frac{1}{2x\sqrt{2+\ln x}}$$

Therefore $\boxed{f'(x) = \dfrac{1}{2x\sqrt{2+\ln x}}}$

The function $f(x)$ is defined only if $2 + \ln x \geq 0$

$$\Rightarrow \quad \ln x \geq -2$$

$$\Rightarrow \quad x \geq e^{-2}$$

Therefore $\boxed{\text{domain of } f = \left(e^{-2}, \infty\right)}$

Answer 33E.

Consider the function,

$$f(x) = \ln\left(x^2 - 2x\right)$$

Differentiate the above equation with respect to x.

$$\frac{d}{dx} f(x) = \frac{d}{dx} \ln\left(x^2 - 2x\right)$$

$$f'(x) = \frac{1}{x^2 - 2x} \cdot \frac{d}{dx}\left(x^2 - 2x\right) \qquad \text{Since } \frac{d}{dx}\ln x = \frac{1}{x}$$

$$= \frac{1}{x^2 - 2x}(2x - 2)$$

$$= \frac{2(x-1)}{x(x-2)}$$

It is need to determine domain of $f(x)$.

Since $f(x) = \ln\left(x^2 - 2x\right)$ is defined for non-zero positive values,

$$x^2 - 2x \geq 0$$

$$x(x-2) \geq 0 \quad \text{...... (1)}$$

The roots of above equation are 0, 2.

Equation (1) is defined for $x < 0$ or $x > 2$.

Therefore,

Domain of $f(x)$ is $\boxed{(-\infty, 0) \cup (2, \infty)}$.

Answer 34E.

We have to differentiate $f(x) = \ln\ln\ln x$

Differentiating both sides with respect to x,

$$\frac{d}{dx}f(x) = \frac{d}{dx}\ln\ln\ln x$$

$$\Rightarrow \quad f'(x) = \frac{1}{\ln\ln x}\cdot\frac{d}{dx}(\ln\ln x) \qquad \text{Using chain rule.}$$

$$= \frac{1}{\ln\ln x}\cdot\frac{1}{\ln x}\frac{d}{dx}\ln x \text{ Using chain rule.}$$

$$= \frac{1}{\ln\ln x}\cdot\frac{1}{\ln x}\cdot\frac{1}{x}$$

$$= \frac{1}{x\ln x\cdot\ln\ln x}$$

Hence,

$$\boxed{f'(x) = \frac{1}{x\ln x\ln\ln x}}$$

Now we find the domain of $f(x) = \ln\ln\ln x$.

The domain of f is the set of those values of x for which f is defined.

For ln x to be defined x > 0

For ln ln x to be defined.

$$\ln x > 0$$

$$\Rightarrow \quad x > e^0 \Rightarrow \quad x > 1$$

Also for ln ln ln x to be defined.

$$\ln\ln x > 0$$

$$\Rightarrow \quad \ln x > e^0$$

$$\Rightarrow \quad \ln x > 1$$

$$\Rightarrow \quad x > e^1$$

$$\Rightarrow \quad x > e$$

Therefore, the domain of $f = (e,\infty)$

Hence,

$$\boxed{\text{The domain of f is } (e,\infty)}$$

Answer 35E.

Consider the following function:

$$f(x) = \frac{\ln x}{1+x^2}$$

Apply quotient rule to differentiate the function.

Quotient rule:

If f and g are two differentiable functions, then the value of $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ is as follows:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

Differentiate as shown below:

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left[\frac{\ln x}{1+x^2}\right]$$

$$f'(x) = \frac{(1+x^2)\frac{d}{dx}\ln x - \ln x\cdot\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)\frac{1}{x} - \ln x\cdot 2x}{(1+x^2)^2}$$

Substitute $x=1$ in $f'(x)$ to find the value of $f'(1)$.

$$f'(1)=\frac{(1+1^2)\frac{1}{1}-\ln 1\cdot 2(1)}{(1+1^2)^2}$$

$$=\frac{2}{4} \qquad \text{Since } \ln 1=0$$

$$=\boxed{\frac{1}{2}}$$

Answer 36E.

Consider the following function:

$$f(x)=\ln(1+e^{2x})$$

Differentiate the above equation on both sides with respect to x.

$$\frac{d}{dx}f(x)=\frac{d}{dx}\ln(1+e^{2x})$$

$$f'(x)=\frac{1}{1+e^{2x}}\cdot\frac{d}{dx}(1+e^{2x}) \qquad \text{Since } \frac{d}{dx}\ln x=\frac{1}{x}$$

$$=\frac{1}{1+e^{2x}}(e^{2x})\frac{d}{dx}2x$$

$$=\frac{1}{1+e^{2x}}(e^{2x})2$$

$$=\frac{2e^{2x}}{1+e^{2x}}$$

Substitute, $x=0$ in $f'(x)$ to find $f'(0)$.

$$f'(0)=\frac{2e^{2\cdot 0}}{1+e^{2\cdot 0}}$$

$$=\frac{2(1)}{1+1}$$

$$=1$$

Therefore, $\boxed{f'(0)=1}$.

Answer 37E.

Given $y=\ln(x^2-3x+1)$

And point $=(3,0)$

Then $\frac{dy}{dx}=\frac{d}{dx}\left(\ln(x^2-3x+1)\right)$

$$=\frac{1}{x^2-3x+1}(2x-3)$$

$$=\frac{2x-3}{x^2-3x+1}$$

Therefore slope of the tangent at $(3,0)$ to the curve y is

$$\left(\frac{dy}{dx}\right)_{(3,0)}=\left.\frac{2x-3}{x^2-3x+1}\right|_{x=3}$$

$$=\frac{3}{1}$$

$$=3$$

Therefore equation of tangent at $(3,0)$ is

$$y-0=\left(\frac{dy}{dx}\right)_{x=3}(x-3)$$

$\Rightarrow \quad y=3(x-3)$

$\Rightarrow \quad 3x-y=9$

Therefore $\boxed{\text{equation of the tangent is } 3x-y=9}$

Answer 38E.

Given $y=x^2\ln x$

And point $=(1,0)$

Then $\frac{dy}{dx}=\frac{d}{dx}\left(x^2\ln x\right)$

$$=x^2\cdot\frac{1}{x}+2x\cdot\ln x$$

$$=x+2x\ln x$$

Therefore slope of the tangent at $(1,0)$ to the curve y is

$$\left.\frac{dy}{dx}\right|_{(1,0)}=\left(x+2x\ln x\right)_{x=1}$$

$$=1$$

Therefore equation of the tangent at $(1,0)$ is

$$y-0=\left.\frac{dy}{dx}\right|_{x=1}(x-1)$$

$\Rightarrow \quad y=1(x-1)$

$\Rightarrow \quad x-y=1$

Therefore $\boxed{\text{Equation of the tangent is } x-y=1}$

Answer 39E.

Consider the function,

$$f(x)=\ln\left(1+e^{2x}\right)$$

Differentiate the above equation on both sides with respect to x.

$$\frac{d}{dx}f(x)=\frac{d}{dx}(\sin x+\ln x)$$

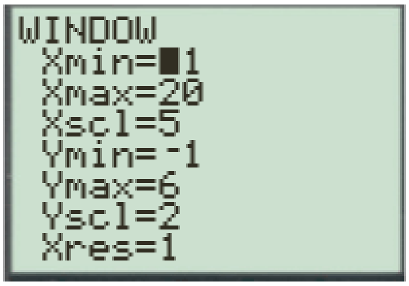
$$f'(x)=\frac{d}{dx}\sin x+\frac{d}{dx}\ln x \qquad \text{Since } \frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

$$=\cos x+\frac{1}{x}$$

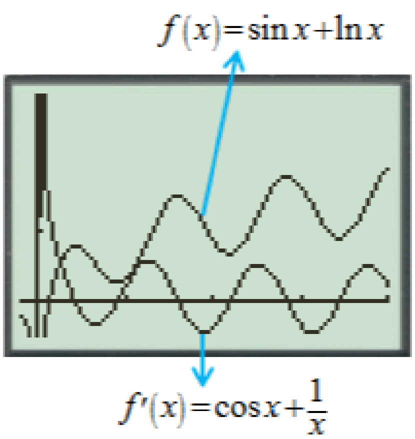
Therefore,

$$f'(x)=\boxed{\cos x+\frac{1}{x}}$$

To draw the graphs of f, f' using graphing calculator, first set the window as follows:



Now press the button $\boxed{Y=}$ then enter the functions as $Y_1 = \sin x + \ln x, Y_2 = \cos x + \frac{1}{x}$. Finally press $\boxed{GRAPH}$. The graph is shown below:



Answer 40E.

Given curve is $y = \frac{\ln x}{x}$.

Then $y' = \frac{dy}{dx}$

$$= \frac{d}{dx}\left(\frac{\ln x}{x}\right)$$

$$= \frac{x \cdot \frac{1}{x} - \ln x}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

At the point $(1,0)$, $y' = (1 - \ln 1) = 1$

At the point $(e, 1/e)$, $y' = 0$

Hence equation of the tangent line at $(1,0)$ is

$$y - 0 = (1 - \ln 1)(x - 1)$$

$$y - 0 = (1)(x - 1)$$

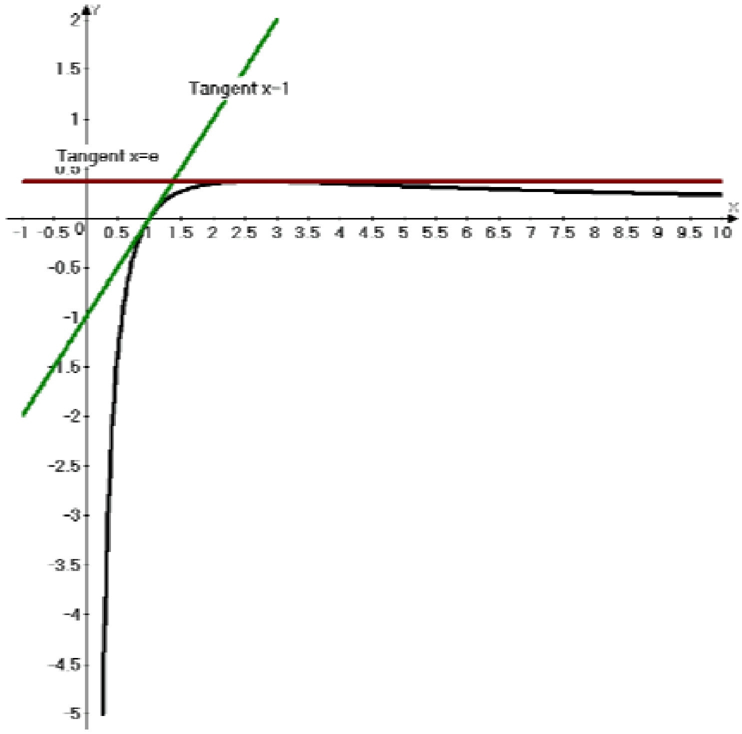
$$\Rightarrow \quad \boxed{y = (x - 1)}$$

The equation of the tangent line at $(e, 1/e)$ is

$$y - 1/e = 0(x - e)$$

$$\Rightarrow \quad \boxed{y = 1/e}$$

The graphs of the tangent lines are



Answer 41E.

Given $f(x)=\left(cx+\ln(\cos x)\right)$

On differentiation

$$f'(x)=\frac{d}{dx}\left(cx+\ln(\cos x)\right)$$
$$=\frac{d}{dx}(cx)+\frac{d}{dx}\left(\ln(\cos x)\right)$$
$$=c+\frac{1}{\cos x}(-\sin x)$$
$$=c-\tan x$$

Given that $f'(\pi/4)=6$

$$\Rightarrow \quad c-\tan\frac{\pi}{4}=6$$
$$\Rightarrow \quad c-1=6$$
$$\Rightarrow \quad c=7$$

Therefore $\boxed{c=7}$

Answer 42E.

Given $f(x)=\log_a\left(3x^2-2\right)$

$$=\frac{1}{\log e^a}\ln\left(3x^2-2\right)$$

On differentiation,

$$f'(x)=\frac{d}{dx}\left(f(x)\right)$$
$$=\frac{1}{\ln a}\cdot\frac{d}{dx}\left(\ln\left(3x^2-2\right)\right)$$
$$=\frac{1}{\ln a}\cdot\frac{1}{3x^2-2}6x$$
$$=\frac{6x}{\left(3x^2-2\right)\ln a}$$

Given that $f'(1)=3$

Therefore $3=\frac{6}{\ln a}$

$\Rightarrow \quad \ln a = 2$

$\Rightarrow \quad a = e^2$

Therefore $\boxed{a=e^2}$

Answer 43E.

Given $y=\left(x^2+2\right)^2\left(x^4+4\right)^4$

Taking logarithms on both sides,

$$\ln y=\ln\left[\left(x^2+2\right)^2\left(x^4+4\right)^4\right]$$

$$=\ln\left(x^2+2\right)^2+\ln\left(x^4+4\right)^4$$

$$=2\ln\left(x^2+2\right)+4\ln\left(x^4+4\right)$$

On differentiation,

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}\left[2\ln\left(x^2+2\right)+4\ln\left(x^4+4\right)\right]$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx}=\frac{2}{x^2+2}(2x)+\frac{4}{x^4+4}\left(4x^3\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=y\left[\frac{4x}{x^2+2}+\frac{16x^3}{x^4+4}\right]$$

i.e., $\boxed{\frac{dy}{dx}=\left(x^2+2\right)^2\left(x^2+4\right)^4\left[\frac{4x}{x^2+2}+\frac{16x^3}{x^4+4}\right]}$

Answer 44E.

Given $y=\frac{e^{-x}\cos^2 x}{x^2+x+1}$

Taking logarithms on both sides,

$$\ln y=\ln\left(e^{-x}\cos^2 x\right)-\ln\left(x^2+x+1\right)$$

On differentiation,

$$\frac{1}{y}\frac{dy}{dx}=\frac{d}{dx}\left[\ln\left(e^{-x}\cos^2 x\right)-\ln\left(x^2+x+1\right)\right]$$

$$=\frac{1}{e^{-x}\cos^2 x}\frac{d}{dx}\left(e^{-x}\cos^2 x\right)-\frac{1}{x^2+x+1}\frac{d}{dx}\left(x^2+x+1\right)$$

$$=\frac{e^{-x}\left(-2\cos x\sin x\right)-e^{-x}\cos^2 x}{e^{-x}\cos^2 x}-\frac{2x+1}{x^2+x+1}$$

$$=\frac{-\sin 2x-\cos^2 x}{\cos^2 x}-\frac{2x+1}{x^2+x+1}$$

$$=-2\tan x-1-\frac{2x+1}{x^2+x+1}$$

Therefore $\boxed{\frac{dy}{dx}=\frac{e^{-x}\cos^2 x}{x^2+x+1}\left(-2\tan x-1-\frac{2x+1}{x^2+x+1}\right)}$

Answer 45E.

