Chapter

8

Differential Equations

CONTENTS

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One of the principal languages of science is that of differential equations. Interprestingly, the date of birth of differential equations is taken to be **November 11, 1675**, *when* **Gottfried Wilthelm Freiherr Leibnitz (1646-1716)** *first put in black and white the identity* $\int y\,dy=\frac{1}{2}y^2$ *thereby introducing both the symbols* $\int$ *and dy. Leibnitz was actually interested in the problem of finding a curve whose tangents were prescribed. This led him to discover the* **'method of separation of variables'** *in* **1691**. *A year later he formulated the* **'method of solving the homogeneous differential equations of the first order'**. *He went further in a very short time to the discovery of the* **'method of solving a linear differential equation of the first-order'**. *How surprising is it that all these methods came from a single man and that too within 25 years of the birth of differential equations.*

Differential Equations

Many of the practical problems in physics and engineering can be converted into differential equations. The solution of differential equations is, therefore of paramount importance. This chapter deals with some elementary aspects of differential equations. These are addressed through simple application of differential and integral calculus.

There are two important aspects of differential equation, which have just been touched in this chapter. How to formulate a problem as a differential equation is the one, and the other is how to solve it.

8.1 Definition

An equation involving independent variable x, dependent variable y and the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots$ is called differential equation.

Examples : (i) $\frac{dy}{dx} = 1 + x + y$ (ii) $\frac{dy}{dx} + xy = \cot x$

(iii) $\left(\frac{d^4y}{dx^4}\right)^3 - 4\frac{dy}{dx} + 4y = 5\cos 3x$ (iv) $x^2\frac{d^2y}{dx^2} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 0$

(1) **Order of a differential equation :** The order of a differential equation is the order of the highest derivative occurring in the differential equation. For example, the order of above differential equations are 1,1,4 and 2 respectively.

Note : ❑ The order of a differential equation is a positive integer. To determine the order of a differential equation, it is not needed to make the equation free from radicals.

(2) **Degree of a differential equation :** The degree of a differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions. In other words, the degree of a differential equation is the power of the highest order derivative occurring in differential equation when it is written as a polynomial in differential coefficients.

Note : ❑ The definition of degree does not require variables x, y, t etc. to be free from radicals and fractions. The degree of above differential equations are 1, 1, 3 and 2 respectively.

Example: 1 The order and degree of the differential equation $y = x\frac{dy}{dx} + \sqrt{a^2\left(\frac{dy}{dx}\right)^2 + b^2}$ are

(a) 1, 2 (b) 2, 1 (c) 1, 1 (d) 2, 2

Solution: (a) Clearly, highest order derivative involved is $\frac{dy}{dx}$, having order 1.

Expressing the above differential equation as a polynomial in derivative, we have $\left(y - x\frac{dy}{dx}\right)^2 = a^2\left(\frac{dy}{dx}\right)^2 + b^2$

i.e., $(x^2 - a^2)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + y^2 - b^2 = 0$

In this equation, the power of highest order derivative is 2. So its degree is 2.

Example: 2 The order and degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{\frac{1}{3}} + x^{\frac{1}{4}} = 0$, are respectively

(a) 2, 3 (b) 3, 3 (c) 2, 6 (d) 2, 4

Solution: (a) The highest order derivative involved is $\frac{d^2y}{dx^2}$ which is the 2nd order derivative. Hence order of the differential equation is 2. Making the above equation free from radical, as far as the derivatives are concerned, we have

$$\left(\frac{d^2y}{dx^2} + x^{\frac{1}{4}}\right)^3 = -\frac{dy}{dx} \quad i.e. \quad \left(\frac{d^2y}{dx^2} + x^{\frac{1}{4}}\right)^3 + \frac{dy}{dx} = 0$$

The exponent of highest order derivative $\frac{d^2y}{dx^2}$ will be 3. Hence degree of the differential equation is 3.

Example: 3 The degree of the differential equation $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x^2\log\left(\frac{d^2y}{dx^2}\right)$ is

(a) 1 (b) 2 (c) 3 (d) None of these

Solution: (d) The above equation cannot be written as a polynomial in derivatives due to the term $x^2\log\left(\frac{d^2y}{dx^2}\right)$.

Hence degree of the differential equation is 'not defined'.

Example: 4 The order of the differential equation whose solution is $x^2 + y^2 + 2gx + 2fy + c = 0$, is

(a) 1 (b) 2 (c) 3 (d) 4

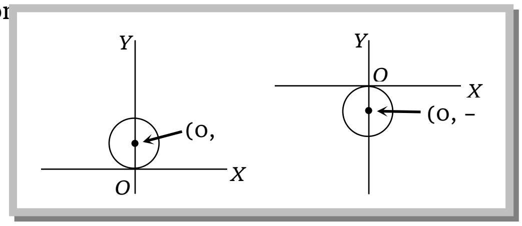
Solution: (c) To eliminate the arbitrary constants g, f and c, we need 3 more equations, that by differentiating the equation 3 times. Hence highest order derivative will be $\frac{d^3y}{dx^3}$. Hence order of the differential equation will be 3.

Example: 5 The order of the differential equation of all circles of radius r, having centre on y-axis and passing through the origin is

(a) 1 (b) 2 (c) 3 (d) 4

Solution: (a) The equation of the circle will be $x^2 + (y \pm r)^2 = r^2$. Hence the constant to be eliminated is r.

$\therefore$ order of differential equatior



Example: 6 The order of the differential equation, whose general solution is $y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} + c_4 e^{x+c_5}$, where c_1, c_2, c_3, c_4, c_5 are arbitrary constants is

(a) 5 (b) 4 (c) 3 (d) None of these

Solution: (c) Rewriting the given general solution, we have

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} + c_4 e^x . e^{c_5}$$

$$= (c_1 + c_4 . e^{c_5}) e^x + c_2 e^{2x} + c_3 e^{3x} = c_1' e^x + c_2 e^{2x} + c_3 e^{3x}$$

where $c_1' = c_1 + c_4 . e^{c_5}$. So there are 3 arbitrary constant associated with different terms. Hence the order of the differential equation formed, will be 3.

Example: 7 The degree of the differential equation satisfying $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ is

(a) 1 (b) 2 (c) 3 (d) None of these

Solution: (a) To eliminate a the above equation is differentiated once and exponent of $\frac{dy}{dx}$ will be1. Hence degree is 1

Example: 8 The order and degree of $y = 1 + \frac{dy}{dx} + \frac{1}{2!}\left(\frac{dy}{dx}\right)^2 + \frac{1}{3!}\left(\frac{dy}{dx}\right)^3 + ...$ is

(a) 1, 2 (b) 1, 1

(c) Order 1, degree not defined (d) None of these

Solution: (b) The given differential equation can be re-written as $y = e^{\frac{dy}{dx}} \Rightarrow \ln y = \frac{dy}{dx}$

This is a polynomial in derivative. Hence order is 1 and degree 1.

Example: 9 The order and degree of $\frac{d^2y}{dx^2} = \sin\left(\frac{dy}{dx}\right) + x$ is

(a) 2, 1 (b) Order 2, degree not defined

(c) 2, 0 (d) None of these

Solution: (b) As the highest order derivative involved is $\frac{d^2y}{dx^2}$. Hence order is 2.

The given differential equation cannot be written as a polynomial in derivatives, the degree is not defined.

8.2 Formation of Differential Equation

Formulating a differential equation from a given equation representing a family of curves means finding a differential equation whose solution is the given equation. If an equation, representing a family of curves, contains n arbitrary constants, then we differentiate the given equation n times to obtain n more equations. Using all these equations, we eliminate the constants. The equation so obtained is the differential equation of order n for the family of given curves.

