Exercise 6.7

Answer 1E.

(a)

Consider the expression,

$\sinh(0)$.

Recollect the definition of the hyperbolic sine function,

$\sinh(x) = \frac{e^x - e^{-x}}{2}$.

Calculate the value of the expression using the definition of the hyperbolic sine function as follows:

$\sinh(0) = \frac{e^0 - e^{-0}}{2}$

$= \frac{e^0 - e^0}{2}$ Since, $-0 = 0$

$= \frac{1-1}{2}$ Since, $e^0 = 1$

$= \frac{0}{2}$ Subtracting

$= 0$ Simplifying

Therefore, the value of the given expression is $\boxed{\sinh(0) = 0}$.

(a)

Consider the expression,

$\cosh(0)$.

Recollect the definition of the hyperbolic cosine function,

$\cosh(x) = \frac{e^x + e^{-x}}{2}$.

Calculate the value of the expression using the definition of the hyperbolic cosine function as follows:

$\cosh(0) = \frac{e^0 + e^{-0}}{2}$

$= \frac{e^0 + e^0}{2}$ Since, $-0 = 0$

$= \frac{1+1}{2}$ Since, $e^0 = 1$

$= \frac{2}{2}$ Adding

$= 1$ Simplifying

Therefore, the value of the given expression is $\boxed{\cosh(0) = 1}$.

Answer 2E.

(A)
$$\tanh 0 = \frac{\sinh 0}{\cosh 0}$$
$$= \frac{\left(e^0 - e^0\right)/2}{\left(e^0 + e^0\right)/2}$$
$$= \frac{\left(e^0 - e^0\right)}{\left(e^0 + e^0\right)}$$
$$= \frac{0}{2}$$
Thus $\boxed{\tanh 0 = 0}$

(B)
$$\tanh 1 = \frac{\sinh 1}{\cosh 1}$$
$$= \frac{\left(e^1 - e^{-1}\right)/2}{\left(e^1 + e^{-1}\right)/2}$$
$$= \frac{\left(e - e^{-1}\right)}{\left(e + e^{-1}\right)}$$
$$= \frac{e^{-1}\left(e^2 - 1\right)}{e^{-1}\left(e^2 + 1\right)}$$
$$= \frac{e^2 - 1}{e^2 + 1} \approx 0.76159$$
Thus $\boxed{\tanh 1 \approx 0.76159}$

Answer 3E.

(A)
$$\sinh\left(\ln 2\right) = \frac{e^{\ln 2} - e^{-\ln 2}}{2}$$
$$= \frac{e^{\ln 2} - 1/e^{\ln 2}}{2}$$
$$= \frac{2 - 1/2}{2}$$
$$= \boxed{3/4}$$

(B)
$$\sinh 2 = \frac{e^2 - e^{-2}}{2}$$
$$= \boxed{\frac{e^4 - 1}{2e^2} \approx 3.626860}$$

Answer 4E.

(A) $\cosh 3 = \dfrac{e^3 + e^{-3}}{2}$

$\Rightarrow \boxed{\cosh 3 \approx 10.06766}$

(B) $\cosh(\ln 3) = \dfrac{e^{\ln 3} + e^{-\ln 3}}{2}$

$= \dfrac{e^{\ln 3} + 1/e^{\ln 3}}{2}$

$= \dfrac{3 + 1/3}{2}$

$= \dfrac{10}{6} = \boxed{5/3}$

Answer 5E.

(A) $\sec h(0) = \dfrac{1}{\cosh 0}$

$= \dfrac{1}{\dfrac{e^0 + e^0}{2}}$

$= \dfrac{2}{1+1}$

Thus $\boxed{\sec h(0) = 1}$

(B) Since $\cosh 0 = 1$

Then $\boxed{\cosh^{-1} 1 = 0}$

Answer 6E.

(A) We have $\sin hx = \dfrac{e^x - e^{-x}}{2}$

Putting $x = 1$

$\sin h1 = \dfrac{e^1 - e^{-1}}{2}$

$\Rightarrow \sin h1 = \dfrac{1}{2}\left(e - \dfrac{1}{e}\right) \approx 1.175$

So $\boxed{\sin h1 \approx 1.175}$

(B) We have $\sin h^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$, $x \in R$

Putting x = 1

$\sin h^{-1} 1 = \ln\left(1 + \sqrt{1+1}\right)$

Or $\sin h^{-1} 1 = \ln\left(1 + \sqrt{2}\right)$

Or $\boxed{\sin h^{-1} 1 \approx 0.881}$

Answer 7E.

By the definition of hyperbolic function, we have

$\sinh(x) = \frac{e^x - e^{-x}}{2}$. Now replacing x by −x we get

$$\sinh(-x) = \frac{e^{(-x)} - e^{-(-x)}}{2}$$
$$= \frac{e^{-x} - e^{x}}{2}$$
$$= -\frac{(e^x - e^{-x})}{2}$$
$$= \boxed{-\sinh x}.$$

Thus $\boxed{\sinh(-x) = -\sinh x}$. Therefore sinhx is an odd function.

Answer 8E.

Consider the following expression:

$\tan(\sin^{-1} x)$

The objective is to simplify the expression.

Let $y = \sin^{-1}(x)$ then $\sin y = x$.

The domain of the inverse sine function is $-1 \le x \le 1$ and range is $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$.

To find $\tan y$ but it is easier to find first $\cos y$.

Use the trigonometric identity.

$$\sin^2 y + \cos^2 y = 1$$
$$\cos^2 y = 1 - \sin^2 y$$
$$= 1 - x^2$$
$$\cos y = \sqrt{1 - x^2} \qquad \left(\text{Since } \cos y > 0 \text{ for } \frac{-\pi}{2} \le y \le \frac{\pi}{2}\right)$$

Find $\tan(\sin^{-1} x)$.

$$\tan(\sin^{-1} x) = \tan y$$
$$= \frac{\sin y}{\cos y} \qquad \left(\text{since } \sin y = x, \cos y = \sqrt{1 - x^2}\right)$$
$$= \frac{x}{\sqrt{1 - x^2}}$$

Therefore, $\boxed{\tan(\sin^{-1} x) = \frac{x}{\sqrt{1 - x^2}}}$.

Answer 9E.

$$\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$$
$$= \frac{e^x + e^{-x} + e^x - e^{-x}}{2}$$
$$= \frac{2e^x}{2}$$

Therefore

$\boxed{\cosh x + \sinh x = e^x}$

Answer 10E.

$$\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}$$
$$= \frac{e^x + e^{-x} - e^x + e^{-x}}{2}$$
$$= \frac{2e^{-x}}{2}$$

Therefore,

$$\boxed{\cosh x - \sinh x = e^{-x}}$$

Answer 11E.

Consider the identity,

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

Prove the identity as follows:

Since $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh y = \frac{e^y - e^{-y}}{2}$

Then,

$$\sinh x \cosh y + \cosh x \sinh y$$
$$= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$$
$$= \frac{1}{4}\left(\left(e^x - e^{-x}\right)\left(e^y + e^{-y}\right)\right) + \frac{1}{4}\left(\left(e^x + e^{-x}\right)\left(e^y - e^{-y}\right)\right)$$
$$= \frac{1}{4}\left[e^x \cdot e^y - \cancel{e^{-x} \cdot e^y} + \cancel{e^x \cdot e^{-y}} - e^{-x} \cdot e^{-y} + e^x \cdot e^y + \cancel{e^{-x} \cdot e^y} - \cancel{e^x \cdot e^{-y}} - e^{-x} \cdot e^{-y}\right]$$
$$= \frac{1}{4}\left[2e^x \cdot e^y - 2e^{-x} \cdot e^{-y}\right]$$

$= \frac{1}{4}\left[2e^{x+y} - 2e^{-x-y}\right]$ Use $x^m \cdot x^n = x^{m+n}$

$= \frac{1}{\cancel{4}_2} \cdot \cancel{2}\left[e^{x+y} - e^{-x-y}\right]$ Factor out 2 and simplify

$$= \frac{e^{x+y} - e^{-(x+y)}}{2}$$

$= \sinh(x+y)$ Since $\sinh x = \frac{e^x - e^{-x}}{2}$

Hence,

$\boxed{\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y}$.

Answer 12E.

We have $\cos hx = \frac{e^x + e^{-x}}{2}, \sin hx = \frac{e^x - e^{-x}}{2}$

And so $\cos hy = \frac{e^y + e^{-y}}{2}, \sin hy = \frac{e^y - e^{-y}}{2}$

Then

$$\cos hx.\cos hy+\sin hx.\sin hy=\left(\frac{e^x+e^{-x}}{2}\right).\left(\frac{e^y+e^{-y}}{2}\right)+\left(\frac{e^x-e^{-x}}{2}\right).\left(\frac{e^y-e^{-y}}{2}\right)$$

$$=\frac{e^xe^y+\cancel{e^{-x}e^y}+\cancel{e^xe^{-y}}+e^{-x}e^{-y}+e^xe^y-\cancel{e^{-x}e^y}-\cancel{e^xe^{-y}}+e^{-x}e^{-y}}{4}$$

$$=\frac{2e^xe^y+2e^{-x}e^{-y}}{4}$$

$$=\frac{2\left(e^{(x+y)}+e^{-(x+y)}\right)}{4} \qquad \left[e^xe^y=e^{x+y}\right]$$

$$=\frac{e^{(x+y)}+e^{-(x+y)}}{2}$$

$$=\cos h(x+y) \qquad \text{(By the definition)}$$

So we have $\boxed{\cos h(x+y)=\cos hx.\cos hx+\sin hx.\sin hy}$

Answer 13E.

We have $\cot hx=\dfrac{\cos hx}{\sin hx}$

Then $\cot h^2x=\dfrac{\cos h^2x}{\sin h^2x}$

Then $\cot h^2x-1=\dfrac{\cos h^2x}{\sin h^2x}-1$

$$=\frac{\left(\frac{e^x+e^{-x}}{2}\right)^2}{\left(\frac{e^x-e^{-x}}{2}\right)^2}-1$$

$$=\frac{\left(e^x+e^{-x}\right)^2-\left(e^x-e^{-x}\right)^2}{\left(e^x-e^{-x}\right)^2}$$

$$=\frac{\cancel{e^{2x}}+2+\cancel{e^{-2x}}-\cancel{e^{2x}}-\cancel{e^{-2x}}+2}{\left(e^x-e^{-x}\right)^2}$$

Therefore

$$\coth^2x-1=\frac{4}{\left(e^x-e^{-x}\right)^2}$$

$$=\left[\frac{2}{e^x-e^{-x}}\right]^2$$

$$=\left[\frac{1}{\sin hx}\right]^2$$

$$=\csc h^2x \qquad \left[\csc hx=\frac{1}{\sin hx}\right]$$

So $\boxed{\cot h^2x-1=\csc h^2x}$

Answer 14E.

Another approach

First take left hand side

$$\tanh(x+y)=\frac{\sinh(x+y)}{\cosh(x+y)}$$

$$=\frac{\left[e^{(x+y)}-e^{-(x+y)}\right]/2}{\left[e^{(x+y)}+e^{-(x+y)}\right]/2}$$

$$=\frac{e^{(x+y)}-e^{-(x+y)}}{e^{(x+y)}+e^{-(x+y)}} \qquad \ldots\ldots\ldots\ldots (1)$$

Now we take right hand side

$$\frac{\tanh x+\tanh y}{1+\tanh x\tanh y}=\frac{\frac{\sinh x}{\cosh x}+\frac{\sinh y}{\cosh y}}{1+\frac{\sinh x}{\cosh x}\frac{\sinh y}{\cosh y}}$$

$$=\frac{\frac{\sinh x\cosh y+\cosh x\sinh y}{\cosh x\cosh y}}{\frac{\cosh x\cosh y+\sinh x\sinh y}{\cosh x\cosh y}}$$

$$=\frac{\sinh x\cosh y+\cosh x\sinh y}{\cosh x\cosh y+\sinh x\sinh y}$$

$$=\frac{\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^y+e^{-y}}{2}\right)+\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^y-e^{-y}}{2}\right)}{\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^y+e^{-y}}{2}\right)+\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^y-e^{-y}}{2}\right)}$$

$$=\frac{\left(e^x-e^{-x}\right)\left(e^y+e^{-y}\right)+\left(e^x+e^{-x}\right)\left(e^y-e^{-y}\right)}{\left(e^x+e^{-x}\right)\left(e^y+e^{-y}\right)+\left(e^x-e^{-x}\right)\left(e^y-e^{-y}\right)}$$

$$=\frac{\left(e^xe^y-e^{-x}e^y+e^xe^{-y}-e^{-x}e^{-y}\right)+\left(e^xe^y+e^{-x}e^y-e^xe^{-y}-e^{-x}e^{-y}\right)}{\left(e^xe^y+e^{-x}e^y+e^xe^{-y}+e^{-x}e^{-y}\right)+\left(e^xe^y-e^{-x}e^y-e^xe^{-y}+e^{-x}e^{-y}\right)}$$

$$=\frac{2e^xe^y-2e^{-x}e^{-y}}{2e^xe^y+2e^{-x}e^{-y}}$$

$$=\frac{e^{(x+y)}-e^{-(x+y)}}{e^{(x+y)}+e^{-(x+y)}} \qquad \ldots\ldots\ldots (2)$$

From (1) and (2), we have

$$\tanh(x+y)=\frac{\tanh x+\tanh y}{1+\tanh x\tanh y}$$

Answer 15E.

Consider the following identity:

$$\sinh 2x=2\sinh x\cosh x$$

The objective is to show that the identity, use the formula.

$$\sinh x=\frac{e^x-e^{-x}}{2},\cosh x=\frac{e^x+e^{-x}}{2}$$

Find $\sinh 2x$:

$$\sinh 2x=\frac{e^{2x}-e^{-2x}}{2}$$

$$=\frac{\left(e^x\right)^2-\left(e^{-x}\right)^2}{2}$$

$$=\left(\frac{\left(e^x-e^{-x}\right)\left(e^x+e^{-x}\right)}{2}\right) \quad \left(a^2-b^2=(a-b)(a+b)\right)$$

$$=\frac{2}{2}\left(\frac{\left(e^x-e^{-x}\right)\left(e^x+e^{-x}\right)}{2}\right) \quad (\text{Multiply and divided by } 2)$$

$$=2\left(\frac{\left(e^x-e^{-x}\right)\left(e^x+e^{-x}\right)}{2\cdot 2}\right)$$

$$=2\left(\frac{e^x-e^{-x}}{2}\cdot\frac{e^x+e^{-x}}{2}\right)$$

$$=2\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^x+e^{-x}}{2}\right) \quad (\text{use the above formulas})$$

$$=2\sinh x\cosh x$$

Therefore, $\boxed{\sinh 2x=2\sinh x\cosh x}$

Hence, the identity is proved.

Answer 16E.

We have $\cosh 2x = \dfrac{e^{2x}+e^{-2x}}{2}$ [By definition]

Using the identity

$$2\left(A^2+B^2\right)=\left(A+B\right)^2+\left(A-B\right)^2$$

We have, $\cosh 2x = \dfrac{\left(e^{x}+e^{-x}\right)^2+\left(e^{x}-e^{-x}\right)^2}{4}$

$$=\frac{\left(e^{x}+e^{-x}\right)^2}{4}+\frac{\left(e^{x}-e^{-x}\right)^2}{4}$$

$$=\left[\frac{e^{x}+e^{-x}}{2}\right]^2+\left[\frac{e^{x}-e^{-x}}{2}\right]^2$$

$=\left[\cosh x\right]^2+\left[\sinh x\right]^2$ [By definition]

Or $\boxed{\cosh 2x=\cosh^2 x+\sinh^2 x}$

Answer 17E.

We have $\tanh x = \dfrac{\sinh x}{\cosh x}$

So $\tanh(\ln x)=\dfrac{\sinh(\ln x)}{\cosh(\ln x)}$

$$=\frac{\dfrac{\left(e^{\ln x}-e^{-\ln x}\right)}{2}}{\dfrac{\left(e^{\ln x}+e^{-\ln x}\right)}{2}}$$

$$=\frac{e^{\ln x}-e^{-\ln x}}{e^{\ln x}+e^{-\ln x}}$$

$$=\frac{\cancel{e^{-\ln x}}\left(e^{2\ln x}-1\right)}{\cancel{e^{-\ln x}}\left(e^{2\ln x}+1\right)}$$

$=\dfrac{e^{2\ln x}-1}{e^{2\ln x}+1}$ $\left[r\ln x=\ln x^r\right]$

$=\dfrac{x^2-1}{x^2+1}$ $\left[e^{\ln x}=x\right]$

So $\boxed{\tanh(\ln x)=\dfrac{x^2-1}{x^2+1}}$

Answer 18E.