Given $y=\sqrt{\frac{x-1}{x^4+1}}$

Taking logarithms on both sides

$$\ln y=\ln\left(\frac{x-1}{x^4+1}\right)^{1/2}$$

$$=\frac{1}{2}\left[\ln(x-1)-\ln\left(x^4+1\right)\right]$$

Differentiating the above equation with respect to x,

$$\frac{1}{y}\frac{dy}{dx}$$

$$=\frac{1}{2}\frac{d}{dx}\left[\ln(x-1)-\ln\left(x^4+1\right)\right]$$

$$=\frac{1}{2}\left[\frac{1}{x-1}-\frac{4x^3}{x^4+1}\right]$$

Therefore $\frac{dy}{dx}=\frac{y}{2}\left[\frac{1}{x-1}-\frac{4x^3}{x^4+1}\right]$

i.e; $\boxed{\frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{x-1}{x^4+1}}\left[\frac{1}{x-1}-\frac{4x^3}{x^4+1}\right]}$

Answer 46E.

Given $y=\sqrt{x}e^{x^2-x}(x+1)^{2/3}$

Taking logarithms both sides,

$$\ln y=\ln\left(\sqrt{x}e^{x^2-x}(x+1)^{2/3}\right)$$

$$=\ln\left(\sqrt{x}\right)+\ln\left(e^{x^2-x}\right)+\ln(x+1)^{2/3}$$

$$=\frac{1}{2}\ln x+\left(x^2-x\right)+\frac{2}{3}\ln(x+1)$$

Differentiating the above equations

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\cdot\frac{1}{x}+(2x-1)+\frac{2}{3}\frac{1}{x+1}$$

Therefore $\frac{dy}{dx}=y\left[\frac{1}{2x}+2x-1+\frac{2}{3(x+1)}\right]$

$$\boxed{\frac{dy}{dx}=\sqrt{x}\,e^{x^2-x}(x+1)^{2/3}\left[\frac{1}{2x}+2x-1+\frac{2}{3(x+1)}\right]}$$

Answer 47E.

$y=x^x$

Taking logarithm on both sides:-

$\ln y=x\ln x$

Differentiating with respect to x by product rule

$$\frac{d}{dx}[\ln y]=\frac{d}{dx}[x\ln x]$$

Or $\frac{1}{y}\frac{dy}{dx}=x\frac{d}{dx}\ln x+\ln x\frac{d}{dx}x$

Or $\frac{dy}{dx}=y\left[x\frac{1}{x}+\ln x\right]$

Or $\boxed{\frac{dy}{dx}=x^x[1+\ln x]}$

Answer 48E.

Given $y=x^{\cos x}$

Applying logarithms on both sides

$\ln y=\ln\left(x^{\cos x}\right)$ $\left[\text{Since } \ln\left(a^b\right)=b\ln a\right]$

$\Rightarrow\ln y=(\cos x)(\ln(x))$

On differentiation with respect to x, we get

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}((\cos x)(\ln x))$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx}=(\cos x)\cdot\frac{d}{dx}(\ln x)+(\ln x)\cdot\frac{d}{dx}(\cos x) \qquad \left[\text{Since } \frac{d}{dx}(\ln x)=\frac{1}{x} \text{ and } \frac{d}{dx}(\cos x)=-\sin x\right]$$

$$=(\cos x)\left(\frac{1}{x}\right)+(\ln x)(-\sin x)$$

$$=\frac{\cos x}{x}-(\ln x)(\sin x)$$

$$\Rightarrow \frac{dy}{dx}=y\frac{(\cos x-x(\ln x)(\sin x))}{x}$$

$$\Rightarrow \boxed{\frac{dy}{dx}=\frac{x^{\cos x}(\cos x-x(\ln x)(\sin x))}{x}}$$

Answer 49E.

$y=x^{\sin x}$

Taking logarithm on both sides

$\ln y=\ln x^{\sin x}$

$\ln y=\sin x\ln x \qquad \left(\text{since } \ln\left(x^n\right)=n\ln x\right)$

Differentiating with respect to x by product rule

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}(\sin x.\ln x)$$

$$\frac{1}{y}\cdot\frac{dy}{dx}=\sin x\frac{d}{dx}\ln x+\ln x\cdot\frac{d}{dx}\sin x \qquad \left(y=uv\Rightarrow\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$\frac{dy}{dx}=y\left[\sin x\cdot\frac{1}{x}+\ln x\cdot\cos x\right]$$

$$\boxed{\frac{dy}{dx}=x^{\sin x}\left[\frac{\sin x}{x}+\ln x\cdot\cos x\right]}$$

Answer 50E.

Consider the following function:

$$y=\sqrt{x^x}$$

Take logarithm on both sides of the equation.

$$\ln y=\ln\sqrt{x^x}$$

$$=\ln x^{\frac{x}{2}}$$

$$=\frac{x}{2}(\ln x) \qquad \text{Since } \ln x^n=n\ln(x)$$

Differentiate both sides of above equation with respect to x.

$$\frac{d}{dx}\ln y=\frac{d}{dx}\left[\frac{x}{2}(\ln x)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{x}{2}\cdot\frac{d}{dx}(\ln x)+\ln x\cdot\frac{d}{dx}\left(\frac{x}{2}\right) \qquad \text{Product rule}$$

$$=\frac{x}{2}\cdot\frac{1}{x}+\ln x\cdot\frac{1}{2}$$

$$=\frac{1}{2}[1+\ln x]$$

It can be written as shown below:

$$\frac{dy}{dx}=\frac{y}{2}[1+\ln x]$$

$$=\boxed{\frac{\sqrt{x^x}}{2}[1+\ln x]}$$

Answer 51E.

Consider the following function:

$$y=(\cos x)^{x}$$

Take logarithm on both sides of the equation.

$$\ln y=\ln(\cos x)^{x}$$
$$=x\cdot\ln(\cos x) \qquad \text{Since } \ln y^{b}=b\cdot\ln y$$

Differentiate the above equation with respect to x.

$$\frac{d}{dx}\ln y=\frac{d}{dx}\left[x\cdot\ln(\cos x)\right]$$
$$\frac{1}{y}\frac{dy}{dx}=x\cdot\frac{d}{dx}\left[\ln(\cos x)\right]+\ln(\cos x)\cdot\frac{d}{dx}x \qquad \text{Product rule}$$
$$=x\cdot\frac{1}{\cos x}\cdot\frac{d}{dx}\cos x+\ln(\cos x) \qquad \text{Since } \frac{d}{dx}\ln x=\frac{1}{x}$$
$$=\frac{x}{\cos x}(-\sin x)+\ln(\cos x)$$
$$=-x\tan x+\ln(\cos x)$$

The above equation can be written as shown below:

$$\frac{dy}{dx}=y\left[-x\tan x+\ln(\cos x)\right]$$
$$=\boxed{(\cos x)^{x}\left[-x\tan x+\ln(\cos x)\right]}.$$

Answer 52E.

Consider the following function:

$$y=(\sin x)^{\ln x}$$

Take logarithm on both sides of the equation.

$$\ln y=\ln(\sin x)^{\ln x}$$
$$=\ln x\cdot\ln(\sin x)$$

Differentiate the above equation with respect to x.

$$\frac{d}{dx}\ln y=\frac{d}{dx}\left[\ln x\cdot\ln(\sin x)\right]$$
$$\frac{1}{y}\frac{dy}{dx}=\ln x\cdot\frac{d}{dx}\ln(\sin x)+\ln(\sin x)\cdot\frac{d}{dx}\ln x$$
$$=\ln x\cdot\frac{1}{\sin x}\frac{d}{dx}\sin x+\ln(\sin x)\cdot\frac{1}{x}$$
$$=\ln x\cdot\frac{1}{\sin x}\cdot\cos x+\ln(\sin x)\cdot\frac{1}{x}$$
$$=\ln x\cdot\cot x+\frac{\ln(\sin x)}{x}$$

The above equation can be written as shown below:

$$\frac{dy}{dx}=y\left[\ln x\cdot\cot x+\frac{\ln(\sin x)}{x}\right]$$
$$=\boxed{(\sin x)^{\ln x}\left[\ln x\cdot\cot x+\frac{\ln(\sin x)}{x}\right]}.$$

Answer 53E.

Consider the equation,

$$y=(\tan x)^{\frac{1}{x}}.$$

The objective is to find the derivative of the function.

Take the logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y=\ln\left[(\tan x)^{\frac{1}{x}}\right]$$

$$\ln y=\frac{1}{x}\ln(\tan x) \qquad \text{Use } \ln\left(m^r\right)=r\ln(m)$$

Recollect, the **derivative of natural logarithmic function is**

$$\frac{d}{dx}\left[\ln(f(x))\right]=\frac{1}{f(x)}\frac{d}{dx}f(x)$$

Recollect, the **product rule**

Suppose *f* and *g* are both differentiable functions, then

$$\frac{d}{dx}\left[f(x)g(x)\right]=f(x)\frac{d}{dx}\left[g(x)\right]+g(x)\frac{d}{dx}\left[f(x)\right]$$

Differentiate $\ln y=\frac{1}{x}\ln(\tan x)$ with respect to *x*:

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}\left[\frac{1}{x}\ln(\tan x)\right]$$

$$\frac{1}{y}\frac{d}{dx}(y)=\frac{1}{x}\frac{d}{dx}\left[\ln(\tan x)\right]+\ln(\tan x)\frac{d}{dx}\left(\frac{1}{x}\right) \qquad \text{Use product rule}$$

$$\frac{(dy/dx)}{y}=\frac{1}{x}\frac{d}{dx}\left[\ln(\tan x)\right]+\ln(\tan x)\left(-\frac{1}{x^2}\right) \quad \ldots\ldots (1)$$

To find the remaining derivative $\frac{d}{dx}\left[\ln(\tan x)\right]$ in (1), use formula of derivative of natural logarithmic function:

$$\frac{d}{dx}\left[\ln(\tan x)\right]=\frac{1}{\tan x}\cdot\frac{d}{dx}(\tan x)$$

$$=\frac{1}{\tan x}\left(\sec^2 x\right) \qquad \text{Use } \frac{d}{dx}(\tan x)=\sec^2 x$$

$$=\frac{\sec^2 x}{\tan x}$$

Substitute $\frac{\sec^2 x}{\tan x}$ for $\frac{d}{dx}\left[\ln(\tan x)\right]$ in (1), then

$$\frac{(dy/dx)}{y}=\frac{1}{x}\left[\frac{\sec^2 x}{\tan x}\right]+\left[\ln(\tan x)\right]\cdot\left(-\frac{1}{x^2}\right)$$

$$=\frac{\sec^2 x}{x\tan x}-\frac{1}{x^2}\left[\ln(\tan x)\right]$$

$$\frac{dy}{dx}=y\left[\frac{\sec^2 x}{x\tan x}-\frac{1}{x^2}\left[\ln(\tan x)\right]\right] \qquad \text{Multiply both sides by } y$$

$$=(\tan x)^{\frac{1}{x}}\left[\frac{\sec^2 x}{x\tan x}-\frac{1}{x^2}\left[\ln(\tan x)\right]\right] \qquad \text{Substitute } (\tan x)^{\frac{1}{x}} \text{ for } y$$

Therefore,

$$y'=\boxed{(\tan x)^{\frac{1}{x}}\left[\frac{\sec^2 x}{x\tan x}-\frac{1}{x^2}\left[\ln(\tan x)\right]\right]}.$$

Answer 54E.

$y=(\ln x)^{\cos x}$

Taking logarithm on both sides:-

$\ln y=\cos x\ln(\ln x)$

Differentiating with respect to x by chain rule together with product rule

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}\left[\cos x\ln(\ln x)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\cos x\frac{d}{dx}\ln(\ln x)+\ln(\ln x)\frac{d}{dx}\cos x$$

$$\frac{dy}{dx}=y\left[\frac{\cos x}{\ln x}.\frac{1}{x}+\ln(\ln x)(-\sin x)\right]$$

$$\boxed{\frac{dy}{dx}=(\ln x)^{\cos x}\left[\frac{\cos x}{x\ln x}-\sin x\ln(\ln x)\right]}$$

Answer 55E.

$y=\ln\left(x^2+y^2\right)$

Differentiating with respect to x by chain rule

$$\frac{d}{dx}y=\frac{d}{dx}\left[\ln\left(x^2+y^2\right)\right]$$

Or $$\frac{dy}{dx}=\frac{1}{\left(x^2+y^2\right)}\frac{d}{dx}\left[x^2+y^2\right]$$

Or $$\frac{dy}{dx}=\frac{1}{\left(x^2+y^2\right)}\left[2x+2y\frac{dy}{dx}\right]$$

Or $$\frac{dy}{dx}=\frac{2x}{\left(x^2+y^2\right)}+\frac{2y}{\left(x^2+y^2\right)}\frac{dy}{dx}$$

Or $$\frac{dy}{dx}\left[1-\frac{2y}{\left(x^2+y^2\right)}\right]=\frac{2x}{x^2+y^2}$$

Or $$\frac{dy}{dx}=\frac{\frac{2x}{\left(x^2+y^2\right)}}{\left(\frac{x^2+y^2-2y}{\left(x^2+y^2\right)}\right)}$$

Or $$\boxed{\frac{dy}{dx}=\frac{2x}{x^2+y^2-2y}}$$

Answer 56E.

We have to find y' if $x^y=y^x$

We have $x^y=y^x$

Taking logarithms of both sides, we get,

$$\ln x^y=\ln y^x$$

$\Rightarrow$ $y\ln x=x\ln y$ Since $\ln x^y=y\ln x$

Differentiating both sides implicitly with respect to x, we get,

$$\frac{d}{dx}[y\ln x]=\frac{d}{dx}[x\ln y]$$

$\Rightarrow$ $y\left[\frac{d}{dx}(\ln x)\right]+(\ln x)\left(\frac{dy}{dx}\right)=x\left(\frac{d}{dx}\ln y\right)+\ln y\left(\frac{d}{dx}x\right)$ Using product rule.

$\Rightarrow$ $y.\frac{1}{x}+(\ln x)\frac{dy}{dx}=x.\frac{1}{y}\left(\frac{dy}{dx}\right)+(\ln y).1$ Using chain rule.

$\Rightarrow$ $\frac{y}{x}+\ln x\frac{dy}{dx}=\frac{x}{y}\frac{dy}{dx}+\ln y$

Now to find y' we will collect the terms containing $\frac{dy}{dx}$ on one side. So we have

$$\ln x\frac{dy}{dx}-\frac{x}{y}\frac{dy}{dx}=\ln y-\frac{y}{x}.$$

$$\Rightarrow \quad \left(\ln x-\frac{x}{y}\right)\frac{dy}{dx}=\left(\ln y-\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{\left(\ln y-\frac{y}{x}\right)}{\left(\ln x-\frac{x}{y}\right)}$$

$$=\frac{(x\ln y-y)/x}{(y\ln x-x)/y}$$

$$=\frac{y(x\ln y-y)}{x(y\ln x-x)}$$

Hence

$$\boxed{y'=\frac{dy}{dx}=\frac{y(x\ln y-y)}{x(y\ln x-x)}}$$

Answer 57E.