Consider a family of curves $f(x, y, a_1, a_2 ....., a_n) = 0$ ......(i)

where $a_1, a_2, ...a_n$ are n independent parameters.

Equation (i) is known as an n parameter family of curves *e.g.* $y = mx$ is a one-parameter family of straight lines. $x^2 + y^2 + ax + by = 0$ is a two parameters family of circles.

If we differentiate equation (i) n times *w.r.t.* x, we will get n more relations between $x, y, a_1, a_2, ...a_n$ and derivatives of y *w.r.t.* x. By eliminating $a_1, a_2, ..., a_n$ from these n relations and equation (i), we get a differential equation.

Clearly order of this differential equation will be n *i.e.* equal to the number of independent parameters in the family of curves.

Algorithm for formation of differential equations

Step (i) **:** Write the given equation involving independent variable x (say), dependent variable y (say) and the arbitrary constants.

Step (ii) **:** Obtain the number of arbitrary constants in step (i). Let there be n arbitrary constants.

Step (iii) **:** Differentiate the relation in step (i) n times with respect to x.

Step (iv) **:** Eliminate arbitrary constants with the help of n equations involving differential coefficients obtained in step (iii) and an equation in step (i). The equation so obtained is the desired differential equation.

Example: 10 Differential equation whose general solution is $y = c_1 x + \frac{c_2}{x}$ for all values of c_1 and c_2 is

(a) $\frac{d^2y}{dx^2} + \frac{x^2}{y} + \frac{dy}{dx} = 0$ (b) $\frac{d^2y}{dx^2} + \frac{y}{x^2} - \frac{dy}{dx} = 0$ (c) $\frac{d^2y}{dx^2} - \frac{1}{2x}\frac{dy}{dx} = 0$ (d) $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 0$

Solution: (d) $y = c_1 x + \frac{c_2}{x}$ .....(i)

There are two arbitrary constants. To eliminate these constants, we need to differentiate (i) twice.

Differentiating (i) with respect to x,

$$\frac{dy}{dx} = c_1 - \frac{c_2}{x^2} \qquad .....(ii)$$

Again differentiating with respect to x,

$$\frac{d^2y}{dx^2} = \frac{2c_2}{x^3} \qquad ......(iii)$$

From (iii), $c_2 = \frac{x^3}{2}\frac{d^2y}{dx^2}$ and from (ii), $c_1 = \frac{dy}{dx} + \frac{c_2}{x^2}$;

$\therefore\ c_1 = \frac{dy}{dx} + \frac{x}{2}\frac{d^2y}{dx^2}$

From (i), $y = \left(\frac{dy}{dx} + \frac{x}{2}\cdot\frac{d^2y}{dx^2}\right)x + \frac{x^2}{2}\cdot\frac{d^2y}{dx^2} \Rightarrow y = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}$

$\therefore\ \frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 0$

Example: 11 $y = \frac{x}{x+1}$ is a solution of the differential equation

(a) $y^2\frac{dy}{dx} = x^2$ (b) $x^2\frac{dy}{dx} = y^2$ (c) $y\frac{dy}{dx} = x$ (d) $x\frac{dy}{dx} = y$

Solution: (b) We have $y = \frac{x}{x+1} \Rightarrow \frac{1}{y} = \frac{x+1}{x} = 1 + \frac{1}{x}$

Differentiating *w.r.t.* x,

$$-\frac{1}{y^2}\frac{dy}{dx} = 0 - \frac{1}{x^2}$$

$\therefore\ x^2\frac{dy}{dx} = y^2$

Example: 12 The differential equation of all parabolas whose axes are parallel to y axis is

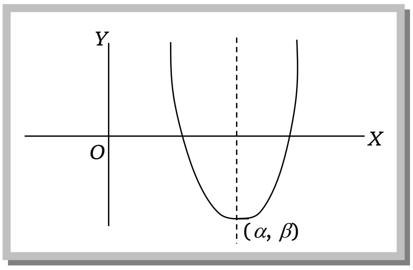
(a) $\frac{d^3y}{dx^3}=0$ (b) $\frac{d^2x}{dy^2}=c$ (c) $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}=0$ (d) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=c$

Solution: (a) The equation of a parabola whose axis is parallel to y-axis may be expressed as

$$(x-\alpha)^2=4a(y-\beta) \qquad \text{........(i)}$$

There are three arbitrary constants α, β and a.

We need to differentiate (i) 3 times



Differentiating (i) *w.r.t.* x, $2(x-\alpha)=4a\frac{dy}{dx}$

Again differentiating *w.r.t.* x,

$$2=4a\frac{d^2y}{dx^2} \Rightarrow \frac{d^2y}{dx^2}=\frac{1}{2a}$$

Differentiating *w.r.t.* x,

$$\frac{d^3y}{dx^3}=0$$

Example: 13 The differential equation of family of curves whose tangent form an angle of $\pi/4$ with the hyperbola $xy=c^2$ is

(a) $\frac{dy}{dx}=\frac{x^2+c^2}{x^2-c^2}$ (b) $\frac{dy}{dx}=\frac{x^2-c^2}{x^2+c^2}$ (c) $\frac{dy}{dx}=-\frac{c^2}{x^2}$ (d) None of these

Solution: (b) The slope of the tangent to the family of curves is $m_1=\frac{dy}{dx}$

Equation of the hyperbola is $xy=c^2 \Rightarrow y=\frac{c^2}{x}$

$$\therefore \frac{dy}{dx}=-\frac{c^2}{x^2}$$

$\therefore$ Slope of tangent to $xy=c^2$ is $m_2=-\frac{c^2}{x^2}$

Now $\tan\frac{\pi}{4}=\frac{m_1-m_2}{1+m_1m_2} \Rightarrow 1=\dfrac{\frac{dy}{dx}+\frac{c^2}{x^2}}{1-\frac{c^2}{x^2}\frac{dy}{dx}} \Rightarrow \frac{dy}{dx}\left(1+\frac{c^2}{x^2}\right)=\left(1-\frac{c^2}{x^2}\right)$

$$\therefore \frac{dy}{dx}=\frac{x^2-c^2}{x^2+c^2}$$

8.3 Variable Separable type Differential Equation

(1) **Solution of differential equations :** If we have a differential equation of order 'n' then by solving a differential equation we mean to get a family of curves with n parameters whose differential equation is the given differential equation. Solution or integral of a differential equation is a relation between the variables, not involving the differential coefficients such that this relation and the derivatives obtained from it satisfy the given differential equation. The solution of a differential equation is also called its primitive.

For example $y=e^x$ is a solution of the differential equation $\frac{dy}{dx}=y$.

(i) **General solution :** The solution which contains as many as arbitrary constants as the order of the differential equation is called the general solution of the differential equation. For example, $y = A\cos x + B\sin x$ is the general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$. But $y = A\cos x$ is not the general solution as it contains one arbitrary constant.

(ii) **Particular solution** : Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution. For example, $y = 3\cos x + 2\sin x$ is a particular solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$

(2) **Differential equations of first order and first degree :** A differential equation of first order and first degree involves x, y and $\frac{dy}{dx}$. So it can be put in any one of the following forms: $\frac{dy}{dx} = f(x,y)$ or $f\left(x, y, \frac{dy}{dx}\right) = 0$ or $f(x,y)dx + g(x,y)dy = 0$ where $f(x,y)$ and $g(x,y)$ are obviously the functions of x and y.

(3) **Geometrical interpretation of the differential equations of first order and first degree :** The general form of a first order and first degree differential equation is $f\left(x, y, \frac{dy}{dx}\right) = 0$ .....(i)

We know that the direction of the tangent of a curve in Cartesian rectangular coordinates at any point is given by $\frac{dy}{dx}$, so the equation in (i) can be known as an equation which establishes the relationship between the coordinates of a point and the slope of the tangent *i.e.*, $\frac{dy}{dx}$ to the integral curve at that point. Solving the differential equation given by (i) means finding those curves for which the direction of tangent at each point coincides with the direction of the field. All the curves represented by the general solution when taken together will give the locus of the differential equation. Since there is one arbitrary constant in the general solution of the equation of first order, the locus of the equation can be said to be made up of single infinity of curves.