We have $\tanh x = \dfrac{\sinh x}{\cosh x}$

Then $\dfrac{1+\tanh x}{1-\tanh x}=\dfrac{1+\dfrac{\sinh x}{\cosh x}}{1-\dfrac{\sinh x}{\cosh x}}$

$$=\frac{\cosh x+\sinh x}{\cosh x-\sinh x}$$

Using the definitions of sinh and cosh functions.

$$\frac{1+\tan hx}{1-\tan hx}=\frac{\frac{e^x+e^{-x}}{2}+\frac{e^x-e^{-x}}{2}}{\frac{e^x+e^{-x}}{2}-\frac{e^x-e^{-x}}{2}}$$

$$=\frac{e^x+e^{-x}+e^x-e^{-x}}{e^x+e^{-x}-e^x+e^{-x}}$$

$$=\frac{2e^x}{2e^{-x}}$$

$$=e^x.e^x$$

$$=e^{2x}$$

So $\boxed{\frac{1+\tan hx}{1-\tan hx}=e^{2x}}$

Answer 19E.

$$(\cos hx+\sin hx)^n=\left(\frac{e^x+e^{-x}}{2}+\frac{e^x-e^{-x}}{2}\right)^n$$

$$=\left(\frac{e^x+e^{-x}+e^x-e^{-x}}{2}\right)^n$$

$$=\left(\frac{2e^x}{2}\right)^n$$

$$=e^{nx}$$

$$=\frac{2e^{nx}}{2}$$ [Divide and multiply by 2]

Adding and subtracting e^{-nx} in numerator

$$(\cos hx+\sin hx)^n=\frac{2e^{nx}+e^{-nx}-e^{-nx}}{2}$$

$$=\frac{e^{nx}+e^{-nx}+e^{nx}-e^{-nx}}{2}$$

$$=\frac{e^{nx}+e^{-nx}}{2}+\frac{e^{nx}-e^{-nx}}{2}$$

$$=\cos hnx+\sin hnx$$ [By definition]

So $\boxed{(\cos hx+\sin hx)^n=\cos hnx+\sin hnx}$

Answer 20E.

Consider the value of the hyperbolic function,

$$\tanh x=\frac{12}{13}$$

The objective is to find the values of the other hyperbolic functions at *x*.

Use the trigonometric identity,

$$\operatorname{sech}^2 x=1-\tanh^2 x$$

$$\operatorname{sech}^2 x=1-\frac{144}{169} \qquad \left(\text{since } \tanh x=\frac{12}{13}\right)$$

$$=\frac{25}{169}$$

$$\operatorname{sech} x=\frac{5}{13}$$

Hence, the value of the hyperbolic function is $\boxed{\operatorname{sech} x=\frac{5}{13}}$

The hyperbolic function $\cosh x$ can be written as,

$$\cosh x = \frac{1}{\operatorname{sech} x}$$

$$= \frac{13}{5} \qquad \left(\text{since } \operatorname{sech} x = \frac{5}{13}\right)$$

Hence, the value of the hyperbolic function is $\boxed{\cosh x = \frac{13}{5}}$

Use the trigonometric identity,

$$\sinh^2 x = \cosh^2 x - 1$$

$$\sinh^2 x = \frac{169}{25} - 1 \qquad \left(\text{since } \cosh x = \frac{13}{5}\right)$$

$$= \frac{144}{25}$$

$$\sinh x = \frac{12}{5}$$

Hence, the value of the hyperbolic function is $\boxed{\sinh x = \frac{12}{5}}$

The hyperbolic function $\operatorname{csch} x$ can be written as,

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

$$= \frac{5}{12} \qquad \left(\sinh x = \frac{12}{5}\right)$$

Hence, the value of the hyperbolic function is $\boxed{\operatorname{csch} x = \frac{5}{12}}$

The hyperbolic function $\coth x$ can be written as,

$$\coth x = \frac{1}{\tanh x}$$

$$= \frac{15}{12} \qquad \left(\text{since } \tanh x = \frac{12}{15}\right)$$

Hence, the value of the hyperbolic function is $\boxed{\coth x = \frac{15}{12}}$.

Answer 21E.

Consider

$$\cosh x = \frac{5}{3} \text{ and } x > 0$$

Need to find the other hyperbolic functions.

The easiest other hyperbolic function to find is $\operatorname{sech} x$.

The hyperbolic functions $\cosh x$ and $\operatorname{sech} x$ are reciprocals.

Thus,

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$= \frac{1}{5/3}$$

$$= \frac{3}{5}$$

Since $x > 0$, all the hyperbolic functions at *x* are positive.

Now to find the other hyperbolic functions, use hyperbolic identities. Start with the identity $\cosh^2 x - \sinh^2 x = 1$.

$$\sinh^2 x = \left(\frac{5}{3}\right)^2 - 1$$

$$\sinh^2 x = \frac{25}{9} - 1$$

$$\sinh^2 x = \frac{16}{9}$$

$$\sinh x = \frac{4}{3}$$

The hyperbolic functions $\sinh x$ and $\operatorname{csch} x$ are also reciprocals.

Thus,

$$\begin{aligned}\operatorname{csch} x &= \frac{1}{\sinh x} \\ &= \frac{1}{4/3} \\ &= \frac{3}{4}\end{aligned}$$

Next use the definition $\tanh x = \dfrac{\sinh x}{\cosh x}$.

$$\begin{aligned}\tanh x &= \frac{4/3}{5/3} \\ &= \frac{4}{5}\end{aligned}$$

Finally, use the definition $\coth x = \cosh x / \sinh x$ to find $\coth x$.

$$\begin{aligned}\coth x &= \frac{5/3}{4/3} \\ &= \frac{5}{4}\end{aligned}$$

Therefore, the other hyperbolic functions at *x* are

$\sinh x = \frac{4}{3}$	$\operatorname{csch} x = \frac{3}{4}$
$\cos x = \frac{5}{3}$	$\operatorname{sech} x = \frac{3}{5}$
$\tanh x = \frac{4}{5}$	$\coth x = \frac{5}{4}$

Answer 22E.

(A)

To sketch the curve of $y = \csc hx = \dfrac{1}{\sin hx}$

We have the graph of $y = \sin hx$ as follows

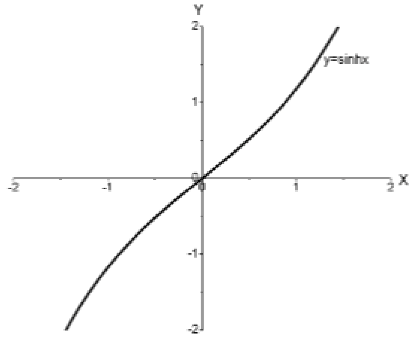


Fig.1

We see that $\sin hx$ is a increasing function so $y = \csc hx$ be a decreasing function at $x = 0$; $\sin hx = 0$ then $\csc hx$ is not defined at $x = 0$

Since $\sin hx \to \infty$ as $x \to \infty$ so $\csc hx \to 0$ as $x \to \infty$

And $\sin hx \to -\infty$ as $x \to -\infty$ so $\csc hx \to 0$ as $x \to -\infty$

Now we can sketch the graph of $y = \csc hx$ as follows

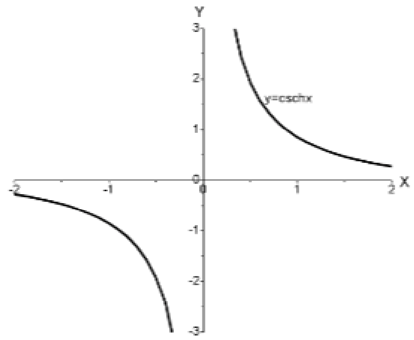


Fig.2

To sketch the curve of $y = \sec hx = \dfrac{1}{\cos hx}$

We have the graph of $y = \cos hx$ as follows

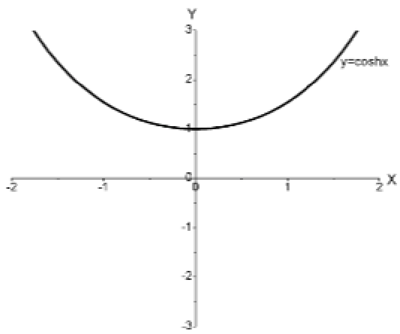


Fig.3

We see that $y = \cos hx$ is decreasing for $x < 0$ and increasing, $x > 0$

So $y = \sec hx$ be increasing for $x < 0$ and decreasing for $x > 0$

And $\cos hx \to \infty$ as $x \to \infty$ so $\sec hx \to 0$ as $x \to \infty$

$\cos hx \to \infty$ as $x \to -\infty$ so $\sec hx \to 0$ as $x \to \infty$

And $\cos hx = 1$ at, $x = 0$ so $\sec hx = 1$ at $x = 0$

Since $\cos hx = 1$ is the absolute minimum of the curve $y = \cos hx$

So $\sec hx = 1$ be the absolute maximum of the curve $y = \sec hx$

Now we can sketch the graph of $y = \sec hx$ as follows

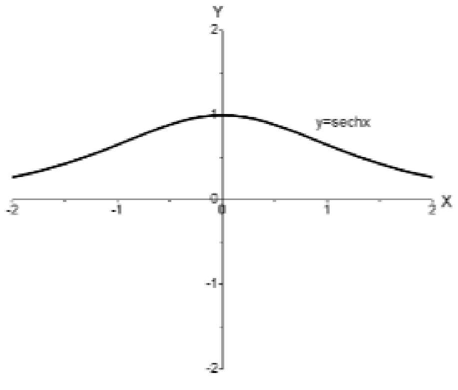


Fig.4

To sketch the curve of $y = \cot hx = \dfrac{1}{\tan hx}$

We have the graph of $y = \tan hx$

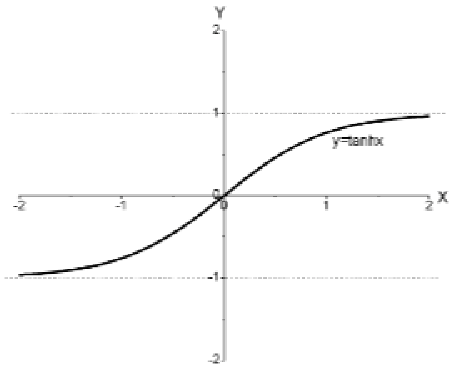


Fig.5

We see that $\tan hx = 0$ at x = 0 so at x = 0, the function $\cot hx$ be not defined

Since $\tan hx$ is increasing form -1 to 1 so $\cot hx$ be decreasing function

$\tan hx \to 1$ As $x \to \infty$ so $\cot hx \to 1$ as $x \to \infty$

And $\tan hx \to -1$ As $x \to -\infty$ so $\cot hx \to -1$ as $x \to -\infty$

Now we can sketch the curve of $y = \cot hx$

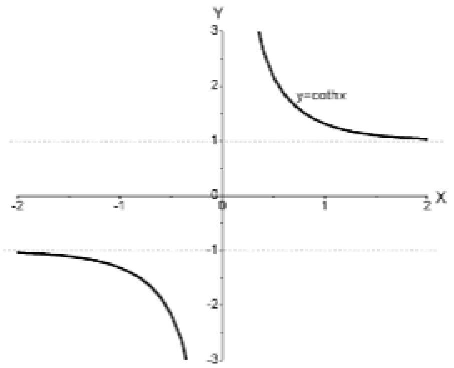


Fig.6

(B) Now we sketch the curve $y = \csc hx$, $y = \sec hx$ and $y = \cot hx$ with the help of computer in figure 1, figure 2 and figure 3 respectively, which are same as part (A).

Answer 23E.

(A) We have $\tan hx = \dfrac{\sin hx}{\cos hx}$

$$\Rightarrow \tan hx = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Taking the limit as $x \to \infty$

$$\lim_{x\to\infty} \tan hx = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$= \lim_{x\to\infty} \frac{\left(1-e^{-2x}\right)e^x}{\left(1+e^{-2x}\right)e^x}$$

$$= \lim_{x\to\infty} \frac{1-e^{-2x}}{1+e^{-2x}}$$

Since $e^{-2x} \to 0$ as $x \to \infty$

So $\lim_{x\to\infty} \tan hx = \dfrac{1-0}{1+0} = \dfrac{1}{1}$

Or $\boxed{\lim_{x\to\infty} \tan hx = 1}$

(B) We have $\tan hx = \dfrac{\sin hx}{\cos hx}$

$$\Rightarrow \tan hx = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Taking the limit as $x \to -\infty$

$$\lim_{x\to-\infty} \tan hx = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$= \lim_{x\to-\infty} \frac{\left(e^{2x}-1\right)e^{-x}}{\left(e^{2x}+1\right)e^{-x}}$$

$$= \lim_{x\to-\infty} \frac{e^{2x}-1}{e^{2x}+1}$$

Since $e^{2x} \to 0$ as $x \to -\infty$ so we have

So $\lim_{x\to-\infty} \tan hx = \dfrac{0-1}{0+1} = \dfrac{-1}{1}$

Or $\boxed{\lim_{x\to\infty} \tan hx = -1}$

(C) We have $\sin hx = \dfrac{e^x - e^{-x}}{2}$

Taking limit as $x \to \infty$

$$\lim_{x\to\infty} \sin hx = \lim_{x\to\infty} \frac{e^x - e^{-x}}{2}$$

$$= \lim_{x\to\infty} \frac{e^x}{2} - \lim_{x\to\infty} \frac{e^{-x}}{2}$$

$$= \frac{1}{2}\lim_{x\to\infty} e^x - \frac{1}{2}\lim_{x\to\infty} e^{-x}$$

Since $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} e^{-x} = 0$

So $\lim_{x\to\infty} \sin hx = \infty - 0 = \infty$

So $\boxed{\lim_{x\to\infty} \sin hx = \infty}$

(D) We have $\sin hx = \dfrac{e^x - e^{-x}}{2}$

Taking limit as $x \to -\infty$

$$\lim_{x\to-\infty} \sin hx = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{2}$$
$$= \lim_{x\to-\infty} \frac{e^x}{2} - \lim_{x\to-\infty} \frac{e^{-x}}{2}$$
$$= \frac{1}{2}\lim_{x\to-\infty} e^x - \frac{1}{2}\lim_{x\to-\infty} e^{-x}$$

Since $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to-\infty} e^{-x} = \infty$

So $\lim_{x\to-\infty} \sin hx = 0 - \infty = -\infty$

So $\boxed{\lim_{x\to-\infty} \sin hx = -\infty}$

(E) We have $\sec hx = \dfrac{1}{\cos hx}$

$$= \frac{2}{e^x + e^{-x}}$$

Taking limit as $x \to \infty$

$$\lim_{x\to\infty} \sec hx = \lim_{x\to\infty} \frac{2}{e^x + e^{-x}}$$
$$= \lim_{x\to\infty} \frac{2/e^x}{1+e^{-2x}}$$ [Divide numerator and denominator by e^x]
$$= \frac{\lim_{x\to\infty}\left(2/e^x\right)}{\lim_{x\to\infty} 1 + \lim_{x\to\infty}\left(1/e^{2x}\right)}$$
$$= \frac{2\lim_{x\to\infty} e^{-x}}{1+\lim_{x\to\infty} e^{-2x}}$$

Since $e^{-x} \to 0$ as $x \to \infty$ and $e^{-2x} \to 0$ as $x \to \infty$

So $\lim_{x\to\infty} \sec hx = \dfrac{0}{1+0} = 0$

So $\boxed{\lim_{x\to\infty} \sec hx = 0}$

(F) We have $\cot hx = \dfrac{\cos hx}{\sin hx}$

Or $\cot hx = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

Taking limit as $x \to +\infty$

$$\lim_{x\to\infty} \cot hx = \lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}}$$
$$= \lim_{x\to\infty} \frac{\left(1+e^{-2x}\right)/e^x}{\left(1-e^{-2x}\right)/e^x}$$
$$= \lim_{x\to\infty} \frac{1+e^{-2x}}{1-e^{-2x}}$$

Since $e^{-2x} \to 0$ as $x \to \infty$

So $\lim_{x\to\infty} \cot hx = \dfrac{1+0}{1-0} = \dfrac{1}{1}$

Or $\boxed{\lim_{x\to\infty} \cot hx = 1}$

(G) We have $\cot hx = \dfrac{\cos hx}{\sin hx}$

Or $\cot hx = \dfrac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$

Taking limit as $x \to 0^{+}$

$$\lim_{x\to 0^{+}} \cot hx = \lim_{x\to 0^{+}} \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$$

$$= \lim_{x\to 0^{+}} \frac{\left(e^{2x}+1\right)e^{-x}}{\left(e^{2x}-1\right)e^{-x}}$$

[Taking e^{x} common in numerator and denominator]

$$= \lim_{x\to 0^{+}} \frac{e^{2x}+1}{e^{2x}-1}$$

Since $e^{2x}+1 \to 2^{+}$ as $x \to 0^{+}$

And $e^{2x}-1 \to 0^{+}$ as $x \to 0^{+}$ [Since $e^{2x} \to 1^{+}$ as $x \to 0^{+}$]

So $\boxed{\lim_{x\to 0^{+}} \cot hx = +\infty}$

(H) We have $\cot hx = \dfrac{\cos hx}{\sin hx}$

Taking limit as $x \to 0^{-}$

$$\lim_{x\to 0^{-}} \cot hx = \lim_{x\to 0^{-}} \frac{\cos hx}{\sin hx}$$

Since x < 0 so $\sin hx = -\sin hx$

Then we have

$$\lim_{x\to 0^{-}} \cot hx = \lim_{x\to 0}\left[-\frac{\cos hx}{\sin hx}\right]$$

<xiv

Since $\sin hx \to 0$ and $\cos hx \to 1$ as $x \to 0$

So $\boxed{\lim_{x\to 0^{-}} \cot hx = -\infty}$

<xv

(I) We have $\lim_{x\to -\infty} \csc hx = \lim_{x\to -\infty} \dfrac{1}{\sin hx}$

Since $\sin hx \to -\infty$ as $x \to -\infty$

So $\boxed{\lim_{x\to -\infty} \csc hx = 0}$

<xvi

Answer 24E.