We have to find $f^{(n)}(x)$ if $f(x)=\ln(x-1)$

For this we will find first three or four derivatives and then guess nth derivative by inspection.

We have $f(x)=\ln(x-1)$

Differentiating both sides with respect to x,

We get $f'(x)=\frac{d}{dx}f(x)=\frac{d}{dx}\ln(x-1)$

$$=\frac{1}{(x-1)}(1-0)$$

$$=\frac{1}{(x-1)}$$

We have $f'(x)=\frac{1}{(x-1)}$

Differentiating with respect to x, we get,

$$f''(x)=\frac{d}{dx}f'(x)=\frac{d}{dx}\frac{1}{(x-1)}$$

$$=\frac{d}{dx}(x-1)^{-1}$$

$$=(-1)(x-1)^{-2}\frac{d}{dx}(x-1) \qquad \text{Using chain rule.}$$

$$=(-1)(x-1)^{-2}(1-0)$$

$$=(-1)(x-1)^{-2}$$

We have $f''(x)=(-1)(x-1)^{-2}$

Differentiating with respect to x, we get

$$f'''(x)=\frac{d}{dx}f''(x)=\frac{d}{dx}(-1)(x-1)^{-2}$$

$$=(-1)(-2)(x-1)^{-2-1}\frac{d}{dx}(x-1) \qquad \text{Using chain rule}$$

$$=(-1)^2 1.2(x-1)^{-3}(1-0)$$

$$=(-1)^2 1.2.(x-1)^{-3}$$

We have got $f'''(x) = (-1)^2 1.2.(x-1)^{-3}$

Differentiating with respect to x, we get

$$f^{(iv)}(x) = \frac{d}{dx} f'''(x) = \frac{d}{dx}(-1)^2 1.2.(x-1)^{-3}$$
$$= (-1)^2 1.2.\frac{d}{dx}(x-1)^{-3}$$
$$= (-1)^2 1.2.(-3)(x-1)^{-4} \frac{d}{dx}(x-1)$$
$$= (-1)^3 1.2.3(x-1)^{-4}(1-0)$$
$$= (-1)^3 1.2.3.(x-1)^{-4}$$

Proceeding in the same way differentiating $f(x)$ n times, we get

$$f^n(x) = (-1)^{n-1} 1.2.3 \ldots\ldots\ldots (n-1)(x-1)^{-n}$$
$$= \boxed{\frac{(-1)^{n-1}(n-1)!}{(x-1)^n}}$$

Hence

$$\boxed{f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(x-1)^n}}$$

Answer 58E.

Evaluate $\frac{d^9}{dx^9}(x^8 \ln x)$

Let $y = x^8 \ln x$

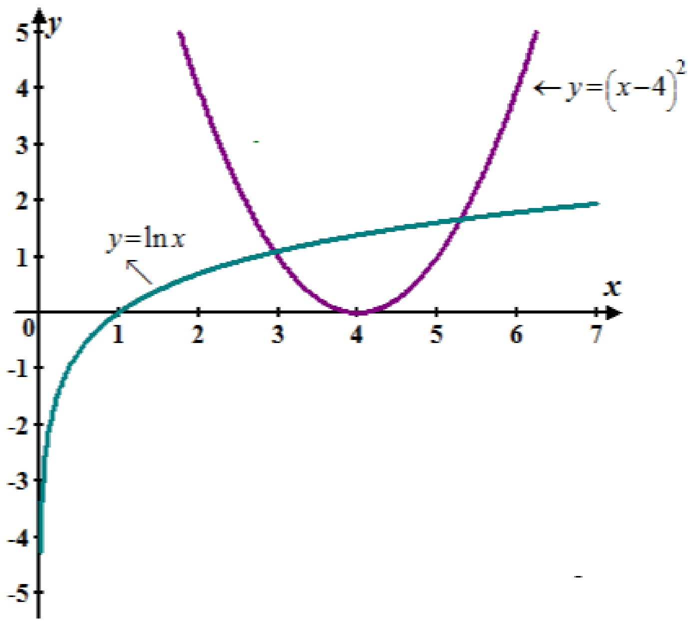
$$y' = x^8 \cdot \frac{1}{x} + 8x^7 \ln x$$
$$y' = x^7 + 8x^7 \ln x$$
$$y'' = 7x^6 + 8x^7 \cdot \frac{1}{x} + 8 \cdot 7 \cdot x^6 \ln x$$
$$= 15x^6 + 8 \cdot 7 \cdot x^6 \ln x$$
$$y''' = 90x^5 + 8 \cdot 7 \cdot x^6 \cdot \frac{1}{x} + 8 \cdot 7 \cdot 6 \cdot x^5 \ln x$$
$$= 146x^5 + 8 \cdot 7 \cdot 6x^5 \ln x$$
$$y^{(4)} = Ax^4 + 8 \cdot 7 \cdot 6 \cdot 5x^4 \ln x$$
$$y^{(5)} = Bx^3 + 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4x^3 \ln x$$
$$y^{(6)} = Cx^2 + 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3x^2 \ln x$$
$$y^{(7)} = Dx^1 + 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2x^1 \ln x$$
$$y^{(8)} = Ex^0 + 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1x^0 \ln x$$
$$y^{(8)} = E + (8!) \ln x$$
$$y^{(9)} = 0 + \frac{8!}{x}$$

Therefore,

$$\frac{d^9}{dx^9}(x^8 \ln x) = \boxed{\frac{8!}{x}}$$

Answer 59E.

First we sketch the curve $y = (x-4)^2$ and $y = \ln x$ on the same screen with the help of computer (figure 1) and move the cursor to the points of intersection of these curves, we see that x-coordinates of the points of intersection are about 3.0 and 5.3.



First we take initial approximation $x_1 = 3.0$

Since $f(x) = (x-4)^2 - \ln x$

Then $f'(x) = 2(x-4) - \frac{1}{x}$ $\left[\frac{d}{dx}\ln x = \frac{1}{x}\right]$

Then Newton's formula becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n-4)^2 - \ln x_n}{2(x_n-4) - \frac{1}{x_n}}$$

$x_1 = 3.0$

Then $x_2 = x_1 - \dfrac{(x_1-4)^2 - \ln x_1}{2(x_1-4) - \left(\frac{1}{x_1}\right)}$

$$= 3 - \frac{(3-4)^2 - \ln 3}{2(3-4) - \left(\frac{1}{3}\right)}$$

Or $x_2 \approx 2.957738$

Similarly $x_3 \approx 2.958516$

$x_4 \approx 2.958516$ $[x_3 \approx x_4]$

So first root of the equation is $\boxed{x \approx 2.958516}$

Now we take initial approximation $x_1 = 5.3$

Then $x_2 = x_1 - \dfrac{(x_1-4)^2 - \ln x_1}{2(x_1-4) - \left(\frac{1}{x_1}\right)}$

$$= 5.3 - \frac{(5.3-4)^2 - \ln(5.3)}{2(5.3-4) - \left(\frac{1}{5.3}\right)}$$

Or $x_2 \approx 5.290755$

Similarly $x_3 \approx 5.290718$

$x_4 \approx 5.290718$

Here $x_3 \approx x_4$ (correct up to six decimal places)

So the second root of the equation is $\boxed{x \approx 5.290718}$

The roots of the equation $(x-4)^2 = \ln x$, are $\boxed{x \approx 2.958516,\ \ 5.290718}$

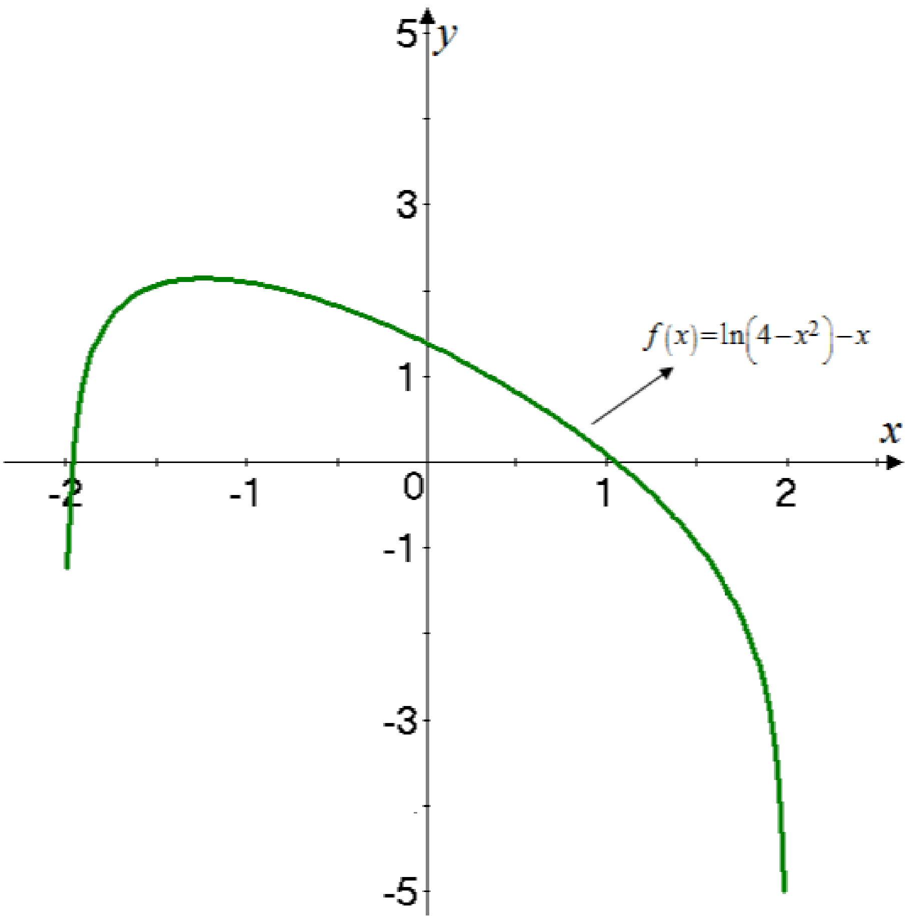
Answer 60E.

Consider the equation $\ln(4-x^2) = x.$

Rewrite the equation as $f(x) = \ln(4-x^2) - x.$

Need to find all the roots of the equation correct to eight decimal places using Newton's method.

The sketch of the graph $f(x) = \ln(4-x^2) - x$ is shown below:



Differentiate the function $f(x)$ with respect to x. get

$$f(x) = \ln(4-x^2) - x$$

$$f'(x) = \frac{-2x}{4-x^2} - 1$$

In Newton's method, *n*th approximation is x_n and $f'(x_n) \neq 0,$ then the next approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

From the graph observe that the equation has two roots.

First find the root located at left side of the origin.

Choose $x_1 = -1.94$ with $n = 1$ then the second approximation is

$$\begin{aligned}
x_2 &= x_{1+1} \\
&= x_1 - \frac{f(x_1)}{f'(x_1)} \\
&= x_1 - \frac{\ln\left(4 - x_1^{\,2}\right) - x_1}{\frac{-2x_1}{4 - x_1^{\,2}} - 1}
\end{aligned}$$

$$= -1.94 - \frac{\ln\left(4 - (-1.94)^2\right) - (-1.94)}{\frac{-2(-1.94)}{4 - (-1.94)^2} - 1}$$

$$\approx -1.94 - \frac{0.497770007}{15.41285956}$$

$$\approx -1.94 - 0.03229575959$$

$$\approx -1.97229576$$

And

$$\begin{aligned}
x_3 &= x_{2+1} \\
&= x_2 - \frac{f(x_2)}{f'(x_2)} \\
&= x_2 - \frac{\ln\left(4 - x_2^{\,2}\right) - x_2}{\frac{-2x_2}{4 - x_2^{\,2}} - 1}
\end{aligned}$$

$$= -1.97229576 - \frac{\ln\left(4 - (-1.97229576)^2\right) - (-1.97229576)}{\frac{-2(-1.97229576)}{4 - (-1.97229576)^2} - 1}$$

$$\approx -1.97229576 - \frac{-0.234529845}{34.84381437}$$

$$\approx -1.97229576 + 0.006730888947$$

$$\approx -1.965564871$$

Similarly, we obtain

$$x_4 \approx -1.964648471$$

$$x_5 \approx -1.9646356$$

$$x_6 \approx -1.964635597$$

$$x_7 \approx -1.964635597$$

Since x_6 and x_7 agree to eight decimal places, we conclude that one root of the equation, correct to eight decimal places, is $x \approx -1.964635597.$

Now find the root located at right side of the origin.

Choose $x_1 = 1$ with $n = 1$ then the second approximation is

$$\begin{aligned} x_2 &= x_{1+1} \\ &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= x_1 - \frac{\ln(4 - x_1^2) - x_1}{\frac{-2x_1}{4 - x_1^2} - 1} \end{aligned}$$

$$= 1 - \frac{\ln(4 - (1)^2) - (1)}{\frac{-2(1)}{4 - (1)^2} - 1}$$

$$\approx 1 - \frac{0.098612289}{-1.666666667}$$

$$\approx 1 + 0.05916737339$$

$$\approx 1.059167373$$

And

$$\begin{aligned} x_3 &= x_{2+1} \\ &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ &= x_2 - \frac{\ln(4 - x_2^2) - x_2}{\frac{-2x_2}{4 - x_2^2} - 1} \end{aligned}$$

$$= 1.059167373 - \frac{\ln(4 - (1.059167373)^2) - (1.059167373)}{\frac{-2(1.059167373)}{4 - (1.059167373)^2} - 1}$$

$$\approx 1.059167373 - \frac{-0.002014617}{-1.736001978}$$

$$\approx 1.059167373 - 0.001160492341$$

$$\approx 1.058006881$$

Similarly, we obtain

$$x_4 \approx 1.058006401$$

$$x_5 \approx 1.058006401$$

Since x_4 and x_5 agree to eight decimal places, we conclude that another root of the equation, correct to eight decimal places, is $x \approx 1.058006401$.

Answer 61E.