(4) **Solution of first order and first degree differential equations :** A first order and first degree differential equation can be written as

$$f(x, y)dx + g(x, y)dy = 0$$

$$\text{or } \frac{dy}{dx} = \frac{f(x,y)}{g(x,y)'} \text{ or } \frac{dy}{dx} = \phi(x,y)$$

Where $f(x,y)$ and $g(x,y)$ are obviously the functions of x and y. It is not always possible to solve this type of equations. The solution of this type of differential equations is possible only when it falls under the category of some standard forms.

(5) **Equations in variable separable form :** If the differential equation of the form

$$f_1(x)dx = f_2(y)dy \qquad .....(i)$$

where f_1 and f_2 being functions of x and y only. Then we say that the variables are separable in the differential equation.

Thus, integrating both sides of (i), we get its solution as $\int f_1(x)dx = \int f_2(y)dy + C$,

where c is an arbitrary constant.

There is no need of introducing arbitrary constants to both sides as they can be combined together to give just one.

(i) **Differential equations of the type $\frac{dy}{dx} = f(x)$**

To solve this type of differential equations we integrate both sides to obtain the general solution as discussed following : $\frac{dy}{dx} = f(x) \Leftrightarrow dy = f(x)dx$

Integrating both sides, we obtain, $\int dy = \int f(x)dx + C$ or $y = \int f(x)dx + C$.

(ii) **Differential equations of the type $\frac{dy}{dx} = f(y)$**

To solve this type of differential equations we integrate both sides to obtain the general solution as discussed following :

$$\frac{dy}{dx} = f(y) \Rightarrow \frac{dx}{dy} = \frac{1}{f(y)} \Rightarrow dx = \frac{1}{f(y)}dy$$

Integrating both sides, we obtain, $\int dx = \int \frac{1}{f(y)}dy + C$ or $x = \int \frac{1}{f(y)}dy + C$.

(6) **Equations reducible to variable separable form**

(i) Differential equations of the form $\frac{dy}{dx} = f(ax + by + c)$ can be reduced to variable separable form by the substitution $ax + by + c = Z$

$$\therefore \quad a + b\frac{dy}{dx} = \frac{dZ}{dx}$$

$$\therefore \quad \left(\frac{dZ}{dx} - a\right)\frac{1}{b} = f(Z) \Rightarrow \frac{dZ}{dx} = a + bf(Z).$$

This is variable separable form.

(ii) **Differential equation of the form**

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}, \text{ where } \frac{a}{A} = \frac{b}{B} = K \text{ (say)}$$

$$\therefore \frac{dy}{dx} = \frac{K(Ax + By) + C}{Ax + By + C}$$

Put $Ax + By = Z$

$$\therefore \quad A + B\frac{dy}{dx} = \frac{dZ}{dx}, \quad \therefore \quad \left[\frac{dZ}{dx} - A\right]\frac{1}{B} = \frac{KZ + C}{Z + C} \Rightarrow \frac{dZ}{dx} = A + B\frac{KZ + C}{Z + C}$$

This is variable separable form and can be solved.

Example: 14 The solution of the differential equation $(1+x^2)\frac{dy}{dx}=x(1+y^2)$ is **[AISSE 1983]**

(a) $2\tan^{-1}y=\log(1+x^2)+c$ (b) $\tan^{-1}y=\log(1+x^2)+c$

(c) $2\tan^{-1}y+\log(1+x^2)+c=0$ (d) None of these

Solution: (a) Separating the variables, we can re-write the given differential equation as

$\frac{xdx}{1+x^2}=\frac{dy}{1+y^2} \Rightarrow \int\frac{2x\,dx}{1+x^2}=2\int\frac{dy}{1+y^2} \Rightarrow 2\tan^{-1}y=\log_e(1+x^2)+c$

Example: 15 The solution of the differential equation $\frac{dy}{dx}=x^2+\sin 3x$ is **[DSSE 1981]**

(a) $y=\frac{x^3}{3}+\frac{\cos 3x}{3}+c$ (b) $y=\frac{x^3}{3}-\frac{\cos 3x}{3}+c$ (c) $y=\frac{x^3}{3}+\sin 3x+c$ (d) None of these

Solution: (b) We have $dy=(x^2+\sin 3x)dx \Rightarrow \int dy=\int(x^2+\sin 3x)dx \Rightarrow y=\frac{x^3}{3}-\frac{\cos 3x}{3}+c$

Example: 16 The solution of $\frac{dy}{dx}=\frac{1}{y^2+\sin y}$ is

(a) $x=\frac{y^3}{3}-\cos y+c$ (b) $y + \cos y = x + c$ (c) $x=\frac{y^3}{3}+\cos y+c$ (d) None of these

Solution: (a) Given equation may be re-written as $dx=(y^2+\sin y)dy$

Integrating, $\int dx=\int(y^2+\sin y)\,dy$

$\therefore\ x=\frac{y^3}{3}-\cos y+c$

Example: 17 The solution of the differential equation $\frac{dy}{dx}=(4x+y+1)^2$ is

(a) $4x - y + 1 = 2\tan(2x - 2c)$ (b) $4x - y - 1 = 2\tan(2x - 2c)$

(c) $4x + y + 1 = 2\tan(2x + 2c)$ (d) None of these

Solution: (c) Let $4x + y + 1 = z \Rightarrow 4+\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{dy}{dx}=\frac{dz}{dx}-4$

$\therefore\ \frac{dy}{dx}=(4x+y+1)^2$

$\Rightarrow \frac{dz}{dx}-4=z^2 \Rightarrow \frac{dz}{dx}=z^2+4 \Rightarrow \frac{dz}{z^2+4}=dx \Rightarrow \frac{1}{2}\tan^{-1}\frac{z}{2}=x+c \Rightarrow \tan^{-1}\left(\frac{4x+y+1}{2}\right)=2x+2c$

$\therefore\ 4x + y + 1 = 2\tan(2x + 2c)$

Example: 18 Solution of the differential equation $\frac{dy}{dx}=\frac{x+y+7}{2x+2y+3}$ is

(a) $6(x + y) + 11\log(3x + 3y + 10) = 9x + c$ (b) $6(x + y) - 11\log(3x + 3y + 10) = 9x + c$

(c) $6(x+y)-11\log\left(x+y+\frac{10}{3}\right)=9x+c$ (d) None of these

Solution: (b, c) Given equation may be re-written as $\frac{dy}{dx}=\frac{x+y+7}{2(x+y)+3}$

Let $x + y = z$

$$\Rightarrow 1+\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{dy}{dx}=\frac{dz}{dx}-1$$

$$\therefore \frac{dz}{dx}-1=\frac{z+7}{2z+3}$$

$$\Rightarrow \frac{dz}{dx}=1+\frac{z+7}{2z+3}=\frac{3z+10}{2z+3} \Rightarrow \frac{2z+3}{3z+10}dz=dx \Rightarrow \frac{\frac{2}{3}(3z+10)-\frac{11}{3}}{3z+10}dz=dx \Rightarrow \int\frac{2}{3}dz-\frac{11}{9}\int\frac{3dz}{3z+10}=\int dx$$

$$\Rightarrow \frac{2}{3}z-\frac{11}{9}\log(3z+10)=x+c_1 \Rightarrow 6z-11\log(3z+10)=9x+9c_1$$

$$\therefore 6(x+y)-11\log(3x+3y+10)=9x+c \qquad [9c_1=c]$$

$$\Rightarrow 6(x+y)-11\log 3\left(x+y+\frac{10}{3}\right)=9x+c \Rightarrow 6(x+y)-11\log\left(x+y+\frac{10}{3}\right)=9x+(c+11\log 3)$$

$$\therefore 6(x+y)-11\log\left(x+y+\frac{10}{3}\right)=9x+k \qquad (k=c+11\log 3)$$

8.4 Homogeneous Differential Equation

(1) **Homogeneous differential equation :** A function $f(x, y)$ is called a homogeneous function of degree n if $f(\lambda x,\lambda y)=\lambda^n f(x,y)$.

