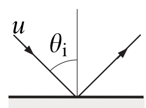
7.8 Waves in and out of media

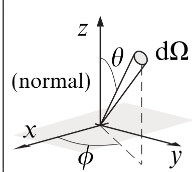
Waves in lossless media

Electric field	$\nabla^2 \boldsymbol{E} = \mu\epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$	(7.193)	$\boldsymbol{E}$ electric field μ permeability ($=\mu_0\mu_r$) ϵ permittivity ($=\epsilon_0\epsilon_r$)
Magnetic field	$\nabla^2 \boldsymbol{B} = \mu\epsilon \frac{\partial^2 \boldsymbol{B}}{\partial t^2}$	(7.194)	$\boldsymbol{B}$ magnetic flux density t time
Refractive index	$\eta = \sqrt{\epsilon_r \mu_r}$	(7.195)	
Wave speed	$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\eta}$	(7.196)	v wave phase speed η refractive index c speed of light
Impedance of free space	$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \simeq 376.7\,\Omega$	(7.197)	Z_0 impedance of free space
Wave impedance	$Z = \frac{E}{H} = Z_0 \sqrt{\frac{\mu_r}{\epsilon_r}}$	(7.198)	Z wave impedance H magnetic field strength

Radiation pressure[a]

Radiation momentum density	$\boldsymbol{G} = \frac{\boldsymbol{N}}{c^2}$	(7.199)	$\boldsymbol{G}$ momentum density $\boldsymbol{N}$ Poynting vector c speed of light
Isotropic radiation	$p_n = \frac{1}{3} u(1+R)$	(7.200)	p_n normal pressure u incident radiation energy density R (power) reflectance coefficient
Specular reflection	$p_n = u(1+R)\cos^2\theta_i$ $p_t = u(1-R)\sin\theta_i\cos\theta_i$	(7.201) (7.202)	p_t tangential pressure θ_i angle of incidence
From an extended source[b]	$p_n = \frac{1+R}{c} \iint I_\nu(\theta,\phi)\cos^2\theta \, d\Omega \, d\nu$	(7.203)	I_ν specific intensity ν frequency Ω solid angle θ angle between $d\Omega$ and normal to plane
From a point source,[c] luminosity L	$p_n = \frac{L(1+R)}{4\pi r^2 c}$	(7.204)	L source luminosity (i.e., radiant power) r distance from source

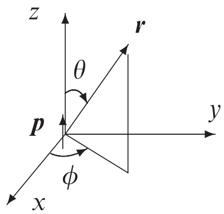




[a]On an opaque surface.
[b]In spherical polar coordinates. See page 120 for the meaning of specific intensity.
[c]Normal to the plane.

Antennas

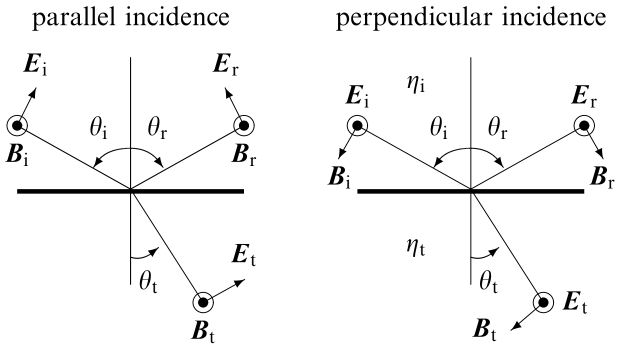
Spherical polar geometry:



Field from a short ($l \ll \lambda$) electric dipole in free space[a]	$E_r = \frac{1}{2\pi\epsilon_0}\left(\frac{[\dot{p}]}{r^2c} + \frac{[p]}{r^3}\right)\cos\theta$ (7.205) $E_\theta = \frac{1}{4\pi\epsilon_0}\left(\frac{[\ddot{p}]}{rc^2} + \frac{[\dot{p}]}{r^2c} + \frac{[p]}{r^3}\right)\sin\theta$ (7.206) $B_\phi = \frac{\mu_0}{4\pi}\left(\frac{[\ddot{p}]}{rc} + \frac{[\dot{p}]}{r^2}\right)\sin\theta$ (7.207)	r distance from dipole θ angle between $\boldsymbol{r}$ and $\boldsymbol{p}$ $[p]$ retarded dipole moment $[p] = p(t - r/c)$ c speed of light
Radiation resistance of a short electric dipole in free space	$R = \frac{\omega^2 l^2}{6\pi\epsilon_0 c^3} = \frac{2\pi Z_0}{3}\left(\frac{l}{\lambda}\right)^2$ (7.208) $\simeq 789\left(\frac{l}{\lambda}\right)^2$ ohm (7.209)	l dipole length ($\ll \lambda$) ω angular frequency λ wavelength Z_0 impedance of free space
Beam solid angle	$\Omega_A = \int_{4\pi} P_n(\theta,\phi)\,d\Omega$ (7.210)	Ω_A beam solid angle P_n normalised antenna power pattern $P_n(0,0) = 1$ $d\Omega$ differential solid angle
Forward power gain	$G(0) = \frac{4\pi}{\Omega_A}$ (7.211)	G antenna gain
Antenna effective area	$A_e = \frac{\lambda^2}{\Omega_A}$ (7.212)	A_e effective area
Power gain of a short dipole	$G(\theta) = \frac{3}{2}\sin^2\theta$ (7.213)	
Beam efficiency	$\text{efficiency} = \frac{\Omega_M}{\Omega_A}$ (7.214)	Ω_M main lobe solid angle
Antenna temperature[b]	$T_A = \frac{1}{\Omega_A}\int_{4\pi} T_b(\theta,\phi)P_n(\theta,\phi)\,d\Omega$ (7.215)	T_A antenna temperature T_b sky brightness temperature

