Theorem 2.1 : *Two distinct lines cannot have more than one point in common.*

Proof : Here we are given two lines l and m. We need to prove that they have only one point in common.

For the time being, let us suppose that the two lines intersect in two distinct points, say P and Q. So, you have two lines passing through two distinct points P and Q. But this assumption clashes with the axiom that only one line can pass through two distinct points. So, the assumption that we started with, that two lines can pass through two distinct points is wrong.

From this, what can we conclude? We are forced to conclude that two distinct lines cannot have more than one point in common.

EXERCISE 2.1

1. Which of the following statements are true and which are false? Give reasons for your answers.

(i) Only one line can pass through a single point.

(ii) There are an infinite number of lines which pass through two distinct points.

(iii) A terminated line can be produced indefinitely on both the sides.

(iv) If two circles are equal, then their radii are equal.

(v) In Fig. 2.9, if AB = PQ and PQ = XY, then AB = XY.

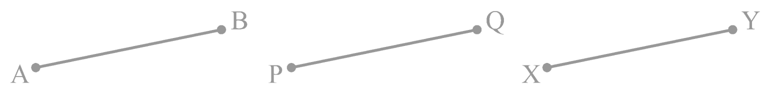


Fig. 2.9

2. Give a definition for each of the following terms. Are there other terms that need to be defined first? What are they, and how might you define them?

(i) parallel lines (ii) perpendicular lines (iii) line segment

(iv) radius of a circle (v) square

3. Consider two 'postulates' given below:

(i) Given any two distinct points A and B, there exists a third point C which is in between A and B.

(ii) There exist at least three points that are not on the same line.

Do these postulates contain any undefined terms? Are these postulates consistent? Do they follow from Euclid's postulates? Explain.

4. If a point C lies between two points A and B such that AC = BC, then prove that AC = $\frac{1}{2}$ AB. Explain by drawing the figure.
5. In Question 4, point C is called a mid-point of line segment AB. Prove that every line segment has one and only one mid-point.
6. In Fig. 2.10, if AC = BD, then prove that AB = CD.

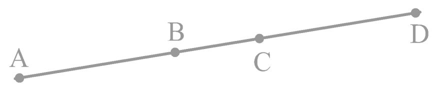


Fig. 2.10

7. Why is Axiom 5, in the list of Euclid's axioms, considered a 'universal truth'? (Note that the question is not about the fifth postulate.)

2.3 Equivalent Versions of Euclid's Fifth Postulate

Euclid's fifth postulate is very significant in the history of mathematics. Recall it again from Section 2.2. We see that by implication, no intersection of lines will take place when the sum of the measures of the interior angles on the same side of the falling line is exactly 180°. There are several equivalent versions of this postulate. One of them is 'Playfair's Axiom' (given by a Scottish mathematician John Playfair in 1729), as stated below:

'*For every line l and for every point* P *not lying on l, there exists a unique line m passing through* P *and parallel to l'*.

From Fig. 2.11, you can see that of all the lines passing through the point P, only line *m* is parallel to line *l*.

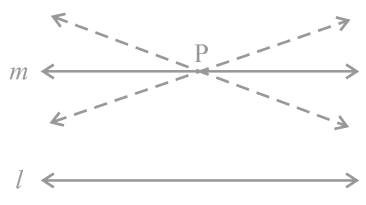


Fig. 2.11

This result can also be stated in the following form:

Two distinct intersecting lines cannot be parallel to the same line.

Euclid did not require his fifth postulate to prove his first 28 theorems. Many mathematicians, including him, were convinced that the fifth postulate is actually a theorem that can be proved using just the first four postulates and other axioms. However, all attempts to prove the fifth postulate as a theorem have failed. But these efforts have led to a great achievement – the creation of several other geometries. These geometries are quite different from Euclidean geometry. They are called *non-Euclidean geometries*. Their creation is considered a landmark in the history of thought because till then everyone had believed that Euclid's was the only geometry and the world itself was Euclidean. Now the geometry of the universe we live in has been shown to be a non-Euclidean geometry. In fact, it is called *spherical geometry*. In spherical geometry, lines are not straight. They are parts of great circles (i.e., circles obtained by the intersection of a sphere and planes passing through the centre of the sphere).

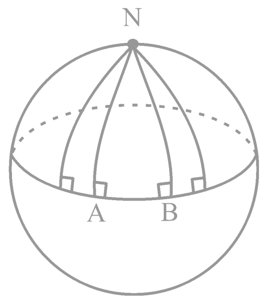


Fig. 2.12

In Fig. 2.12, the lines AN and BN (which are parts of great circles of a sphere) are perpendicular to the same line AB. But they are meeting each other, though the sum of the angles on the same side of line AB is not less than two right angles (in fact, it is $90° + 90° = 180°$). Also, note that the sum of the angles of the triangle NAB is greater than $180°$, as $\angle A + \angle B = 180°$. Thus, Euclidean geometry is valid only for the figures in the plane. On the curved surfaces, it fails.

Now, let us consider an example.

Example 3 : Consider the following statement : There exists a pair of straight lines that are everywhere equidistant from one another. Is this statement a direct consequence of Euclid's fifth postulate? Explain.

Solution : Take any line l and a point P not on l. Then, by Playfair's axiom, which is equivalent to the fifth postulate, we know that there is a unique line m through P which is parallel to l.

Now, the *distance of a point from a line* is the length of the perpendicular from the point to the line. This distance will be the same for any point on m from l and any point on l from m. So, these two lines are everywhere equidistant from one another.

Remark : The geometry that you will be studying in the next few chapters is Euclidean Geometry. However, the axioms and theorems used by us may be different from those of Euclid's.

EXERCISE 2.2

1. How would you rewrite Euclid's fifth postulate so that it would be easier to understand?
2. Does Euclid's fifth postulate imply the existence of parallel lines? Explain.

2.4 Summary

In this chapter, you have studied the following points:

1. Though Euclid defined a point, a line, and a plane, the definitions are not accepted by mathematicians. Therefore, these terms are now taken as undefined.
2. Axioms or postulates are the assumptions which are obvious universal truths. They are not proved.
3. Theorems are statements which are proved, using definitions, axioms, previously proved statements and deductive reasoning.
4. Some of Euclid's axioms were :

   (1) Things which are equal to the same thing are equal to one another.

   (2) If equals are added to equals, the wholes are equal.

   (3) If equals are subtracted from equals, the remainders are equal.

   (4) Things which coincide with one another are equal to one another.

   (5) The whole is greater than the part.

   (6) Things which are double of the same things are equal to one another.

   (7) Things which are halves of the same things are equal to one another.
5. Euclid's postulates were :

   Postulate 1 : A straight line may be drawn from any one point to any other point.

   Postulate 2 : A terminated line can be produced indefinitely.

   Postulate 3 : A circle can be drawn with any centre and any radius.

   Postulate 4 : All right angles are equal to one another.

   Postulate 5 : If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of angles is less than two right angles.
6. Two equivalent versions of Euclid's fifth postulate are:

   (i) 'For every line l and for every point P not lying on l, there exists a unique line m passing through P and parallel to l'.

   (ii) Two distinct intersecting lines cannot be parallel to the same line.
7. All the attempts to prove Euclid's fifth postulate using the first 4 postulates failed. But they led to the discovery of several other geometries, called non-Euclidean geometries.

CHAPTER 3

LINES AND ANGLES

3.1 Introduction

In Chapter 2, you have studied that a minimum of two points are required to draw a line. You have also studied some axioms and, with the help of these axioms, you proved some other statements. In this chapter, you will study the properties of the angles formed when two lines intersect each other, and also the properties of the angles formed when a line intersects two or more parallel lines at distinct points. Further you will use these properties to prove some statements using deductive reasoning (see Appendix 1). You have already verified these statements through some activities in the earlier classes.

In your daily life, you see different types of angles formed between the edges of plane surfaces. For making a similar kind of model using the plane surfaces, you need to have a thorough knowledge of angles. For instance, suppose you want to make a model of a hut to keep in the school exhibition using bamboo sticks. Imagine how you would make it? You would keep some of the sticks parallel to each other, and some sticks would be kept slanted. Whenever an architect has to draw a plan for a multistoried building, she has to draw intersecting lines and parallel lines at different angles. Without the knowledge of the properties of these lines and angles, do you think she can draw the layout of the building?

