# 10. Plane

#### • Equation of a plane in normal form:

- **Vector form:** Equation of a plane which is at a distance of d from the origin, and the unit vector  $\hat{n}$  normal to the plane through the origin is  $\vec{r} \cdot \hat{n} = d$ , where  $\vec{r}$  is the position vector of a point in the plane
- Cartesian form: Equation of a plane which is at a distance d from the origin and the d.c.'s of the normal to the plane as l, m, n is lx + my + nz = d

### • Equation of a plane perpendicular to a given vector and passing through a given point:

- Vector form: Equation of a plane through a point whose position vector is  $\vec{a}$  and perpendicular to the vector  $\vec{N}_{1s}$  ( $\vec{r} \vec{a}$ )  $\vec{N} = 0$ , where  $\vec{r}$  is the position vector of a point in the plane
- Cartesian form: Equation of plane passing through the point  $(x_1, y_1, z_1)$  and perpendicular to a given line whose d.r.'s are A, B, C is  $A(x x_1) + B(y y_1) + C(z z_1) = 0$

#### • Equation of a plane passing through three non-collinear points:

• Cartesian form: Equation of a plane passing through three non-collinear points  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, y_3, z_4)$ 

$$z_2$$
), and  $(x_3, y_3, z_3)$  is 
$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

• Vector form: Equation of a plane that contains three non-collinear points having position vectors  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$  is  $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$ , where  $\vec{r}$  is the position vector of a point in the plane

### • Planes passing through the intersection of two planes:

- Vector form: Equation of the plane passing through intersection of two planes  $\vec{r} \cdot \vec{n_1} = \vec{d_1}$  and  $\vec{r} \cdot \vec{n_2} = \vec{d_2}$  is given by,  $\vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$ , where  $\lambda$  is a non-zero constant
- Cartesian form: Equation of a plane passing through the intersection of two planes  $A_1x + B_1y + C_1z + D_1 = 0$  and  $A_2x + B_2y + C_2z + D_2 = 0$ , is given by,  $(A_1x + B_1y + C_1z + D_1) + \lambda (A_2x + B_2y + C_2z + D_2) = 0$ , where  $\lambda$  is a non-zero constant

## • Angle between two planes: The angle between two planes is defined as the angle between their normals.

• Vector form: If  $\theta$  is the angle between the two planes  $\vec{r} \cdot \vec{n_1} = d_1$  and  $\vec{r} \cdot \vec{n_2} = d_2$ , then

$$\cos \theta = \left| \frac{\vec{n_1} \cdot \vec{n_2}}{\left| \vec{n_1} \right| \left| \vec{n_2} \right|} \right|$$

Note that if two planes are perpendicular to each other, then  $\vec{n_1} \cdot \vec{n_2} = 0$ ; and if they are parallel to each other, then  $\vec{n_1}$  is parallel to  $\vec{n_2}$ .

• Cartesian form: If  $\theta$  is the angle between the two planes

$$A_1x + B_1y + C_1z + D_1 = 0 \text{ and } A_2x + B_2y + C_2z + D_2 = 0, \text{ then } \cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}$$

Note that if two planes are perpendicular to each other, then  $A_1A_2 + B_1B_2 + C_1C_2 = 0$ ; and if they are parallel to each other, then  $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$ 

• Angle between a line and a plane: The angle  $\Phi$  between a line  $\vec{r} = \vec{a} + \lambda \vec{b}$  and the plane  $\vec{r} = \vec{n} = d$  is the complement of the angle between the line and the normal to the plane and is given by

$$\Phi = \sin^{-1} \left| \frac{\overrightarrow{b} \cdot \overrightarrow{n}}{\left| \overrightarrow{b} \right| \left| \overrightarrow{n} \right|} \right|.$$

- Co-planarity of two lines
  - Vector form: Two lines  $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$  and  $\vec{r} = \vec{a_2} + \mu \vec{b_2}$  are co-planar, if  $(\vec{a_2} \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0$
  - Cartesian form: Two lines  $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$  and  $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$  and  $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$  and  $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$  are co-planar, if  $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$
- Distance of a point from a plane:
  - Vector form: The distance of a point, whose position vector is  $\vec{a}$ , from the plane  $\vec{r} : \hat{n} = d$  is  $|d \vec{a} : \hat{n}|$

Note:

• If the equation of the plane is in the form of  $\vec{r} \cdot \vec{N} = d$ , where  $\vec{N}$  is the normal to the plane, then the perpendicular distance is  $\frac{|\vec{a} \cdot \vec{N} - d|}{|\vec{N}|}$ .

• Cartesian form: The distance from a point  $(x_1, y_1, z_1)$  to the plane Ax + By + Cz + D = 0 is

$$\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}.$$