2.3 Series, summations, and progressions

Progressions and summations

Arithmetic progression	$S_n = a+(a+d)+(a+2d)+\cdots + [a+(n-1)d]$ (2.102) $= \frac{n}{2}[2a+(n-1)d]$ (2.103) $= \frac{n}{2}(a+l)$ (2.104)	n number of terms S_n sum of n successive terms a first term d common difference l last term
Geometric progression	$S_n = a+ar+ar^2+\cdots+ar^{n-1}$ (2.105) $= a\frac{1-r^n}{1-r}$ (2.106) $S_\infty = \frac{a}{1-r} \quad (\lvert r\rvert<1)$ (2.107)	r common ratio
Arithmetic mean	$\langle x\rangle_{\mathrm{a}} = \frac{1}{n}(x_1+x_2+\cdots+x_n)$ (2.108)	$\langle . \rangle_{\mathrm{a}}$ arithmetic mean
Geometric mean	$\langle x\rangle_{\mathrm{g}} = (x_1x_2x_3\ldots x_n)^{1/n}$ (2.109)	$\langle . \rangle_{\mathrm{g}}$ geometric mean
Harmonic mean	$\langle x\rangle_{\mathrm{h}} = n\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)^{-1}$ (2.110)	$\langle . \rangle_{\mathrm{h}}$ harmonic mean
Relative mean magnitudes	$\langle x\rangle_{\mathrm{a}} \geq \langle x\rangle_{\mathrm{g}} \geq \langle x\rangle_{\mathrm{h}} \quad \text{if } x_i>0 \text{ for all } i$ (2.111)	
Summation formulas	$\sum_{i=1}^{n} i = \frac{n}{2}(n+1)$ (2.112) $\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$ (2.113) $\sum_{i=1}^{n} i^3 = \frac{n^2}{4}(n+1)^2$ (2.114) $\sum_{i=1}^{n} i^4 = \frac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$ (2.115) $\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots = \ln 2$ (2.116) $\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2i-1} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots = \frac{\pi}{4}$ (2.117) $\sum_{i=1}^{\infty} \frac{1}{i^2} = 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots = \frac{\pi^2}{6}$ (2.118)	i dummy integer
Euler's constant[a]	$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)$ (2.119)	γ Euler's constant

[a] $\gamma \simeq 0.577215664\ldots$

Power series

Binomial series[a]	$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$	(2.120)
Binomial coefficient[b]	${}^nC_r \equiv \binom{n}{r} \equiv \frac{n!}{r!(n-r)!}$	(2.121)
Binomial theorem	$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$	(2.122)
Taylor series (about a)[c]	$f(a+x) = f(a) + xf^{(1)}(a) + \frac{x^2}{2!}f^{(2)}(a) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(a) + \cdots$	(2.123)
Taylor series (3-D)	$f(\boldsymbol{a}+\boldsymbol{x}) = f(\boldsymbol{a}) + (\boldsymbol{x}\cdot\nabla)f\vert_{\boldsymbol{a}} + \frac{(\boldsymbol{x}\cdot\nabla)^2}{2!}f\vert_{\boldsymbol{a}} + \frac{(\boldsymbol{x}\cdot\nabla)^3}{3!}f\vert_{\boldsymbol{a}} + \cdots$	(2.124)
Maclaurin series	$f(x) = f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \cdots$	(2.125)

[a] If n is a positive integer the series terminates and is valid for all x. Otherwise the (infinite) series is convergent for $|x| < 1$.
[b] The coefficient of x^r in the binomial series.
[c] $xf^{(n)}(a)$ is x times the nth derivative of the function $f(x)$ with respect to x evaluated at a, taken as well behaved around a. $(\boldsymbol{x}\cdot\nabla)^n f|_{\boldsymbol{a}}$ is its extension to three dimensions.