In science, you study the properties of light by drawing the ray diagrams. For example, to study the refraction property of light when it enters from one medium to the other medium, you use the properties of intersecting lines and parallel lines. When two or more forces act on a body, you draw the diagram in which forces are represented by directed line segments to study the net effect of the forces on the body. At that time, you need to know the relation between the angles when the rays (or line segments) are parallel to or intersect each other. To find the height of a tower or to find the distance of a ship from the light house, one needs to know the angle formed between the horizontal and the line of sight. Plenty of other examples can be given where lines and angles are used. In the subsequent chapters of geometry, you will be using these properties of lines and angles to deduce more and more useful properties.

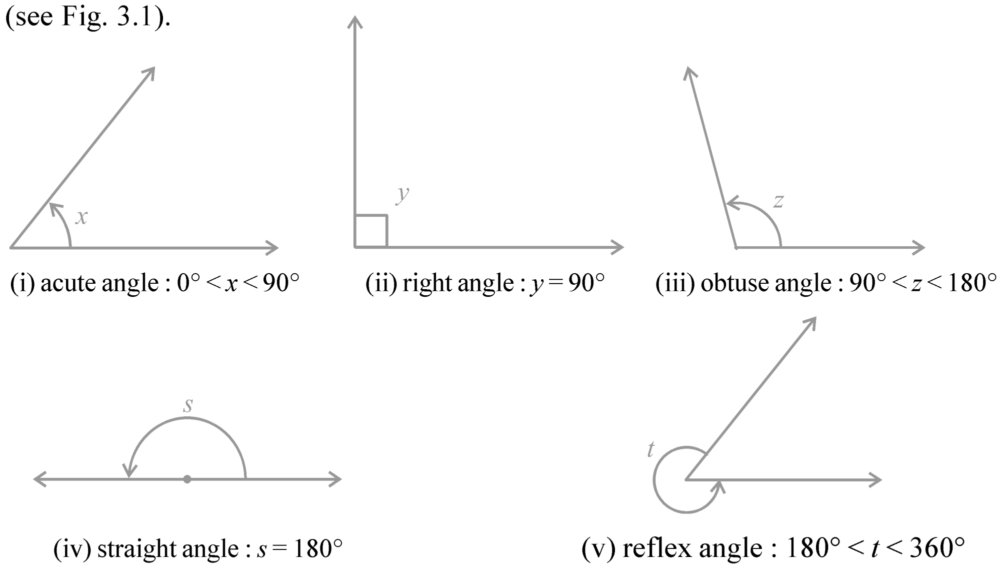
Let us first revise the terms and definitions related to lines and angles learnt in earlier classes.

3.2 Basic Terms and Definitions

Recall that a part (or portion) of a line with two end points is called a **line-segment** and a part of a line with one end point is called a **ray**. Note that the line segment AB is denoted by $\overline{AB}$, and its length is denoted by AB. The ray AB is denoted by $\overrightarrow{AB}$, and a line is denoted by $\overleftrightarrow{AB}$. However, **we will not use these symbols**, and will denote the line segment AB, ray AB, length AB and line AB by the same symbol, AB. The meaning will be clear from the context. Sometimes small letters *l*, *m*, *n*, etc. will be used to denote lines.

If three or more points lie on the same line, they are called **collinear points**; otherwise they are called **non-collinear points**.

Recall that an **angle** is formed when two rays originate from the same end point. The rays making an angle are called the **arms** of the angle and the end point is called the **vertex** of the angle. You have studied different types of angles, such as acute angle, right angle, obtuse angle, straight angle and reflex angle in earlier classes (see Fig. 3.1).



(i) acute angle : $0° < x < 90°$ (ii) right angle : $y = 90°$ (iii) obtuse angle : $90° < z < 180°$

(iv) straight angle : $s = 180°$ (v) reflex angle : $180° < t < 360°$

Fig. 3.1 : Types of Angles

An **acute** angle measures between 0° and 90°, whereas a **right angle** is exactly equal to 90°. An angle greater than 90° but less than 180° is called an **obtuse angle**. Also, recall that a **straight angle** is equal to 180°. An angle which is greater than 180° but less than 360° is called a **reflex angle**. Further, two angles whose sum is 90° are called **complementary angles**, and two angles whose sum is 180° are called **supplementary angles**.

You have also studied about adjacent angles in the earlier classes (see Fig. 3.2). Two angles are **adjacent**, if they have a common vertex, a common arm and their non-common arms are on different sides of the common arm. In Fig. 3.2, ∠ ABD and ∠ DBC are adjacent angles. Ray BD is their common arm and point B is their common vertex. Ray BA and ray BC are non common arms. Moreover, when two angles are adjacent, then their sum is always equal to the angle formed by the two non-common arms. So, we can write

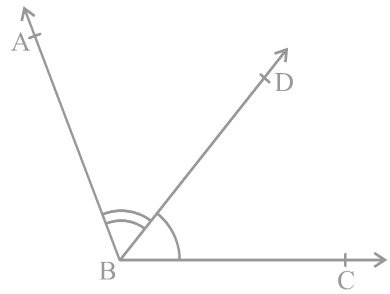


Fig. 3.2 : Adjacent angles

$$\angle ABC = \angle ABD + \angle DBC.$$

Note that ∠ ABC and ∠ ABD are not adjacent angles. Why? Because their non-common arms BD and BC lie on the same side of the common arm BA.

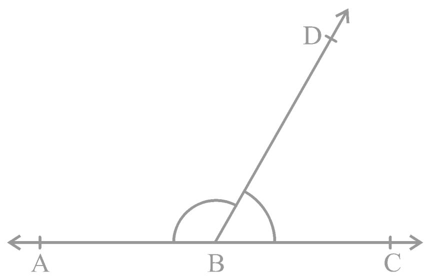


Fig. 3.3 : Linear pair of angles

If the non-common arms BA and BC in Fig. 3.2, form a line then it will look like Fig. 3.3. In this case, ∠ ABD and ∠ DBC are called **linear pair of angles**.

You may also recall the **vertically opposite angles** formed when two lines, say AB and CD, intersect each other, say at the point O (see Fig. 3.4). There are two pairs of vertically opposite angles.

One pair is ∠AOD and ∠BOC. Can you find the other pair?

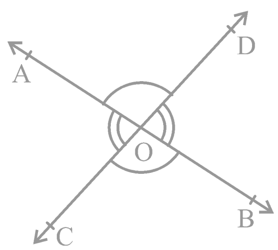


Fig. 3.4 : Vertically opposite angles

3.3 Intersecting Lines and Non-intersecting Lines

Draw two different lines PQ and RS on a paper. You will see that you can draw them in two different ways as shown in Fig. 3.5 (i) and Fig. 3.5 (ii).

(i) Intersecting lines (ii) Non-intersecting (parallel) lines

Fig. 3.5 : Different ways of drawing two lines

Recall the notion of a line, that it extends indefinitely in both directions. Lines PQ and RS in Fig. 3.5 (i) are intersecting lines and in Fig. 3.5 (ii) are parallel lines. Note that the lengths of the common perpendiculars at different points on these parallel lines is the same. This equal length is called the *distance between two parallel lines*.

3.4 Pairs of Angles

In Section 3.2, you have learnt the definitions of some of the pairs of angles such as complementary angles, supplementary angles, adjacent angles, linear pair of angles, etc. Can you think of some relations between these angles? Now, let us find out the relation between the angles formed when a ray stands on a line. Draw a figure in which a ray stands on a line as shown in Fig. 3.6. Name the line as AB and the ray as OC. What are the angles formed at the point O? They are $\angle$ AOC, $\angle$ BOC and $\angle$ AOB.

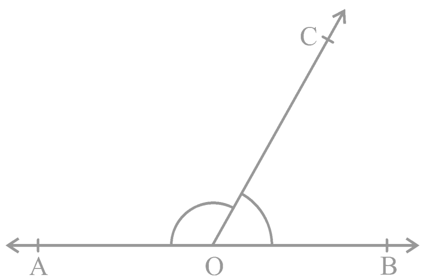


Fig. 3.6 : Linear pair of angles

Can we write $\angle$ AOC + $\angle$ BOC = $\angle$ AOB? (1)

Yes! (Why? Refer to adjacent angles in Section 6.2)

What is the measure of $\angle$ AOB? It is 180°. (Why?) (2)

From (1) and (2), can you say that $\angle$ AOC + $\angle$ BOC = 180°? Yes! (Why?)