Recall that $\cos hx = \frac{e^x + e^{-x}}{2}$

(a) Need to prove $\frac{d}{dx}(\cos hx) = \sin hx$

Take $\cos hx = \frac{e^x + e^{-x}}{2}$

Differentiating with respect to x

$$\frac{d}{dx}(\cos hx) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$\frac{d}{dx}(\cos hx) = \frac{d}{dx}\left(\frac{e^x}{2}\right) + \frac{d}{dx}\left(\frac{e^{-x}}{2}\right)$$

Since $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$

$$= \frac{1}{2}\frac{d}{dx}(e^x) + \frac{1}{2}\frac{d}{dx}(e^{-x})$$

$= \frac{1}{2}e^x - \frac{1}{2}e^{-x}$ Since $\frac{d}{dx}(e^x) = e^x, \frac{d}{dx}(e^{-x}) = e^{-x}$

$= \frac{e^x - e^{-x}}{2}$ Simplify

$= \sin hx$ Since $\sin hx = \frac{e^x - e^{-x}}{2}$

Therefore,

$$\frac{d}{dx}(\cos hx) = \boxed{\sin hx}.$$

(b)

Need to prove $\frac{d}{dx}(\tan hx) = \sec h^2 x$

Recall that $\tan hx = \frac{\sin hx}{\cos hx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Differentiating with respect to x

$$\frac{d}{dx}(\tan hx) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$$

By Quotient rule $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}$

$$\frac{d}{dx}(\tan hx) = \frac{(e^x + e^{-x})\frac{d}{dx}(e^x - e^{-x}) - (e^x - e^{-x})\frac{d}{dx}(e^x + e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

Since $\frac{d}{dx}(e^x) = e^x, \frac{d}{dx}(e^{-x}) = e^{-x}$

$$= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{(e^x + e^{-x})^2}$$

Continuation to the above steps,

$$\frac{d}{dx}(\tan hx)=\frac{4}{\left(e^{x}+e^{-x}\right)^{2}}$$

$$=\left[\frac{2}{e^{x}+e^{-x}}\right]^{2}$$ Rewrite

$$=\left[\frac{1}{\cos hx}\right]^{2}$$ Since $\cos hx=\frac{e^{x}+e^{-x}}{2}$

$$=\sec h^{2}x$$ Since $\sec hx=\frac{1}{\cos hx}$

Therefore,

$$\frac{d}{dx}(\tan hx)=\boxed{\sec h^{2}x}.$$

(c)

Need to prove $\frac{d}{dx}(\csc h)=-\cot hx.\csc hx$

Recall that $\csc hx=\frac{1}{\sin hx}$

Or $\csc h=\frac{2}{e^{x}-e^{-x}}$ Since $\sin hx=\frac{e^{x}-e^{-x}}{2}$

Differentiating with respect to x

$$\frac{d}{dx}(\csc h)=\frac{d}{dx}\left(\frac{2}{e^{x}-e^{-x}}\right)$$

$$\frac{d}{dx}(\csc h)=2\frac{d}{dx}\left(\frac{1}{e^{x}-e^{-x}}\right)$$ Since $\int cf(x)dx=c\int f(x)dx$

$$=2.\frac{-1}{\left(e^{x}-e^{-x}\right)^{2}}.\frac{d}{dx}\left(e^{x}-e^{-x}\right)$$ Since $\frac{d}{dx}\left(x^{-n}\right)=-n\cdot x^{-n-1}\frac{d}{dx}(x)$

$$=\frac{-2.\left(e^{x}+e^{-x}\right)}{\left(e^{x}-e^{-x}\right)^{2}}$$ Since $\frac{d}{dx}\left(e^{x}\right)=e^{x},\frac{d}{dx}\left(e^{-x}\right)=e^{-x}$

$$=-\frac{\left(e^{x}+e^{-x}\right)}{\left(e^{x}-e^{-x}\right)}.\frac{2}{\left(e^{x}-e^{-x}\right)}$$ Rewrite

$$=-\frac{\left(e^{x}+e^{-x}\right)/2}{\left(e^{x}-e^{-x}\right)/2}.\frac{1}{\left(e^{x}-e^{-x}\right)/2}$$ Divide numerator and denominator by 4

$$=-\frac{\cos hx}{\sin hx}.\frac{1}{\sin hx}$$ Since $\sin hx=\frac{e^{x}-e^{-x}}{2}$ and $\cos hx=\frac{e^{x}+e^{-x}}{2}$

$$=-\cot hx.\csc hx$$ Since $\csc hx=\frac{1}{\sin hx}$

Therefore,

$$\frac{d}{dx}(\csc h)=\boxed{-\cot hx.\csc hx}.$$

(d)

Need to prove $\frac{d}{dx}(\sec hx) = -\sec hx.\tan hx$

Recall that $\sec hx = \frac{1}{\cos hx}$

Or $\sec hx = \frac{2}{e^x + e^{-x}}$ Since $\cos hx = \frac{e^x + e^{-x}}{2}$

Differentiating with respect to x

$$\frac{d}{dx}(\sec hx) = \frac{d}{dx}\left(\frac{2}{e^x + e^{-x}}\right)$$

$$\frac{d}{dx}(\sec hx) = 2\frac{d}{dx}\left(\frac{1}{e^x + e^{-x}}\right)$$ Since $\int cf(x)dx = c\int f(x)dx$

$$\frac{d}{dx}(\sec hx) = -\frac{2}{(e^x + e^{-x})^2}.(e^x + e^{-x})$$ Since $\frac{d}{dx}(x^{-n}) = -n \cdot x^{-n-1}\frac{d}{dx}(x)$

$$= -\frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2}$$ Since $\frac{d}{dx}(e^x) = e^x, \frac{d}{dx}(e^{-x}) = e^{-x}$

Continuation to the above steps,

$$= -\frac{(e^x - e^{-x})}{(e^x + e^{-x})}.\frac{2}{(e^x + e^{-x})}$$ Rewrite

$$= -\frac{(e^x - e^{-x})/2}{(e^x + e^{-x})/2}.\frac{1}{(e^x + e^{-x})/2}$$

Divide numerator and denominator by 4

$$= -\frac{\sin hx}{\cos hx}.\frac{1}{\cos hx}$$

Since $\sin hx = \frac{e^x - e^{-x}}{2}$ and $\cos hx = \frac{e^x + e^{-x}}{2}$

$$= -\tan hx.\sec hx$$ Since $\sec hx = \frac{1}{\cos hx}$

Therefore,

$$\frac{d}{dx}(\sec hx) = \boxed{-\sec hx.\tan hx}.$$

(e)

Need to prove $\frac{d}{dx}(\cot\ hx) = -\csc h^2 x$

Recall that $\cot\ hx = \frac{\cos hx}{\sin hx} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Differentiating with respect to x

$$\frac{d}{dx}(\cot\ hx) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)$$

By Quotient rule $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2}$

$$\frac{d}{dx}(\cot\ hx) = \frac{(e^x - e^{-x})\frac{d}{dx}(e^x + e^{-x}) - (e^x + e^{-x})\frac{d}{dx}(e^x - e^{-x})}{(e^x - e^{-x})^2}$$

Continuation to the above steps,

$$=\frac{(e^x-e^{-x})(e^x-e^{-x})-(e^x+e^{-x})(e^x+e^{-x})}{(e^x-e^{-x})^2}$$

Since $\frac{d}{dx}(e^x)=e^x, \frac{d}{dx}(e^{-x})=e^{-x}$

$$=\frac{e^{2x}-2+e^{-2x}-e^{2x}-2-e^{-2x}}{(e^x-e^{-x})^2}$$

$$=-\frac{4}{(e^x-e^{-x})^2}$$

$$=-\left[\frac{2}{(e^x-e^{-x})}\right]^2$$ Rewrite

$\frac{d}{dx}(\cot hx)=-\csc h^2x$ Since $\csc hx=\frac{1}{\sin hx}$

Therefore,

$$\frac{d}{dx}(\cot hx)=\boxed{-\csc h^2x}.$$

Answer 25E.

Let $y=\sin h^{-1}x$

Then $x=\sin hy$ --- (1)

We have $\cos h^2x-\sin h^2x=1$

Replacing x by y

$\cos h^2y=1+\sin h^2y$

Or $\cosh y=\sqrt{1+\sin h^2y}$

Or $\cosh y=\sqrt{1+x^2}$ --- (2) [From (1)]

Now we have

$\cos hx+\sin hx=e^x$

Replacing x by y

$\Rightarrow \sin hy+\cos hy=e^y$

$\Rightarrow x+\sqrt{1+x^2}=e^y$

$\Rightarrow \boxed{y=\ln\left(x+\sqrt{1+x^2}\right)}$ $[e^y=x\Leftrightarrow \ln x=y]$

Or $\boxed{\sin h^{-1}x=\ln\left(x+\sqrt{1+x^2}\right)}$

Answer 26E.

Let $y=\cos h^{-1}x$

Then $x=\cos hy$ --- (1)

We have $\cos h^2x-\sin h^2x=1$

Replacing x by y,

$\cos h^2y-\sin h^2y=1$

Or $\sin h^2y=\cos h^2y-1$

Or $\sin hy=\sqrt{\cos h^2y-1}$

Or $\sin hy=\sqrt{x^2-1}$ --- (2) [From (1)]

Now we have

$$\sin hx + \cos hx = e^x$$

Replacing x by y,

$$\Rightarrow \sin hy + \cos hy = e^y$$
$$\Rightarrow \sqrt{x^2-1} + x = e^y$$
$$\Rightarrow e^y = x + \sqrt{x^2-1}$$
$$\Rightarrow y = \ln\left(x + \sqrt{x^2-1}\right)$$

Or $\boxed{\cos h^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)}$

Answer 27E.

(A) Let $\tan h^{-1} x = y$

Then $x = \tan hy$

$$\Rightarrow x = \frac{\sin hy}{\cos hy} = \frac{\left(e^y - e^{-y}\right)/2}{\left(e^y + e^{-y}\right)/2}$$
$$\Rightarrow x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$
$$\Rightarrow xe^y + xe^{-y} - e^y + e^{-y} = 0$$
$$\Rightarrow (x-1)e^y + (x+1)e^{-y} = 0$$

Multiplying by e^y

$$\Rightarrow (x-1)e^{2y} + (x+1) = 0$$
$$\Rightarrow e^{2y} = -\frac{(1+x)}{(x-1)}$$
$$\Rightarrow e^{2y} = \frac{(1+x)}{(1-x)}$$
$$\Rightarrow 2y = \ln\left(\frac{1+x}{1-x}\right)$$
$$\Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \left[e^x = y \Leftrightarrow \ln y = x\right]$$
$$\Rightarrow \boxed{\tan h^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)}$$

(B) Let $\tan h^{-1} x = y$

$$\Rightarrow \tan hy = x \qquad \text{--- (1)}$$

We have

$$e^{2x} = \frac{1+\tan hx}{1-\tan hx}$$

Replacing x, by y

$$e^{2y} = \frac{1+\tan hy}{1-\tan hy}$$
$$\Rightarrow e^{2y} = \frac{1+x}{1-x} \qquad \text{[From (1)]}$$

Then $2y = \ln\left(\frac{1+x}{1-x}\right) \qquad \left[e^y = x \Leftrightarrow \ln x = y\right]$

$$\Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$
$$\Rightarrow \boxed{\tan h^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)}$$

Answer 28E.

(A)

(i) From the graph of $\operatorname{csc}hx$ we see that

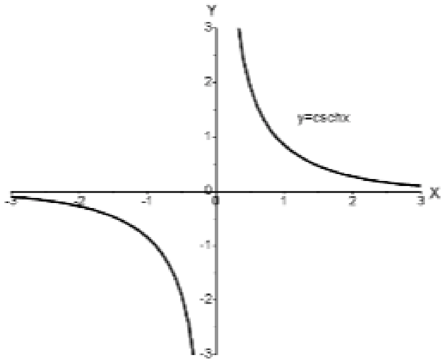


Fig.1

$\operatorname{csc}h$ is one to one function so it has inverse function denoted by $\operatorname{csc}h^{-1}x$

$$y = \operatorname{csc}h^{-1}x \Leftrightarrow \operatorname{csc}hy = x$$

(ii) We sketch the curve of $\operatorname{csc}h^{-1}x$

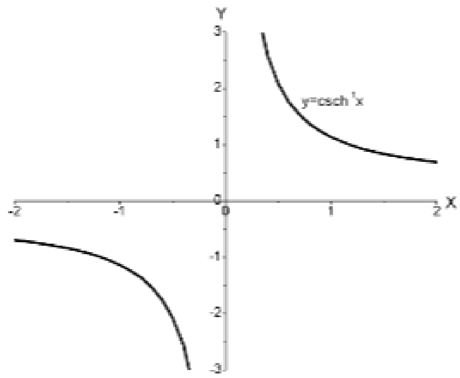


Fig.2

$\Rightarrow$ Domain $= (-\infty, 0) \cup (0, \infty)$ and range $= (-\infty, 0) \cup (0, \infty) = \{x / x \neq 0\}$

(iii) Let $\operatorname{csc}h^{-1}x = y$

$$\Rightarrow \operatorname{csc}hy = x \qquad \text{--- (1)}$$

$$\Rightarrow \frac{1}{\sin hy} = x$$

$$\Rightarrow \frac{2}{e^y - e^{-y}} = x$$

$$\Rightarrow 2 = xe^y - xe^{-y}$$

$$\Rightarrow 2e^y = xe^{2y} - x \qquad \text{[Multiplying by } e^y\text{]}$$

$$\Rightarrow xe^{2y} - 2e^y - x = 0$$

This is a quadratic equation in e^y

So $$e^y = \frac{2 \pm \sqrt{(-2)^2 - 4(-x)(x)}}{2x}$$

$$\Rightarrow e^y = \frac{2 \pm \sqrt{4 + 4x^2}}{2x}$$

$$\Rightarrow e^y = \frac{1 \pm \sqrt{1 + x^2}}{x}$$

Since $e^y > 0$ so $\dfrac{1-\sqrt{1+x^2}}{x} > 0$ for x < 0

And $\dfrac{1-\sqrt{1+x^2}}{x} > 0$ for x > 0

So $$y = \boxed{\operatorname{csc}h^{-1}x = \begin{cases} \ln\left(\dfrac{1-\sqrt{1+x^2}}{x}\right) & for\ x<0 \\ \ln\left(\dfrac{1+\sqrt{1+x^2}}{x}\right) & for\ x<0 \end{cases}}$$

(B)

(i) From the graph of $\sec hx$ we see that $\sec hx$ is not a one to one function but when we restrict it domain $[0,\infty)$ then it becomes one to one function and then this function denoted by $\sec h^{-1}x$

$$\Rightarrow y = \sec h^{-1}x \Leftrightarrow x = \sec h^{-1}x \qquad \text{and} \qquad y \geq 0$$

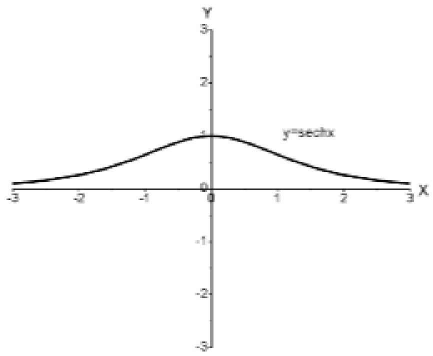
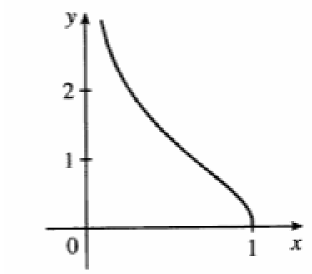