For example, $f(x,y)=x^2-y^2+3xy$ is a homogeneous function of degree 2, because $f(\lambda x,\lambda y)=\lambda^2x^2-\lambda^2y^2+3\lambda x.\lambda y=\lambda^2 f(x,y)$. A homogeneous function $f(x, y)$ of degree n can always be written as $f(x,y)=x^n f\left(\frac{y}{x}\right)$ or $f(x,y)=y^n f\left(\frac{x}{y}\right)$. If a first-order first degree differential equation is expressible in the form $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ where $f(x, y)$ and $g(x, y)$ are homogeneous functions of the same degree, then it is called a homogeneous differential equation. Such type of equations can be reduced to variable separable form by the substitution $y=vx$. The given differential equation can be written as $\frac{dy}{dx}=\frac{x^n f(y/x)}{x^n g(y/x)}=\frac{f(y/x)}{g(y/x)}=F\left(\frac{y}{x}\right)$. If $y=vx$, then $\frac{dy}{dx}=v+x\frac{dv}{dx}$. Substituting the value of $\frac{dy}{dx}=F\left(\frac{y}{x}\right)$, we get $v+x\frac{dv}{dx}=F(v)\Rightarrow\frac{dv}{F(v)-v}=\frac{dx}{x}$. On integration, $\int\frac{1}{F(v)-v}dv=\int\frac{dx}{x}+C$ where C is an arbitrary constant of integration. After integration, v will be replaced by $\frac{y}{x}$ in complete solution.

(2) **Algorithm for solving homogeneous differential equation**

Step (i) **:** Put the differential equation in the form $\frac{dy}{dx}=\frac{\phi(x,y)}{\psi(x,y)}$

Step (ii) **:** Put $y = vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in the equation in step (i) and cancel out x from the right hand side. The equation reduces to the form $v+x\frac{dv}{dx}=F(v)$.

Step (iii) **:** Shift v on RHS and separate the variables v and x

Step (iv) **:** Integrate both sides to obtain the solution in terms of v and x.

Step (v) **:** Replace v by $\frac{y}{x}$ in the solution obtained in step (iv) to obtain the solution in terms of x and y.

(3) **Equation reducible to homogeneous form**

A first order, first degree differential equation of the form

$$\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}, \text{ where } \frac{a_1}{a_2}\neq\frac{b_1}{b_2} \quad .....(i)$$

This is non-homogeneous.

It can be reduced to homogeneous form by certain substitutions. Put $x = X+h, y = Y+k$

Where h and k are constants, which are to be determined.

$$\therefore \frac{dy}{dx}=\frac{dy}{dY}.\frac{dY}{dX}.\frac{dX}{dx}=\frac{dY}{dX}$$

Substituting these values in (i), we have $\frac{dY}{dX}=\frac{(a_1X+b_1Y)+a_1h+b_1k+c_1}{(a_2X+b_2Y)+a_2h+b_2k+c_2}$ .....(ii)

Now h, k will be chosen such that $\left.\begin{array}{l} a_1h+b_1k+c_1=0 \\ a_2h+b_2k+c_2=0\end{array}\right]$

.....(iii)

i.e. $\frac{h}{b_1c_2-b_2c_1}=\frac{k}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$ .....(iv)

For these values of h and k the equation (ii) reduces to $\frac{dY}{dX}=\frac{a_1X+b_1Y}{a_2X+b_2Y}$ which is a homogeneous differential equation and can be solved by the substitution $Y = vX$. Replacing X and Y in the solution so obtained by $x-h$ and $y-k$ respectively, we can obtain the required solution in terms of x and y.

Example: 19 The solution of the differential equation $x\frac{dy}{dx}=y(\log y-\log x+1)$ is **[IIT 1986]**

(a) $y=xe^{cx}$ (b) $y+xe^{cx}=0$ (c) $y+e^x=0$ (d) None of these

Solution: (a) Given equation may be expressed as $\frac{dy}{dx}=\frac{y}{x}\left[\log\left(\frac{y}{x}\right)+1\right]$ ....(i)

Let $\frac{y}{x}=v \Rightarrow y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$\therefore$ From (i), $v+x\frac{dv}{dx}=v(\log v+1) \Rightarrow x\frac{dv}{dx}=v\log v \Rightarrow \frac{dv}{v\log v}=\frac{dx}{x} \Rightarrow \int\frac{1}{\log v}d(\log v)=\int\frac{dx}{x}$

$\therefore \log(\log v)=\log x+\log c \Rightarrow \log(\log v)=\log(cx) \Rightarrow \log v=cx \Rightarrow v=e^{cx} \Rightarrow \frac{y}{x}=e^{cx}$, $\therefore y=xe^{cx}$

Example: 20 The solution of differential equation $yy'=x\left(\frac{y^2}{x^2}+\frac{\phi(y^2/x^2)}{\phi'(y^2/x^2)}\right)$ is

(a) $\phi(y^2/x^2)=cx^2$ (b) $x^2\phi(y^2/x^2)=c^2y^2$ (c) $x^2\phi(y^2/x^2)=c$ (d) $\phi(y^2/x^2)=\frac{cy}{x}$

Solution: (a) Given equation may be re-written as $\frac{y}{x}\cdot\frac{dy}{dx}=\left(\frac{y}{x}\right)^2+\frac{\phi((y/x)^2)}{\phi'((y/x)^2)}$ .....(i)

Let $y = vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$ and $\frac{y}{x}=v$

$\therefore$ From (i), $v\left(v+x\frac{dv}{dx}\right)=v^2+\frac{\phi(v^2)}{\phi'(v^2)} \Rightarrow vx\frac{dv}{dx}=\frac{\phi(v^2)}{\phi'(v^2)} \Rightarrow \frac{\phi'(v^2)(2v\,dv)}{\phi(v^2)}=2\frac{dx}{x}$

Integrating, $\ln(\phi(v^2))=2\ln x+\ln c \Rightarrow \phi(v^2)=cx^2$

$\therefore\ \phi(y^2/x^2)=cx^2$

Example: 21 The solution of $\frac{dy}{dx}=\frac{y^3+2x^2y}{x^3+2xy^2}$ is

(a) $(x^2-y^2)^3=Bx^2y^2$ (b) $(x^2+y^2)^3=Bx^2y^2$ (c) $(x^2-y^2)^3=x^2y^2$ (d) None of these

Solution: (a) Given equation is homogeneous. Let $y = vx$ $\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$

$\Rightarrow \frac{y^3+2x^2y}{x^3+2xy^2}=v+x\frac{dv}{dx} \Rightarrow \frac{(y/x)^3+2(y/x)}{1+2(y/x)^2}=v+x\frac{dv}{dx} \Rightarrow \frac{v^3+2v}{1+2v^2}=v+x\frac{dv}{dx} \Rightarrow x\frac{dv}{dx}=v\left\{\frac{v^2+2}{1+2v^2}-1\right\}=v\left\{\frac{1-v^2}{1+2v^2}\right\}$