From the above discussion, we can state the following Axiom:

Axiom 3.1 : *If a ray stands on a line, then the sum of two adjacent angles so formed is* 180°.

Recall that when the sum of two adjacent angles is 180°, then they are called a **linear pair of angles**.

In Axiom 3.1, it is given that 'a ray stands on a line'. From this 'given', we have concluded that 'the sum of two adjacent angles so formed is 180°'. Can we write Axiom 3.1 the other way? That is, take the 'conclusion' of Axiom 3.1 as 'given' and the 'given' as the 'conclusion'. So it becomes:

(A) If the sum of two adjacent angles is 180°, then a ray stands on a line (that is, the non-common arms form a line).

Now you see that the Axiom 6.1 and statement (A) are in a sense the reverse of each others. We call each as converse of the other. We do not know whether the statement (A) is true or not. Let us check. Draw adjacent angles of different measures as shown in Fig. 3.7. Keep the ruler along one of the non-common arms in each case. Does the other non-common arm also lie along the ruler?

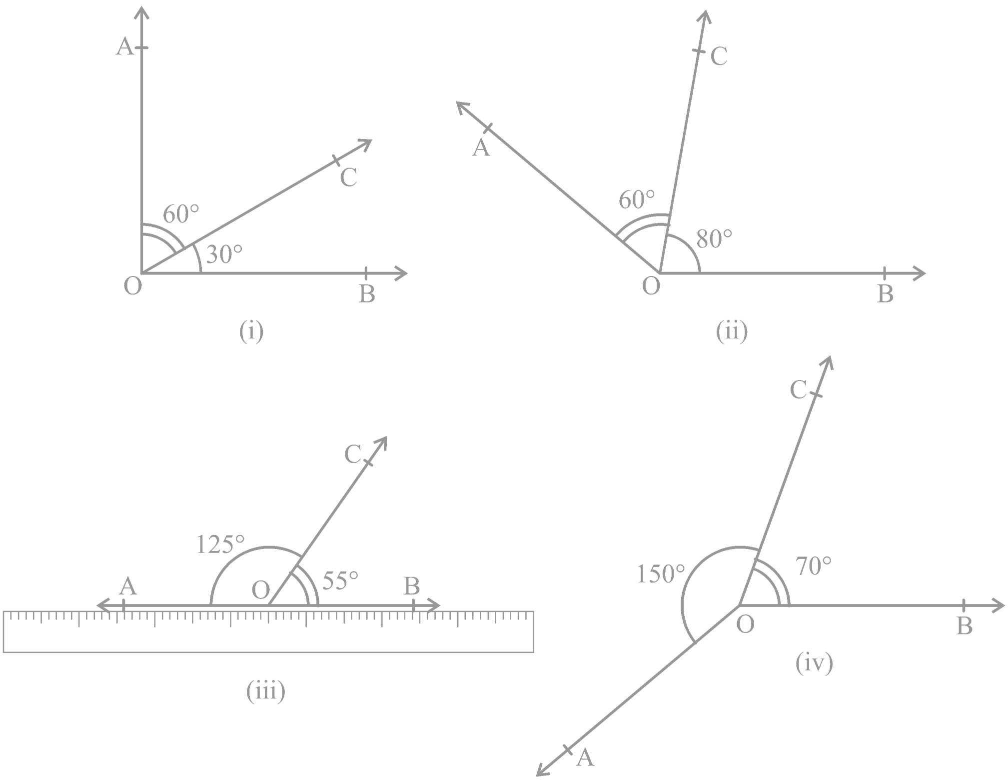


Fig. 3.7 : Adjacent angles with different measures

You will find that only in Fig. 3.7 (iii), both the non-common arms lie along the ruler, that is, points A, O and B lie on the same line and ray OC stands on it. Also see that ∠ AOC + ∠ COB = 125° + 55° = 180°. From this, you may conclude that statement (A) is true. So, you can state in the form of an axiom as follows:

Axiom 3.2 : *If the sum of two adjacent angles is* 180°, *then the non-common arms of the angles form a line*.

For obvious reasons, the two axioms above together is called the **Linear Pair Axiom**.

Let us now examine the case when two lines intersect each other.

Recall, from earlier classes, that when two lines intersect, the vertically opposite angles are equal. Let us prove this result now. See Appendix 1 for the ingredients of a proof, and keep those in mind while studying the proof given below.

Theorem 3.1 : *If two lines intersect each other, then the vertically opposite angles are equal*.

Proof : In the statement above, it is given that 'two lines intersect each other'. So, let AB and CD be two lines intersecting at O as shown in Fig. 3.8. They lead to two pairs of vertically opposite angles, namely,

(i) ∠ AOC and ∠ BOD (ii) ∠ AOD and ∠ BOC.

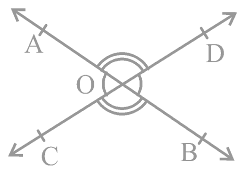


Fig. 3.8 : Vertically opposite angles

We need to prove that ∠ AOC = ∠ BOD and ∠ AOD = ∠ BOC.

Now, ray OA stands on line CD.

Therefore, ∠ AOC + ∠ AOD = 180° (Linear pair axiom) (1)

Can we write ∠ AOD + ∠ BOD = 180°? Yes! (Why?) (2)

From (1) and (2), we can write

∠ AOC + ∠ AOD = ∠ AOD + ∠ BOD

This implies that ∠ AOC = ∠ BOD (Refer Section 2.2, Axiom 3)

Similarly, it can be proved that ∠AOD = ∠BOC

Now, let us do some examples based on Linear Pair Axiom and Theorem 3.1.

Example 1 : In Fig. 3.9, lines PQ and RS intersect each other at point O. If $\angle$ POR : $\angle$ ROQ = 5 : 7, find all the angles.

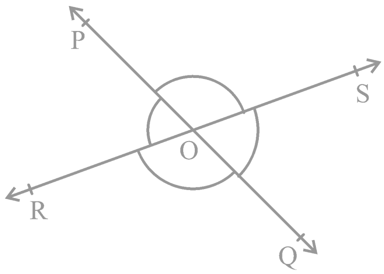


Fig. 3.9

Solution : $\angle$ POR + $\angle$ ROQ = 180° (Linear pair of angles)

But $\angle$ POR : $\angle$ ROQ = 5 : 7 (Given)

Therefore, $\angle \text{POR} = \frac{5}{12} \times 180° = 75°$

Similarly, $\angle \text{ROQ} = \frac{7}{12} \times 180° = 105°$

Now, $\angle$ POS = $\angle$ROQ = 105° (Vertically opposite angles)

and $\angle$ SOQ = $\angle$POR = 75° (Vertically opposite angles)

Example 2 : In Fig. 3.10, ray OS stands on a line POQ. Ray OR and ray OT are angle bisectors of $\angle$ POS and $\angle$ SOQ, respectively. If $\angle$ POS = x, find $\angle$ ROT.

Solution : Ray OS stands on the line POQ.

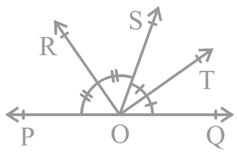


Fig. 3.10

Therefore, $\angle$ POS + $\angle$ SOQ = 180°

But, $\angle$ POS = x

Therefore, x + $\angle$ SOQ = 180°

So, $\angle$ SOQ = 180° – x

Now, ray OR bisects $\angle$ POS, therefore,

$$\angle \text{ROS} = \frac{1}{2} \times \angle \text{POS}$$

$$= \frac{1}{2} \times x = \frac{x}{2}$$

Similarly, $$\angle \text{SOT} = \frac{1}{2} \times \angle \text{SOQ}$$

$$= \frac{1}{2} \times (180° - x)$$

$$= 90° - \frac{x}{2}$$

Now, $\angle \text{ROT} = \angle \text{ROS} + \angle \text{SOT}$

$$= \frac{x}{2} + 90° - \frac{x}{2}$$

$$= 90°$$

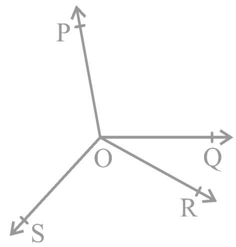


Fig. 3.11

Example 3 : In Fig. 3.11, OP, OQ, OR and OS are four rays. Prove that $\angle \text{POQ} + \angle \text{QOR} + \angle \text{SOR} + \angle \text{POS} = 360°$.