Fig.3

(ii) We sketch the function $\sec h^{-1}x$



Domain is $=(0,1]$ Range $=[0,\infty)$

(iii) Let $\sec h^{-1}x = y$

$\Rightarrow x = \sec hy$ --- (1)

$\Rightarrow x = \dfrac{2}{e^y + e^{-y}}$

$\Rightarrow xe^y + xe^{-y} = 2$

Multiplying e^y

$xe^{2y} + x = 2e^y$

$\Rightarrow xe^{2y} - 2e^y + x = 0$

This is a quadratic equation in e^y

Then $e^y = \dfrac{2 \pm \sqrt{4-4x^2}}{2x}$

$\Rightarrow e^y = \dfrac{1 \pm \sqrt{1-x^2}}{x}$

Since $e^y > 0$ so $\frac{1+\sqrt{1-x^2}}{x} > 0$ for x > 0 and $x \leq 1$

And $\frac{1-\sqrt{1-x^2}}{x} > 0$ for x < 0 and $x \geq -1$

Then $y = \sec h^{-1} x = \begin{cases} \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) & for\ 0 < x \leq 1 \\ \ln\left(\frac{1-\sqrt{1-x^2}}{x}\right) & for\ -1 < x \leq 0 \end{cases}$

(C)

(i) From the graph of $\cot hx$ we see that

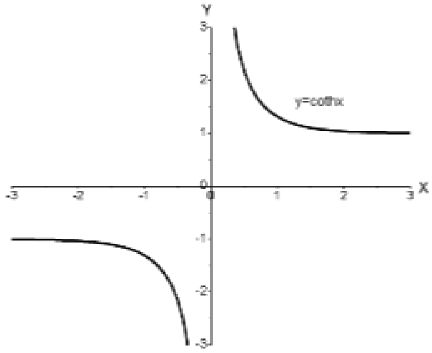


Fig.5

$\cot hx$ is one to one function, so it inverse is denoted by $\cot h^{-1} x$

$y = \cot h^{-1} x \Rightarrow x = \cot hy$

(ii)

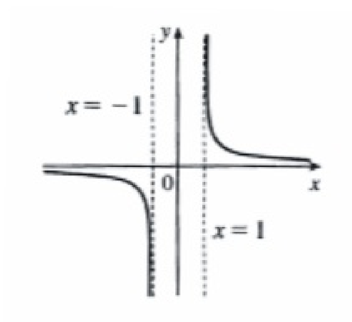


Fig. 6

Domain $= (-\infty,-1) \cup (1,\infty)$ Range $= R = (-\infty,\infty)$

(iii) Let $y = \cot h^{-1} x$

$\Rightarrow x = \cot hy$

$\Rightarrow x = \frac{\cos hy}{\sin hy}$

$\Rightarrow x = \frac{e^y + e^{-y}}{e^y - e^{-y}}$

$\Rightarrow xe^y - xe^{-y} = e^y + e^{-y}$

$\Rightarrow xe^y - e^y - xe^{-y} - e^{-y} = 0$

$\Rightarrow (x-1)e^y - (x+1)e^{-y} = 0$

$$\Rightarrow (x-1)e^{2y} = (x+1)$$

$$\Rightarrow e^{2y} = \frac{x+1}{x-1}$$

Then $2y = \ln\left(\frac{x+1}{x-1}\right)$

Or $y = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$

Or $\boxed{\cot h^{-1}x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)}$

Answer 29E.

(a). Let $y = \cos h^{-1}x$

Then $\cos hy = x$

Differentiating this equation implicitly with respect to x we get

$$\frac{d}{dy}(\cos hy)\frac{dy}{dx} = 1 \quad \text{By chain rule}$$

$$\Rightarrow \sin hy.\frac{dy}{dx} = 1 \quad \left[\because \frac{d}{dx}(\cos hx) = \sin hx\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin hy}$$

Or $\frac{d}{dx}(\cos h^{-1}x) = \frac{1}{\sqrt{\cos h^2 y - 1}}$ $\quad [\cos h^2 x - \sin h^2 x = 1]$

Or $\boxed{\frac{d}{dx}(\cos h^{-1}x) = \frac{1}{\sqrt{x^2-1}}}$

(b). Let $y = \tan h^{-1}x$

Then $\tan hy = x$

Differentiating this equation implicitly with respect to x

$$\frac{d}{dy}(\tan hy)\frac{dy}{dx} = 1 \quad \text{(By chain rule)}$$

$$\Rightarrow \sec h^2 y\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec h^2 y}$$

We have $\sec h^2 y = 1 - \tan h^2 y$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1-\tan h^2 y}$$

$$\Rightarrow \boxed{\frac{d}{dx}(\tan h^{-1}x) = \frac{1}{1-x^2}}$$

(c). Let $y = \csc h^{-1}x$

Then $\csc hy = x$

Differentiating this equation implicitly with respect to x we get

$$\frac{d}{dy}(\csc hy)\frac{dy}{dx} = 1 \quad \text{[By chain rule]}$$

$$\Rightarrow -\csc hy.\cot hy.\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\csc hy.\cot hy}$$

Since $\coth y = \pm\sqrt{1+\operatorname{csch}^2 y} = \pm\sqrt{x^2+1}$

If x > 0 the $\coth y > 0$ and so $\coth y = \sqrt{x^2+1}$

And x < 0 then $\coth y = -\sqrt{x^2+1}$

So we have

$$\frac{dy}{dx} = \frac{-1}{-(x)\sqrt{x^2+1}} \quad \text{for } x < 0$$

And $$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2+1}} \quad \text{for } x > 0$$

So we have

$$\boxed{\frac{d}{dx}\left(\operatorname{csch}^{-1}x\right) = \frac{-1}{|x|\sqrt{x^2+1}}}$$

(d) Let $y = \operatorname{sech}^{-1}x$

Then $\operatorname{sech} y = x$

Differentiating this equation implicitly with respect to x we get

$$\frac{d}{dy}(\operatorname{sech} y)\frac{dy}{dx} = 1 \quad \text{by Chain Rule}$$

$$\Rightarrow -\operatorname{sech} y.\tanh y.\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\operatorname{sech} y.\tanh y}$$

We have $1-\tanh^2 y = \operatorname{sech}^2 y$

Or $\tanh^2 y = 1-\operatorname{sech}^2 y$

Or $\tanh y = \sqrt{1-x^2}$

Then $$\frac{dy}{dx} = \frac{-1}{x\sqrt{1-x^2}}$$

Or $$\boxed{\frac{d}{dx}\left(\operatorname{sech}^{-1}x\right) = \frac{-1}{x\sqrt{1-x^2}}}$$

(e). Let $\coth^{-1}x = y$

$\Rightarrow \coth y = x$

Differentiating with respect to x

$$\frac{d}{dy}(\coth y)\frac{dy}{dx} = 1 \quad \text{(By chain rule)}$$

$$\Rightarrow -\operatorname{csch}^2 y\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\operatorname{csch}^2 y}$$

We have

$$\coth^2 y - 1 = \operatorname{csc}^2 y$$

Then $$\frac{dy}{dx} = \frac{-1}{\operatorname{csch}^2 y - 1}$$

$$= \frac{-1}{x^2-1}$$

Or $$\boxed{\frac{d}{dx}\left(\coth^{-1}x\right) = \frac{1}{1-x^2}}$$

Answer 30E.

Consider the function

$$f(x)=\tanh\left(1+e^{2x}\right)$$

Need to find the derivative of the function $f(x)$.

Use the Chain Rule which says:

$$\frac{d}{dx}\tanh u=\operatorname{sech}^2 u\cdot\frac{du}{dx}.$$

Derivative with respect to x the function $f(x)$, to get

$$\frac{d}{dx}(f(x))=\frac{d}{dx}\left(\tanh\left(1+e^{2x}\right)\right)$$

$$f'(x)=\operatorname{sech}^2\left(1+e^{2x}\right)\frac{d}{dx}\left(1+e^{2x}\right) \quad \text{Here use } \frac{d}{dx}(\tanh(x))=\operatorname{sech}^2(x)$$

$$=\operatorname{sech}^2\left(1+e^{2x}\right)\left[\frac{d}{dx}(1)+\frac{d}{dx}\left(e^{2x}\right)\right]$$

Now to apply the Chain Rule for the exponential function which is

$$\frac{d}{dx}e^u=e^u\frac{du}{dx}.$$

Therefore,

$$f'(x)=\operatorname{sech}^2\left(1+e^{2x}\right)\left[0+2e^{2x}\right]$$

Thus, the final derivative is $\boxed{f'(x)=2e^{2x}\operatorname{sech}^2\left(1+e^{2x}\right)}$

Answer 31E.

Consider the function

$$f(x)=x\sinh x-\cosh x$$

Need to find the derivative of the function $f(x)$.

Derivative with respect to x the function $f(x)$, to get

$$\frac{d}{dx}(f(x))=\frac{d}{dx}(x\sinh x-\cosh x)$$

$$f'(x)=\frac{d}{dx}(x\sinh x)-\frac{d}{dx}\cosh x$$

For the first term, $x\sinh x$, use the Product Rule which says:

$$\frac{d}{dx}(u\cdot v)=u\cdot\frac{d}{dx}(v)+v\cdot\frac{d}{dx}(u).$$

$$\frac{d}{dx}(x\sinh x)=x\cdot\frac{d}{dx}(\sinh x)+\sinh x\cdot\frac{d}{dx}(x)$$

$$\frac{d}{dx}(x\sinh x)=x\cdot\cosh x+1\cdot\sinh x$$

Here $u=x$ and $v=\sinh x$

Use this value for $\frac{d}{dx}(x\sinh x)$ and insert into $f'(x)$.

$$f'(x)=\frac{d}{dx}(x\sinh x)-\frac{d}{dx}\cosh x$$

$$=(\sinh x+x\cosh x)-\sinh x$$

$$=x\cosh x$$

Thus, the final derivative is $\boxed{f'(x)=x\cosh x}$.

Answer 32E.

Consider the function

$g(x) = \cosh(\ln x)$

Need to find the derivative of the function $g(x)$.

Recollect the Chain Rule which says:

$$\frac{d}{dx}\cosh u = \sinh u \cdot \frac{du}{dx}.$$

Derivative with respect to x the function $g(x)$, to get

$$\frac{d}{dx}(g(x)) = \frac{d}{dx}(\cosh(\ln x))$$

$$\begin{aligned} g'(x) &= \sinh(\ln x) \cdot \frac{d}{dx}(\ln x) \\ &= \sinh(\ln x) \cdot \frac{1}{x} \\ &= \frac{\sinh(\ln x)}{x} \end{aligned}$$

Thus, the final derivative is $\boxed{g'(x) = \frac{\sinh(\ln x)}{x}}$.

Answer 33E.

Consider the function

$h(x) = \ln(\cosh x)$

Need to find the derivative of the function $h(x)$.

Recollect the Chain Rule which says:

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}.$$

Derivative with respect to x the function $h(x)$, to get

$$\frac{d}{dx}(h(x)) = \frac{d}{dx}(\ln(\cosh x))$$

$$\begin{aligned} h'(x) &= \frac{1}{\cosh x}\frac{d}{dx}(\cosh x) \\ &= \frac{1}{\cosh x} \cdot \sinh x \\ &= \frac{\sinh x}{\cosh x} \\ &= \tanh x \end{aligned}$$

The last step used the definition of hyperbolic tangent: $\frac{\sinh x}{\cosh x} = \tanh x$

Thus, the final derivative is $\boxed{h'(x) = \tanh x}$.

Answer 34E.

Consider the function

$$y = x\coth(1+x^2)$$

Need to find the derivative of the function y.

Derivative with respect to x the function y, to get

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(x\coth(1+x^2)\right)$$

For $x\coth(1+x^2)$, use the Product Rule:

$$\frac{d}{dx}(u \cdot v) = u' \cdot v + u \cdot v'.$$

Let $u = x$ and $v = \coth(1+x^2)$. Then $u' = 1$.

Take the derivative of $v = \coth(1+x^2)$, by using the Chain Rule:

$$\frac{d}{dx}\coth u = -\operatorname{csch}^2 u \cdot \frac{du}{dx}$$

$$v' = -\operatorname{csch}^2(1+x^2)2x$$

To summarize, $u = x$, $v = \coth(1+x^2)$, $u' = 1$ and $v' = -2x\operatorname{csch}^2(1+x^2)$.

Apply these values to the Product Rule: $\frac{d}{dx}(u \cdot v) = u' \cdot v + u \cdot v'$

$$\begin{aligned}\frac{d}{dx}x\coth(1+x^2) &= 1 \cdot \coth(1+x^2) + x(-2x)\operatorname{csch}^2(1+x^2)\\ &= \coth(1+x^2) - 2x^2\operatorname{csch}^2(1+x^2)\end{aligned}$$

Thus, the final derivative is $\boxed{y' = \coth(1+x^2) - 2x^2\operatorname{csch}^2(1+x^2)}$

Answer 35E.

We have $y = e^{\cosh 3x}$

Differentiating with respect to x by chain rule

$$\begin{aligned}y' &= e^{\cosh 3x}\frac{d}{dx}(\cosh 3x) && \text{[Chain rule]}\\ &= e^{\cosh 3x}\sinh 3x\frac{d}{dx}(3x) && \text{[Chain rule]}\end{aligned}$$

Thus $\boxed{y' = 3e^{\cosh(3x)}\sinh(3x)}$

Answer 36E.

Consider the function

$$y = \operatorname{csch} t(1 - \ln \operatorname{csch} t)$$

Need to find the derivative of the function y.

Derivative with respect to t the function y, to get

$$\frac{d}{dx}(y) = \frac{d}{dx}(\operatorname{csch} t(1 - \ln \operatorname{csch} t))$$
$$= \operatorname{csch} t \cdot \frac{d}{dx}(1 - \ln \operatorname{csch} t) + (1 - \ln \operatorname{csch} t) \cdot \frac{d}{dx}(\operatorname{csch} t)$$

Here using product rule:

$$\frac{d}{dt}(u \cdot v) = u \cdot \frac{d}{dt}(v) + v \cdot \frac{d}{dt}(u)$$

In this case $u = \operatorname{csch} t$ and $v = 1 - \ln \operatorname{csch} t$.

Their respective derivatives are $u' = -\operatorname{csch} t \coth t$ and use the Chain Rule for v which says

$$\frac{d}{dt}(\ln u) = \frac{1}{u}\frac{du}{dt}.$$

$$v' = -\frac{1}{\operatorname{csch} t} \cdot \frac{d}{dt}(\operatorname{csch} t)$$
$$= -\frac{1}{\operatorname{csch} t}(-\operatorname{csch} t \coth t)$$
$$= \coth t$$

To summarize, $u = \operatorname{csch} t$, $v = 1 - \ln \operatorname{csch} t$, $u' = -\operatorname{csch} t \coth t$ and $v' = \coth t$.

Apply these to the Product Rule.

$$y' = \frac{d}{dt}(\operatorname{csch} t(1 - \ln(\operatorname{csch} t)))$$
$$= \operatorname{csch} t \cdot \frac{d}{dx}(1 - \ln(\operatorname{csch} t)) + (1 - \ln(\operatorname{csch} t)) \cdot \frac{d}{dx}(\operatorname{csch} t)$$
$$= \operatorname{csch} t \coth t + (1 - \ln(\operatorname{csch} t))(-\operatorname{csch} t \cdot \coth t)$$
$$= \operatorname{csch} t \coth t - \operatorname{csch} t \cdot \coth t + \ln(\operatorname{csch} t) \cdot \operatorname{csch} t \cdot \coth t$$
$$= \ln(\operatorname{csch} t) \cdot \operatorname{csch} t \cdot \coth t$$

Thus, the final derivative is $\boxed{y' = \operatorname{csch} t \coth t \ln \operatorname{csch} t}$.

Answer 37E.

Consider the following function:

$$f(t) = \sec h^2(e^t)$$

Find the derivative of this function using the Chain Rule.

Differentiate the function with respect to t.

$$\frac{d}{dt}[f(t)] = \frac{d}{dt}\left[\sec h^2(e^t)\right].$$

Use $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$ to $\frac{d}{dt}[f(t)] = \frac{d}{dt}\left[\sec h^2(e^t)\right]$.

$$f'(t) = 2\sec h(e^t) \cdot \left[\frac{d}{dx}\sec h(e^t)\right] \qquad \text{since } \frac{d}{dx}(u^n) = nu^n \frac{d}{dx}(x)$$
$$= 2\sec h(e^t) \cdot \left[-\sec h(e^t)\tanh(e^t) \cdot \frac{d}{dt}(e^t)\right]$$
$$= -2e^t \sec h(e^t) \sec h(e^t) \tanh(e^t)$$
$$= -2e^t \sec h^2(e^t) \tanh(e^t)$$

Therefore, the derivative of $f(t) = \sec h^2(e^t)$ is $\boxed{f'(t) = -2e^t \sec h^2(e^t)\tanh(e^t)}$.