[a] All field components propagate with a further phase factor equal to $\exp\mathrm{i}(kr - \omega t)$, where $k = 2\pi/\lambda$.
[b] The brightness temperature of a source of specific intensity I_ν is $T_b = \lambda^2 I_\nu/(2k_B)$.

Reflection, refraction, and transmission[a]



E	electric field
B	magnetic flux density
η_i	refractive index on incident side
η_t	refractive index on transmitted side
θ_i	angle of incidence
θ_r	angle of reflection
θ_t	angle of refraction

Law of reflection $\quad \theta_i = \theta_r \quad$ (7.216)

Snell's law[b] $\quad \eta_i \sin\theta_i = \eta_t \sin\theta_t \quad$ (7.217)

Brewster's law $\quad \tan\theta_B = \eta_t/\eta_i \quad$ (7.218)

θ_B Brewster's angle of incidence for plane-polarised reflection ($r_\parallel = 0$)

Fresnel equations of reflection and refraction

$$r_\parallel = \frac{\sin 2\theta_i - \sin 2\theta_t}{\sin 2\theta_i + \sin 2\theta_t} \quad (7.219)$$

$$t_\parallel = \frac{4\cos\theta_i \sin\theta_t}{\sin 2\theta_i + \sin 2\theta_t} \quad (7.220)$$

$$R_\parallel = r_\parallel^2 \quad (7.221)$$

$$T_\parallel = \frac{\eta_t \cos\theta_t}{\eta_i \cos\theta_i} t_\parallel^2 \quad (7.222)$$

$$r_\perp = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} \quad (7.223)$$

$$t_\perp = \frac{2\cos\theta_i \sin\theta_t}{\sin(\theta_i + \theta_t)} \quad (7.224)$$

$$R_\perp = r_\perp^2 \quad (7.225)$$

$$T_\perp = \frac{\eta_t \cos\theta_t}{\eta_i \cos\theta_i} t_\perp^2 \quad (7.226)$$

Coefficients for normal incidence[c]

$$R = \frac{(\eta_i - \eta_t)^2}{(\eta_i + \eta_t)^2} \quad (7.227)$$

$$T = \frac{4\eta_i \eta_t}{(\eta_i + \eta_t)^2} \quad (7.228)$$

$$R + T = 1 \quad (7.229)$$

$$r = \frac{\eta_i - \eta_t}{\eta_i + \eta_t} \quad (7.230)$$

$$t = \frac{2\eta_i}{\eta_i + \eta_t} \quad (7.231)$$

$$t - r = 1 \quad (7.232)$$

$\parallel$	electric field parallel to the plane of incidence	$\perp$	electric field perpendicular to the plane of incidence
R	(power) reflectance coefficient	r	amplitude reflection coefficient
T	(power) transmittance coefficient	t	amplitude transmission coefficient

[a]For the plane boundary between lossless dielectric media. All coefficients refer to the electric field component and whether it is parallel or perpendicular to the plane of incidence. Perpendicular components are out of the paper.
[b]The incident wave suffers total internal reflection if $\frac{\eta_i}{\eta_t}\sin\theta_i > 1$.
[c]I.e., $\theta_i = 0$. Use the diagram labelled "perpendicular incidence" for correct phases.