Solution : In Fig. 3.11, you need to produce any of the rays OP, OQ, OR or OS backwards to a point. Let us produce ray OQ backwards to a point T so that TOQ is a line (see Fig. 3.12).

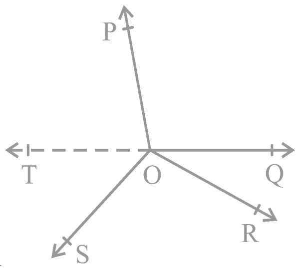


Fig. 3.12

Now, ray OP stands on line TOQ.

Therefore, $\angle \text{TOP} + \angle \text{POQ} = 180°$ (1)

(Linear pair axiom)

Similarly, ray OS stands on line TOQ.

Therefore, $\angle \text{TOS} + \angle \text{SOQ} = 180°$ (2)

But $\angle \text{SOQ} = \angle \text{SOR} + \angle \text{QOR}$

So, (2) becomes

$$\angle \text{TOS} + \angle \text{SOR} + \angle \text{QOR} = 180° \quad (3)$$

Now, adding (1) and (3), you get

$$\angle \text{TOP} + \angle \text{POQ} + \angle \text{TOS} + \angle \text{SOR} + \angle \text{QOR} = 360° \quad (4)$$

But $\angle \text{TOP} + \angle \text{TOS} = \angle \text{POS}$

Therefore, (4) becomes

$$\angle \text{POQ} + \angle \text{QOR} + \angle \text{SOR} + \angle \text{POS} = 360°$$

EXERCISE 3.1

1. In Fig. 3.13, lines AB and CD intersect at O. If $\angle \text{AOC} + \angle \text{BOE} = 70°$ and $\angle \text{BOD} = 40°$, find $\angle \text{BOE}$ and reflex $\angle \text{COE}$.

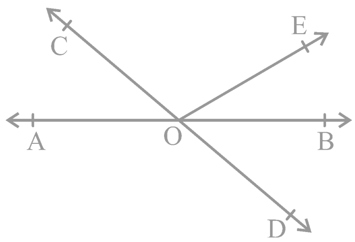


Fig. 3.13

2. In Fig. 3.14, lines XY and MN intersect at O. If $\angle$ POY = 90° and $a : b = 2 : 3$, find c.

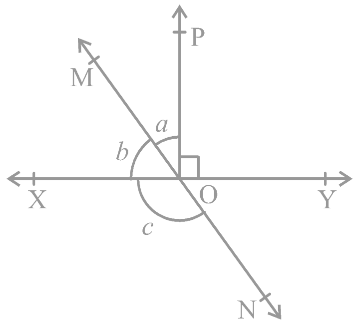


Fig. 3.14

3. In Fig. 3.15, $\angle$ PQR = $\angle$ PRQ, then prove that $\angle$ PQS = $\angle$ PRT.

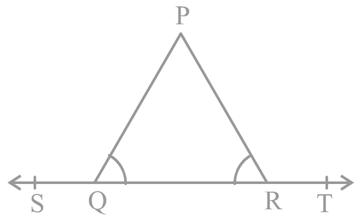


Fig. 3.15

4. In Fig. 3.16, if $x + y = w + z$, then prove that AOB is a line.

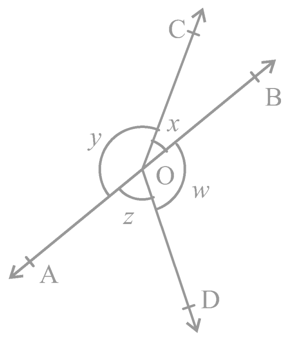


Fig. 3.16

5. In Fig. 3.17, POQ is a line. Ray OR is perpendicular to line PQ. OS is another ray lying between rays OP and OR. Prove that

$$\angle \text{ROS} = \frac{1}{2} (\angle \text{QOS} - \angle \text{POS}).$$

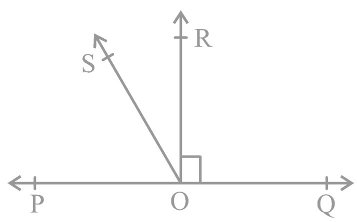


Fig. 3.17

6. It is given that $\angle$ XYZ = 64° and XY is produced to point P. Draw a figure from the given information. If ray YQ bisects $\angle$ ZYP, find $\angle$ XYQ and reflex $\angle$ QYP.

3.5 Parallel Lines and a Transversal

Recall that a line which intersects two or more lines at distinct points is called a **transversal** (see Fig. 3.18). Line l intersects lines m and n at points P and Q respectively. Therefore, line l is a transversal for lines m and n. Observe that four angles are formed at each of the points P and Q.

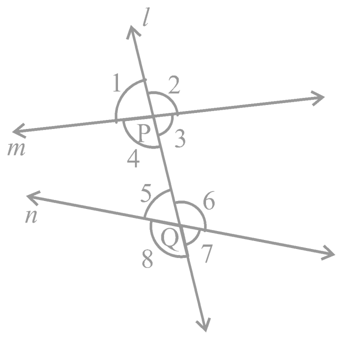


Fig. 3.18

Let us name these angles as $\angle 1, \angle 2, \ldots, \angle 8$ as shown in Fig. 3.18.

$\angle 1$, $\angle 2$, $\angle 7$ and $\angle 8$ are called **exterior angles**, while $\angle 3$, $\angle 4$, $\angle 5$ and $\angle 6$ are called **interior angles**.

Recall that in the earlier classes, you have named some pairs of angles formed when a transversal intersects two lines. These are as follows:

(a) **Corresponding angles :**

(i) $\angle 1$ and $\angle 5$ (ii) $\angle 2$ and $\angle 6$

(iii) $\angle 4$ and $\angle 8$ (iv) $\angle 3$ and $\angle 7$

(b) **Alternate interior angles :**

(i) $\angle 4$ and $\angle 6$ (ii) $\angle 3$ and $\angle 5$

(c) **Alternate exterior angles:**

(i) $\angle 1$ and $\angle 7$ (ii) $\angle 2$ and $\angle 8$

(d) **Interior angles on the same side of the transversal:**

(i) $\angle 4$ and $\angle 5$ (ii) $\angle 3$ and $\angle 6$

Interior angles on the same side of the transversal are also referred to as **consecutive interior** angles or **allied** angles or **co-interior** angles. Further, many a times, we simply use the words alternate angles for alternate interior angles.

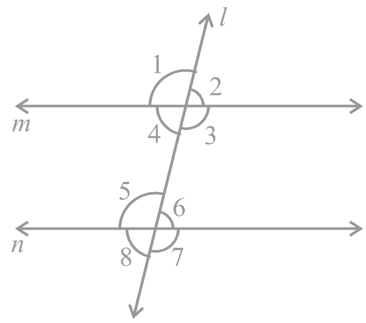


Fig. 3.19

Now, let us find out the relation between the angles in these pairs when line m is parallel to line n. You know that the ruled lines of your notebook are parallel to each other. So, with ruler and pencil, draw two parallel lines along any two of these lines and a transversal to intersect them as shown in Fig. 3.19.

Now, measure any pair of corresponding angles and find out the relation between them. You may find that : ∠ 1 = ∠ 5, ∠ 2 = ∠ 6, ∠ 4 = ∠ 8 and ∠ 3 = ∠ 7. From this, you may conclude the following axiom.

Axiom 3.3 : *If a transversal intersects two parallel lines, then each pair of corresponding angles is equal.*

Axiom 3.3 is also referred to as the **corresponding angles axiom**. Now, let us discuss the converse of this axiom which is as follows:

If a transversal intersects two lines such that a pair of corresponding angles is equal, then the two lines are parallel.

Does this statement hold true? It can be verified as follows: Draw a line AD and mark points B and C on it. At B and C, construct ∠ ABQ and ∠ BCS equal to each other as shown in Fig. 3.20 (i).