The given function is $f(x) = \frac{\ln x}{\sqrt{x}}$

Domain of the function is $(0, \infty)$

Differentiating with respect to x, we get,

$$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{\ln x}{\sqrt{x}}\right)$$

$$= \frac{1}{\sqrt{x}} \frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) \qquad \text{Using product rule.}$$

$$= \frac{1}{\sqrt{x}}\left(\frac{1}{x}\right) + (\ln x) \cdot \left(\frac{d}{dx} x^{-1/2}\right)$$

$$= \frac{1}{x^{3/2}} + \ln x \left(-\frac{1}{2} x^{-\frac{1}{2}-1}\right)$$

$$= \frac{1}{x^{3/2}} - \frac{\ln x}{2x^{3/2}}$$

$$= \frac{2 - \ln x}{2x^{3/2}}$$

We have $f'(x)=\dfrac{2-\ln x}{2x^{3/2}}$

Differentiating with respect to x, we get,

$$f''(x)=\frac{d}{dx}\left[f'(x)\right]=\frac{d}{dx}\left[\frac{(2-\ln x)}{2x^{3/2}}\right]$$

$$=\frac{1}{2}\frac{d}{dx}\left[\frac{2-\ln x}{x^{3/2}}\right]$$

$$=\frac{\frac{1}{2}\left[x^{3/2}\frac{d}{dx}(2-\ln x)-(2-\ln x)\frac{d}{dx}x^{3/2}\right]}{\left(x^{3/2}\right)^2}$$ Using quotient rule.

$$=\frac{\frac{1}{2}\left[x^{3/2}\left(0-\frac{1}{x}\right)-(2-\ln x)\cdot\frac{3}{2}x^{\frac{3}{2}-1}\right]}{x^3}$$

$$=\frac{\frac{1}{2}\left[-x^{1/2}-\frac{3}{2}(2-\ln x)x^{1/2}\right]}{x^3}$$

$$=\frac{\frac{1}{4}x^{1/2}\left[-2-6+3\ln x\right]}{x^3}$$

$$=\frac{\left[3\ln x-8\right]}{4x^{5/2}}$$

For point of inflection $f''(x)=0$

$\Rightarrow\quad \dfrac{3\ln x-8}{4x^{5/2}}=0$

$\Rightarrow\quad 3\ln x-8=0$

$\Rightarrow\quad \ln x=\dfrac{8}{3}$

$\Rightarrow\quad x=e^{8/3}$

Also, when $x=e^{8/3}$

$$y=f(x)=\frac{\ln x}{\sqrt{x}}$$

$$=\frac{\ln e^{8/3}}{\sqrt{e^{8/3}}}$$

$$=\frac{8/3}{\left[e^{(8/3)}\right]^{1/2}}$$ Since $\ln e^x=x$

$$=\frac{8}{3e^{4/3}}=\frac{8}{3}e^{-4/3}$$

Therefore, point of inflection is

$$\left(e^{8/3},\frac{8}{3}e^{-4/3}\right)$$

Let us check for concavity of f(x) The given function f(x) will be concave upward if $f''(x)>0$

$\Rightarrow\quad \dfrac{3\ln x-8}{4x^{5/2}}>0$

$\Rightarrow\quad 3\ln x-8>0$

$\Rightarrow\quad 3\ln x>8$

$\Rightarrow\quad \ln x>\dfrac{8}{3}$

$\Rightarrow\quad x>e^{8/3}$

Thus f(x) is concave upward for $x\in\left(e^{8/3},\infty\right)$

The given function f(x) will be concave downward if $f''(x)<0$

$\Rightarrow\quad \dfrac{3\ln x-8}{4x^{5/2}}<0$

$\Rightarrow\quad 3\ln x-8<0$

$\Rightarrow\quad 3\ln x<8$

$\Rightarrow\quad \ln x<8/3$

$\Rightarrow \quad x < e^{8/3}$

Thus f(x) is concave downward for $x \in \left(0, e^{8/3}\right)$

Hence,

$f(x)$ is concave upward on $\left(e^{8/3}, \infty\right)$
$f(x)$ is concave downward on $\left(0, e^{8/3}\right)$
Point of inflection is $\left(e^{8/3}, \frac{8}{3}e^{-4/3}\right)$

Answer 62E.

We have to find the absolute minimum value of the function $f(x) = x\ln x$

The domain of $f(x) = x\ln x$ is $(0, \infty)$ as ln x is not defined for $x \le 0$

Given, $f(x) = x\ln x$

Differentiating with respect to x,

$$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx}(x\ln x)$$

$$= x\frac{d}{dx}(\ln x) + (\ln x)\frac{d}{dx}x \qquad \text{Using product rule.}$$

$$= x \cdot \frac{1}{x} + (\ln x) \cdot 1$$

$$= 1 + \ln x$$

For critical values $f'(x) = 0$

$\Rightarrow \quad 1 + \ln x = 0$

$\Rightarrow \quad \ln x = -1$

$\Rightarrow \quad x = e^{-1}$

$\Rightarrow \quad x = 1/e$

Now, $f'(x) = 1 + \ln x$

For, $f'(x) > 0$

$\Rightarrow \quad 1 + \ln x > 0$

$\Rightarrow \quad \ln x > -1$

$\Rightarrow \quad x > e^{-1}$

i.e. $\quad x > \frac{1}{e}$

$\Rightarrow \quad f(x)$ is increasing when $x > \frac{1}{e}$

For, $f'(x) < 0$

We have,

$1 + \ln x < 0$

$\Rightarrow \quad \ln x < -1$

$\Rightarrow \quad x < e^{-1}$

$\Rightarrow \quad x < 1/e$

$\Rightarrow \quad f(x)$ is decreasing for $x < \frac{1}{e}$

Here f(x) is decreasing for $x < 1/e$ and increasing for $x > 1/e$
So by first derivative test for absolute extreme values, f(x) has absolute minimum value at $x = 1/e$.
And absolute minimum value of f(x) is

$$f(x) = x\ln x$$
$$= \frac{1}{e}\ln\frac{1}{e}$$
$$= \frac{1}{e}\ln e^{-1}$$
$$= -\frac{1}{e}\ln e \qquad \text{Since } \ln x^y = y\ln x$$
$$= -\frac{1}{e}\cdot 1 \qquad \ln e = 1$$
$$= -\frac{1}{e}$$

Hence,

Absolute minimum value $= -1/e$

Answer 63E.

Consider the curve $y = \ln(\sin x)$

Rewrite the curve as $f(x) = \ln(\sin x)$

A. The domain is

$$D = \{x \mid \sin x > 0\} = (0,\pi)\cup(2\pi,3\pi)\cup(4\pi,5\pi)\ldots$$

$= \{x \mid x \in (2n\pi,(2n+1)\pi)\}$ *n* is an integer including 0

B. *x*-intercept occur when $\ln(\sin x) = 0$

$$\Rightarrow \sin x = 1$$

$$\Rightarrow \boxed{x = \frac{\pi}{2} + 2n\pi}$$

No *y*-intercept since at $x = 0$, $f(x)$ is not defined

C. Since $f(-x)$ is not defined so it is not symmetric

$$f(x+2\pi) = \ln(\sin(x+2\pi)) = \ln(\sin x)$$

So it is a periodic function and has a period of 2π

D. We look for vertical asymptotes at the end points of the domain.

$$\lim_{x\to n\pi^-} \ln(\sin x) = -\infty \text{ and } \lim_{x\to n\pi^+} \ln(\sin x) = -\infty$$

So vertical asymptotes are $x = n\pi$

There is no horizontal asymptote

E. Since $f(x) = \ln(\sin x)$.

Then $f'(x) = \frac{1}{\sin x}(\cos x) = \cot x$

Since $f'(x) > 0$ on the interval $\left(2n\pi, 2n\pi + \frac{\pi}{2}\right)$

And $f'(x) < 0$ on $\left(2n\pi + \frac{\pi}{2}, (2n+1)\pi\right)$

So $f(x)$ is increasing on $\left(2n\pi, 2n\pi + \frac{\pi}{2}\right)$

And decreasing on $\left(2n\pi + \frac{\pi}{2}, (2n+1)\pi\right)$

F. $f(x)$ has local maximum $f\left(2n\pi+\frac{\pi}{2}\right)=0$

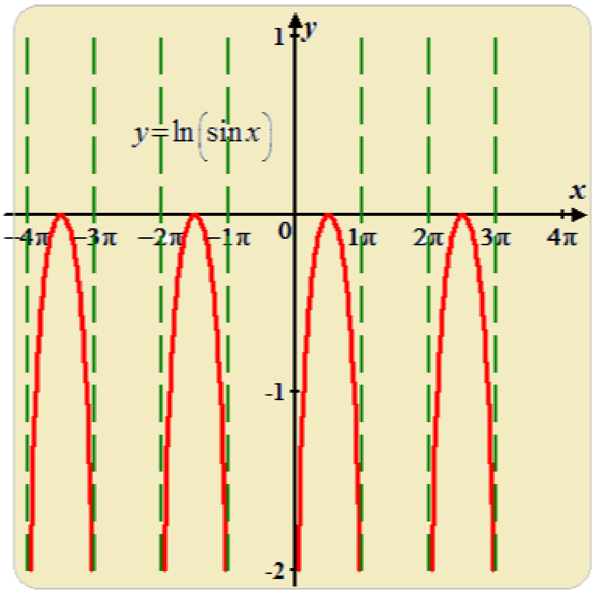
G. Since $f'(x)=\cot x$

Then $f''(x)=-\csc^2 x$

Since $f''(x)<0$ for all x

So $f(x)$ is concave downward on its entire domain

H. Using this information, we sketch the curve



Answer 64E.

Consider the equation of the curve.

$y=\ln\left(\tan^2 x\right)$

Find the domain of the function.

Domain is the set of values of x for which $f(x)$ is defined.

As the domain of $\ln(x)$ is the set of all positive real numbers.

So, the domain of $y=\ln\left(\tan^2 x\right)$ is a set of positive values of $\tan^2 x$ for which $\tan^2 x$ is defined.

As $\tan^2 x$ gives positive values, so find the set values of x for which $\tan^2 x$ must not be equal to zero.

The domain of the function, $y=\ln\left(\tan x^2\right)$ is as follows:

$$\left\{x \mid x\in\left(n\pi,n\pi+\frac{\pi}{2}\right)\cup\left(n\pi,(n+1)\pi\right)\right\}$$

Set $y=0$ to find x-intercept.

$$\ln\left(\tan^2 x\right)=0$$
$$\tan^2 x=e^0$$
$$(\tan x)^2=1$$
$$\tan x=1$$
$$x=\tan^{-1}(1)$$
$$x=\tan^{-1}\left(\tan\left(n\pi-\frac{\pi}{4}\right)\right) \qquad \text{For } n\in Z$$
$$x=\left(n\pi-\frac{\pi}{4}\right) \qquad \text{For } n\in Z$$

Set $y=0$ in the equation $y=\ln\left(\tan^2 x\right)$ to find y-intercept .

$$y=\ln\left(\tan^2 0\right)$$
$$y=\ln(0)$$
$$y=-\infty \qquad \text{undefined.}$$

The function $y=\ln\left(\tan^2 x\right)$ that has periodicity is π

The function has no horizontal asymptotes.

The vertical asymptotes are the following:

$$\lim_{x\to 0}\ln\left(\tan^2 x\right)=-\infty$$
$$\lim_{x\to n\pi}\ln\left(\tan^2 x\right)=-\infty \qquad \text{for every } n$$

Therefore, the vertical asymptotes are $x=0, n\pi$ for every n .

Find the derivative of the function $y=\ln\left(\tan^2 x\right)$ with respect to the variable x

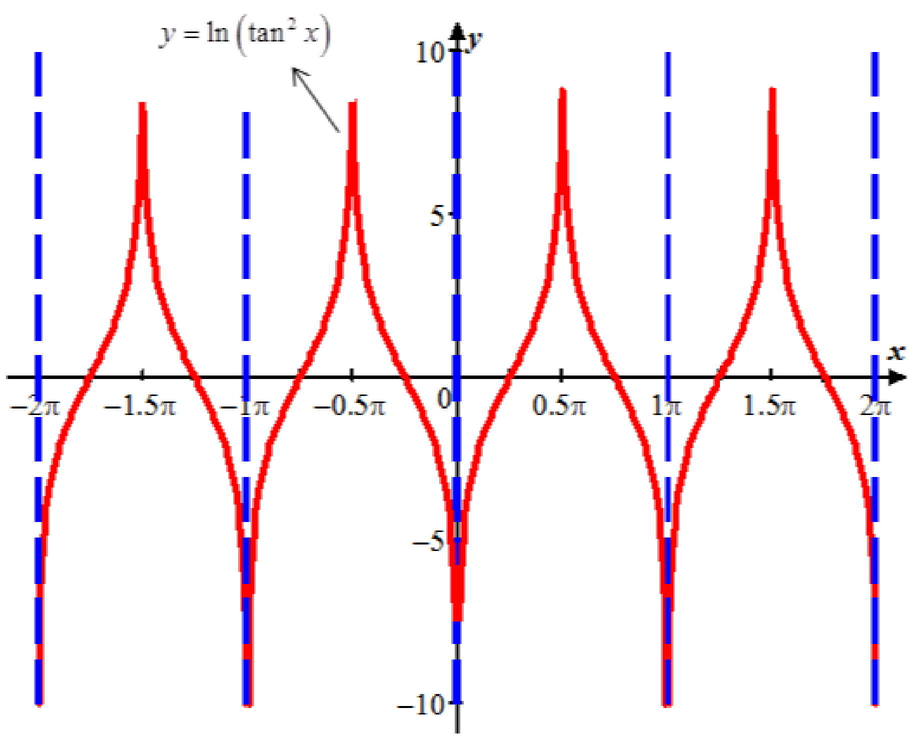
$$\frac{dy}{dx}=\frac{1}{\tan^2 x}\frac{d}{dx}\left(\tan^2 x\right)$$
$$=\frac{1}{\tan^2 x}\left(2\tan x\cdot\sec^2 x\right)$$
$$=\frac{2}{\sin x\cos x}$$

Set $\frac{dy}{dx}=0$ and solve for x ,

$$\frac{2}{\sin x\cos x}=0$$
$$2=0 \qquad \text{Which is false.}$$

Therefore, there are no critical points and the graph of the function $y=\ln\left(\tan^2 x\right)$ has no local minimum and no local maximum.

Sketch the graph of the function $y=\ln\left(\tan^2 x\right)$:



Answer 65E.

Consider the curve $y=\ln(1+x^2)$

Rewrite the curve as $f(x)=\ln(1+x^2)$

A. $y=f(x)=\ln(1+x^2)$

Domain is a set of real number R

B. For x-intercept $\ln(1+x^2)=0$

Or $1+x^2=1$ or $x=0$

And for y-intercept $y=\ln(1+0)=0$

So x and y intercept are 0, 0

C. Since $f(-x)=f(x)$ so it is an even function and symmetric about y – axis

D. $\lim_{x\to\pm\infty}\ln(1+x^2)=\infty$ so there is no asymptote

E. Since $f(x)=\ln(1+x^2)$

Then $f'(x)=\frac{2x}{(1+x^2)}$ [By chain rule]

Since $f'(x)>0$ for x > 0 so $f(x)$ is increasing on $(0,\infty)$

And $f'(x)<0$ for x < 0 so $f(x)$ is decreasing on $(-\infty,0)$

F. So $f(0)=0$ is the local and absolute minimum

G. $f''(x)=\frac{(1+x^2).2-(2x)(2x)}{(1+x^2)^2}$

$=\frac{2(1+x^2)-4x^2}{(1+x^2)^2}$

$=\frac{2(1-x^2)}{(1+x^2)^2}$

$f''(x)=0$ *when* $1-x^2=0$ or $x=\pm1$

So $f''(x)<0$ when $-\infty<x<-1$ and $1<x<\infty$

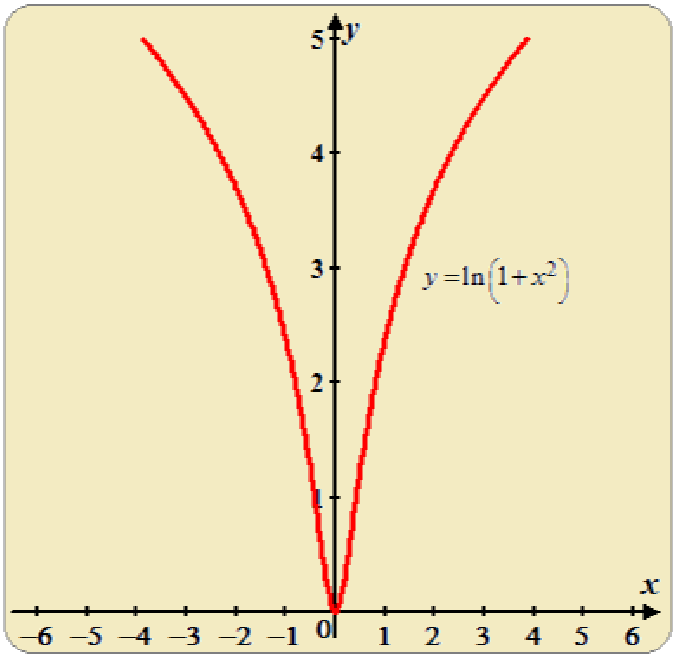
And $f''(x)>0$ when $-1<x<1$

So $f(x)$ is concave upward on (-1, 1)

And concave downward on $(-\infty,-1)$ and $(1,\infty)$

Inflection points are $(-1,\ln 2)$ and $(1,\ln 2)$

H. Using this information, we sketch the curve



Answer 66E.