$\Rightarrow \frac{1+2v^2}{v(1-v^2)}dv=\frac{dx}{x} \Rightarrow \frac{1+2v^2}{v(1-v)(1+v)}dv=\frac{dx}{x} \Rightarrow \left(\frac{A}{v}+\frac{B}{1-v}+\frac{D}{1+v}\right)dv=\frac{dx}{x}$

where $A(1-v)(1+v)+Bv(1+v)+Dv(1-v)=1+2v^2$

Putting $v = 0$, $A = 1$

$v = 1$, $B=\frac{3}{2}$

$v = -1$, $D=-\frac{3}{2}$

$\therefore \left(\frac{1}{v}+\frac{3}{2}\frac{1}{1-v}-\frac{3}{2}\frac{1}{1+v}\right)dv=\frac{dx}{x}$

Integrating both side, we get $\ln v+\frac{3}{2}\frac{\ln(1-v)}{-1}-\frac{3}{2}\ln(1+v)=\ln x+\ln c \Rightarrow \ln v-\frac{3}{2}\ln(1-v)-\frac{3}{2}\ln(1+v)=\ln cx$

$\Rightarrow v/\{(1-v)1+v\}^{3/2}=cx \Rightarrow \left(\frac{v}{cx}\right)^2=(1-v^2)^3 \Rightarrow \left(\frac{y}{cx^2}\right)^2=\left(1-\frac{y^2}{x^2}\right)^3 \Rightarrow (x^2-y^2)^3=\frac{x^2y^2}{c^2}$

$\therefore (x^2-y^2)^3=Bx^2y^2$, $\left(\because \frac{1}{c^2}=B\right)$

Example: 22 The solution of $\frac{dy}{dx}=\frac{x-3y+2}{3x-y+6}$ is

(a) $y^2+6(x+2)y+(x+2)^2=c$ (b) $y^2-6(x+2)y+(x+2)^2=c$

(c) $y^2-6(y+2)x+x^2=c$ (d) None of these

Solution: (b) Given equation is non-homogeneous

Let $x = X + h$, $y = Y + k$

$\Rightarrow \frac{dy}{dx}=\frac{dY}{dX}$

$\therefore \frac{dY}{dX}=\frac{(X+h)-3(Y+k)+2}{3(X+h)-(Y+k)+6}=\frac{X-3Y+(h-3k+2)}{3X-Y+(3h-k+6)}$

Let us select h and k so that $h - 3k + 2 = 0$ and $3h - k + 6 = 0$

Solving, $k = 0$, $h = -2$ $\therefore X = x - h = x + 2$, $Y=y-k=y$

$\therefore \frac{dY}{dX}=\frac{X-3Y}{3X-Y}$, which is homogeneous

Now, let $Y = vX$

$$\Rightarrow \frac{dY}{dX} = v + X\frac{dv}{dX} \Rightarrow \frac{X-3Y}{3X-Y} = v + X\frac{dv}{dX} \Rightarrow \frac{1-3(Y/X)}{3-(Y/X)} = v + X\frac{dv}{dX} \Rightarrow \frac{1-3v}{3-v} = v + X\frac{dv}{dX}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{1-3v}{3-v} - v = \frac{v^2-6v+1}{3-v} \Rightarrow \frac{(3-v)dv}{v^2-6v+1} = \frac{dX}{X} \Rightarrow \frac{2v-6}{v^2-6v+1}dv = -2\frac{dX}{X}$$

Integrating, $\ln(v^2-6v+1) = -2\ln X + \ln c \Rightarrow \ln(v^2-6v+1) + \ln X^2 = \ln c \Rightarrow X^2(v^2-6v+1) = c \Rightarrow$ $Y^2 - 6XY + X^2 = c$

$\therefore\ y^2 - 6(x+2)y + (x+2)^2 = c$

8.5 Exact Differential Equation

(1) **Exact differential equation :** If M and N are functions of x and y, the equation $Mdx + Ndy = 0$ is called exact when there exists a function $f(x, y)$ of x and y such that

$d[f(x, y)] = Mdx + Ndy$ *i.e.*, $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$

where $\frac{\partial f}{\partial x}$ = Partial derivative of $f(x, y)$ with respect to x (keeping y constant)

$\frac{\partial f}{\partial y}$ = Partial derivative of $f(x, y)$ with respect to y (treating x as constant)

Note : ❑ An exact differential equation can always be derived from its general solution directly by differentiation without any subsequent multiplication, elimination etc.

(2) **Theorem :** The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ *i.e.*, partial derivative of $M(x, y)$ *w.r.t.* y = Partial derivative of $N(x, y)$ *w.r.t.* x

(3) **Integrating factor :** If an equation of the form $Mdx + Ndy = 0$ is not exact, it can always be made exact by multiplying by some function of x and y. Such a multiplier is called an integrating factor.

(4) **Working rule for solving an exact differential equation :**

Step (i) **:** Compare the given equation with $Mdx + Ndy = 0$ and find out M and N. Then find out $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact.

Step (ii) **:** Integrate M with respect to x treating y as a constant.

Step (iii) **:** Integrate N with respect to y treating x as constant and omit those terms which have been already obtained by integrating M.

Step (iv) **:** On adding the terms obtained in steps (ii) and (iii) and equating to an arbitrary constant, we get the required solution.

In other words, solution of an exact differential equation is $\underset{\substack{\text{Regarding } y \\ \text{as constant}}}{\int} Mdx + \underset{\substack{\text{Only those terms} \\ \text{not containing } x}}{\int} Ndy = c$

(5) **Solution by inspection :** If we can write the differential equation in the form $f(f_1(x,y))d(f_1(x,y)) + \phi(f_2(x,y))d(f_2(x,y)) + ... = 0$, then each term can be easily integrated separately. For this the following results must be memorized.

(i) $d(x + y) = dx + dy$ (ii) $d(xy) = xdy + ydx$

(iii) $d\left(\frac{x}{y}\right)=\frac{ydx-xdy}{y^2}$

(iv) $d\left(\frac{y}{x}\right)=\frac{xdy-ydx}{x^2}$

(v) $d\left(\frac{x^2}{y}\right)=\frac{2xydx-x^2dy}{y^2}$

(vi) $d\left(\frac{y^2}{x}\right)=\frac{2xydy-y^2dx}{x^2}$

(vii) $d\left(\frac{x^2}{y^2}\right)=\frac{2xy^2dx-2x^2ydy}{y^4}$

(viii) $d\left(\frac{y^2}{x^2}\right)=\frac{2x^2ydy-2xy^2dx}{x^4}$

(ix) $d\left(\tan^{-1}\frac{x}{y}\right)=\frac{ydx-xdy}{x^2+y^2}$

(x) $d\left(\tan^{-1}\frac{y}{x}\right)=\frac{xdy-ydx}{x^2+y^2}$

(xi) $d[\ln(xy)]=\frac{xdy+ydx}{xy}$

(xii) $d\left(\ln\left(\frac{x}{y}\right)\right)=\frac{ydx-xdy}{xy}$

(xiii) $d\left[\frac{1}{2}\ln(x^2+y^2)\right]=\frac{xdx+ydy}{x^2+y^2}$

(xiv) $d\left[\ln\left(\frac{y}{x}\right)\right]=\frac{xdy-ydx}{xy}$

(xv) $d\left(-\frac{1}{xy}\right)=\frac{xdy+ydx}{x^2y^2}$

(xvi) $d\left(\frac{e^x}{y}\right)=\frac{ye^xdx-e^xdy}{y^2}$

(xvii) $d\left(\frac{e^y}{x}\right)=\frac{xe^ydy-e^ydx}{x^2}$

(xviii) $d(x^my^n)=x^{m-1}y^{n-1}(mydx+nxdy)$

(xix) $d\left(\sqrt{x^2+y^2}\right)=\frac{xdx+y\,dy}{\sqrt{x^2+y^2}}$

(xx) $d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right)=\frac{x\,dy-y\,dx}{x^2-y^2}$