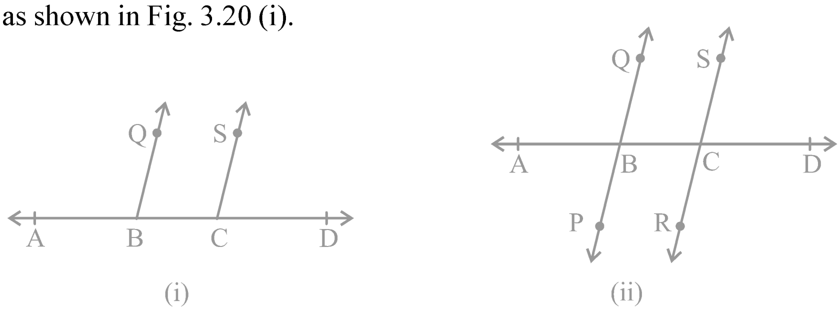


Fig. 3.20

Produce QB and SC on the other side of AD to form two lines PQ and RS [see Fig. 3.20 (ii)]. You may observe that the two lines do not intersect each other. You may also draw common perpendiculars to the two lines PQ and RS at different points and measure their lengths. You will find it the same everywhere. So, you may conclude that the lines are parallel. Therefore, the converse of corresponding angles axiom is also true. So, we have the following axiom:

Axiom 3.4 : *If a transversal intersects two lines such that a pair of corresponding angles is equal, then the two lines are parallel to each other.*

Can we use corresponding angles axiom to find out the relation between the alternate interior angles when a transversal intersects two parallel lines? In Fig. 3.21, transveral PS intersects parallel lines AB and CD at points Q and R respectively.

Is ∠ BQR = ∠ QRC and ∠ AQR = ∠ QRD?

You know that ∠ PQA = ∠ QRC (1)

(Corresponding angles axiom)

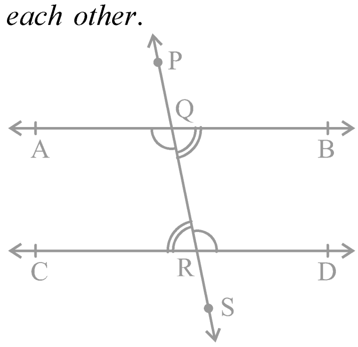


Fig. 3.21

Is $\angle$ PQA = $\angle$ BQR? Yes! (Why ?) (2)

So, from (1) and (2), you may conclude that

$\angle$ BQR = $\angle$ QRC.

Similarly, $\angle$ AQR = $\angle$ QRD.

This result can be stated as a theorem given below:

Theorem 3.2 : *If a transversal intersects two parallel lines, then each pair of alternate interior angles is equal.*

Now, using the converse of the corresponding angles axiom, can we show the two lines parallel if a pair of alternate interior angles is equal? In Fig. 3.22, the transversal PS intersects lines AB and CD at points Q and R respectively such that $\angle$ BQR = $\angle$ QRC.

Is AB || CD?

$\angle$ BQR = $\angle$ PQA (Why?) (1)

But, $\angle$ BQR = $\angle$ QRC (Given) (2)

So, from (1) and (2), you may conclude that

$\angle$ PQA = $\angle$ QRC

But they are corresponding angles.

So, AB || CD (Converse of corresponding angles axiom)

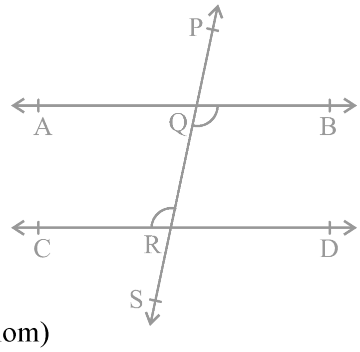


Fig. 3.22

This result can be stated as a theorem given below:

Theorem 3.3 : *If a transversal intersects two lines such that a pair of alternate interior angles is equal, then the two lines are parallel.*

In a similar way, you can obtain the following two theorems related to interior angles on the same side of the transversal.

Theorem 3.4 : *If a transversal intersects two parallel lines, then each pair of interior angles on the same side of the transversal is supplementary.*

Theorem 3.5 : *If a transversal intersects two lines such that a pair of interior angles on the same side of the transversal is supplementary, then the two lines are parallel.*

You may recall that you have verified all the above axioms and theorems in earlier classes through activities. You may repeat those activities here also.

3.6 Lines Parallel to the Same Line

If two lines are parallel to the same line, will they be parallel to each other? Let us check it. See Fig. 3.23 in which line m || line l and line n || line l.

Let us draw a line t transversal for the lines, l, m and n. It is given that line m || line l and line n || line l.

Therefore, $\angle 1 = \angle 2$ and $\angle 1 = \angle 3$

(Corresponding angles axiom)

So, $\angle 2 = \angle 3$ (Why?)

But $\angle 2$ and $\angle 3$ are corresponding angles and they are equal.

Therefore, you can say that

Line m || Line n

(Converse of corresponding angles axiom)

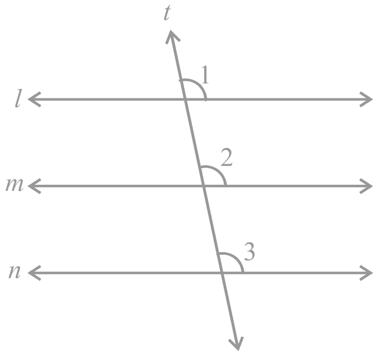


Fig. 3.23

This result can be stated in the form of the following theorem:

Theorem 3.6 : *Lines which are parallel to the same line are parallel to each other.*

Note : The property above can be extended to more than two lines also.

Now, let us solve some examples related to parallel lines.

Example 4 : In Fig. 3.24, if PQ || RS, $\angle$ MXQ = 135° and $\angle$ MYR = 40°, find $\angle$ XMY.

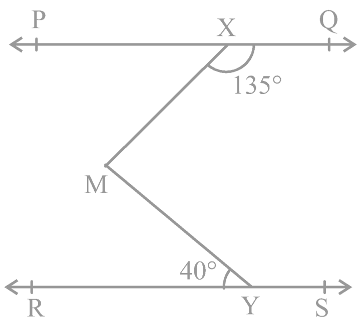


Fig. 3.24

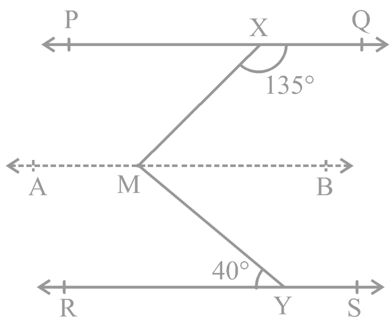


Fig. 3.25

Solution : Here, we need to draw a line AB parallel to line PQ, through point M as shown in Fig. 3.25. Now, AB || PQ and PQ || RS.

Therefore, AB || RS (Why?)

Now, $\angle$ QXM + $\angle$ XMB = 180°

(AB || PQ, Interior angles on the same side of the transversal XM)

But $\angle$ QXM = 135°

So, 135° + $\angle$ XMB = 180°

Therefore, $\angle$ XMB = 45° (1)

Now, $\angle$ BMY = $\angle$ MYR (AB || RS, Alternate angles)

Therefore, $\angle$ BMY = 40° (2)

Adding (1) and (2), you get

$\angle$ XMB + $\angle$ BMY = 45° + 40°

That is, $\angle$ XMY = 85°

Example 5 : If a transversal intersects two lines such that the bisectors of a pair of corresponding angles are parallel, then prove that the two lines are parallel.

Solution : In Fig. 3.26, a transversal AD intersects two lines PQ and RS at points B and C respectively. Ray BE is the bisector of $\angle$ ABQ and ray CG is the bisector of $\angle$ BCS; and BE || CG.

We are to prove that PQ || RS.

It is given that ray BE is the bisector of $\angle$ ABQ.

Therefore, $\angle \text{ABE} = \frac{1}{2} \angle \text{ABQ}$ (1)

Similarly, ray CG is the bisector of $\angle$ BCS.

Therefore, $\angle \text{BCG} = \frac{1}{2} \angle \text{BCS}$ (2)

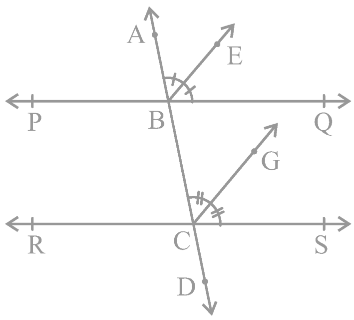


Fig. 3.26

But BE || CG and AD is the transversal.