Consider the curve $y = \ln(x^2 - 3x + 2)$

Rewrite the curve as $f(x) = \ln(x^2 - 3x + 2)$

A. $y = f(x) = \ln(x^2 - 3x + 2)$

Domain is $\{x \mid x^2 - 3x + 2 > 0\} \Rightarrow D\{x \mid (x-1)(x-2) > 0\}$

Since $(x-1)(x-2) > 0$ when both factors are having same sing so

$D = \{x \mid x \not\in [1,2]\}$

B. For x-intercept

$\ln(x^2 - 3x + 2) = 0$

$x^2 - 3x + 2 = e^0$

$x^2 - 3x + 2 = 1$

$x^2 - 3x + 1 = 0$

$x = \dfrac{3 \pm \sqrt{9-4}}{2}$ or $x = \dfrac{3 \pm \sqrt{5}}{2}$

So x – intercepts are $\dfrac{3-\sqrt{5}}{2}$ *and* $\dfrac{3+\sqrt{5}}{2}$

And for *y*-intercept $y = \ln(2)$

So *y*-intercept is $\ln(2)$

C. $\lim_{x \to \pm\infty} \ln(x^2 - 3x + 2) = \infty$ so there is no horizontal asymptotes

$\lim_{x \to 1^-} \ln(x^2 - 3x + 2) = -\infty$ and $\lim_{x \to 2^+} \ln(x^2 - 3x + 2) = -\infty$

so vertical asymptotes are x = 1, 2

D. $f(-x) \neq f(x) \neq -f(x)$ So this function is not symmetric

E. Since $f(x)=\ln(x^2-3x+2)$

So $f'(x)=\dfrac{2x-3}{x^2-3x+2}$

$f'(x)=0$ When $2x-3=0$ or $x=\dfrac{3}{2}$

And $f'(x)$ is not defined when $x^2-3x+2=0 \Rightarrow (x-2)(x-1)=0$

$x=2 \ or \ x=1$

But f(x) is also not defined for x=1,2 and 3/2 is also not in the domain.

$f'(x)<0$ on $(-\infty,1)$ and $f'(x)>0$ on $(2,\infty)$

So $f(x)$ is increasing on $(2,\infty)$

And $f(x)$ is decreasing on $(-\infty,1)$

F. $f(x)$ has no local maximum or minimum

G. $f'(x)=\dfrac{(2x-3)}{(x^2-3x+2)}$

Then

$$f''(x)=\frac{(x^2-3x+2)(2)-(2x-3)(2x-3)}{(x^2-3x+2)^2}$$

$$=\frac{2x^2-6x+4-4x^2-9+12x}{(x^2-3x+2)^2}$$

$$=\frac{-2x^2+6x-5}{(x^2-3x+2)^2}$$

Since $f''(x)=0$ when $-2x^2+6x-5=0$

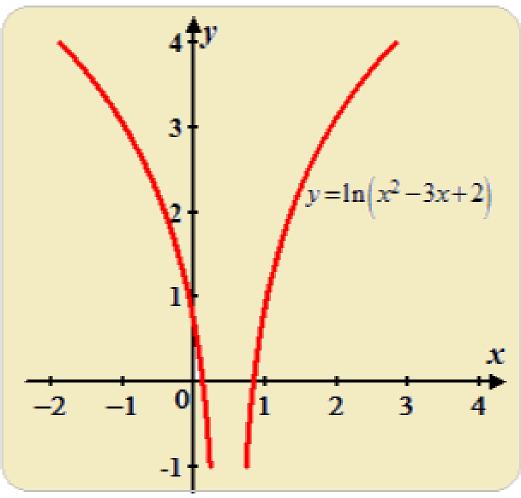
Or $2x^2-6x+5=0$ which has no real roots

$f''(x)$ is not defined when $x^2-3x+2=0 \ or \ x=1, or \ 2$

So $f''(x)<0$ for $-\infty<x<1$ and $2<x<\infty$

So $f(x)$ is concave downward on $(-\infty,1)$ and $(2,\infty)$

H. Using this information, sketch the curve



Answer 67E.

Consider the function $f(x)=\ln(2x+x\sin x)$

Find the first derivative of the function $f(x)=\ln(2x+x\sin x)$ with respect the variable x using CAS.

The maple commands are:

Fx:=ln(2*x+x*sin(x));

```
> Fx := ln(2*x + x*sin(x));
```

$\ln(2x+x\sin(x))$

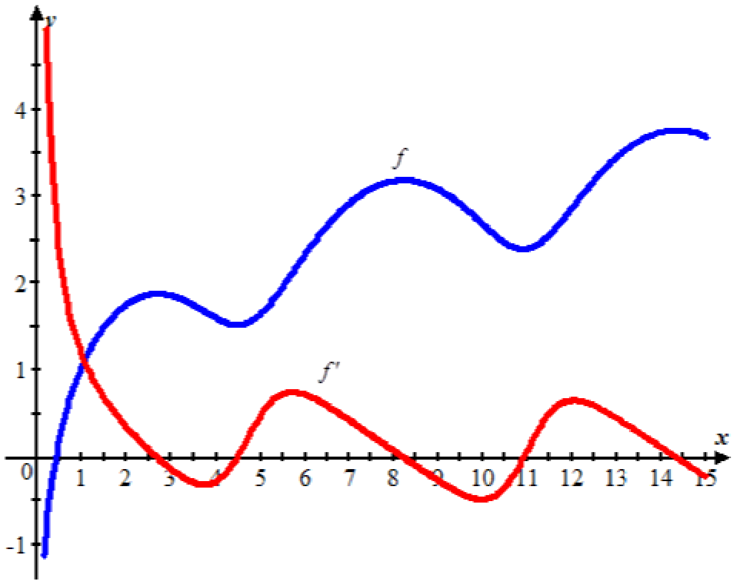
diff(Fx,x);

```
> diff(Fx, x);
```

$$\frac{2+\sin(x)+x\cos(x)}{2x+x\sin(x)}$$

Therefore, $f'(x)=\dfrac{2+\sin x+x\sin x}{2x+x\sin x}$

Sketch the graph of f and f' .



The function f is increasing on the interval, if $f'(x)>0$ on that an interval.

The function f is decreasing on the interval, if $f'(x)<0$ on that an interval.

Observe the graph of f' , it is positive on the intervals $(0,2.7),(4.5,8.2),$ and $(10.9,14.3)$

So, the function f is increasing on those intervals $(0,2.7),(4.5,8.2),$ and $(10.9,14.3)$.

Find the second derivative of the function $f(x)=\ln(2x+x\sin x)$ with respect the variable x using CAS.

The maple commands are:

Fx:=ln(2*x+x*sin(x));

```
> Fx := ln(2*x + x*sin(x));
```

$\ln(2x+x\sin(x))$

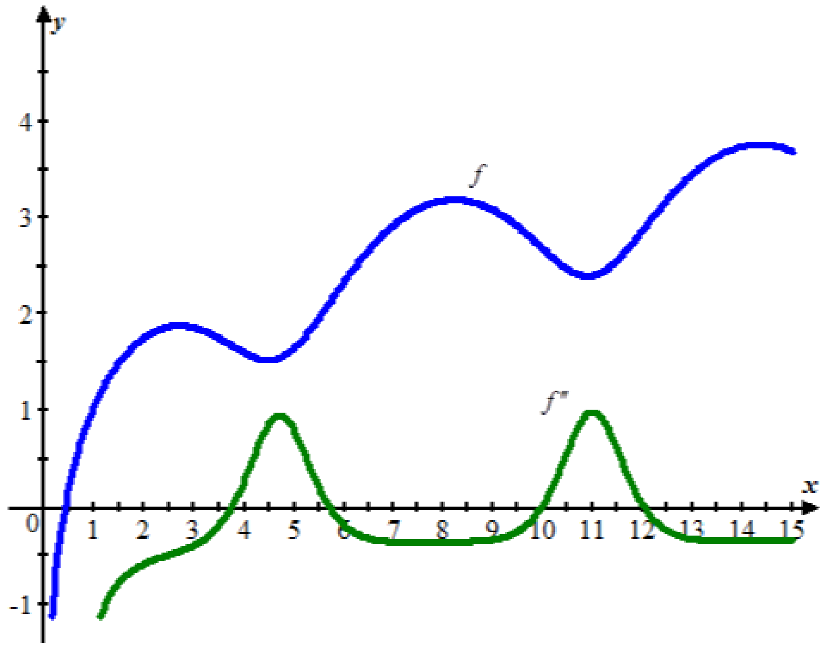
diff(Fx,x,x);

```
> diff(Fx, x, x);
```

$$\frac{2\cos(x)-x\sin(x)}{2x+x\sin(x)}-\frac{(2+\sin(x)+x\cos(x))^2}{(2x+x\sin(x))^2}$$

Therefore, $f''(x)=\dfrac{2\cos x-x\sin x}{2x+x\sin x}-\dfrac{(2+\sin x+x\cos x)^2}{(2x+x\sin x)^2}$

Sketch the graph of f and f'' together.



The graph of f'' has zeros approximately at $x = 3.8, x = 5.7, x = 10,$ and $x = 12.$

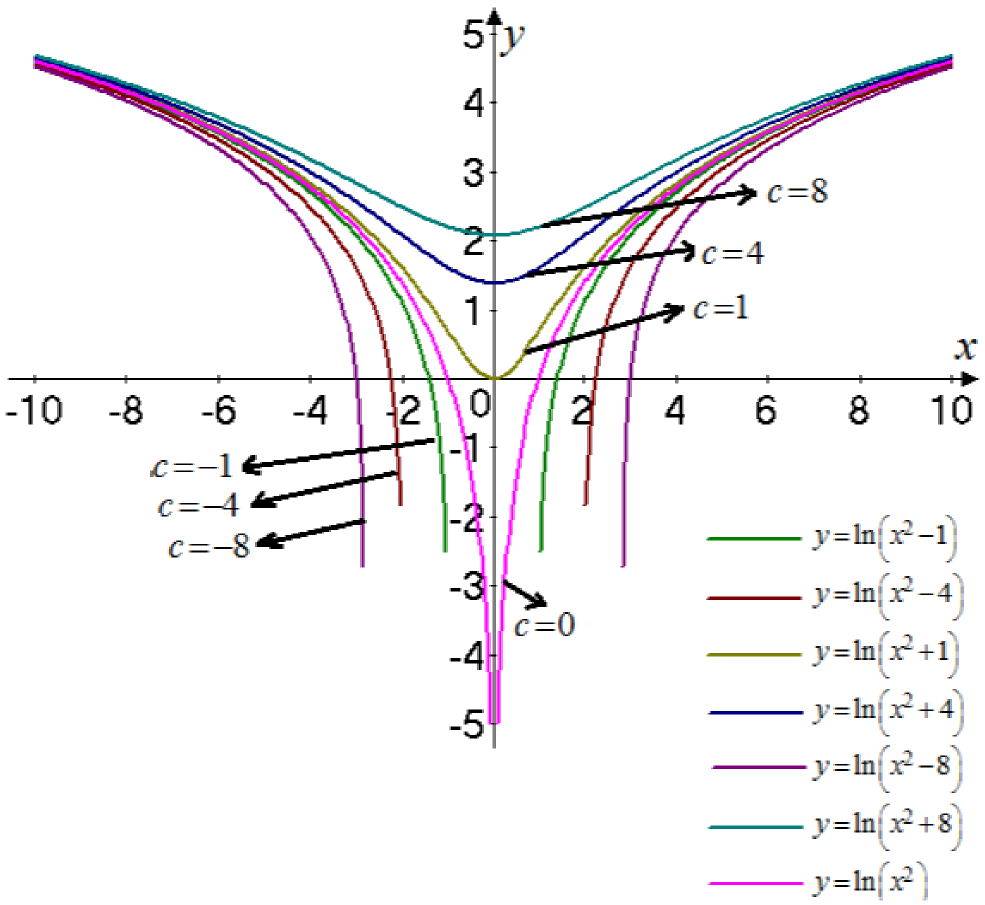
Looking back to the graph of f at these points $x = 3.8, x = 5.7, x = 10,$ and $x = 12$ to get y-coordinates.

Therefore, the Inflection points are $\boxed{(3.8,1.7),(5.7,2.1),(10,2.7), \text{ and } (12,2.9)}$.

Answer 68E.

Need to graph the members of the family of functions $f(x) = \ln(x^2 + c)$ for several values of $c.$

The graphs of the members of the family of functions are shown below:



From the graphs observe that for $c>0,$ the function $f(x)$ is defined everywhere.

As c increases, the dip at $x=0$ becomes deeper.

For $c \le 0,$

$$\lim_{x \to \pm\sqrt{c}^-} \ln\left(x^2+c\right) = -\infty.$$

So the graph has asymptotes at $x=\pm\sqrt{c}.$

Consider the data describe the charge Q remaining on the capacitor (measured in microcoulombs, μQ) at time t (measured in seconds)

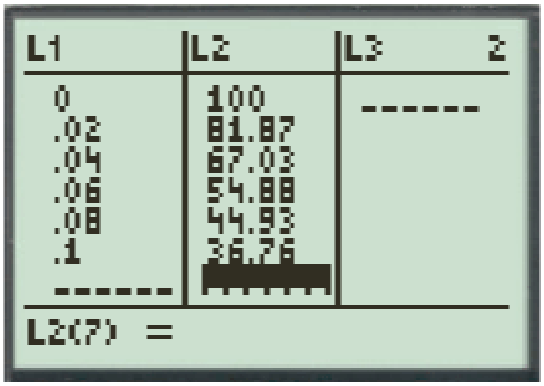
t	0.00	0.02	0.04	0.06	0.08	0.10
Q	100.00	81.87	67.03	54.88	44.93	36.76

(a)

By using TI-83 calculator

Press STAT to view the statistics editing options.