Answer 38E.

We have $y = \sinh(\cosh x)$

Differentiating with respect to x by chain rule

$$\frac{dy}{dx} = \cosh(\cosh x)\frac{d}{dx}(\cosh x)$$

Then $\boxed{y' = \sinh x \cosh(\cosh x)}$

Answer 39E.

We differentiate this function with respect to x,

We get,

$$g'(x) = \frac{d}{dx}\left(\frac{1-\cosh x}{1+\cosh x}\right)$$

$$g'(x) = \frac{(1+\cosh x)\frac{d(1-\cosh x)}{dx} - (1-\cosh x)\frac{d}{dx}(1+\cosh x)}{(1+\cosh x)^2}$$

$$g'(x) = \frac{(1+\cosh x)(-\sinh x) - (1-\cosh x)(\sinh x)}{(1+\cosh x)^2}$$

$$g'(x) = \frac{-\sinh x - \sinh x\cosh x - \sinh x + \sinh x\cosh x}{(1+\cosh x)^2}$$

Thus,

$$g'(x) = \frac{-2\sinh x}{(1+\cosh x)^2}$$

This is the required derivative

Answer 40E.

Given $y = \sinh^{-1}(\tan x)$

On differentiation

$$y' = \frac{dy}{dx}$$

$$= \frac{d}{dx}\left[\sinh'(\tan x)\right]$$

$$= \frac{1}{\sqrt{1+\tan^2 x}} \cdot \frac{d}{dx}(\tan x)$$

$$= \frac{1}{\sec x} \cdot \sec^2 x$$

$$= \sec x$$

Therefore $\boxed{y' = \sec x}$

Answer 41E.

Given $y = \cosh^{-1}\sqrt{x}$

On differentiation

$$y' = \frac{dy}{dx}$$

$$= \frac{d}{dx}\left[\cosh^{-1}\sqrt{x}\right]$$

$$= \frac{1}{\sqrt{x-1}} \frac{d}{dx}\left(\sqrt{x}\right)$$
$$= \frac{1}{\sqrt{x-1}} \times \frac{1}{2\sqrt{x}}$$
$$= \frac{1}{2}\sqrt{x(x-1)}$$

Therefore $\boxed{y' = \frac{1}{2\sqrt{x(x-1)}}}$

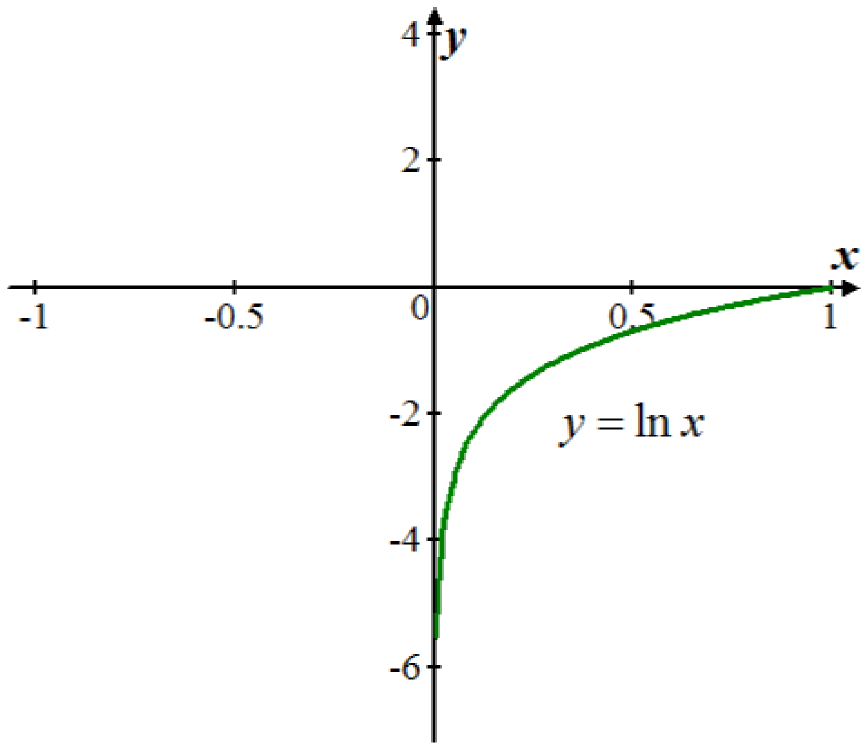
Answer 42E.

Consider the following limit of the function:

$$\lim_{x \to 0^+} \tan^{-1}(\ln x)$$

The objective is to find the limit of the above function.

Sketch the graph of $y = \ln x$ as shown below:



From the graph, observe that,

As x tends to 0 from the right side, the function value $\ln(x)$ tends to $-\infty$.

That is, as $x \to 0^+$ $\ln(x) \to -\infty$.

Then,

$$\lim_{x \to 0^+} \tan^{-1}(\ln x) = \tan^{-1}(-\infty)$$
$$= \tan^{-1}\left(\tan\left(\frac{-\pi}{2}\right)\right)$$
$$= -\frac{\pi}{2}$$

Thus, the limit of the function is, $\lim_{x \to 0^+} \tan^{-1}(\ln x) = \boxed{-\frac{\pi}{2}}$.

Answer 43E.

Consider the following function provided in the text-book:

$$y = x\sinh^{-1}\left(\frac{x}{3}\right) - \sqrt{9+x^2}$$

Differentiate with respect to x by product rule followed by the chain rule as follows:

$$\begin{aligned} y' &= x\frac{d}{dx}\left\{\sinh^{-1}\left(\frac{x}{3}\right)\right\} + \left\{\sinh^{-1}\left(\frac{x}{3}\right)\right\}\frac{d}{dx}(x) - \frac{d}{dx}\sqrt{9+x^2} \\ &= \frac{x \times \frac{1}{3}}{\sqrt{1+\left(\frac{x}{3}\right)^2}} + \sinh^{-1}\left(\frac{x}{3}\right) - \frac{2x}{2\sqrt{9+x^2}} \\ &= \frac{x}{3\sqrt{\frac{9+x^2}{9}}} + \sinh^{-1}\left(\frac{x}{3}\right) - \frac{x}{\sqrt{9+x^2}} \end{aligned}$$

Solve further,

$$\begin{aligned} y' &= \frac{x}{\sqrt{9+x^2}} + \sinh^{-1}\left(\frac{x}{3}\right) - \frac{x}{\sqrt{9+x^2}} \\ &= \sinh^{-1}\left(\frac{x}{3}\right) \end{aligned}$$

Hence, the derivative of $y = x\sinh^{-1}\left(\frac{x}{3}\right) - \sqrt{9+x^2}$ is $\boxed{\sinh^{-1}\left(\frac{x}{3}\right)}$.

Answer 44E.

Given $y = \sec h^{-1}\left(e^{-x}\right)$

On differentiation,

$$\begin{aligned} y' &= \frac{dy}{dx} \\ &= \frac{d}{dx}\left[\sec h^{-1}\left(e^{-x}\right)\right] \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{e^{-x}\sqrt{1-e^{-2x}}} \cdot \frac{d}{dx}\left(e^{-x}\right) \\ &= -\frac{e^{x}}{\sqrt{1-e^{-2x}}} \times e^{-x} \times \frac{d}{dx}(-x) \\ &= -\frac{e^{x}}{\sqrt{1-e^{-2x}}} \times e^{-x} \times -1 \\ &= \frac{1}{\sqrt{1-e^{-2x}}} \end{aligned}$$

Therefore $\boxed{y' = \frac{1}{\sqrt{1-e^{-2x}}}}$

Answer 45E.

Given $y = \cot h^{-1}(\sec x)$

On differentiation

$$\begin{aligned} y' &= \frac{dy}{dx} \\ &= \frac{d}{dx}\left[\cot h'(\sec x)\right] \end{aligned}$$

$$= \frac{1}{1-\sec^2 x} \cdot \frac{d}{dx}(\sec x)$$

$$= \frac{\sec x \cdot \tan x}{1-\sec^2 x}$$

$$= \frac{\sec x \cdot \tan x}{-\tan^2 x}$$

$$= -\frac{\sec x}{\tan x}$$

$$= -\frac{1}{\cos x} \times \frac{\cos x}{\sin x}$$

$$= -\csc x$$

Therefore $\boxed{y' = -\csc x}$

Answer 46E.

Consider $\frac{d}{dx}\sqrt{\frac{1+\tanh x}{1-\tanh x}} = \frac{d}{dx}\left(\frac{1+\tanh x}{1-\tanh x}\right)^{\frac{1}{2}}$

$$= \frac{d}{dx}\left[\frac{1+\tanh x}{1-\tanh x}\right]^{1/2}$$

$$= \frac{d}{dx}\left[\frac{1+\frac{\sinh x}{\cosh x}}{1-\frac{\sinh x}{\cosh x}}\right]^{1/2}$$

$$= \frac{d}{dx}\left[\frac{\cosh x+\sinh x}{\cosh x-\sinh x}\right]^{1/2}$$

$$= \frac{d}{dx}\left[\frac{(\cosh x+\sinh x)(\cosh x+\sinh x)}{(\cosh x-\sinh x)(\cosh x+\sinh x)}\right]^{1/2}$$

$$= \frac{d}{dx}\left[\frac{(\cosh x+\sinh x)^2}{\cosh^2 x-\sinh^2 x}\right]^{1/2}$$

$$= \frac{d}{dx}\left[\frac{(\cosh x+\sinh x)^2}{1}\right]^{1/2}$$

$$= \frac{d}{dx}[\cosh x+\sinh x]$$

$$= \sinh x+\cosh x$$

$$= \frac{e^x-e^{-x}}{2}+\frac{e^x+e^{-x}}{2}$$

$$= \frac{2e^x}{2}$$

$$= e^x$$

Therefore $\boxed{\frac{d}{dx}\sqrt{\frac{1+\tanh x}{1-\tanh s}} = e^x}$

Answer 47E.

Consider the left hand side function

$$\frac{d}{dx}\left[\arctan(\tanh x)\right]$$

To obtain the Right hand side, differentiate this as follows:

LHS :

$$\frac{d}{dx}\left[\arctan(\tanh x)\right] = \frac{d}{dx}\left[\tan^{-1}(\tanh x)\right] \quad \text{Write } \arctan = \tan^{-1}$$

$$= \frac{1}{1+\tanh^2 x}\cdot\frac{d}{dx}(\tanh x) \quad \text{Since } \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$= \left(\frac{1}{1+\tanh^2 x}\right)\cdot \operatorname{sech}^2 x \quad \text{Use } \frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$= \frac{\operatorname{sech}^2 x}{1+\tanh^2 x} \quad \text{Simplify}$$

$$= \frac{1-\tanh^2 x}{1+\tanh^2 x} \quad \text{Since } 1-\tanh^2 x = \operatorname{sech}^2 x$$

Continuation to the above steps:

$$\frac{d}{dx}\left[\arctan(\tanh x)\right] = \frac{1-\tanh^2 x}{1+\tanh^2 x}$$

$$= \frac{1-\frac{\sinh^2 x}{\cosh^2 x}}{1+\frac{\sinh^2 x}{\cosh^2 x}} \quad \text{Since } \tanh x = \frac{\sinh x}{\cosh x}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x + \sinh^2 x} \quad \text{Do LCM}$$

$$= \frac{1}{\cosh 2x} \quad \text{Since } \begin{bmatrix} \cosh^2 x - \sinh^2 x = 1 \\ \cosh^2 x + \sinh^2 x = \cosh 2x \end{bmatrix}$$

$$= \operatorname{sech} 2x$$

$= RHS$

Therefore

$$\boxed{\frac{d}{dx}\left[\arctan(\tanh x)\right] = \operatorname{sech} 2x}$$

Answer 48E.

The gateway Arch in St. Louis was designed by E S and was constructed using equation

$y = 211.49 - 20.96\cosh(0.03291765x)$

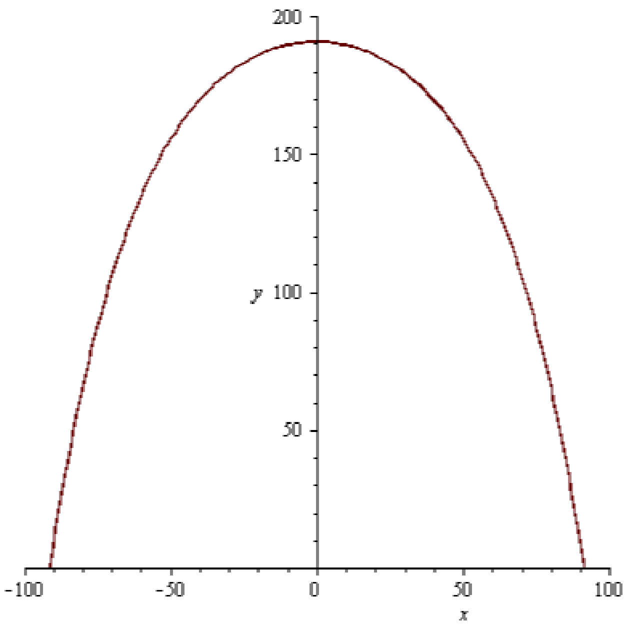
(a)

The graph of the equation is as shown below.

Input:

> $plot(211.49 - 20.96\cdot\cosh(0.03291765\cdot x), x = -100..100, y = 0..200)$

Output:



(b)

To find the height of the arch at its center, substitute the value $x = 0$ into the arch.

The height of the arch at its center is,

$$\begin{aligned} y &= 211.49 - 20.96\cosh(0.03291765x) \\ &= 211.49 - 20.96\cosh(0.03291765\cdot 0) \\ &= 211.49 - 20.96\cosh(0) \\ &= 190.53 \text{ meters} \end{aligned}$$

Hence, the height of the arch at its center is $\boxed{y = 190.53 \text{ meters}}$.

(c)

To find the point at the height is 100 meters, equate the equation to 100 meters.

The equation is,

$$\begin{aligned} y &= 211.49 - 20.96\cosh(0.03291765x) \\ 0 &= 211.49 - 20.96\cosh(0.03291765x) \\ \frac{211.49}{20.96} &= \cosh(0.03291765x) \\ 0.03291765x &= \cosh^{-1}\left(\frac{211.49}{20.96}\right) \end{aligned}$$

$$\begin{aligned} x &= \frac{\cosh^{-1}\left(\frac{211.49}{20.96}\right)}{0.03291765} \\ &\approx 71.56m \end{aligned}$$

Hence, the height to the arch would be 100 meter above the ground at

$\boxed{x = \pm 71.56 \text{ meters (both sides of the arch)}}$.

(d)

To find the slope of the arch at the points, find the derivative of the equation and then substitute the points.

The slope of the arch is,

$$y = 211.49 - 20.96\cosh(0.03291765x)$$
$$y' = -0.6899\sinh(0.03291765x)$$

The slope of the arch at $x = 71.56$ meters is,

$$\begin{aligned} y' &= -0.68995\sinh(0.03291765 \cdot 7156) \\ &= -0.68995\sinh(2.355587034) \\ &= (-0.68995)\cdot(5.2247395502942) \\ &= -3.6048 \end{aligned}$$

The slope of the arch at $x = -71.56$ meters is,

$$\begin{aligned} y' &= -0.68995\sinh(0.03291765 \cdot -7156) \\ &= -0.68995\sinh(-2.355587034) \\ &= (-0.68995)\cdot(-5.2247395502942) \\ &= 3.6048 \end{aligned}$$

Hence, the slope of the arch at the points $x = \pm 71.56$ meters is $\boxed{y' = \pm 3.6048}$.

Answer 49E.