Therefore, $\angle$ ABE = $\angle$ BCG

(Corresponding angles axiom) (3)

Substituting (1) and (2) in (3), you get

$$\frac{1}{2} \angle \text{ABQ} = \frac{1}{2} \angle \text{BCS}$$

That is, $\angle$ ABQ = $\angle$ BCS

But, they are the corresponding angles formed by transversal AD with PQ and RS; and are equal.

Therefore, PQ || RS

(Converse of corresponding angles axiom)

Example 6 : In Fig. 3.27, AB || CD and CD || EF. Also EA ⊥ AB. If ∠ BEF = 55°, find the values of x, y and z.

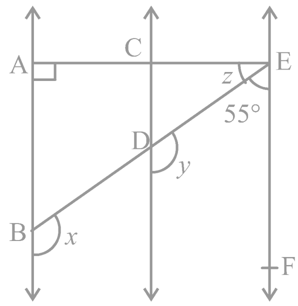


Fig. 3.27

Solution : $y + 55° = 180°$

(Interior angles on the same side of the transversal ED)

Therefore, $y = 180° – 55° = 125°$

Again $x = y$

(AB || CD, Corresponding angles axiom)

Therefore $x = 125°$

Now, since AB || CD and CD || EF, therefore, AB || EF.

So, ∠ EAB + ∠ FEA = 180° (Interior angles on the same side of the transversal EA)

Therefore, $90° + z + 55° = 180°$

Which gives $z = 35°$

EXERCISE 3.2

1. In Fig. 3.28, find the values of x and y and then show that AB || CD.

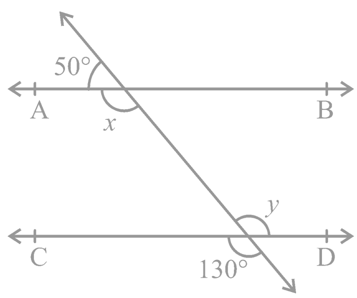


Fig. 3.28

2. In Fig. 3.29, if AB || CD, CD || EF and $y : z = 3 : 7$, find x.

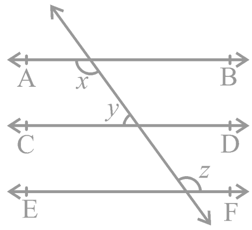


Fig. 3.29

3. In Fig. 3.30, if AB || CD, EF ⊥ CD and ∠ GED = 126°, find ∠ AGE, ∠ GEF and ∠ FGE.

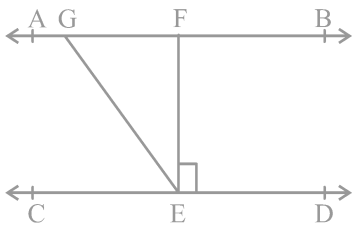


Fig. 3.30

4. In Fig. 3.31, if PQ || ST, ∠ PQR = 110° and ∠ RST = 130°, find ∠ QRS.

[**Hint :** Draw a line parallel to ST through point R.]

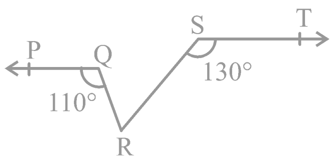


Fig. 3.31

5. In Fig. 3.32, if AB || CD, ∠ APQ = 50° and ∠ PRD = 127°, find x and y.

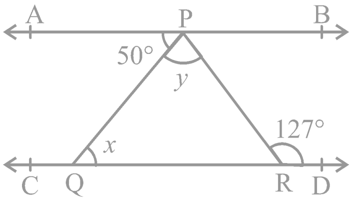


Fig. 3.32

6. In Fig. 3.33, PQ and RS are two mirrors placed parallel to each other. An incident ray AB strikes the mirror PQ at B, the reflected ray moves along the path BC and strikes the mirror RS at C and again reflects back along CD. Prove that AB || CD.

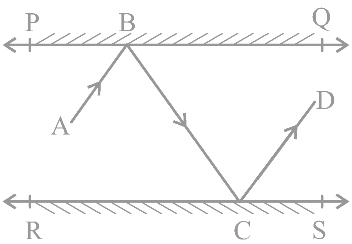


Fig. 3.33

3.7 Angle Sum Property of a Triangle

In the earlier classes, you have studied through activities that the sum of all the angles of a triangle is 180°. We can prove this statement using the axioms and theorems related to parallel lines.

Theorem 3.7 : *The sum of the angles of a triangle is 180º.*

Proof : Let us see what is given in the statement above, that is, the hypothesis and what we need to prove. We are given a triangle PQR and ∠ 1, ∠ 2 and ∠ 3 are the angles of Δ PQR (see Fig. 3.34).

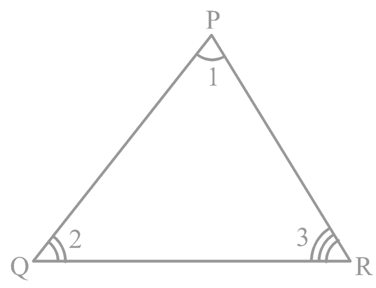


Fig. 3.34

We need to prove that ∠ 1 + ∠ 2 + ∠ 3 = 180°. Let us draw a line XPY parallel to QR through the opposite vertex P, as shown in Fig. 3.35, so that we can use the properties related to parallel lines.

Now, XPY is a line.

Therefore, ∠ 4 + ∠ 1 + ∠ 5 = 180° (1)

But XPY || QR and PQ, PR are transversals.

So, ∠ 4 = ∠ 2 and ∠ 5 = ∠ 3
(Pairs of alternate angles)

Substituting ∠ 4 and ∠ 5 in (1), we get

∠ 2 + ∠ 1 + ∠ 3 = 180°

That is, ∠ 1 + ∠ 2 + ∠ 3 = 180°

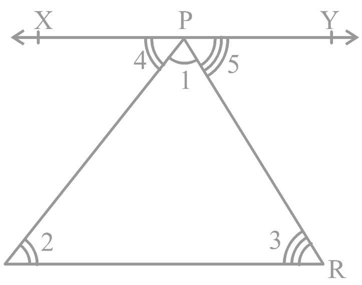


Fig. 3.35

Recall that you have studied about the formation of an exterior angle of a triangle in the earlier classes (see Fig. 3.36). Side QR is produced to point S, ∠ PRS is called an exterior angle of ΔPQR.

Is ∠ 3 + ∠ 4 = 180°? (Why?) (1)

Also, see that

∠ 1 + ∠ 2 + ∠ 3 = 180° (Why?) (2)

From (1) and (2), you can see that

∠ 4 = ∠ 1 + ∠ 2.

This result can be stated in the form of a theorem as given below:

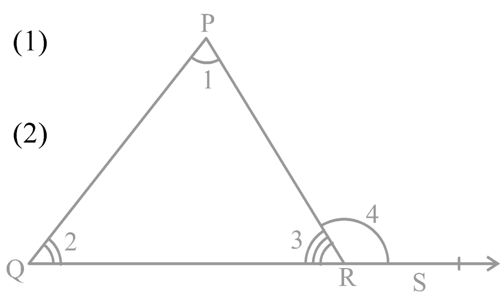


Fig. 3.36

Theorem 3.8 : *If a side of a triangle is produced, then the exterior angle so formed is equal to the sum of the two interior opposite angles.*

It is obvious from the above theorem that an *exterior angle of a triangle is greater than either of its interior apposite angles.*

Now, let us take some examples based on the above theorems.

Example 7 : In Fig. 3.37, if QT ⊥ PR, ∠ TQR = 40° and ∠ SPR = 30°, find x and y.