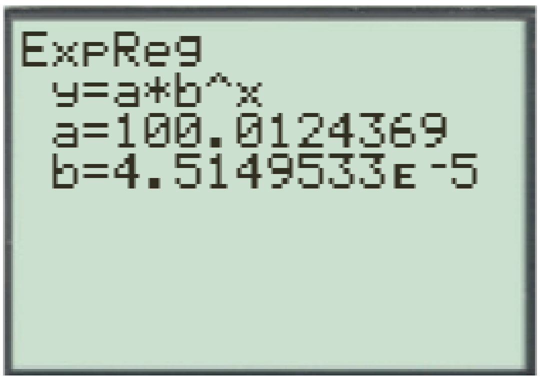
Press the number associated with an option or use the cursor keys to select an option and press enter and press 1 to select edit which displays the lists of data.



Press STAT ▹ to view the statistics calculation options.

Press the number associated with an option or use the cursor keys to select an option and press.

Press 0 to select exponential regression which finds the best-fit exponential equation of the form $y=ab^x$

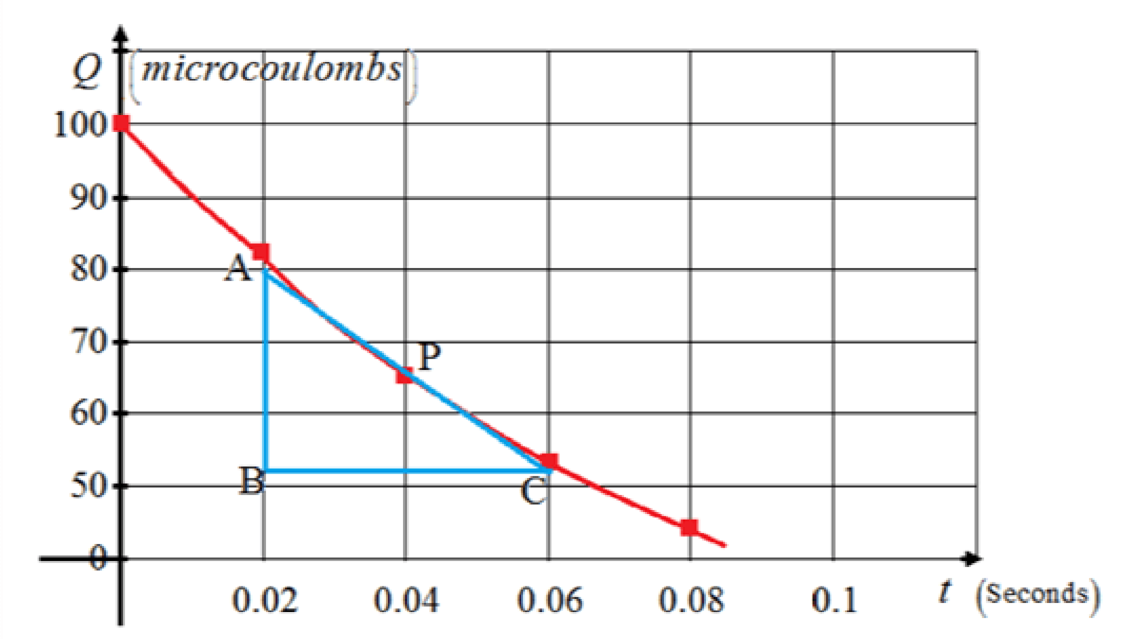


So, the obtain the model $Q=ab^t$ with $\boxed{a \approx 100.0124369}$ and $\boxed{b \approx 0.000045145933}$

(b)

The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The data in the table describe the charge Q remaining on the capacitor at time t. Use the data to draw the graph of this function and estimate the slope of the tangent line at the point where $t = 0.04$.

So, the graph of the function.



The points $P(0.04, 67.03)$ and $R(0.00, 100.00)$ on the graph, to find that the slope of the secant line PR is

$$m_{PR} = \frac{100.00 - 67.03}{0.00 - 0.04}$$
$$= \frac{32.97}{0.04}$$
$$= -824.25$$

The table at the left shows the result of similar calculations for the slopes of other secant lines. From this table the slope of the tangent line at $t = 0.04$ to lie somewhere between -742 and -607.5. In fact, the average of the slope of the two closest secant lines is

$$\frac{1}{2}(-742 - 607.5) = \frac{1}{2}(-1349.5)$$
$$= -674.75$$

So, by this method to estimate the slope of the tangent line to be -675

Another method is to draw an approximation to the tangent line at P and measure the sides of the triangle ABC in the above diagram. So that an estimate of the slope of the tangent line as

$$-\frac{|AB|}{|BC|} \approx -\frac{80.4 - 53.6}{0.06 - 0.02}$$
$$= \boxed{-670}$$

Answer 70E.

Consider the data,

Year	Population
1790	3,929,000
1800	5,308,000
1810	7,240,000
1820	9,639,000
1830	12,861,000
1840	17,063,000
1850	23,192,000
1860	31,443,000

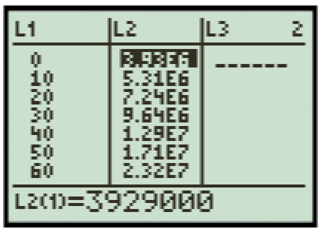
(a)

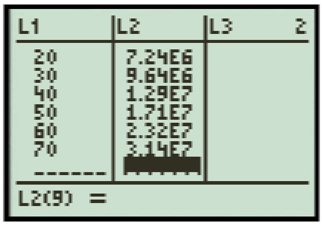
Find an exponential model for the above data:

Let t represents the year, with $t=0$ corresponds to 1790.

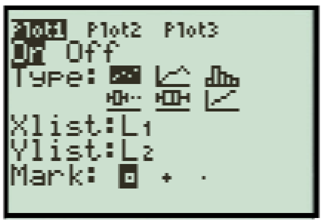
Step1: Press STAT 1 for the EDIT screen; enter values of years into L_1 and sales into L_2.

The display as shown below:

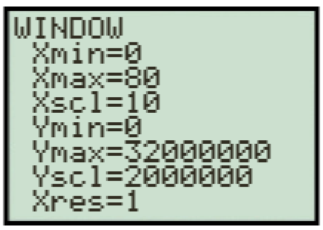




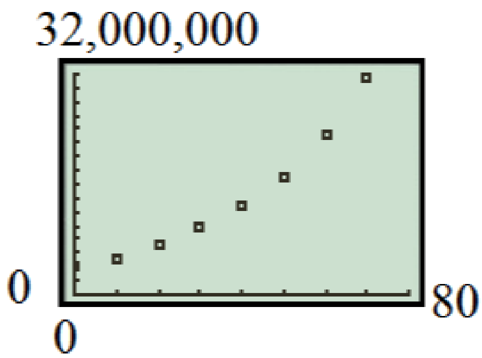
Step2: Press 2nd Y= and select Plot 1. Then set as follows



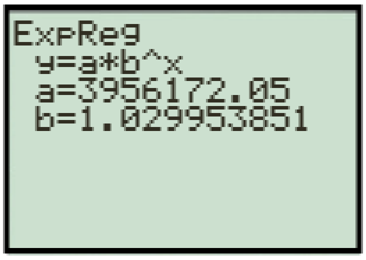
Step3: Press WINDOW key then set as follows



Step4: To view the scatter plot of the data, Press GRAPH key.



Step5: Press 2nd MODE then press STAT chooses CALC sub menu and press 0 then Press 2nd L_1 , 2nd L_2 and press ENTER key.



Therefore, the Exponential Model: $P = \boxed{3956172.05(1.030)^t}$

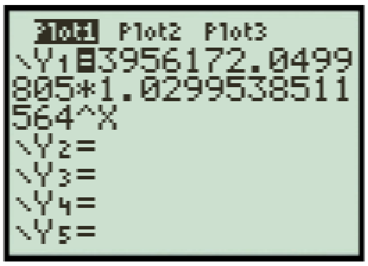
Now, graph this regression equation with the data points.

To paste the regression equation into the function editor:

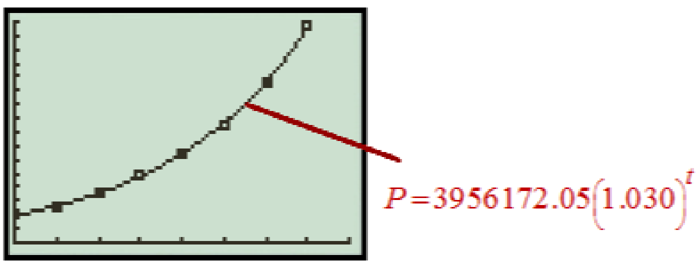
Press Y= and move the cursor to Y1, then press **[VARS] [5] [<]**

[<] EQ, select RegEq **[ENTER]**.

The regression equation should be in the function editor, **[Y=]**, as shown below.



Press GRAPH key and the regression line will be shown with the data points



In exponential model, the data is either increasing or decreasing.

From the scatterplot, it appears that an exponential model is a good fit.

(b)

One way to estimate the rate of change of population growth in 1800 is to take the average of the slopes of the secant lines before and after $t = 1800$.

The slope of the secant line from $1790 \text{ and } 1800$ is,

$$
\begin{aligned}
&= \frac{\text{change in population}}{\text{change in time}} \\
&= \frac{5,308,000 - 3,929,000}{1800 - 1790} \\
&= \frac{1,379,000}{10} \\
&= 137,900
\end{aligned}
$$

The slope of the secant line from $1800 \text{ and } 1810$ is,

$$
\begin{aligned}
&= \frac{\text{change in population}}{\text{change in time}} \\
&= \frac{7,240,000 - 5,308,000}{1810 - 1800} \\
&= \frac{1,932,000}{10} \\
&= 193,200
\end{aligned}
$$

So, the rate of population growth in 1800 would be exactly half way between 137,900 and 193,200.

$$
\begin{aligned}
\frac{137,900 + 193,200}{2} &= \frac{331100}{2} \\
&= 165,550
\end{aligned}
$$

Therefore, the rate of population growth in 1800 is $\boxed{165,550}$

One way to estimate the rate of change of population growth in 1850 is to take the average of the slopes of the secant lines before and after $t = 1850$.

The slope of the secant line from $1840 \text{ and } 1850$ is,

$$
\begin{aligned}
&= \frac{\text{change in population}}{\text{change in time}} \\
&= \frac{23,192,000 - 17,063,000}{1850 - 1840} \\
&= \frac{6,129,000}{10} \\
&= 612,900
\end{aligned}
$$

The slope of the secant line from $1850 \text{ and } 1860$ is,

$$
\begin{aligned}
&= \frac{\text{change in population}}{\text{change in time}} \\
&= \frac{31,443,000 - 23,192,000}{1860 - 1850} \\
&= \frac{8,251,000}{10} \\
&= 825,100
\end{aligned}
$$

So, the rate of population growth in 1850 would be exactly half way between 612,900 and 825,100.

$$
\begin{aligned}
\frac{612,900 + 825,100}{2} &= \frac{1,438,000}{2} \\
&= 719,000
\end{aligned}
$$

Therefore, the rate of population growth in 1850 is $\boxed{719,000}$

(c)

From part (a), the exponential model is $P = 3956172.05(1.030)^t$.

Use this model and then estimate the population in 1800 and 1850.

Since $t = 0$ corresponds to the year 1790, so $t = 10$ corresponds to the year 1800

The rate of growth in population in 1800 is,

$$P = 3956172.05(1.030)^{10}$$
$$\approx \boxed{5,316,764}$$

The answer is 32 times larger than the answer in part (b).

Since $t = 0$ corresponds to the year 1790, so $t = 60$ corresponds to the year 1850

The rate of growth in population in 1850 is,

$$P = 3956172.05(1.030)^{50}$$
$$\approx \boxed{17,343,486}$$

The answer is 24 times larger than the answer in part (b).

(d)

From part (a), the exponential model is $P = 3956172.05(1.030)^t$.

Use this model and then estimate the population in 1870.

Since $t = 0$ corresponds to the year 1790, so $t = 70$ corresponds to the year 1870

The rate of growth in population in 1870 is,

$$P = 3956172.05(1.030)^{70}$$
$$\approx \boxed{31,324,266}$$

But the actual population in 1870 is 38,558,000.

There is a large difference between the population estimated using the exponential model in 1870 and the actual population.

Answer 71E.

We have
$$\int_2^4 \frac{3}{x}dx = 3\int_2^4 \frac{1}{x}dx$$
$$= 3\left[\ln|x|\right]_2^4$$
$$= 3\left[\ln(4) - \ln(2)\right]$$
$$= 3\left[\ln\left(\frac{4}{2}\right)\right]$$
$$= \boxed{3\ln(2)}.$$

Answer 72E.

Consider $\int_0^3 \frac{dx}{5x+1}$

$$= \left[\frac{\ln(5x+1)}{5}\right]_0^3$$
$$= \frac{1}{5}(\ln 16 - \ln 1)$$
$$= \frac{1}{5}(\ln 2^4 - 0)$$
$$= \frac{1}{5}\cdot 4\ln 2$$
$$= \frac{4}{5}\ln 2$$

Therefore $\boxed{\int_0^3 \frac{dx}{5x+1} = \frac{4}{5}\ln 2}$

Answer 73E.

We have to evaluate $\int_1^2 \frac{dt}{8-3t}$

Let $8-3t=x$ then $-3dt=dx \Leftrightarrow -\frac{dx}{3}=dt$

When $t=1 \Rightarrow x=5$ and when $t=2 \Rightarrow x=2$

Therefore

$$\int_1^2 \frac{dt}{8-3t}=\int_5^2 \frac{-1}{3x}dx$$

$$=\frac{1}{3}\int_2^5 \frac{1}{x}dx \qquad \left(\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx\right)$$

$$=\frac{1}{3}\left[\ln|x|\right]_2^5$$

$$=\frac{1}{3}\left[\ln 5-\ln 2\right]$$

$$=\frac{1}{3}\ln\left(\frac{5}{2}\right) \qquad \left(\ln\left(\frac{m}{n}\right)=\ln(m)-\ln(n)\right)$$

Therefore $\int_1^2 \frac{dt}{8-3t}=\boxed{\frac{1}{3}\ln\left(\frac{5}{2}\right)}$

Answer 74E.