(xxi) $\frac{d\,[f(x,y)]^{1-n}}{1-n}=\frac{f'(x,y)}{(f(x,y))^n}$

Example: 23 The general solution of the differential equation $(x+y)dx+xdy=0$ is **[MP PET 1994, 95]**

(a) $x^2+y^2=c$ (b) $2x^2-y^2=c$ (c) $x^2+2xy=c$ (d) $y^2+2xy=c$

Solution: (c) We have $xdx+(ydx+xdy)=0 \Rightarrow xdx+d(xy)=0$

Integrating, $\frac{x^2}{2}+xy=\frac{c}{2}$

$\therefore\ x^2+2xy=c$

Example: 24 Solution of $y(2xy+e^x)dx=e^xdy$ is

(a) $yx^2+e^x=cy$ (b) $xy^2+e^x=cx$ (c) $xy^2+e^{-x}=c$ (d) None of these

Solution: (a) Re-writing the given equation,

$2xy^2dx+ye^xdx=e^xdy \Rightarrow 2x\,dx+\frac{ye^xdx-e^xdy}{y^2}=0 \Rightarrow d(x^2)+d\left(\frac{e^x}{y}\right)=0$

Integrating, $x^2+\frac{e^x}{y}=c$

$\therefore\ yx^2+e^x=cy$

Example: 25 Solution of $(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0$ is

(a) $x^3+y^3-6xy(x+y)=c$ (b) $x^3+y^3+6xy(x-y)=c$

(c) $x^3+y^3+6xy(x+y)=c$ (d) $x^3+y^3-6xy(x-y)=c$

Solution: (a) Comparing given equation with $Mdx + Ndy = 0$,

We get, $M = x^2 - 4xy - 2y^2$, $N = y^2 - 4xy - 2x^2$

$$\frac{\partial M}{\partial y} = -4x - 4y$$

$$\frac{\partial N}{\partial x} = -4y - 4x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So the given differential equation is exact.

Integrating *m w.r.t. x,* treating *y* as constant,

$$\int Mdx = \int (x^2 - 4xy - 2y^2)dx = \frac{x^3}{3} - 2x^2y - 2y^2x$$

Integrating *N w.r.t. y,* treating *x* as constant,

$$\int Ndy = \int (y^2 - 4xy - 2x^2)dy = \frac{y^3}{3} - 2xy^2 - 2x^2y = \frac{y^3}{3}$$; (omitting $-2xy^2 - 2x^2y$ which already occur in $\int Mdx$)

$\therefore$ Solution of the given equation is $\frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = \lambda \Rightarrow x^3 + y^3 - 6xy(x+y) = 3\lambda$

$\therefore$ $x^3 + y^3 - 6xy(x+y) = c$ $(3\lambda = c)$

8.6 Linear Differential Equation

(1) **Linear and non-linear differential equations** : A differential equation is a linear differential equation if it is expressible in the form $P_o \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + ... + P_{n-1}\frac{dy}{dx} + P_n y = Q$ where $P_0, P_1, P_2, ..., P_{n-1}, P_n$ and Q are either constants or functions of independent variable x.

Thus, if a differential equation when expressed in the form of a polynomial involves the derivatives and dependent variable in the first power and there are no product of these, and also the coefficient of the various terms are either constants or functions of the independent variable, then it is said to be linear differential equation. Otherwise, it is a non linear differential equation.

It follows from the above definition that a differential equation will be non-linear differential equation if (i) its degree is more than one (ii) any of the differential coefficient has exponent more than one. (iii) exponent of the dependent variable is more than one. (iv) products containing dependent variable and its differential coefficients are present.

(2) **Linear differential equation of first order :** The general form of a linear differential equation of first order is

$$\frac{dy}{dx} + Py = Q \quad .....(i)$$

Where P and Q are functions of x (or constants)

For example, $\frac{dy}{dx} + xy = x^3$, $x\frac{dy}{dx} + 2y = x^3$, $\frac{dy}{dx} + 2y = \sin x$ etc. are linear differential equations. This type of differential equations are solved when they are multiplied by a factor, which is called integrating factor, because by multiplication of this factor the left hand side of the differential equation (i) becomes exact differential of some function.

Multiplying both sides of (i) by $e^{\int P dx}$, we get $e^{\int P dx}\left(\frac{dy}{dx}+Py\right)=Q\,e^{\int P dx} \Rightarrow \frac{d}{dx}\left\{y\,e^{\int P dx}\right\}=Q\,e^{\int P dx}$

On integrating both sides *w. r. t. x*, we get ; $y\,e^{\int P dx} = \int Q\,e^{\int P dx}dx + C$

........(ii)

which is the required solution, where C is the constant of integration. $e^{\int P dx}$ is called the integrating factor. The solution (ii) in short may also be written as $y.(I.F.) = \int Q.(I.F.)dx + C$

(3) **Algorithm for solving a linear differential equation :**

Step (i) **:** Write the differential equation in the form $\frac{dy}{dx}+Py=Q$ and obtain P and Q.

Step (ii) **:** Find integrating factor (I.F.) given by $I.F. = e^{\int P dx}$.

Step (iii) **:** Multiply both sides of equation in step (i) by I.F.

Step (iv) **:** Integrate both sides of the equation obtained in step (iii) *w. r. t.* x to obtain $y(I.F.) = \int Q(I.F.)dx + C$

This gives the required solution.

(4) **Linear differential equations of the form** $\frac{dx}{dy}+Rx=S$. Sometimes a linear differential equation can be put in the form $\frac{dx}{dy}+Rx=S$ where R and S are functions of y or constants. Note that y is independent variable and x is a dependent variable.

(5) **Algorithm for solving linear differential equations of the form** $\frac{dx}{dy}+Rx=S$

Step (i) **:** Write the differential equation in the form $\frac{dx}{dy}+Rx=S$ and obtain R and S.

Step (ii) **:** Find I.F. by using $I.F. = e^{\int R\,dy}$

Step (iii) **:** Multiply both sides of the differential equation in step (i) by *I.F.*

Step (iv) **:** Integrate both sides of the equation obtained in step (iii) *w. r. t. y* to obtain the solution given by

$x(I.F.) = \int S\,(I.F.)dy + C$ where C is the constant of integration.

(6) **Equations reducible to linear form (Bernoulli's differential equation) :** The differential equation of type $\frac{dy}{dx}+Py=Qy^n$

......(i)

Where P and Q are constants or functions of x alone and n is a constant other than zero or unity, can be reduced to the linear form by dividing by y^n and then putting $y^{-n+1}=v$, as explained below.

Dividing both sides of (i) by y^n, we get $\quad y^{-n}\dfrac{dy}{dx}+Py^{-n+1}=Q$

Putting $y^{-n+1}=v$ so that $(-n+1)y^{-n}\dfrac{dy}{dx}=\dfrac{dv}{dx}$, we get $\dfrac{1}{-n+1}\dfrac{dv}{dx}+Pv=Q \Rightarrow \dfrac{dv}{dx}+(1-n)Pv=(1-n)Q$ which is a linear differential equation.

Remark : If $n=1$, then we find that the variables in equation (i) are separable and it can be easily integrated by the method discussed in variable separable from.

(7) **Differential equation of the form :** $\dfrac{dy}{dx}+P\phi(y)=Q\psi(y)$

where P and Q are functions of x alone or constants.

Dividing by $\psi(y)$, we get $\dfrac{1}{\psi(y)}\dfrac{dy}{dx}+\dfrac{\phi(y)}{\psi(y)}P=Q$

Now put $\dfrac{\phi(y)}{\psi(y)}=v$, so that $\dfrac{d}{dx}\left\{\dfrac{\phi(y)}{\psi(y)}\right\}=\dfrac{dv}{dx}$ or $\dfrac{dv}{dx}=k\cdot\dfrac{1}{\psi(y)}\dfrac{dy}{dx}$, where k is constant

We get $\dfrac{dv}{dx}+kPv=kQ$

Which is linear differential equation.