Solution : In Δ TQR, 90° + 40° + x = 180°

(Angle sum property of a triangle)

Therefore, $x = 50°$

Now, $y = \angle \text{SPR} + x$ (Theorem 3.8)

Therefore, $y = 30° + 50°$

$= 80°$

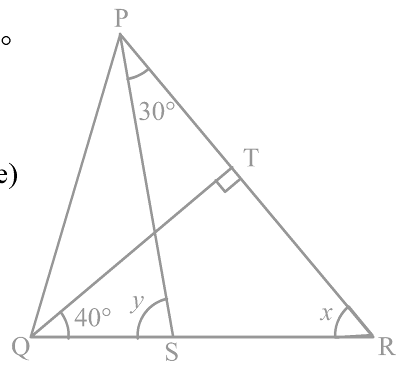


Fig. 3.37

Example 8 : In Fig. 3.38, the sides AB and AC of ΔABC are produced to points E and D respectively. If bisectors BO and CO of ∠ CBE and ∠ BCD respectively meet at point O, then prove that $\angle \text{BOC} = 90° - \frac{1}{2} \angle \text{BAC}$.

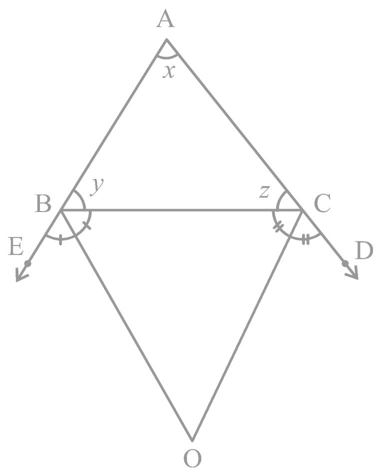


Fig. 3.38

Solution : Ray BO is the bisector of ∠ CBE.

Therefore, $\angle \text{CBO} = \frac{1}{2} \angle \text{CBE}$

$= \frac{1}{2} (180° - y)$

$= 90° - \frac{y}{2}$ (1)

Similarly, ray CO is the bisector of ∠ BCD.

Therefore, $\angle \text{BCO} = \frac{1}{2} \angle \text{BCD}$

$= \frac{1}{2} (180° - z)$

$= 90° - \frac{z}{2}$ (2)

In Δ BOC, $\angle$ BOC + $\angle$ BCO + $\angle$ CBO = 180° (3)

Substituting (1) and (2) in (3), you get

$$\angle \text{ BOC} + 90° - \frac{z}{2} + 90° - \frac{y}{2} = 180°$$

So, $$\angle \text{ BOC} = \frac{z}{2} + \frac{y}{2}$$

or, $$\angle \text{ BOC} = \frac{1}{2}(y + z) \quad (4)$$

But, $x + y + z = 180°$ (Angle sum property of a triangle)

Therefore, $y + z = 180° - x$

Therefore, (4) becomes

$$\angle \text{ BOC} = \frac{1}{2}(180° - x)$$

$$= 90° - \frac{x}{2}$$

$$= 90° - \frac{1}{2}\angle \text{ BAC}$$

EXERCISE 3.3

1. In Fig. 3.39, sides QP and RQ of Δ PQR are produced to points S and T respectively. If $\angle$ SPR = 135° and $\angle$ PQT = 110°, find $\angle$ PRQ.
2. In Fig. 3.40, $\angle$ X = 62°, $\angle$ XYZ = 54°. If YO and ZO are the bisectors of $\angle$ XYZ and $\angle$ XZY respectively of Δ XYZ, find $\angle$ OZY and $\angle$ YOZ.
3. In Fig. 3.41, if AB || DE, $\angle$ BAC = 35° and $\angle$ CDE = 53°, find $\angle$ DCE.

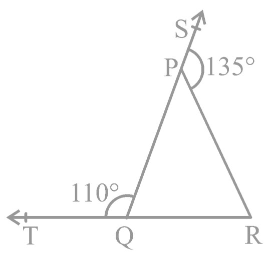


Fig. 3.39

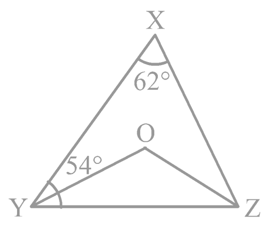


Fig. 3.40

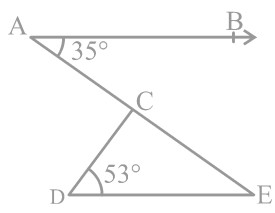


Fig. 3.41

4. In Fig. 3.42, if lines PQ and RS intersect at point T, such that $\angle$ PRT = 40°, $\angle$ RPT = 95° and $\angle$ TSQ = 75°, find $\angle$ SQT.

5. In Fig. 3.43, if PQ $\perp$ PS, PQ $\parallel$ SR, $\angle$ SQR = 28° and $\angle$ QRT = 65°, then find the values of x and y.

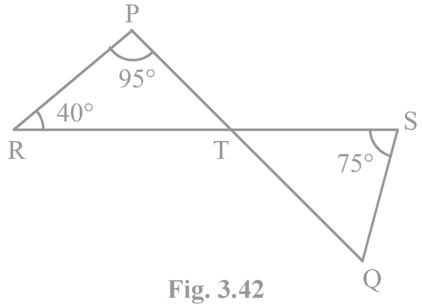


Fig. 3.42

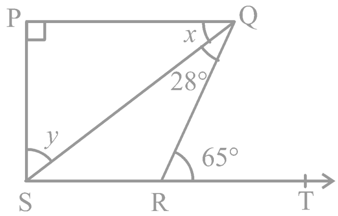


Fig. 3.43

6. In Fig. 3.44, the side QR of Δ PQR is produced to a point S. If the bisectors of $\angle$ PQR and $\angle$ PRS meet at point T, then prove that $\angle QTR = \frac{1}{2}\angle QPR$.

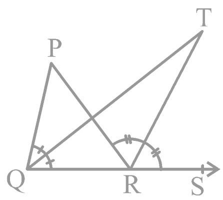


Fig. 3.44

3.8 Summary

In this chapter, you have studied the following points:

1. If a ray stands on a line, then the sum of the two adjacent angles so formed is 180° and vice-versa. This property is called as the Linear pair axiom.

2. If two lines intersect each other, then the vertically opposite angles are equal.

3. If a transversal intersects two parallel lines, then
 (i) each pair of corresponding angles is equal,
 (ii) each pair of alternate interior angles is equal,
 (iii) each pair of interior angles on the same side of the transversal is supplementary.

4. If a transversal intersects two lines such that, either
 (i) any one pair of corresponding angles is equal, or
 (ii) any one pair of alternate interior angles is equal, or
 (iii) any one pair of interior angles on the same side of the transversal is supplementary, then the lines are parallel.

5. Lines which are parallel to a given line are parallel to each other.

6. The sum of the three angles of a triangle is 180°.

7. If a side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.

CHAPTER 4

POLYNOMIALS

4.1 Introduction

You have studied algebraic expressions, their addition, subtraction, multiplication and division in earlier classes. You also have studied how to factorise some algebraic expressions. You may recall the algebraic identities :

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

and $$x^2 - y^2 = (x + y)(x - y)$$

and their use in factorisation. In this chapter, we shall start our study with a particular type of algebraic expression, called *polynomial,* and the terminology related to it. We shall also study the *Remainder Theorem* and *Factor Theorem* and their use in the factorisation of polynomials. In addition to the above, we shall study some more algebraic identities and their use in factorisation and in evaluating some given expressions.

4.2 Polynomials in One Variable

Let us begin by recalling that a variable is denoted by a symbol that can take any real value. We use the letters x, y, z, etc. to denote variables. Notice that $2x$, $3x$, $-x$, $-\frac{1}{2}x$ are algebraic expressions. All these expressions are of the form (a constant) × x. Now suppose we want to write an expression which is (a constant) × (a variable) and we do not know what the constant is. In such cases, we write the constant as a, b, c, etc. So the expression will be ax, say.

However, there is a difference between a letter denoting a constant and a letter denoting a variable. The values of the constants remain the same throughout a particular situation, that is, the values of the constants do not change in a given problem, but the value of a variable can keep changing.

Now, consider a square of side 3 units (see Fig. 4.1). What is its perimeter? You know that the perimeter of a square is the sum of the lengths of its four sides. Here, each side is 3 units. So, its perimeter is 4×3, i.e., 12 units. What will be the perimeter if each side of the square is 10 units? The perimeter is 4×10, i.e., 40 units. In case the length of each side is x units (see Fig. 4.2), the perimeter is given by $4x$ units. So, as the length of the side varies, the perimeter varies.