We have to evaluate the integral $\int_4^9 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$

We have $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2=\left(\sqrt{x}\right)^2+\left(\frac{1}{\sqrt{x}}\right)^2+2.\sqrt{x}.\frac{1}{\sqrt{x}}$

$$=x+\frac{1}{x}+2$$

Therefore $\int_4^9 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx=\int_4^9\left(x+\frac{1}{x}+2\right)dx$

$$=\int_4^9 x\,dx+\int_4^9\frac{1}{x}dx+2\int_4^9 dx$$

$$=\left[\frac{x^2}{2}\right]_4^9+\left[\ln x\right]_4^9+2\left[x\right]_4^9$$

$$=\frac{1}{2}\left[9^2-4^2\right]+\left[\ln 9-\ln 4\right]+2\left[9-4\right]$$

$$=\frac{1}{2}(65)+\ln\frac{9}{4}+10$$

$$=\frac{85}{2}+\ln\frac{9}{4}$$

Hence

$$\boxed{\int_4^9 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx=\frac{85}{2}+\ln\frac{9}{4}}$$

Answer 75E.

We have to evaluate the integral $\int_1^e \left(\frac{x^2+x+1}{x}\right)dx$

We have $\frac{x^2+x+1}{x}=\frac{x^2}{x}+\frac{x}{x}+\frac{1}{x}$

$$=x+1+\frac{1}{x}$$

Therefore $\int_1^e \left(\frac{x^2+x+1}{x}\right)dx = \int_1^e \left(x+1+\frac{1}{x}\right)dx$

$$= \int_1^e x\,dx + \int_1^e dx + \int_1^e \frac{1}{x}dx$$

$$= \left[\frac{x^2}{2}\right]_1^e + [x]_1^e + [\ln x]_1^e$$

$$= \frac{1}{2}\left(e^2-1^2\right)+(e-1)+(\ln e-\ln 1)$$

$$= \frac{1}{2}\left(e^2-1\right)+e-1+1-0 \quad \text{Since } \begin{matrix}\ln e = 1\\ \ln 1 = 0\end{matrix}$$

$$= \frac{1}{2}\left(e^2-1\right)+e$$

$$= \frac{e^2-1+2e}{2}$$

Hence

$$\boxed{\int_1^e \left(\frac{x^2+x+1}{x}\right)dx = \frac{e^2-1+2e}{2}}$$

Answer 76E.

Consider the following integral:

$$\int \frac{\sin(\ln x)}{x}dx$$

Since $\frac{1}{x}dx = du$, substitute $\ln x = u$ as shown below:

$$\int \frac{\sin(\ln x)}{x}dx = \int \sin(u)du$$

$$= -\cos(u)+C$$

$$= \boxed{-\cos(\ln x)+C}$$

Answer 77E.

We have to evaluate the integral $\int \frac{(\ln x)^2}{x}dx$

For this let $\ln x = t$

Differentiating both sides, we get,

$$\frac{1}{x}dx = dt$$

Now $\int \frac{(\ln x)^2}{x}dx = \int t^2 dt$

$$= \frac{t^3}{3}+c \qquad \text{Where c is a constant.}$$

$$= \frac{(\ln x)^3}{3}+c \qquad \text{Since } t = \ln x$$

Hence

$$\boxed{\int \frac{(\ln x)^2}{x}dx = \frac{(\ln x)^3}{3}+c}$$

Answer 78E.

We have to evaluate $\int \frac{\cos x}{2+\sin x}dx$

Let $2+\sin x = t$

Then $\cos x dx = dt$

Therefore $\int \frac{\cos x}{2+\sin x}dx = \int \frac{dt}{t}$

$$= \ln|t| + C$$

$$= \boxed{\ln|2+\sin x| + C}$$

Answer 79E.

There are various techniques to evaluate the integral.

One of the techniques is the method of substitution.

In the method of substitution, an expression is substituted for one variable in order to simplify the integrand so that it is brought into a form that is easy to evaluate.

Consider the integral:

$$\int \frac{\sin 2x}{1+\cos^2 x}dx$$

Make the substitution as shown below:

$$\begin{aligned} \cos^2 x &= u \\ 2\cos x(-\sin x)dx &= du \\ -2\sin x \cos x dx &= du \\ \sin 2x dx &= -du \end{aligned}$$

Evaluate the integral as shown below:

$$\begin{aligned} \int \frac{\sin 2x}{1+\cos^2 x}dx &= -\int \frac{1}{1+u}du \\ &= -\ln|1+u| + C \\ &= -\ln|1+\cos^2 x| + C \\ &= -\ln(1+\cos^2 x) + C \end{aligned}$$

Hence, the final value of the integral is $\boxed{-\ln(1+\cos^2 x)+C}$.

Answer 80E.

We have to evaluate $\int \frac{e^x}{e^x+1}dx$

Let $e^x + 1 = t$

Then $e^x dx = dt$

Therefore

$$\int \frac{e^x}{e^x+1}dx = \int \frac{dt}{t}$$

$$= \ln|t| + C$$

$$= \boxed{\log(e^x+1)+C} \qquad \text{Since } e^x + 1 > 0 \text{ for all } x.$$

Answer 81E.

We have to evaluate the integral $\int_1^2 10^t\,dt$

Now $\int_1^2 10^t\,dt = \left[\frac{10^t}{\ln 10}\right]_1^2$ Since $\int a^x dx = \frac{a^x}{\ln a}$

$$= \frac{1}{\ln 10}\left[10^t\right]_1^2$$
$$= \frac{1}{\ln 10}\left[10^2 - 10^1\right]$$
$$= \frac{1}{\ln 10}[100-10]$$
$$= \frac{90}{\ln 10}$$

Hence

$$\boxed{\int_1^2 10^t\,dt = \frac{90}{\ln 10}}$$

Answer 82E.

We have to evaluate the integral $\int x 2^{x^2} dx$

For this let $x^2 = t$

On differentiating both sides, we get,

$$2xd = dt$$

$$\Rightarrow \quad x\,dx = \frac{dt}{2}$$

Now, $\int x 2^{x^2} dx = \int 2^t \frac{dt}{2}$

$$= \frac{1}{2}\int 2^t dt$$
$$= \frac{1}{2}\frac{2^t}{\ln 2} + c \quad \text{Where c is a constant.}$$
$$= \frac{2^{x^2}}{2\ln 2} + c \quad \text{Since } t = x^2$$

Hence

$$\boxed{\int x 2^{x^2} dx = \frac{2^{x^2}}{2\ln 2} + c}$$

Answer 83E.

(A) We know that if $\frac{d}{dx}\left[f(x)+c\right] = g(x)$

Then $\int g(x)dx = f(x)+c$

Now $\frac{d}{dx}\left[\ln|\sin x| + c\right] = \frac{d}{dx}\ln|\sin x| + \frac{d}{dx}c$

$$= \frac{1}{\sin x}\frac{d}{dx}\sin x + 0$$
$$= \frac{1}{\sin x}\cos x$$
$$= \cot x$$

Therefore $\int \cot x\,dx = \ln|\sin x| + c$

(B) Let us evaluate $\int \cot x dx$ by method of substitution.

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}dx$$

Let $\sin x = t$

On differentiating we get

$$\cos x\,dx = dt$$

Therefore

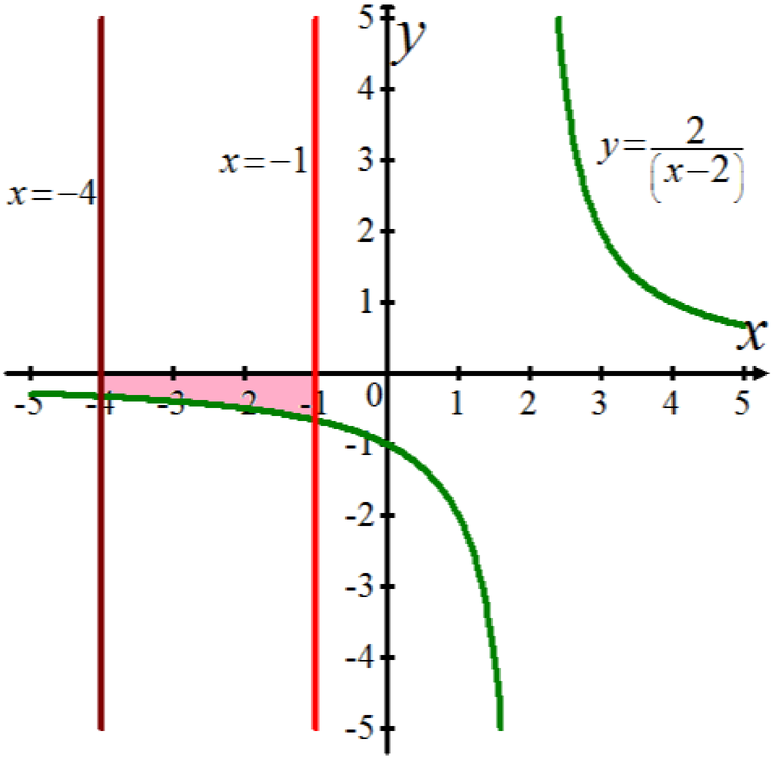
$$\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x}$$
$$= \int \frac{dt}{t}$$
$$= \ln|t| + c \quad \text{Where c is a constant.}$$
$$= \ln|\sin x| + c \qquad \text{Since} \quad t = \sin x.$$

Hence

$$\boxed{\int \cot x\,dx = \ln|\sin x| + c}$$

Answer 84E.

Consider the curve $y = \frac{2}{(x-2)}$ and the lines $x = -4$ and $x = -1$



So, the area of the region the curve $y = \frac{2}{(x-2)}$ and the lines $x = -4$ and $x = -1$

$$A = \int_{-4}^{-1}\left(0 - \frac{2}{(x-2)}\right)dx$$
$$= \int_{-4}^{-1}\left(-\frac{2}{(x-2)}\right)dx$$
$$= -2\left[\ln|x-2|\right]_{-4}^{-1}$$
$$= -2\left[\ln|-1-2| - \ln|-4-2|\right]$$
$$= -2\left[\ln|-3| - \ln|-6|\right]$$
$$= -2\left[\ln(3) - \ln(6)\right]$$
$$= -2\left[\ln\left(\frac{3}{6}\right)\right]$$
$$= -2\left[\ln\left(\frac{1}{2}\right)\right]$$
$$= -2(-0.693147)$$
$$= \boxed{1.386294}$$

Answer 85E.

First we sketch the region under the curve $y = 1/\sqrt{x+1}$ from 0 to 1.

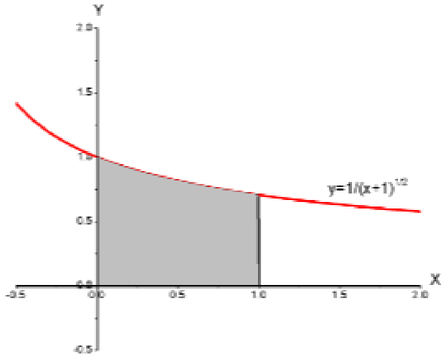


Fig.1

Now we consider a vertical strip with thickness Δx in this region. If we rotate this region about x – axis, then we get a typical disk of radius $y = 1/\sqrt{x+1}$

Then the area of the cross section is $A(x) = \pi\left(1/\sqrt{x+1}\right)^2 = \pi/(x+1)$

And so the volume of approximating disk is $A(x)\,\Delta x = \dfrac{\pi}{(x+1)}\Delta x$

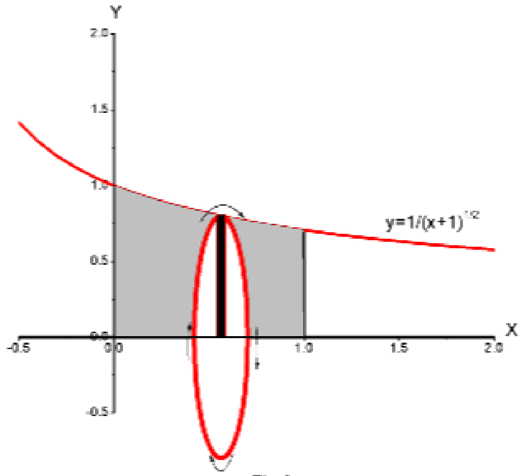


Fig.2

The solid lies between x = 0 and x = 1.
Then the volume of the resulting solid obtained by rotating the region about x – axis is

$$V = \int_0^1 A(x)\,dx$$

$$= \int_0^1 \frac{\pi}{(x+1)}dx$$

$$= \pi\int_0^1 \frac{1}{(x+1)}dx$$

Let $x+1 = u \Leftrightarrow dx = du$
When x = 0, u = 1 and when x = 1, u = 2.
Therefore,

$$V = \pi\int_1^2 \frac{1}{u}du$$

$$= \pi\left[\ln|u|\right]_1^2$$

$$= \pi[\ln 2 - \ln 1]$$

Or,

$\boxed{V = \pi \ln 2}$ [Since ln 1 = 0]

Answer 86E.

First we sketch the region under the curve $y=\dfrac{1}{x^2+1}$ from 0 to 3.

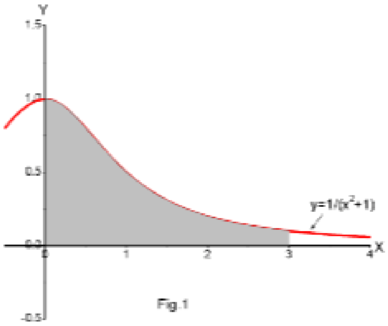


Fig.1

Now we consider vertical strip with thickness Δx at a distance x from the origin in this region. If we rotate this region about y- axis, then we get a cylindrical shell of radius x.

The height of the cylindrical shell is $y=\dfrac{1}{x^2+1}$

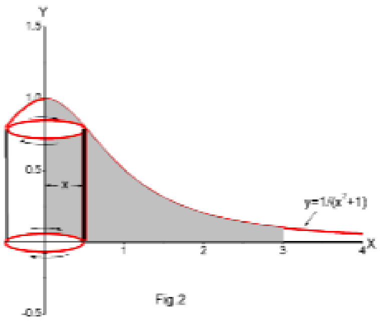


Fig.2

Then the volume of the resulting solid obtained by rotating the region about y-axis is

$$V=\int_0^3(\text{circumference of the shell})(\text{height of the shell})dx$$

$$=\int_0^3(2\pi x)\left(\frac{1}{x^2+1}\right)dx$$

$$=\pi\int_0^3\frac{2x}{\left(x^2+1\right)}dx$$

Let $x^2+1=u \Leftrightarrow 2xdx=du$

When $x=0, u=1$ and when $x=3, u=10$

Therefore

$$V=\pi\int_1^{10}\frac{1}{u}du$$

$$=\pi\left[\ln|u|\right]_1^{10}$$

$$=\pi[\ln 10-\ln 1]$$

Or $\boxed{V=\pi\ln 10}$ [Since $\ln 1=0$]

Answer 87E.

The work W done by the engine when the volume expands from volume V_1 to V_2 is described by the following formula.

$$W=\int_{V_1}^{V_2}P\,dV$$

Here, the pressure P is a function of the volume $P=P(V)$.