Here we have the data on the diameters of 36 rivet heads in 1/100 of an inch.

a)

We need to compute the sample mean and sample standard deviation.
By definitions, the sample mean diameter is given as

$$\begin{aligned} \bar{x} &= \frac{1}{36}[6.72 + 6.77 + 6.82 + \ldots\ldots + 6.76 + 6.72] \\ &= \boxed{6.7261} \end{aligned}$$

By definitions, the sample variance is given as

$$\begin{aligned} s^2 &= \frac{1}{36-1}\left[(6.72 - 6.7261)^2 + (6.77 - 6.7261)^2 + \ldots + (6.72 - 6.7261)^2\right] \\ &= 0.0029 \end{aligned}$$

Thus, the sample standard deviation is given as

$$\begin{aligned} s &= \sqrt{0.0029} \\ &= \boxed{0.0536} \end{aligned}$$

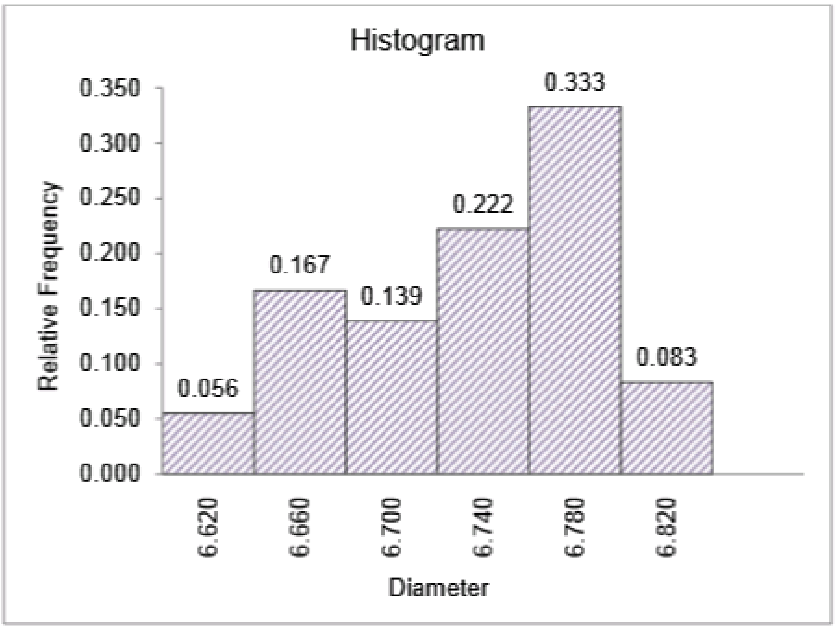
b)

We need to construct the relative frequency histogram of the data.

The relative frequency distribution for the given data is as follows:

Diameter						
lower		*upper*	*midpoint*	*width*	*frequency*	*Relative frequency*
6.600	<	6.640	6.620	0.040	2	0.056
6.640	<	6.680	6.660	0.040	6	0.167
6.680	<	6.720	6.700	0.040	5	0.139
6.720	<	6.760	6.740	0.040	8	0.222
6.760	<	6.800	6.780	0.040	12	0.333
6.800	<	6.840	6.820	0.040	3	0.083
					36	1.000

Using Excel, we construct a relative frequency histogram as follows:



c)

From the relative frequency histogram, we observed that the nature of the distribution is negatively skewed as the left tail of the curve is pronounced than the right tail. Hence the distribution is not symmetric.
The histogram indicates that the sample does not come from a population of a bell-shaped distribution.

Substitute 4.0/s for ω, 10.0cm for A, and 0.161s for t.

$$\begin{aligned} a &= \left(-160\ \text{cm/s}^2\right)\sin\left[4(0.161\text{s})\right] \\ &= -96.0\ \text{cm/s}^2 \end{aligned}$$

Therefore, the speed and acceleration of the particle, when its distance is 6.00cm from equilibrium are $\boxed{32.0\text{cm/s}}$ and $\boxed{-96.0\text{cm/s}^2}$.

(c)

The required time interval for a particle to move 8.00 cm from the origin can be determined as follows:

Use equation (1) to calculate the distance of the particle.

$$x(t) = A\sin\omega t$$

Rearrange the equation for time t.

$$t = \left(\frac{1}{\omega}\right)\sin^{-1}\left(\frac{x}{A}\right)$$

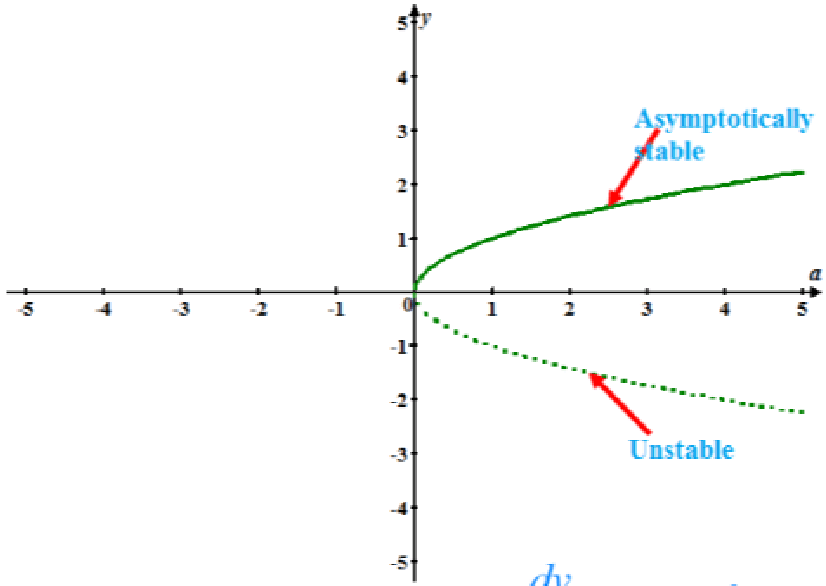
Substitute 10.0cm for A, 8.00cm for x, and 4.00/s for ω.

$$\begin{aligned} t &= \left(\frac{1}{4.00/\text{s}}\right)\sin^{-1}\left(\frac{8.00\text{cm}}{10.0\text{cm}}\right) \\ &= 0.232\text{s} \end{aligned}$$

Therefore, the time interval required for the object to move from $x = 0$ to $x = 8$ cm is $\boxed{0.232\text{s}}$.

(c)

Sketch the bifurcation diagram for the differential equation, $\frac{dy}{dt}=ay-y^3$ by using advanced Grapher and location of the critical points as a function of a in the $ay-$plane as follows:



Bifurcation diagram for $\frac{dy}{dt}=ay-y^3$

Rewrite the equation as

$$\begin{aligned} f(P) &= 0.08P\left(1-\frac{P}{1000}\right) \\ &= \left(0.08P-\frac{0.08P^2}{1000}\right) \\ &= \left(\frac{0.08P\times 1000-0.08P^2}{1000}\right) \\ &= \left(\frac{0.08}{1000}\right)(1000P-P^2) \end{aligned}$$

The reproduction rate approaches maximum value at $f'(P)=0$, so

$$f'(P)=\left(\frac{0.08}{1000}\right)(1000-2P)=0$$

Find the second derivative of $f(P)=0.08P\left(1-\frac{P}{1000}\right)$

$$f''(P)=\left(\frac{0.08}{1000}\right)(-2)<0$$

The second derivative is negative, so the maximum reproduction rate is

$$\begin{aligned} f(500) &= 0.08\times 500\left(1-\frac{500}{1000}\right) \\ &= 20 \end{aligned}$$

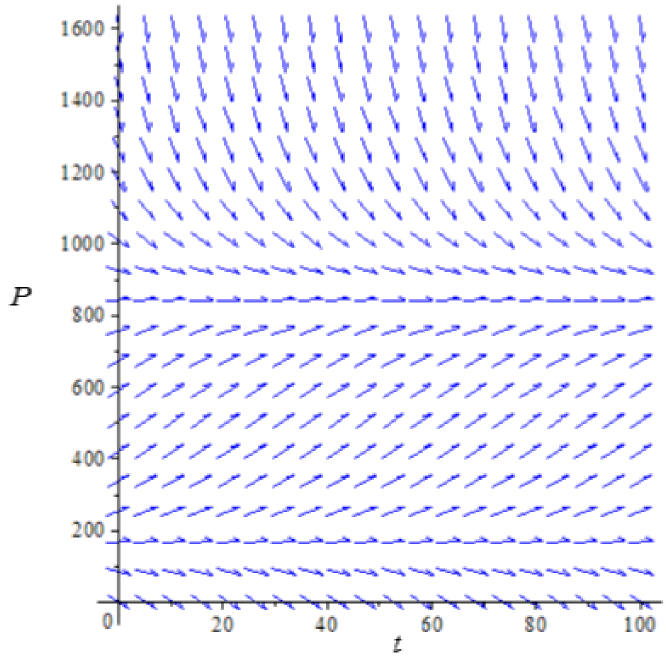
If the value of c exceed this value 20, the production rate P will be negative for every value of P

Therefore, fish population will always die out for $c>20$.

(c)

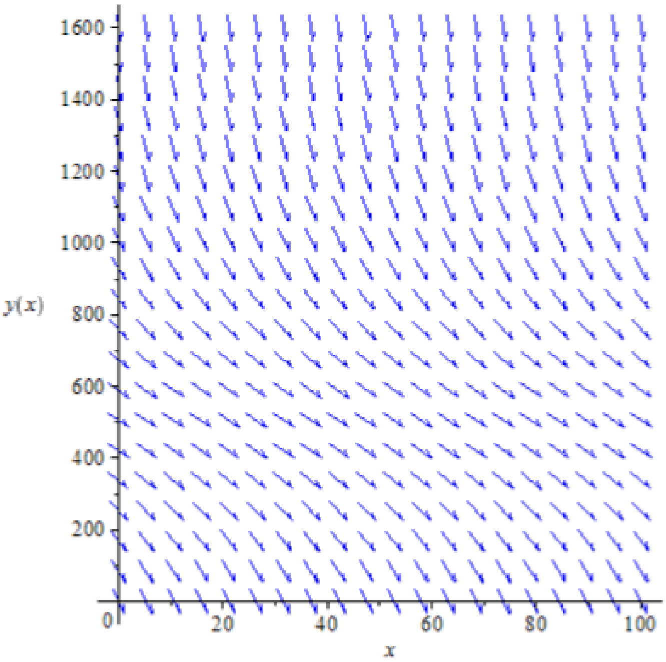
For $c < 20$,

Sketch the direction field for the differential equation $\frac{dP}{dx} = 0.08P\left(1 - \frac{P}{1000}\right) - c$



For $c > 20$,

Sketch the direction field for the differential equation $\frac{dP}{dx} = 0.08P\left(1 - \frac{P}{1000}\right) - c$



(d)

Since the solution of the differential equation $\frac{dP}{dx} = 0.08P\left(1 - \frac{P}{1000}\right) - c$ is

$P(x) = \frac{1000}{1+Ae^{-kt}}$ where $A = \frac{1000 - P_0}{P_0}$

Using the above expression $P(x)$, $\lim_{x\to\infty} P(x) = 1000$ this is to be expected.

Answer 50E.

We sketch the curve $y = a\cos h(x/a)$ for $a = 1,2,3,4,$ and 5

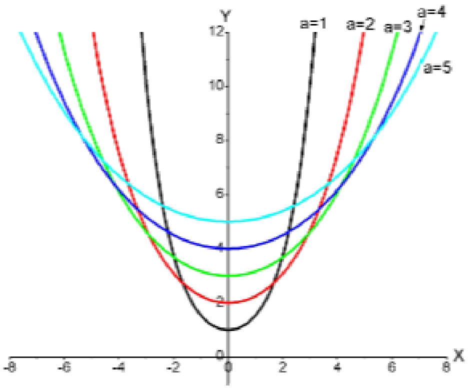


Fig.1

1. For all graphs, the y –intercepts are $y = a$.
2. As a increases, the opening of parabola becomes broader

Answer 51E.

(A)

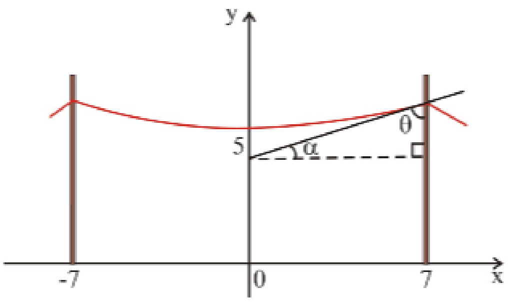


Fig. 1

The shaped of the catenary $y = 20\cos h\left(\frac{x}{20}\right) - 15$

Then slope of the tangent at any point is

$$\frac{dy}{dx} = 20\frac{d}{dx}\cos h\left(\frac{x}{20}\right) - \frac{d}{dx}15$$

$$= 20.\frac{1}{20}\sin h\left(\frac{x}{20}\right) - 0$$

$$= \sin h\left(\frac{x}{20}\right)$$

The right pole is at $x = 7$ so for $x = 7$ the slope of the curve is

$$\frac{dy}{dx} = \sin h\left(\frac{7}{20}\right) \boxed{\approx 0.3572}$$

(B) Let the tangent at $x = 7$, makes an angle ϕ with x – axis

Then $\tan\phi = \frac{dy}{dx}$ (at x = 7)

Or $\tan\phi = \sin h\left(\frac{7}{20}\right)$

Or $\phi = \tan^{-1}\left(\sin h\left(\frac{7}{20}\right)\right) \approx 19.66°$

By the property of right angle triangle $\theta + \phi + 90° = 180°$

So $\theta = 90° - \phi \approx 90° - 19.66°$

Or $\boxed{\theta \approx 70.34°}$

Answer 52E.

We have $y = \frac{T}{\rho g}\cos h\left(\frac{\rho g x}{T}\right)$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{T}{\rho g}\frac{dy}{dx}\cos h\left(\frac{\rho g x}{T}\right)$$

$$= \frac{T}{\rho g}\cdot\frac{\rho g}{T}.\sin h\left(\frac{\rho g x}{T}\right) \qquad \text{[By chain rule]}$$

Or $\frac{dy}{dx} = \sin h\left(\frac{\rho g x}{T}\right)$ --- (1)

Again differentiating (1) with respect to x

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\sin h\left(\frac{\rho g x}{T}\right)$$

Or $\frac{d^2y}{dx^2} = \frac{\rho g}{T}\cos h\left(\frac{\rho g x}{T}\right)$ --- (2) [By chain rule]

$$\frac{\rho g}{T}\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{\rho g}{T}\sqrt{1+\sin h^2\left(\frac{\rho g x}{T}\right)}$$

Using $\cos h^2 x - \sin h^2 x = 1$

Putting $x = \frac{\rho g x}{T} \Rightarrow \quad \cos h^2\left(\frac{\rho g x}{T}\right) - \sin h^2\left(\frac{\rho g x}{T}\right) = 1$

$$\Rightarrow \cos h^2\left(\frac{\rho g x}{T}\right) = 1 + \sin h^2\left(\frac{\rho g x}{T}\right)$$

So $\frac{\rho g}{T}\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{\rho g}{T}\sqrt{\cos h^2\left(\frac{\rho g x}{T}\right)}$

$$= \frac{\rho q}{T}\cos h^2\left(\frac{\rho g x}{T}\right) = \frac{d^2y}{dx^2} \quad \text{(from (2))}$$

Thus $y = \frac{T}{\rho g}\cos h\left(\frac{\rho g x}{T}\right)$ is the solution of given equation

Answer 53E.

A cable linear density $\rho = 2\ \text{kg/m}$ is strung from the tops of two poles that are 200 m apart.

(a)

The shape that this cable would be the function $y = \frac{T}{\rho g}\cosh\left(\frac{\rho g x}{T}\right)$ where T is the tension in the cable in units of Newton's N, and *g* is the acceleration due to gravity, a constant $9.81\ \text{m/s}^2$.

To find the tension T, the cable will make 60 meter at the center 0 i.e. $x = 0$ and $y = 60$.

Substitute these values into the function,

$$y = \frac{T}{\rho g}\cosh\left(\frac{\rho g x}{T}\right)$$

$$60 = \frac{T}{2(9.81)}\cosh\left(\frac{\rho g \cdot 0}{T}\right)$$

$$T = 19.62 \cdot 60$$

$$= 1177.2\text{N}$$

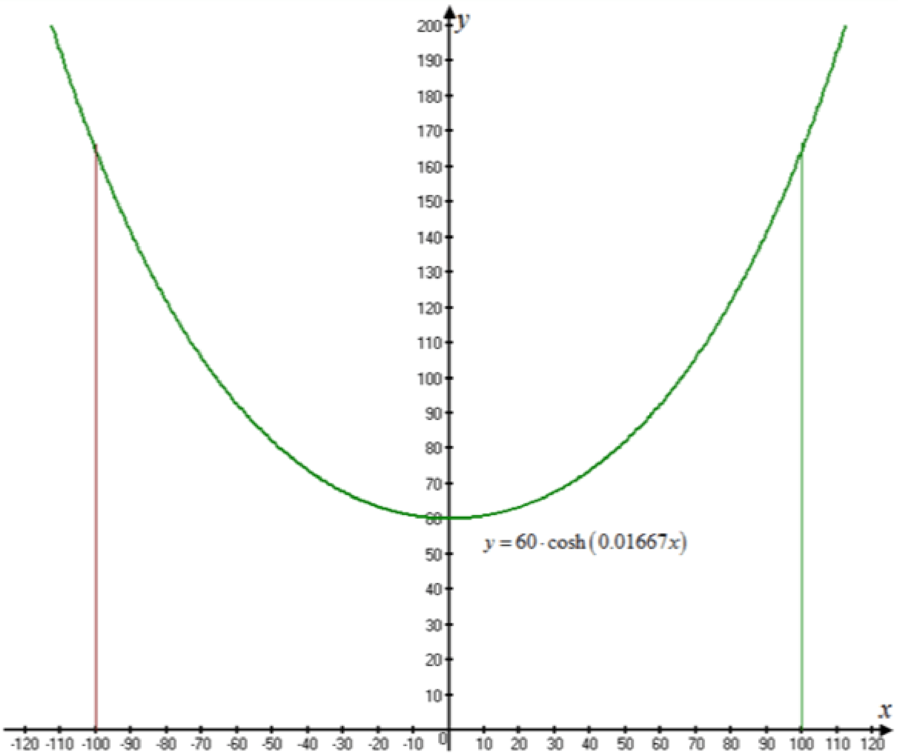
Hence, the value of tension T is $\boxed{T = 1177.2\text{N}}$.