Limits

$n^c x^n \to 0 \quad \text{as} \quad n \to \infty \quad \text{if} \quad \vert x\vert < 1 \quad \text{(for any fixed } c\text{)}$	(2.126)
$x^n/n! \to 0 \quad \text{as} \quad n \to \infty \quad \text{(for any fixed } x\text{)}$	(2.127)
$(1+x/n)^n \to \mathrm{e}^x \quad \text{as} \quad n \to \infty$	(2.128)
$x \ln x \to 0 \quad \text{as} \quad x \to 0$	(2.129)
$\frac{\sin x}{x} \to 1 \quad \text{as} \quad x \to 0$	(2.130)
If $f(a) = g(a) = 0$ or ∞ then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{(1)}(a)}{g^{(1)}(a)}$ (l'Hôpital's rule)	(2.131)

Series expansions

$\exp(x)$	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$	(2.132)	(for all x)
$\ln(1+x)$	$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$	(2.133)	$(-1<x\leq 1)$
$\ln\left(\frac{1+x}{1-x}\right)$	$2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots\right)$	(2.134)	$(\lvert x\rvert<1)$
$\cos(x)$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$	(2.135)	(for all x)
$\sin(x)$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$	(2.136)	(for all x)
$\tan(x)$	$x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}\cdots$	(2.137)	$(\lvert x\rvert<\pi/2)$
$\sec(x)$	$1+\frac{x^2}{2}+\frac{5x^4}{24}+\frac{61x^6}{720}+\cdots$	(2.138)	$(\lvert x\rvert<\pi/2)$
$\csc(x)$	$\frac{1}{x}+\frac{x}{6}+\frac{7x^3}{360}+\frac{31x^5}{15120}+\cdots$	(2.139)	$(\lvert x\rvert<\pi)$
$\cot(x)$	$\frac{1}{x}-\frac{x}{3}-\frac{x^3}{45}-\frac{2x^5}{945}-\cdots$	(2.140)	$(\lvert x\rvert<\pi)$
$\arcsin(x)$[a]	$x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}\cdots$	(2.141)	$(\lvert x\rvert<1)$
$\arctan(x)$[b]	$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$	(2.142)	$(\lvert x\rvert\leq 1)$
	$\frac{\pi}{2}-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}+\cdots$		$(x>1)$
	$-\frac{\pi}{2}-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}+\cdots$		$(x<-1)$
$\cosh(x)$	$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots$	(2.143)	(for all x)
$\sinh(x)$	$x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots$	(2.144)	(for all x)
$\tanh(x)$	$x-\frac{x^3}{3}+\frac{2x^5}{15}-\frac{17x^7}{315}+\cdots$	(2.145)	$(\lvert x\rvert<\pi/2)$

[a] $\arccos(x)=\pi/2-\arcsin(x)$. Note that $\arcsin(x)\equiv\sin^{-1}(x)$ etc.
[b] $\mathrm{arccot}(x)=\pi/2-\arctan(x)$.

Inequalities

Triangle inequality	$\lvert a_1\rvert - \lvert a_2\rvert \le \lvert a_1 + a_2\rvert \le \lvert a_1\rvert + \lvert a_2\rvert$;	(2.146)
	$\left\lvert \sum_{i=1}^{n} a_i \right\rvert \le \sum_{i=1}^{n} \lvert a_i\rvert$	(2.147)
Chebyshev inequality	if $a_1 \ge a_2 \ge a_3 \ge \ldots \ge a_n$	(2.148)
	and $b_1 \ge b_2 \ge b_3 \ge \ldots \ge b_n$	(2.149)
	then $n\sum_{i=1}^{n} a_i b_i \ge \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right)$	(2.150)
Cauchy inequality	$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$	(2.151)
Schwarz inequality	$\left[\int_a^b f(x)g(x)\,\mathrm{d}x\right]^2 \le \int_a^b [f(x)]^2\,\mathrm{d}x \int_a^b [g(x)]^2\,\mathrm{d}x$	(2.152)