First use the equation $PV = C$, where P is pressure and V is volume, to calculate the value of the constant C. The problem states that when the steam starts, it has a pressure of 150 kPa and a volume of 600 cm^3. Use this to calculate C.

$$(150 \text{ kPa})(600 \text{ cm}^3) = C$$

$$90000 \text{ J-cm}^3 \approx C \qquad [\text{Because } 1 \text{ KPa} = 1 \text{J}]$$

Now, write the pressure P as a function of volume.

$$PV \approx 90000$$

$$P \approx \frac{90000}{V}$$

Finally, use the formula for work done to calculate the work done as the steam goes from a volume of 600 cm^3 to a volume of 1000cm^3.

$$\begin{aligned} W &\approx \int_{600}^{1000} \frac{90000}{V} dV \\ &\approx 90000[\ln V]_{600}^{1000} \\ &\approx 90000[\ln 1000 - \ln 600] \\ &\approx 45975 \text{J} \end{aligned}$$

Therefore, about $\boxed{45975 \text{J}}$ of work is done during the cycle.

Answer 88E.

We have to find f.

Given that $f''(x) = x^{-2}$, x>0 , f(1) =0 and f(2) =0

$$\Rightarrow \quad \frac{d}{dx} f'(x) = x^{-2}$$

$$\Rightarrow \quad d[f'(x)] = x^{-2} dx$$

Integrating both sides we get,

$$\int d[f'(x)] = \int x^{-2} dx$$

$$\Rightarrow \quad f'(x) = \frac{x^{-2+1}}{-2+1} + c \qquad [\text{c is a constant}]$$

$$\Rightarrow \quad f'(x) = \frac{-1}{x} + c$$

Also from $f'(x) = \frac{-1}{x} + c$

$$\Rightarrow \quad \frac{d}{dx} f(x) = -\frac{1}{x} + c$$

$$\Rightarrow \quad d[f(x)] = \left(-\frac{1}{x} + c\right) dx$$

Integrating both sides, we get,

$$\int d[f(x)] = \int \left[-\frac{1}{x} + c\right] dx$$

$$\Rightarrow \quad f(x) = -\ln|x| + cx + c_1 \qquad \text{Where } c_1 \text{ is a constant}$$

Given that when $x=1$, $f(1)=0$

Therefore from $f(x)=-\ln x+cx+c_1$ where $x>0$

We have $f(1)=-\ln 1+c.(1)+c_1$

$\Rightarrow \quad 0=0+c+c_1 \qquad [\ln 1=0]$

$\Rightarrow \quad c+c_1=0 \qquad --(i)$

Also when $x=2$, $f(2)=0$

Therefore from $f(x)=-\ln x+cx+c_1$

We have $f(2)=-\ln 2+c.(2)+c_1$

$0=-\ln 2+2c+c_1$

$\Rightarrow \quad 0=-\ln 2+c+(c+c_1)$

$\Rightarrow \quad 0=-\ln 2+c+0 \qquad$ Since $c+c_1=0$ from equation (i)

$\Rightarrow \quad c=\ln 2$

From (i) $c_1=-c$

$\Rightarrow \quad c_1=-\ln 2$

Substituting values of c and c_1 in $f(x)$ we get

$$f(x)=-\ln x+cx+c_1$$
$$=-\ln x+(\ln 2)x-\ln 2$$
$$=(x-1)\ln 2-\ln x$$

Hence

$$\boxed{f(x)=(x-1)\ln 2-\ln x}$$

Answer 89E.

The given function is $f(x)=2x+\ln x$.

Also g is the inverse function of $f(x)=2x+\ln x$.

So $g=f^{-1}$

We have $f(x)=2x+\ln x$

Differentiating with respect to x, we get

$$f'(x)=\frac{d}{dx}f(x)=\frac{d}{dx}(2x+\ln x)$$
$$=\frac{d}{dx}2x+\frac{d}{dx}\ln x$$
$$=2+\frac{1}{x}.$$

$f(x)$ is defined if $x>0$

Here $f'(x)$ is always positive as $x>0$. Therefore $f(x)$ is increasing function.

So $f(x)$ will be one- to – one.

Now, we need to find $g'(2)$ i.e. $(f^{-1})'2$

For this we have

$$f(x)=2x+\ln x$$

Putting $x=1$ we get,

$$f(1)=2\times 1+\ln 1$$

$\Rightarrow \quad f(1)=2+0$

$\Rightarrow \quad f(1)=2 \qquad \Rightarrow \qquad f^{-1}(2)=1 \qquad$ By definition of inverse function.

We know that if f is one – to – one differentiable function with inverse function f^{-1} and $f'\left[f^{-1}(a)\right]\neq 0$ then the inverse function is differentiable at a and

$$\left(f^{-1}\right)'(a)=\frac{1}{f'\left[f^{-1}(a)\right]}$$

Now, we have $f^{-1}(2)=1$

Also $f(x)=2x+\ln x$

And $f'(x)=2+\frac{1}{x}$

So $f'\left[f^{-1}(2)\right]=f'(1)=2+\frac{1}{1}=3$

Therefore $g'(2)=\left(f^{-1}\right)'(2)$

$$=\frac{1}{f'\left[f^{-1}(2)\right]}$$

$$=\frac{1}{f'(1)}$$

$$=\frac{1}{3}$$

Hence

$$\boxed{g'(2)=\frac{1}{3}}$$

Answer 90E.

Given $f(x)=e^x+\ln x$

Differentiating with respect to x, we get,

$$f'(x)=\frac{d}{dx}f(x)=\frac{d}{dx}\left(e^x+\ln x\right)$$

$$=\frac{d}{dx}e^x+\frac{d}{dx}\ln x$$

$$=e^x+\frac{1}{x}$$

Since e^x is always positive and $\frac{1}{x}$ is also always positive for $x>0$.

[Note: for $x\leq 0$ $f(x)$ is not defined]

Therefore $f'(x)>0$

$\Rightarrow$ $f(x)$ is increasing function.

So $f(x)$ will be one – to – one.

Now, we need to find $h'(e)$ i.e. $\left(f^{-1}\right)'(e)$

For this we have

$$f(x)=e^x+\ln x$$

Therefore $f(1)=e^1+\ln 1$

$$=e+0$$

$$=e$$

$\Rightarrow$ $f^{-1}(e)=1$ By definition of inverse function.

We know that if f is one – to – one differentiable function with inverse function f^{-1} and $f'\left[f^{-1}(a)\right]\neq 0$, then the inverse function is differentiable at a and

$$\left(f^{-1}\right)'(a)=\frac{1}{f'\left[f^{-1}(a)\right]}$$

Now, we have $f^{-1}(e)=1$

Also $f(x)=e^x+\ln x$

And $f'(x)=e^x+\dfrac{1}{x}$

So $f'\left[f^{-1}(e)\right]=f'(1)=e^1+\dfrac{1}{1}$

$$=e+1$$

Therefore

$$\left(f^{-1}\right)'(e)=\frac{1}{f'\left[f^{-1}(e)\right]}$$
$$=\frac{1}{f'(1)}$$
$$=\frac{1}{e+1}$$

Hence

$$\boxed{h'(e)=\left(f^{-1}\right)'(e)=\frac{1}{e+1}}$$

Answer 91E.

Consider the curve $y=\dfrac{x}{\left(x^2+1\right)}$ and the line $y=mx.$

$$f(x)=\frac{x}{\left(x^2+1\right)}$$

Differentiate with respect to x.

$$f'(x)=\frac{\left(x^2+1\right)(1)-x(2x)}{\left(x^2+1\right)}$$
$$=\frac{x^2+1-2x^2}{\left(x^2+1\right)}$$
$$=\frac{1-x^2}{\left(x^2+1\right)}$$

Hence, the slope of the curve $y=\dfrac{x}{\left(x^2+1\right)}$ is $f'(x)=\dfrac{\left(1-x^2\right)}{\left(x^2+1\right)}$.

If $x=0$ then $f'(0)=1$. The slope of the line $y=mx$ is , $0<m<f'(0)$ that implies $0<m<1.$

Hence, proceed as follows:

$$mx = \frac{x}{(x^2+1)}$$
$$mx(x^2+1) = x$$
$$x^2+1 = \frac{1}{m}$$
$$x^2+1-\frac{1}{m} = 0$$
$$x^2 = \frac{1}{m}-1$$
$$x = \pm\sqrt{\frac{1}{m}-1}$$

$x = \sqrt{\frac{1}{m}-1}$ and $x = -\sqrt{\frac{1}{m}-1}$

The interval, $-\sqrt{\frac{1}{m}-1} < x < 0$ and $0 < x < \sqrt{\frac{1}{m}-1}$

Hence, the area of the region is as calculated as follows:

$$A = \int_{-\sqrt{\frac{1}{m}-1}}^{0}\left(\frac{x}{(x^2+1)}-mx\right)dx + \int_{0}^{\sqrt{\frac{1}{m}-1}}\left(\frac{x}{(x^2+1)}-mx\right)dx$$

This suggests that one should substitute, $x^2+1=u$ since $2xdx = du.$

The limits are as calculated as follows:

As $x \to -\sqrt{\frac{1}{m}-1}$ then $u \to \frac{1}{m}$

As $x \to 0$ then $u \to 1$

$$A = \int_{-\sqrt{\frac{1}{m}-1}}^{0}\left(\frac{x}{(x^2+1)}-mx\right)dx + \int_{0}^{\sqrt{\frac{1}{m}-1}}\left(\frac{x}{(x^2+1)}-mx\right)dx$$
$$= \boxed{0}$$

Or

$$A = \int_{-\sqrt{\frac{1}{m}-1}}^{0}\left(\frac{x}{(x^2+1)}-mx\right)dx + \int_{0}^{\sqrt{\frac{1}{m}-1}}\left(\frac{x}{(x^2+1)}-mx\right)dx$$
$$= \boxed{m-1-\ln m}$$

If $m=1$ then

$$A = \int_{-\sqrt{\frac{1}{m}-1}}^{0}\left(\frac{x}{(x^2+1)}-mx\right)dx + \int_{0}^{\sqrt{\frac{1}{m}-1}}\left(\frac{x}{(x^2+1)}-mx\right)dx$$
$$= \boxed{0}$$

Answer 92E.

(a)

Consider the function $f(x) = \ln x$

$$f'(x) = \frac{1}{x}$$

$$f'(1) = \frac{1}{1} = 1$$

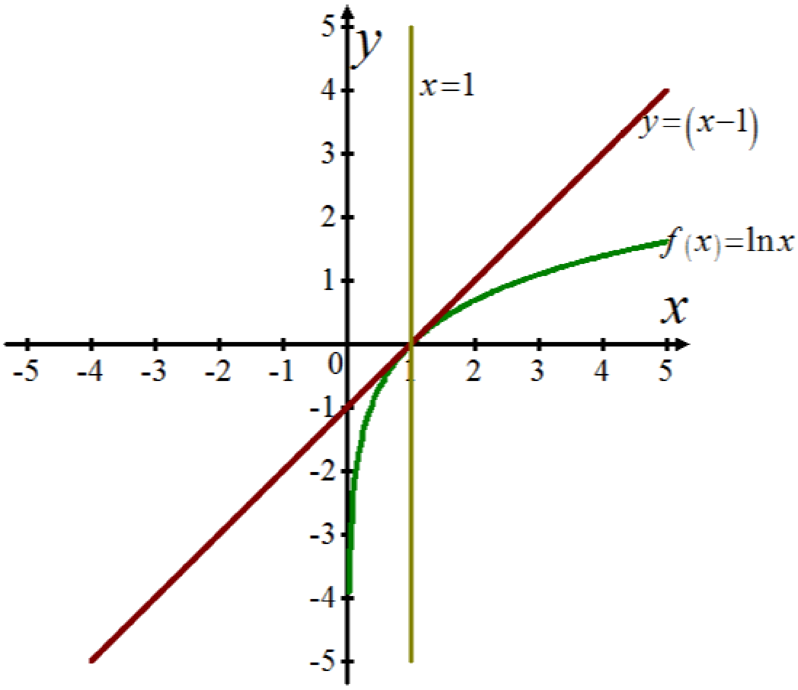
Hence the linear approximation $f_1(x)$ is given by

$$\begin{aligned} f_1(x) &= \ln 1 + f'(1)(x-1) \\ &= 0 + 1(x-1) \\ &= x-1 \end{aligned}$$

Therefore the linear approximation of $f(x) = \ln x$ is $(x-1)$

(b)

The graph of $f(x) = \ln x$ and the linear approximation of $f(x) = \ln x$ is $(x-1)$



(c)

The linear approximation of $f(x) = \ln x$ is $(x-1)$

But the linear approximation accurate to within 0.1

So that

$$\begin{aligned} (x-1) &< 0.1 \\ x &< 0.1+1 \\ x &< 1.1 \\ x &\approx 1 \end{aligned}$$

Answer 93E.

Let $f(x) = \ln x$

Differentiating with respect to x, we get,

$$f'(x) = \frac{d}{dx}\ln x = \frac{1}{x}$$

So $f'(1) = \frac{1}{1} = 1$

Also $f(x+h)=\ln(x+h)$

From the definition of a derivative as a limit, we have

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{f(1+x)-f(1)}{x} \qquad \text{Replacing h by x.}$$

$$=\lim_{x\to 0}\frac{\ln(1+x)-\ln(1)}{x}$$

$$=\lim_{x\to 0}\frac{\ln(1+x)-0}{x}$$

$$=\lim_{x\to 0}\frac{\ln(1+x)}{x}$$

Therefore,

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=f'(1) \qquad \text{Since } f'(1)=1$$

$$=1$$

Answer 94E.

Let $f(y)=\ln y$ Then $f'(y)=\dfrac{1}{y}$

And $f(y+h)=\ln(y+h)$

Now from the definition of the a derivative as a limit, we have,

$$f'(y)=\lim_{h\to 0}\frac{f(y+h)-f(y)}{h}$$

$$=\lim_{h\to 0}\frac{\ln(y+h)-\ln y}{h}$$

$$=\lim_{h\to 0}\frac{\ln\frac{y+h}{y}}{h} \qquad \text{Since } \ln\frac{x}{y}=\ln x-\ln y$$

$$=\lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{y}\right)$$

Since, $f'(y)=\dfrac{1}{y}$

Therefore, we have,

$$\lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{y}\right)=\frac{1}{y}$$

Let us put $h=\dfrac{1}{n}$ and $y=\dfrac{1}{x}$

As $h\to 0$, $n\to\infty$

Therefore, we have,

$$\lim_{n\to\infty}\frac{1}{(1/n)}\ln\left(1+\frac{(1/n)}{(1/x)}\right)=\frac{1}{(1/x)}$$

$$\Rightarrow \quad \lim_{n\to\infty} n\ln\left(1+\frac{x}{n}\right)=x$$

$$\Rightarrow \quad \lim_{n\to\infty}\ln\left(1+\frac{x}{n}\right)^n=x \qquad \text{Since } \ln x^y=y\ln x$$

Applying exponential function to both sides, we get,

$$e^{\lim_{n\to\infty}\ln\left(1+\frac{x}{n}\right)^n}=e^x$$

$$\Rightarrow \quad \lim_{n\to\infty} e^{\ln\left(1+\frac{x}{n}\right)^n}=e^x$$

$$\Rightarrow \quad \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x \qquad \text{Since } e^{\ln x}=x$$

Hence,

$$\boxed{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x}$$