To find the height of the poles, make sure to draw the graph of the function and then substitute x value.

Substitute the values into the function,

$$y = \frac{T}{\rho g}\cosh\left(\frac{\rho g x}{T}\right)$$
$$= \frac{1177.2}{2(9.81)}\cosh\left(\frac{2(9.81)\cdot x}{1177.2}\right)$$
$$y = 60\cdot\cosh(0.01667x)$$

The graph of the function is as shown below.



Observe, the two poles are 200 meters apart, so we need to find the value of the function when $x = 100\,\text{meter}$.

The height of the pole is,

$$y = 60\cdot\cosh(0.01667x)$$
$$= 60\cdot\cosh(0.01667\cdot 100)$$
$$\approx 164.506\text{ meter}$$
$$= 164.5\text{ meter}$$

Hence, the height of the poles is $\boxed{y = 164.5\text{ meter}}$.

(b)

The tension is doubled i.e. $T = 2354.4\text{N}$ $(\text{since } 2(1177.2\text{N}))$.

To find the new low point of the cable, substitute the tension and $x = 0$ into the function

The function is,

$$y = \frac{T}{\rho g}\cosh\left(\frac{\rho g x}{T}\right)$$
$$= \frac{2354.4}{2(9.81)}\cosh\left(\frac{2(9.81)x}{2354.4}\right)$$
$$= 120\cosh(0.008333x)$$

Hence, the function is $y = 120\cosh(0.008333x)$.

The new low point of the cable is,

$$\begin{aligned} y &= 120\cosh(0.008333x) \\ &= 120\cosh(0.008333\cdot 0) \\ &= 120\cosh(0) \\ &= 120 \text{ meter} \end{aligned}$$

Hence, the new low point of the cable is $\boxed{y = 120 \text{ meter}}$.

To find the new height of the pole, substitute the same value $x = 100$ meter in the function as the poles are 200 meters apart.

$$\begin{aligned} y &= 120\cosh(0.008333x) \\ &= 120\cosh(0.008333\cdot 1000) \\ &= 120\cosh(8.333) \\ &= 164.13 \text{ meter} \end{aligned}$$

Hence, the new height of the pole is $\boxed{y = 164.13 \text{ meter}}$ with doubled tension and the poles are 200 meters apart.

Answer 54E.

We have to evaluate $\lim_{x\to\infty} \frac{\sin hx}{e^x}$

We have

$$\begin{aligned} \lim_{x\to\infty} \frac{\sin hx}{e^x} &= \lim_{x\to\infty} \frac{e^x - e^{-x}}{2e^x} \qquad \left[\sin hx = \frac{e^x - e^{-x}}{2}\right] \\ &= \lim_{x\to\infty} \frac{1 - e^{-2x}}{2} \\ &= \frac{1}{2} - \lim_{x\to\infty} \frac{e^{-2x}}{2} \end{aligned}$$

Since $e^{-2x} \to 0$ as $x \to \infty$

So $\lim_{x\to\infty} \frac{\sin hx}{e^x} = \frac{1}{2} - 0$

Or $\boxed{\lim_{x\to\infty} \frac{\sin hx}{e^x} = \frac{1}{2}}$

Answer 55E.

Consider the function,

Given function is $y = A\sin hmx + B\cos hmx$ ...... (1)

Differentiate with respect to x.

$$y' = \frac{dy}{dx} = Am\cos hmx + Bm\sin hmx \text{ ...... (2)}$$

$$\left[\frac{d}{dx}\cos hx = \sin hx, \frac{d}{dx}\sin hx = \cos hx\right]$$

Differentiating again with respect to x

$$y'' = \frac{d^2y}{dx^2} = Am^2 \sin hmx + Bm^2 \cos hmx$$

So that,

$$\begin{aligned} y'' &= m^2 (A\sin hmx + B\cos hmx) \\ &= m^2 y \end{aligned}$$

Therefore the function satisfies the differential equation $y'' = m^2 y$

(b)

Find a function y such that $y'' = 9y$ satisfying $y(0) = -4$ and $y'(0) = 6$

Compare with the differential equation $y'' = m^2 y$ so that,

$$m^2 = 9$$
$$m = 3$$

Use the initial condition $y(0) = -4$ in (1)

$$A\sinh(m0) + B\cosh(m0) = -4$$
$$A\sinh(0) + B\cosh(0) = -4$$
$$A(0) + B(1) = -4 \qquad (\sinh(0) = 0, \cosh(0) = 1)$$
$$B = -4$$

Use the initial condition $y'(0) = 6$ in (2).

$$Am\cosh(m0) + Bm\sinh(m0) = 6$$
$$Am\cosh(0) + Bm\sinh(0) = 6$$
$$Am(1) + B(0) = 6$$
$$Am = 6$$

Find A using the value of m.

$$A = \frac{6}{m}$$
$$= \frac{6}{3} \qquad (m = 3)$$
$$= 2$$

Plugin the values and write the desired function.

$$y = A\sin hmx + B\cos hmx$$
$$= 2\sin h3x - 4\cosh 3x \qquad (A = 2, B = -4, m = 3)$$

Therefore the desired function is,

$$y = \boxed{2\sin h3x - 4\cosh 3x}$$

Answer 56E.

Let $ae^x + be^{-x} = \alpha \sin h(x + \beta) \qquad a \neq 0, b \neq 0$

$$= \alpha.\frac{e^{(x+\beta)} - e^{-(x+\beta)}}{2}$$

$$\Rightarrow 2ae^x + 2be^{-x} = \alpha.e^x.e^\beta - \alpha e^{-x}.e^{-\beta}$$

Comparing both sides, we have

$2a = \alpha e^\beta$ --- (1)

And $2b = -\alpha e^{-\beta}$ --- (2)

From (1)

$$\alpha = 2ae^{-\beta}$$

Then from equation (2), we have

$$2b = -2ae^{-\beta}.e^{-\beta}$$
$$\Rightarrow -\frac{b}{a} = e^{-2\beta}$$

Since $e^{-2\beta} > 0$ for all β, so $-\frac{b}{a}$ must be positive which is possible when b and a both are having opposite signs.

Now we suppose that

$$ae^x + be^{-x} = \alpha \cos h(x+\beta)$$

$$= \alpha.\frac{e^{(x+\beta)} + e^{-(x+\beta)}}{2}$$

$$\Rightarrow 2ae^x + 2be^{-x} = \alpha e^{\beta}.e^x + \alpha e^{-\beta} e^{-x}$$

Comparing both sides, we have

$$2a = \alpha e^{\beta} \qquad \text{--- (3)}$$

And $\quad 2b = \alpha e^{-\beta} \qquad$ --- (4)

From equation (3), $\quad \alpha = 2ae^{-\beta}$

Then from equation (4), we have

$$2b = 2ae^{-2\beta}$$

$$\Rightarrow \frac{b}{a} = e^{-2\beta} \qquad \text{--- (5)}$$

Since $e^{-2\beta} > 0$ so $\frac{b}{a}$ must be positive which is possible when a and b both are having same sign.

$\Rightarrow$ Calculating α and β

Since $\frac{b}{a} = e^{-2\beta}$

$$\Rightarrow -2\beta = \ln\left(\frac{b}{a}\right) \qquad \left[\ln x = y \Leftrightarrow e^y = x\right]$$

$$\Rightarrow \beta = -\frac{1}{2}\ln\left(\frac{b}{a}\right)$$

$$\Rightarrow \boxed{\beta = \ln\sqrt{\frac{a}{b}}} \qquad \left[r\ln x = \ln x^r\right]$$

Now from (5), we have

$$e^{-2\beta} = \frac{b}{a}$$

$$\Rightarrow e^{-\beta} = \sqrt{\frac{b}{a}}$$

Then from (4), we have

$$2b = \alpha.\sqrt{\frac{b}{a}}$$

$$\Rightarrow \alpha = 2b\sqrt{\frac{a}{b}}$$

$$\Rightarrow \boxed{\alpha = 2\sqrt{ab}}$$

So we have two cases as follows:

When a and b have opposite sign $\Rightarrow ae^x + be^{-x} = \alpha \sin h(x+\beta)$

And when a and b have same sign $\Rightarrow ae^x + be^{-x} = \alpha \cos h(x+\beta)$

And $\boxed{\alpha = 2\sqrt{ab}}$ and $\boxed{\beta = \ln\sqrt{\frac{a}{b}}}$.

Answer 57E.

First we find slope of tangent as

$$y' = \frac{d}{dx}\cosh x = \sinh x$$

It is given that slope is 1, thus

$$\sinh x = 1$$

$$x = \sinh^{-1} 1 = \ln\left(1+\sqrt{2}\right)$$

Putting value of x in $\cosh x$, we get $y = \sqrt{2}$

Therefore, at point $\ln\left(1+\sqrt{2}\right), \sqrt{2}$ the tangent have slope 1.

Answer 59E.

The given integral is $\int \sin hx \cos h^2 x\,dx$

Putting $\cos hx = u$

Differentiating both sides, we get

$$\sin hx\,dx = du$$

Therefore,

$$\int \sin hx \cos h^2 x\,dx = \int u^2 du$$
$$= \frac{u^3}{3} + C$$
$$= \frac{\cos h^3 x}{3} + C$$

Hence,

$$\boxed{\int \sin hx \cos h^2 x\,dx = \frac{\cos h^3 x}{3} + C}$$

Answer 60E.

Consider the following integral:

$$\int \sinh(1+4x)\,dx$$

To evaluate the above integral substitute:

$$1+4x = u$$

Then,

$$4dx = du$$
$$dx = \frac{1}{4}du$$

Use the above substitution and evaluate the integral:

$$\int \sinh(1+4x)\,dx = \int \sinh u \cdot \frac{1}{4}du$$

Replace, $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and evaluate:

$$\int \sinh(1+4x)\,dx = \frac{1}{4}\int \frac{\left(e^u - e^{-u}\right)}{2}du$$
$$= \frac{1}{8}\int e^u du - \frac{1}{8}\int e^{-u} du \quad \text{...... (1)}$$
$$= \frac{1}{8}e^u - \frac{1}{8}\int e^{-u} du + C_1$$

To evaluate the integral $\int e^{-u} du$ use the following substitution:

$$-u = t$$
$$-du = dt$$
$$du = -dt$$

Then,

$$\int e^{-u} du = \int e^t (-dt)$$
$$= -\int e^t (dt)$$
$$= -e^t + C_2$$
$$= -e^{-u} + C_2$$

Use this in (1):

$$\int \sinh(1+4x)\,dx = \frac{1}{8}e^u - \frac{1}{8}\int e^{-u} du + C_1$$
$$= \frac{1}{8}e^u - \frac{1}{8}\left[-e^{-u} + C_2\right] + C_1$$
$$= \frac{1}{8}e^u + \frac{1}{8}e^{-u} - \frac{1}{8}C_2 + C_1$$

Simplify,

$$\int \sinh(1+4x)\,dx = \frac{1}{4}\left[\frac{e^u + e^{-u}}{2}\right] + C$$

Where $C = \frac{-1}{8}C_2 + C_1$

Also note that $\cosh u = \left[\frac{e^u + e^{-u}}{2}\right]$ then,

$$\int \sinh(1+4x)\,dx = \frac{1}{4}\cosh u + C$$

As $1+4x = u$ above becomes:

$$\int \sinh(1+4x)\,dx = \frac{1}{4}\cosh(1+4x) + C$$

Therefore the required integral is:

$$\boxed{\int \sinh(1+4x)\,dx = \frac{1}{4}\cosh(1+4x) + C}$$

Answer 61E.

Consider the following integral:

$$\int \frac{\sinh\sqrt{x}}{\sqrt{x}}\,dx$$

The objective is to use the substitution rule to evaluate the integral.

Let $u = \sqrt{x}$ then $du = 1/\left(2\sqrt{x}\right)dx$

$$\Rightarrow 2du = \left(1/\sqrt{x}\right)dx$$

Now the integral becomes,

$$\begin{aligned}
\int \frac{\sinh\sqrt{x}}{\sqrt{x}}\,dx &= \int \sinh u\,(2du) \\
&= 2\int \sinh u\,du \\
&= 2(\cosh u) + C \\
&= 2\cosh\sqrt{x} + C \qquad \text{back substitute } \sqrt{x} \text{ for } u
\end{aligned}$$

Hence, $\int \frac{\sinh\sqrt{x}}{\sqrt{x}}\,dx = \boxed{2\cosh\sqrt{x} + C}$.

Answer 62E.

The given integral is $\int \tan hx\,dx$

From definition of hyperbolic functions,

$$\sin hx = \frac{e^x - e^{-x}}{2} \quad , \qquad \cos hx = \frac{e^x + e^{-x}}{2}$$

and

$$\begin{aligned}
\tan hx = \frac{\sin hx}{\cos hx} &= \frac{\left(e^x - e^{-x}\right)/2}{\left(e^x + e^{-x}\right)/2} \\
&= \frac{e^x - e^{-x}}{e^x + e^{-x}}
\end{aligned}$$

Therefore, $\int \tan hx\,dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$

Let $e^x + e^{-x} = u$

Differentiating we get,

$$\left(e^x - e^{-x}\right)dx = du$$

Now, $\int \tan hx\,dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln\left(e^x + e^{-x}\right) + C \qquad \text{since } \left[\left(e^x + e^{-x}\right) > 0 \quad for \quad all \quad x.\right]$$

Hence,

$$\boxed{\int \tan hx\,dx = \ln\left(e^x + e^{-x}\right) + C}$$

Answer 63E.

The given integral is $\int \frac{\cos hx}{\cos h^2 x - 1}dx$

From hyperbolic identities, we have

$$\cos h^2 x - \sin h^2 x = 1$$

$$\Rightarrow \quad \cos h^2 x - 1 = \sin h^2 x$$

Therefore, $\int \frac{\cos hx\,dx}{\cos h^2 x - 1} = \int \frac{\cos hx\,dx}{\sin h^2 x}$

Let $\sin hx = u$

Differentiating we get,

$$\cos hx\,dx = du$$

So, $\int \frac{\cos hx\,dx}{\sin h^2 x} = \int \frac{du}{u^2}$

$$= \int u^{-2}\,du$$

$$= \frac{u^{-2+1}}{(-2+1)} + C$$

$$= \frac{u^{-1}}{-1} + C$$

$$= -\frac{1}{u} + C$$

$$= -\frac{1}{\sin hx} + C$$

Hence,

$$\boxed{\int \frac{\cos hx\,dx}{\cos h^2 - 1} = \frac{-1}{\sin hx} + C = -\csc hx + C}$$

Answer 64E.

The given integral is $\int \frac{\sec h^2 x\,dx}{2 + \tan hx}$

Let $2 + \tan hx = u$

Differentiating we get,

$$d\left(2 + \tan hx\right) = du$$

$$\Rightarrow \quad \sec h^2 x\,dx = du$$

Therefore,

$$\int \frac{\sec h^2 x\,dx}{2+\tan hx} = \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|(2+\tan hx)| + C$$

$$= \ln(2+\tan hx) + C \qquad \text{since} \begin{bmatrix} -1 < \tanh x < 1\ so \\ (2+\tanh x) > 0\ \ for\ \ all\ \ x. \end{bmatrix}$$

Hence,

$$\boxed{\int \frac{\sec h^2 x\,dx}{2+\tan hx} = \ln(2+\tan hx) + C}$$

Answer 65E.