Example: 26 Which of the following is a linear differential equation

(a) $\left(\dfrac{d^2y}{dx^2}\right)^2+x^2\left(\dfrac{dy}{dx}\right)^2=0$ (b) $y=\dfrac{dy}{dx}+\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$ (c) $\dfrac{dy}{dx}+\dfrac{y}{x}=\log x$ (d) $y\dfrac{dy}{dx}-4=x$

Solution: (c) (a), (b), (d) do not fulfill the criteria of a linear differential equation but (c) do.

$\dfrac{dy}{dx}+\dfrac{y}{x}=\log x$ is a linear differential equation.

Example: 27 Find the integral factor of equation $(x^2+1)\dfrac{dy}{dx}+2xy=x^2-1$ **[UPSEAT 2002]**

(a) x^2+1 (b) $\dfrac{2x}{x^2+1}$ (c) $\dfrac{x^2-1}{x^2+1}$ (d) None of these

Solution: (a) Given equation may be written as $\dfrac{dy}{dx}+\dfrac{2x}{x^2+1}y=\dfrac{x^2-1}{x^2+1}$

Comparing with $\dfrac{dy}{dx}+Py=Q$,

$P=\dfrac{2x}{x^2+1}$

I.F. $=e^{\int Pdx}=e^{\int\frac{2xdx}{1+x^2}}=e^{\ln(1+x^2)}=1+x^2$

Example: 28 The solution of $\dfrac{dy}{dx}+2y\tan x=\sin x$ is

(a) $y\sec^3 x=\sec^2 x+c$ (b) $y\sec^2 x=\sec x+c$ (c) $y\sin x=\tan x+c$ (d) None of these

Solution: (b) Comparing with $\frac{dy}{dx}+Py=Q$,

$P=2\tan x$, $Q=\sin x$

I.F. $=e^{\int 2\tan x dx}=e^{2\ln\sec x}=e^{\ln\sec^2 x}=\sec^2 x$

Multiplying given equation by I.F. and integrating, $y\sec^2 x=\int \sin x.\sec^2 x dx=\int \sec x\tan x dx$

$\therefore\ y\sec^2 x=\sec x+c$

Example: 29 The solution of the differential equation $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$ is **[AIEEE 2003]**

(a) $(x-2)=ke^{\tan^{-1}y}$ (b) $2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+k$

(c) $xe^{\tan^{-1}y}=\tan^{-1}y+k$ (d) $xe^{2\tan^{-1}y}=e^{\tan^{-1}y}+k$

Solution: (b) We have $(x-e^{\tan^{-1}y})\frac{dy}{dx}=-(1+y^2)\Rightarrow\frac{dx}{dy}=-\left(\frac{x-e^{\tan^{-1}y}}{1+y^2}\right)\Rightarrow\frac{dx}{dy}+\frac{1}{1+y^2}x=\frac{e^{\tan^{-1}y}}{1+y^2}$ ....(i)

This is a linear differential equation of the form $\frac{dx}{dy}+R(y).x=S(y)$

$R=\frac{1}{1+y^2}$, $S=\frac{e^{\tan^{-1}y}}{1+y^2}$

Integrating factor $=e^{\int Rdy}=e^{\int\frac{dy}{1+y^2}}=e^{\tan^{-1}y}$

Multiplying (i) by I.F. and integrating, $xe^{\tan^{-1}y}=\int\frac{e^{\tan^{-1}y}}{1+y^2}\cdot e^{\tan^{-1}y}dy=\int\frac{(e^{\tan^{-1}y})^2dy}{1+y^2}=\frac{(e^{\tan^{-1}y})^2}{2}+\frac{k}{2}$

$\therefore\ 2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+k$

Example: 30 Solution of $\frac{dy}{dx}-y\tan x=-y^2\sec x$ is

(a) $y\sec x=\tan x+c$ (b) $\frac{\sec x}{y}=\tan x+c$ (c) $y\cos x=\tan x+c$ (d) None of these

Solution: (b) Re-writing the given equation, $y^{-2}\frac{dy}{dx}-y^{-1}\tan x=-\sec x$

Let $y^{-1}=v\Rightarrow -y^{-2}\frac{dy}{dx}=\frac{dv}{dx}$

$\therefore\ \frac{dv}{dx}+\tan x.v=\sec x$ ....(i)

I.F. $=e^{\int\tan x}=e^{\ln\sec x}=\sec x$

Multiplying (i) by $\sec x$ and integrating, $v\sec x=\int\sec^2 x dx=\tan x+c$

$\therefore\ \frac{\sec x}{y}=\tan x+c$

Example: 31 The solution of $\frac{dz}{dx}+\frac{z}{x}\log z=\frac{z}{x^2}(\log z)^2$ is

(a) $\left(\frac{1}{\log z}\right)x=2-x^2c$ (b) $\left(\frac{1}{\log z}\right)x=2+x^2c$ (c) $\left(\frac{1}{\log z}\right)x=x^2c$ (d) $\left(\frac{1}{\log z}\right)x=\frac{1}{2}+cx^2$

Solution: (d) Dividing the given equation by $z(\log z)^2$, $\frac{1}{z(\log z)^2}\frac{dz}{dx}+\frac{1}{x}\frac{1}{\log z}=\frac{1}{x^2}$

Let $\frac{1}{\log z}=t \Rightarrow -\frac{1}{(\log z)^2}\cdot\frac{1}{z}\frac{dz}{dx}=\frac{dt}{dx}$

$\therefore -\frac{dt}{dx}+\frac{t}{x}=\frac{1}{x^2}$

$\Rightarrow \frac{dt}{dx}-\frac{t}{x}=-\frac{1}{x^2}$ ....(i)

$\text{I.F.}=e^{\int -\frac{dx}{x}}=e^{-\ln x}=e^{\ln\frac{1}{x}}=\frac{1}{x}$

Multiplying (i) by $\frac{1}{x}$ and integrating, $\frac{t}{x}=\int -\frac{1}{x^3}dx=\frac{1}{2x^2}+c \Rightarrow \frac{1}{x\log z}=\frac{1}{2x^2}+c$

$\therefore \left(\frac{1}{\ln z}\right)x=\left(\frac{1}{2}\right)+cx^2$

8.7 Application of Differential Equation

Differential equation is applied in various practical fields of life. It is used to define various physical laws and quantities. It is widely used in physics, chemistry, engineering etc.

Some important fields of application are ;

(i) Rate of change (ii) Geometrical problems etc.

Differential equation is used for finding the family of curves for which some conditions involving the derivatives are given.