Propagation in conducting media[a]

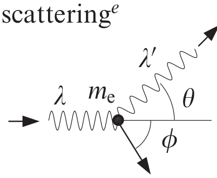
Electrical conductivity ($B=0$)	$\sigma = n_\mathrm{e} e\mu = \frac{n_\mathrm{e} e^2}{m_\mathrm{e}}\tau_\mathrm{c}$	(7.233)
Refractive index of an ohmic conductor[b]	$\eta = (1+\mathbf{i})\left(\frac{\sigma}{4\pi\nu\epsilon_0}\right)^{1/2}$	(7.234)
Skin depth in an ohmic conductor	$\delta = (\mu_0\sigma\pi\nu)^{-1/2}$	(7.235)

σ	electrical conductivity
n_e	electron number density
τ_c	electron relaxation time
μ	electron mobility
B	magnetic flux density
m_e	electron mass
$-e$	electronic charge
η	refractive index
ϵ_0	permittivity of free space
ν	frequency
δ	skin depth
μ_0	permeability of free space

[a]Assuming a relative permeability, μ_r, of 1.
[b]Taking the wave to have an $\mathrm{e}^{-\mathbf{i}\omega t}$ time dependence, and the low-frequency limit ($\sigma \gg 2\pi\nu\epsilon_0$).

Electron scattering processes[a]

Rayleigh scattering cross section[b]	$\sigma_\mathrm{R} = \frac{\omega^4\alpha^2}{6\pi\epsilon_0 c^4}$	(7.236)
Thomson scattering cross section[c]	$\sigma_\mathrm{T} = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\epsilon_0 m_\mathrm{e} c^2}\right)^2$	(7.237)
	$= \frac{8\pi}{3} r_\mathrm{e}^2 \simeq 6.652\times10^{-29}\,\mathrm{m}^2$	(7.238)
Inverse Compton scattering[d]	$P_\mathrm{tot} = \frac{4}{3}\sigma_\mathrm{T} c u_\mathrm{rad}\gamma^2\left(\frac{v^2}{c^2}\right)$	(7.239)
Compton scattering[e]	$\lambda' - \lambda = \frac{h}{m_\mathrm{e} c}(1-\cos\theta)$	(7.240)
	$h\nu' = \frac{m_\mathrm{e} c^2}{1-\cos\theta+(1/\varepsilon)}$	(7.241)
	$\cot\phi = (1+\varepsilon)\tan\frac{\theta}{2}$	(7.242)
Klein–Nishina cross section (for a free electron)	$\sigma_\mathrm{KN} = \frac{\pi r_\mathrm{e}^2}{\varepsilon}\left\{\left[1-\frac{2(\varepsilon+1)}{\varepsilon^2}\right]\ln(2\varepsilon+1)+\frac{1}{2}+\frac{4}{\varepsilon}-\frac{1}{2(2\varepsilon+1)^2}\right\}$	(7.243)
	$\simeq \sigma_\mathrm{T} \quad (\varepsilon \ll 1)$	(7.244)
	$\simeq \frac{\pi r_\mathrm{e}^2}{\varepsilon}\left(\ln 2\varepsilon + \frac{1}{2}\right) \quad (\varepsilon \gg 1)$	(7.245)



σ_R	Rayleigh cross section
ω	radiation angular frequency
α	particle polarisability
ϵ_0	permittivity of free space
σ_T	Thomson cross section
m_e	electron (rest) mass
r_e	classical electron radius
c	speed of light
P_tot	electron energy loss rate
u_rad	radiation energy density
γ	Lorentz factor $=[1-(v/c)^2]^{-1/2}$
v	electron speed
λ,λ'	incident & scattered wavelengths
ν,ν'	incident & scattered frequencies
θ	photon scattering angle
$\frac{h}{m_\mathrm{e} c}$	electron Compton wavelength
ε	$= h\nu/(m_\mathrm{e} c^2)$
σ_KN	Klein–Nishina cross section

[a]For Rutherford scattering see page 72.
[b]Scattering by bound electrons.
[c]Scattering from free electrons, $\varepsilon \ll 1$.
[d]Electron energy loss rate due to photon scattering in the Thomson limit ($\gamma h\nu \ll m_\mathrm{e} c^2$).
[e]From an electron at rest.

Cherenkov radiation

Cherenkov cone angle	$$\sin\theta = \frac{c}{\eta v} \quad (7.246)$$	θ cone semi-angle c (vacuum) speed of light $\eta(\omega)$ refractive index v particle velocity
Radiated power[a]	$$P_{\text{tot}} = \frac{e^2\mu_0}{4\pi} v \int_0^{\omega_c} \left[1 - \frac{c^2}{v^2\eta^2(\omega)}\right] \omega \, \mathrm{d}\omega \quad (7.247)$$ where $\eta(\omega) \geq \frac{c}{v}$ for $0 < \omega < \omega_c$	P_{tot} total radiated power $-e$ electronic charge μ_0 free space permeability ω angular frequency ω_c cutoff frequency

[a]From a point charge, e, travelling at speed v through a medium of refractive index $\eta(\omega)$.