Consider the following definite integral:

$$\int_4^6 \frac{1}{\sqrt{t^2-9}}\,dt$$

To find the value of the above integral, first make the denominator in the standard form:

$$\int_4^6 \frac{1}{\sqrt{t^2-9}}\,dt = \int_4^6 \frac{1}{\sqrt{9\left(\frac{t^2}{9}-1\right)}}\,dt$$

$$= \frac{1}{3}\int_4^6 \frac{1}{\sqrt{\left(\frac{t}{3}\right)^2-1}}\,dt$$

To evaluate above, substitute:

$$\frac{t}{3} = \cosh x$$

Then,

$$\frac{dt}{3} = \sinh x dx$$

Or

$$dt = 3\sinh x dx$$

Also, the limits of the integral become:

When,

$$t = 4$$

Then

$$\cosh x = \frac{4}{3}$$

$$x = \cosh^{-1}\left(\frac{4}{3}\right)$$

And for $t = 6$:

$$\cosh x = \frac{6}{3}$$

$$x = \cosh^{-1}(2)$$

Use all the values determined above and find the integral:

$$\frac{1}{3}\int_4^6 \frac{1}{\sqrt{\left(\frac{t}{3}\right)^2 - 1}} = \frac{1}{3}\int_{\cosh^{-1}(4/3)}^{\cosh^{-1}(2)} \frac{3\sinh x dx}{\sqrt{\cosh^2 x - 1}}$$

$$= \int_{\cosh^{-1}(4/3)}^{\cosh^{-1}(2)} \frac{\sinh x dx}{\sqrt{\sinh^2 x}}$$

Since, $\cos h^2 x - 1 = \sin h^2 x$, further:

$$\frac{1}{3}\int_4^6 \frac{1}{\sqrt{\left(\frac{t}{3}\right)^2 - 1}} = \int_{\cosh^{-1}(4/3)}^{\cosh^{-1}(2)} \frac{\sinh x dx}{\sinh x}$$

$$= \int_{\cosh^{-1}(4/3)}^{\cosh^{-1}(2)} 1 \cdot dx$$

$$= [x]_{\cosh^{-1}(4/3)}^{\cosh^{-1}(2)}$$

$$= \cosh^{-1}(2) - \cosh^{-1}(4/3)$$

Now, note that $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$ when $x \geq 1$ so, the above expression becomes:

$$\frac{1}{3}\int_4^6 \frac{1}{\sqrt{\left(\frac{t}{3}\right)^2 - 1}} = \ln\left[2 + \sqrt{(2)^2 - 1}\right] - \ln\left[\frac{4}{3} + \sqrt{\left(\frac{4}{3}\right)^2 - 1}\right]$$

$$= \ln\left[2 + \sqrt{3}\right] - \ln\left[\frac{4}{3} + \sqrt{\frac{16}{9} - 1}\right]$$

$$= \ln\left[2 + \sqrt{3}\right] - \ln\left[\frac{4 + \sqrt{7}}{3}\right]$$

$$= \ln\frac{\left(2 + \sqrt{3}\right)}{\left(4 + \sqrt{7}\right)/3}$$

Simplify:

$$\frac{1}{3}\int_4^6 \frac{1}{\sqrt{\left(\frac{t}{3}\right)^2 - 1}} = \ln\left(\frac{6 + 3\sqrt{3}}{4 + \sqrt{7}}\right)$$

Hence the required integral is:

$$\boxed{\int_4^6 \frac{1}{\sqrt{t^2 - 9}} dt = \ln\left(\frac{6 + 3\sqrt{3}}{4 + \sqrt{7}}\right)}$$

Answer 66E.

The integral is $\int_0^1 \frac{1}{\sqrt{16t^2 + 1}} dt$.

The objective is to evaluate the integral.

Use the following formula:

$$\int \frac{1}{\sqrt{1 + x^2}} dx = \sinh^{-1} x .$$

Consider the integral,

$$\int_0^1 \frac{1}{\sqrt{16t^2+1}}dt = \int_0^1 \frac{1}{\sqrt{1+16t^2}}dt$$
$$= \int_0^1 \frac{1}{\sqrt{1+(4t)^2}}dt$$

Let $4t = x$

Differentiate both sides.

$$4dt = dx$$
$$dt = \frac{1}{4}dx$$

Limits of integration:

Upper limit:

$$x = 4(1) = 4$$

Lower limit:

$$x = 4(0) = 0$$

The integral becomes,

$$\int_0^1 \frac{1}{\sqrt{16t^2+1}}dt = \int_0^1 \frac{1}{\sqrt{1+(4t)^2}}dt$$
$$= \int_0^4 \frac{1}{\sqrt{1+x^2}}\left(\frac{1}{4}\right)dx$$
$$= \frac{1}{4}\left[\sinh^{-1} x\right]_0^4$$
$$= \frac{1}{4}\left[\sinh^{-1} 4 - \sinh^{-1} 0\right]$$

Answer 67E.

Consider the indefinite integral:

$$\int \frac{e^x}{1-e^{2x}}dx.$$

The objective is to evaluate this integral.

To evaluate this integral, use substitution: $u = e^x$, and $du = e^x\,dx$.

Now, plug in the values of u and du, into the integral, to evaluate it:

$$\int \frac{e^x}{1-e^{2x}}dx = \int \frac{du}{1-(u)^2}$$
$$= \int \frac{du}{1-u^2}$$

$= \tanh^{-1}(u) + C$ Since, the anti-derivative of $\frac{1}{1-u^2}$ is $\tanh^{-1}(u)$.

$= \tanh^{-1}(e^x) + C$ Since, $u = e^x$

Thus, the value of the indicated integral is,

$$\boxed{\int \frac{e^x}{1-e^{2x}}dx = \tanh^{-1}(e^x) + C}.$$

Answer 68E.

The area under the curve $y = \sin hcx$ between $x = 0$ and $x = 1$ is

$$A = \int_0^1 \sin hcx\,dx$$

$$= \int_0^1 \left(\frac{e^{cx} - e^{-cx}}{2}\right) dx$$

$$= \frac{1}{2}\int_0^1 e^{cx}\,dx - \frac{1}{2}\int_0^1 e^{-cx}\,dx$$

$$= \frac{1}{2c}\left[e^{cx}\right]_0^1 + \frac{1}{2c}\left[e^{-cx}\right]_0^1$$

$$= \frac{1}{2c}\left[e^{c} - e^{0}\right] + \frac{1}{2c}\left[e^{-c} - e^{0}\right]$$

$$= \frac{1}{2c}\left[e^{c} - 1\right] + \frac{1}{2c}\left[e^{-c} - 1\right]$$

Or $\quad A = \frac{1}{2c}\left[e^{c} + e^{-c} - 2\right]$

We have $\quad A = 1$

So $\quad \frac{1}{2c}\left[e^{c} + e^{-c} - 2\right] = 1$

Or $\quad e^{c} + e^{-c} - 2 = 2c$

$$\Rightarrow \boxed{e^{c} + e^{-c} = 2(c+1)}$$

For solving this equation we use graphical method. We sketch the curves $y = e^{c} + e^{-c}$ and $y = 2(c+1)$ on the same set of axis and move the cursor at the point of intersection of these curves, we see that x –coordinates of the points are about 0 and 1.6 [figure 1]

But we can not take 0 because $\sin h0 = 0$

So $\quad \boxed{c \approx 1.6}$

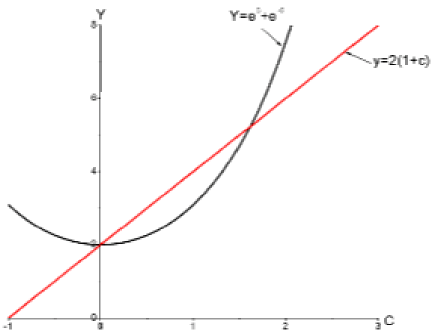


Fig.1

Answer 69E.

(A) First we sketch the curves $y = \cos h2x$ and $y = 1 + \sin hx$ on the same see of axis and move the cursor to the points of intersection, we see that the x-coordinates of the points of intersection are about 0 and 0.48

So the solutions of the equation $\cos h2x = 1 + \sin hx$ are

$$\boxed{x = 0 \text{ and } x \approx 0.48}$$

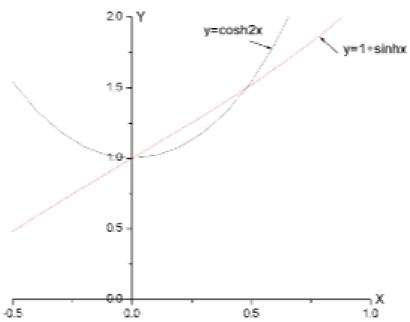


figure1

(B) From figure 1 we see that from $x = 0$ to $x \approx 0.48$ the curve $y = \cos h\, 2x$ lies below the curve $y = 1 + \sin h\, x$.

So the area of the region bounded by the curves $y = \cos h\, 2x$ and $y = 1 + \sin h\, x$ is

$$A = \int_0^{0.48} (1 + \sin hx - \cos h\, 2x)\, dx$$
$$= \int_0^{0.4} \left(1 + \frac{e^x - e^{-x}}{2} - \frac{e^{2x} + e^{-2x}}{2}\right) dx$$
$$= \frac{1}{2} \int_0^{0.4} \left(2 + e^x - e^{-x} - e^{2x} - e^{-2x}\right) dx$$
$$= \frac{1}{2}\left[2x + e^x + e^{-x} - \frac{e^{2x}}{2} + \frac{e^{-2x}}{2}\right]_0^{0.48}$$
$$= \frac{1}{2}\left[2(0.48) + e^{0.48} + e^{-0.48} - \frac{e^{2(0.48)}}{2} + \frac{e^{-2(0.48)}}{2} - 0 - e^0 - e^0 + \frac{e^0}{2} - \frac{e^0}{2}\right]$$
$$\approx \frac{1}{2}(0.08)$$

Or $\boxed{A \approx 0.04}$

Answer 70E.

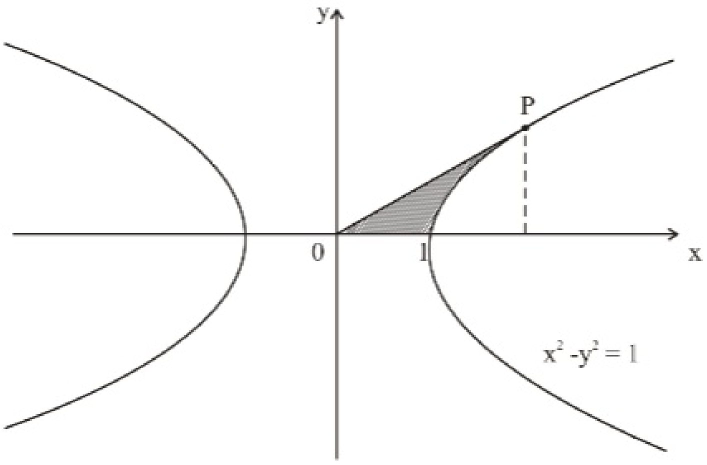


Fig. 1

The equation of the curve is

$$x^2 - y^2 = 1$$
$$\Rightarrow y^2 = x^2 - 1 \Rightarrow y = \sqrt{x^2 - 1}$$

Area of the shaded region is

A = Area of the right triangle – [Area of the region under curve $x^2 - y^2 = 1$ from 1 to $\cos ht$]

$$= \frac{1}{2}(base) \times (height) - \int_1^{\cos ht} \sqrt{x^2 - 1}\, dx$$

Or $$A(t) = \frac{1}{2}(\cos ht)(\sin ht) - \int_1^{\cos ht} \sqrt{x^2 - 1}\, dx$$

Differentiating with respect to t

$$A'(t) = \frac{d}{dt}\left(\frac{1}{2}\cos ht \sin ht\right) - \frac{d}{dt}\int_1^{\cos ht} \sqrt{x^2 - 1}\, dx$$

First we calculate $\frac{d}{dt}\left(\frac{1}{2}\cos ht \sin ht\right)$

$$\frac{d}{dt}\left(\frac{1}{2}\cos ht \sin ht\right) = \frac{1}{2}\left[\left(\frac{d}{dt}(\cos ht)\right).\sin ht + \cos ht \frac{d}{dt}(\sin ht)\right]$$

(By Product rule)

$$= \frac{1}{2}\left[\sin ht . \sin ht + \cos ht . \cos ht\right]$$
$$= \frac{1}{2}\left[\sin h^2 t + \cos^2 ht\right]$$

Now we calculate $\frac{d}{dt}\int_1^{\cos ht}\sqrt{x^2-1}\,dx$

Let $\cos ht = u$ then $\sin ht = \frac{du}{dt}$

So $$\frac{d}{dt}\int_1^{\cos ht}\sqrt{x^2-1}\,dx = \frac{d}{du}\int_1^{u}\sqrt{x^2-1}\,dx.\frac{du}{dt} \qquad \text{(By chain rule)}$$
$$= \sqrt{u^2-1}.(\sin ht) \qquad [\text{F.T.C - 1}]$$
$$= \sqrt{\cos h^2 t-1}.(\sin ht)$$
$$= \sqrt{\sin h^2 t}.(\sin ht) \qquad \left[\sin h^2 t = \cos h^2 t - 1\right]$$
$$= \sin h^2 t$$

Then $$A'(t) = \frac{1}{2}\sin h^2 t + \frac{1}{2}\cos h^2 t - \sin h^2 t$$
$$= \frac{1}{2}\left(\cos h^2 t - \sin h^2 t\right)$$
$$\boxed{A'(t) = \frac{1}{2}}$$

Since derivative of $A(t) = \frac{1}{2}$ so an anti derivative of $A'(t)$ is $A(t) = \frac{1}{2}t$

So $\boxed{A(t) = \frac{1}{2}t}$

Answer 71E.

Let $$ae^x + be^{-x} = \alpha \sin h(x+\beta) \qquad a \neq 0, b \neq 0$$
$$= \alpha.\frac{e^{(x+\beta)} - e^{-(x+\beta)}}{2}$$
$$\Rightarrow 2ae^x + 2be^{-x} = \alpha e^x.e^\beta - \alpha e^{-x}.e^{-\beta}$$

Comparing both sides we have

$$2a = \alpha e^\beta \qquad \text{--- (1)}$$
And $$2b = -\alpha e^{-\beta} \qquad \text{--- (2)}$$

Now we suppose that
$$ae^x + be^{-x} = \alpha \cos h(x+\beta)$$
$$= \alpha.\frac{e^{(x+\beta)} + e^{-(x+\beta)}}{2}$$
$$\Rightarrow 2ae^x + 2be^{-x} = \alpha e^\beta.e^x + \alpha e^{-\beta}e^{-x}$$
Comparing both sides we have
$$2a = \alpha e^\beta \qquad \text{--- (3)}$$
And $$2b = \alpha e^{-\beta} \qquad \text{--- (4)}$$

Now we suppose that
$$ae^x + be^{-x} = \alpha \cos h(x+\beta)$$
$$= \alpha.\frac{e^{(x+\beta)} + e^{-(x+\beta)}}{2}$$
$$\Rightarrow 2ae^x + 2be^{-x} = \alpha e^\beta.e^x + \alpha e^{-\beta}e^{-x}$$
Comparing both sides we have
$$2a = \alpha e^\beta \qquad \text{--- (3)}$$
And $$2b = \alpha e^{-\beta} \qquad \text{--- (4)}$$

From (3) $\alpha = 2ae^{-\beta}$

Then from (4) we have

$$2b = 2ae^{-2\beta}$$

$$\Rightarrow \frac{b}{a} = e^{-2\beta} \qquad \text{--- (5)}$$

Since $e^{-2\beta} > 0$ so $\frac{b}{a}$ must be positive which is possible when a and b both are having same sign.

$\Rightarrow$ Calculating α and β

Since $\frac{b}{a} = e^{-2\beta}$

$$\Rightarrow -2\beta = \ln\left(\frac{b}{a}\right) \qquad \left[\ln x = y \Leftrightarrow e^y = x\right]$$

$$\Rightarrow \beta = -\frac{1}{2}\ln\left(\frac{b}{a}\right)$$

$$\Rightarrow \boxed{\beta = \ln\sqrt{\frac{a}{b}}} \qquad \left[r\ln x = \ln x^r\right]$$

Now from (5) we have

$$e^{-2\beta} = \frac{b}{a}$$

$$\Rightarrow e^{-\beta} = \sqrt{\frac{b}{a}}$$

Then from (4) we have

$$2b = \alpha\sqrt{\frac{b}{a}}$$

$$\Rightarrow \alpha = 2b\sqrt{\frac{a}{b}}$$

$$\Rightarrow \boxed{\alpha = 2\sqrt{ab}}$$

So we have two cases

When and b have opposite sign $\Rightarrow ae^{x} + be^{-x} = \alpha \sin h\left(x+\beta\right)$

And when a and b have same sign $\Rightarrow ae^{x} + be^{-x} = \alpha \cos h\left(x+\beta\right)$

And $\boxed{\alpha = 2\sqrt{ab}}$ and $\boxed{\beta = \ln\sqrt{\frac{a}{b}}}$