Equation of the tangent at a point (x, y) to the curve $y = f(x)$ is given by $Y-y=\frac{dy}{dx}(X-x)$ ....(i)

and equation of normal at (x, y) is $Y-y=-\frac{1}{\left(\frac{dy}{dx}\right)}(X-x)$ ....(ii)

The tangent meets X-axis at $\left(x-\frac{y}{\left(\frac{dy}{dx}\right)},0\right)$ and Y-axis at $\left(0,y-x\frac{dy}{dx}\right)$

The normal meets X-axis at $\left(x+y\frac{dy}{dx},0\right)$ and Y-axis at $\left(0,y+\frac{x}{\left(\frac{dy}{dx}\right)}\right)$

Example: 32 A particle moves in a straight line with a velocity given by $\frac{dx}{dt}=(x+1)$ (x is the distance described). The time taken by a particle to transverse a distance of 99 *metres*

(a) $\log_{10} e$ (b) $2\log_e 10$ (c) $\log_{10} e$ (d) $\frac{1}{2}\log_{10} e$

Solution: (b) We have $\frac{dx}{x+1}=dt$

Integrating, $\int_0^{99} \frac{dx}{x+1} = \int_0^t dt \Rightarrow [\ln(x+1)]_0^{99} = t$

$\therefore\ t = \ln 100 = \log_e(10)^2 = 2\log_e 10$

Example: 33 The slope of the tangent at (x, y) to a curve passing through $\left(1, \frac{\pi}{4}\right)$ as given by $\frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$, then the equation of the curve is **[Kurukshetra CEE 2002]**

(a) $y = \tan^{-1}\left[\log\left(\frac{e}{x}\right)\right]$ (b) $y = x\tan^{-1}\left[\log\left(\frac{x}{e}\right)\right]$ (c) $y = x\tan^{-1}\left[\log\left(\frac{e}{x}\right)\right]$ (d) None of these

Solution: (c) We have $\frac{dy}{dx} = \frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$

Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \Rightarrow v + x\frac{dv}{dx} = v - \cos^2 v \Rightarrow x\frac{dv}{dx} = -\cos^2 v \Rightarrow \sec^2 v\,dv = -\frac{dx}{x} \Rightarrow \tan v = -\ln x + c$

$\Rightarrow \tan(y/x) = -\ln x + c$

For $x = 1$, $y = \pi/4$

$\Rightarrow \tan \pi/4 = -\ln 1 + c \Rightarrow 1 = 0 + c$

$\therefore\ c = 1$

$\therefore\ \tan(y/x) = 1 - \ln x$

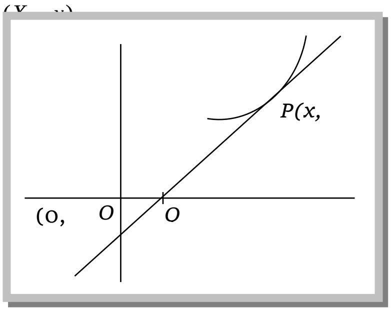
$\Rightarrow y/x = \tan^{-1}(1 - \ln x) = \tan^{-1}(\ln e - \ln x) = \tan^{-1}\left[\ln\left(\frac{e}{x}\right)\right]$

$\therefore\ y = x\tan^{-1}\left[\ln\left(\frac{e}{x}\right)\right]$

Example: 34 The equation of the curve which is such that the portion of the axis of x cut off between the origin and tangent at any point is proportional to the ordinate of that point (b is constant of proportionality)

(a) $y = \frac{x}{(a - b\log x)}$ (b) $\log x = by^2 + a$ (c) $x^2 = y(a - b\log y)$ (d) None of these

Solution: (d) Tangent at $P(x, y)$ to the curve $y = f(x)$ may be expressed as $Y - y = \frac{dy}{dx}(X - x)$



$\therefore\ Q = \left(x - y\frac{dx}{dy}, 0\right)$

As per question, $OQ \propto y$

$\Rightarrow x - y\frac{dx}{dy} \propto y \Rightarrow x - y\frac{dx}{dy} = by \Rightarrow \frac{x}{y} - \frac{dx}{dy} = b$

$\therefore\ \frac{dx}{dy} = \frac{x}{y} - b$

Let $\frac{x}{y} = v \Rightarrow x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy} \Rightarrow \frac{x}{y} - b = v + y\frac{dv}{dy} \Rightarrow v - b = v + y\frac{dv}{dy} \Rightarrow -b = y\frac{dv}{dy} \Rightarrow -b\frac{dy}{y} = dv$

Integrating, $\int dv = -b\int\frac{dy}{y} \Rightarrow v = -b\ln y + a \Rightarrow \frac{x}{y} = a - b\ln y$ (a, an arbitrary constant)

$\therefore\ x = y(a - b\ln y)$

8.8 Miscellaneous Differential Equation

(1) **A special type of second order differential equation :** $\frac{d^2y}{dx^2} = f(x)$ ....(i)

Equation (i) may be re-written as $\frac{d}{dx}\left(\frac{dy}{dx}\right) = f(x) \Rightarrow d\left(\frac{dy}{dx}\right) = f(x)dx$

Integrating, $\frac{dy}{dx} = \int f(x)dx + c_1$ *i.e.* $\frac{dy}{dx} = F(x) + c_1$ ....(ii)

Where $F(x) = \int f(x)dx + c_1 dx$

From (ii), $dy = f(x)dx + c_1 dx$

Integrating, $y = \int F(x)dx + c_1 x + c_2$

$\therefore\ y = H(x) + c_1 x + c_2$

where $\boxed{H(x) = \int F(x)dx}$ c_1 and c_2 are arbitrary constants.

(2) **Particular solution type problems :** To solve such a problem, we proceed according to the type of the problem (*i.e.* variable-separable, linear, exact, homogeneous etc.) and then we apply the given conditions to find the particular values of the arbitrary constants.

Example: 35 The solution of the equation $x^2 \frac{d^2y}{dx^2} = \ln x$ when $x = 1$, $y = 0$ and $\frac{dy}{dx} = -1$ is **[Orissa JEE 2003]**

(a) $\frac{1}{2}(\ln x)^2 + \ln x$ (b) $\frac{1}{2}(\ln x)^2 - \ln x$ (c) $-\frac{1}{2}(\ln x)^2 + \ln x$ (d) $-\frac{1}{2}(\ln x)^2 - \ln x$

Solution: (d) We have $\frac{d^2y}{dx^2} = \frac{\ln x}{x^2} \Rightarrow d\left(\frac{dy}{dx}\right) = \frac{\ln x}{x^2}dx$

Integrating, $\frac{dy}{dx} = \int \ln x\, d\left(-\frac{1}{x}\right) = -\frac{\ln x}{x} + \int \frac{1}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{x} + c \Rightarrow \frac{dy}{dx} = -\frac{1+\ln x}{x} + c$

When $x = 1$, $\frac{dy}{dx} = -1$

$\therefore -1 = -1 + c \Rightarrow c = 0$

$\therefore\ \frac{dy}{dx} = -\frac{1+\ln x}{x} \Rightarrow dy = -\frac{1+\ln x}{x}dx \Rightarrow -\int dy = +\int \frac{dx}{x} + \int \ln x . \frac{1}{x}dx \Rightarrow -y = \ln x + \frac{1}{2}(\ln x)^2 + \lambda$

$y = 0$ when $x = 1$

$\therefore\ 0 = 0 + 0^2 + \lambda \Rightarrow \lambda = 0 \Rightarrow -y = \ln x + \frac{1}{2}(\ln x)^2$

$\therefore\ y = -\frac{1}{2}(\ln x)^2 - \ln x$

Example: 36 A continuously differentiable function $\phi(x)$ in $(0, \pi)$ satisfying $y' = 1 + y^2$, $y(0) = 0 = y(\pi)$ is

(a) $\tan x$ (b) $x(x - \pi)$ (c) $(x-\pi)(1-e^x)$ (d) Not possible

Solution: (d) For $\phi(x) = y$, $y' = 1 + y^2 \Rightarrow \frac{dy}{dx} = 1 + y^2 \Rightarrow \int \frac{dy}{1+y^2} = \int dx \Rightarrow \tan^{-1} y = x + c$

$\therefore\ y = \tan(x + c)$

i.e., $\phi(x) = \tan(x + c)$

As $y(0) = 0$, $0 = \tan c$ and $y(\pi) = 0 \Rightarrow 0 = \tan(\pi + c) = \tan c$

$\therefore\ c = 0$

$\therefore\ \phi(x) = y = \tan x$.

But $\tan x$ is not continuous in $(0, \pi)$

Since $\tan\frac{\pi}{2}$ is not defined.

Hence there exists not a function satisfying the given condition.