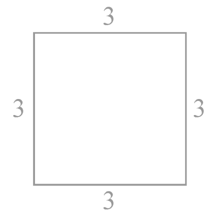


Fig. 4.1

Can you find the area of the square PQRS? It is $x \times x = x^2$ square units. x^2 is an algebraic expression. You are also familiar with other algebraic expressions like $2x$, $x^2 + 2x$, $x^3 - x^2 + 4x + 7$. Note that, all the algebraic expressions we have considered so far have only whole numbers as the exponents of the variable. Expressions of this form are called *polynomials in one variable*. In the examples above, the variable is x. For instance, $x^3 - x^2 + 4x + 7$ is a polynomial in x. Similarly, $3y^2 + 5y$ is a polynomial in the variable y and $t^2 + 4$ is a polynomial in the variable t.

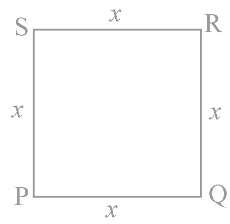


Fig. 4.2

In the polynomial $x^2 + 2x$, the expressions x^2 and $2x$ are called the **terms** of the polynomial. Similarly, the polynomial $3y^2 + 5y + 7$ has three terms, namely, $3y^2$, $5y$ and 7. Can you write the terms of the polynomial $-x^3 + 4x^2 + 7x - 2$? This polynomial has 4 terms, namely, $-x^3$, $4x^2$, $7x$ and -2.

Each term of a polynomial has a **coefficient**. So, in $-x^3 + 4x^2 + 7x - 2$, the coefficient of x^3 is -1, the coefficient of x^2 is 4, the coefficient of x is 7 and -2 is the coefficient of x^0 (Remember, $x^0 = 1$). Do you know the coefficient of x in $x^2 - x + 7$? It is -1.

2 is also a polynomial. In fact, 2, -5, 7, etc. are examples of *constant polynomials*. The constant polynomial 0 is called the **zero polynomial**. This plays a very important role in the collection of all polynomials, as you will see in the higher classes.

Now, consider algebraic expressions such as $x + \frac{1}{x}$, $\sqrt{x} + 3$ and $\sqrt[3]{y} + y^2$. Do you know that you can write $x + \frac{1}{x} = x + x^{-1}$? Here, the exponent of the second term, i.e., x^{-1} is -1, which is not a whole number. So, this algebraic expression is not a polynomial.

Again, $\sqrt{x} + 3$ can be written as $x^{\frac{1}{2}} + 3$. Here the exponent of x is $\frac{1}{2}$, which is not a whole number. So, is $\sqrt{x} + 3$ a polynomial? No, it is not. What about $\sqrt[3]{y} + y^2$? It is also not a polynomial (Why?).

If the variable in a polynomial is x, we may denote the polynomial by $p(x)$, or $q(x)$, or $r(x)$, etc. So, for example, we may write :

$$p(x) = 2x^2 + 5x - 3$$
$$q(x) = x^3 - 1$$
$$r(y) = y^3 + y + 1$$
$$s(u) = 2 - u - u^2 + 6u^5$$

A polynomial can have any (finite) number of terms. For instance, $x^{150} + x^{149} + ... + x^2 + x + 1$ is a polynomial with 151 terms.

Consider the polynomials $2x$, 2, $5x^3$, $-5x^2$, y and u^4. Do you see that each of these polynomials has only one term? Polynomials having only one term are called *monomials* ('mono' means 'one').

Now observe each of the following polynomials:

$p(x) = x + 1$, $\quad q(x) = x^2 - x$, $\quad r(y) = y^{30} + 1$, $\quad t(u) = u^{43} - u^2$

How many terms are there in each of these? Each of these polynomials has only two terms. Polynomials having only two terms are called *binomials* ('bi' means 'two').

Similarly, polynomials having only three terms are called *trinomials* ('tri' means 'three'). Some examples of trinomials are

$p(x) = x + x^2 + \pi$, $\quad q(x) = \sqrt{2} + x - x^2$,

$r(u) = u + u^2 - 2$, $\quad t(y) = y^4 + y + 5$.

Now, look at the polynomial $p(x) = 3x^7 - 4x^6 + x + 9$. What is the term with the highest power of x ? It is $3x^7$. The exponent of x in this term is 7. Similarly, in the polynomial $q(y) = 5y^6 - 4y^2 - 6$, the term with the highest power of y is $5y^6$ and the exponent of y in this term is 6. We call the highest power of the variable in a polynomial as the *degree of the polynomial*. So, the degree of the polynomial $3x^7 - 4x^6 + x + 9$ is 7 and the degree of the polynomial $5y^6 - 4y^2 - 6$ is 6. **The degree of a non-zero constant polynomial is zero**.

Example 1 : Find the degree of each of the polynomials given below:

(i) $x^5 - x^4 + 3$ $\quad$ (ii) $2 - y^2 - y^3 + 2y^8$ $\quad$ (iii) 2

Solution : (i) The highest power of the variable is 5. So, the degree of the polynomial is 5.

(ii) The highest power of the variable is 8. So, the degree of the polynomial is 8.

(iii) The only term here is 2 which can be written as $2x^0$. So the exponent of x is 0. Therefore, the degree of the polynomial is 0.

Now observe the polynomials $p(x) = 4x + 5$, $q(y) = 2y$, $r(t) = t + \sqrt{2}$ and $s(u) = 3 - u$. Do you see anything common among all of them? The degree of each of these polynomials is one. A polynomial of degree one is called a *linear polynomial*. Some more linear polynomials in one variable are $2x - 1$, $\sqrt{2}\,y + 1$, $2 - u$. Now, try and find a linear polynomial in x with 3 terms? You would not be able to find it because a linear polynomial in x can have at most two terms. So, any linear polynomial in x will be of the form $ax + b$, where a and b are constants and $a \neq 0$ (why?). Similarly, $ay + b$ is a linear polynomial in y.

Now consider the polynomials :

$$2x^2 + 5,\ \ 5x^2 + 3x + \pi,\ \ x^2 \text{ and } x^2 + \frac{2}{5}x$$

Do you agree that they are all of degree two? A polynomial of degree two is called a *quadratic polynomial.* Some examples of a quadratic polynomial are $5 - y^2$, $4y + 5y^2$ and $6 - y - y^2$. Can you write a quadratic polynomial in one variable with four different terms? You will find that a quadratic polynomial in one variable will have at most 3 terms. If you list a few more quadratic polynomials, you will find that any quadratic polynomial in x is of the form $ax^2 + bx + c$, where $a \neq 0$ and a, b, c are constants. Similarly, quadratic polynomial in y will be of the form $ay^2 + by + c$, provided $a \neq 0$ and a, b, c are constants.

We call a polynomial of degree three a *cubic polynomial*. Some examples of a cubic polynomial in x are $4x^3$, $2x^3 + 1$, $5x^3 + x^2$, $6x^3 - x$, $6 - x^3$, $2x^3 + 4x^2 + 6x + 7$. How many terms do you think a cubic polynomial in one variable can have? It can have at most 4 terms. These may be written in the form $ax^3 + bx^2 + cx + d$, where $a \neq 0$ and a, b, c and d are constants.

Now, that you have seen what a polynomial of degree 1, degree 2, or degree 3 looks like, can you write down a polynomial in one variable of degree n for any natural number n? A polynomial in one variable x of degree n is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$.

In particular, if $a_0 = a_1 = a_2 = a_3 = \ldots = a_n = 0$ (all the constants are zero), we get the **zero polynomial**, which is denoted by 0. What is the degree of the zero polynomial? The degree of the zero polynomial is *not defined*.

So far we have dealt with polynomials in one variable only. We can also have polynomials in more than one variable. For example, $x^2 + y^2 + xyz$ (where variables are x, y and z) is a polynomial in three variables. Similarly $p^2 + q^{10} + r$ (where the variables are p, q and r), $u^3 + v^2$ (where the variables are u and v) are polynomials in three and two variables, respectively. You will be studying such polynomials in detail later.

EXERCISE 4.1

1. Which of the following expressions are polynomials in one variable and which are not? State reasons for your answer.

(i) $4x^2 - 3x + 7$ (ii) $y^2 + \sqrt{2}$ (iii) $3\sqrt{t} + t\sqrt{2}$ (iv) $y + \frac{2}{y}$

(v) $x^{10} + y^3 + t^{50}$

2. Write the coefficients of x^2 in each of the following:

(i) $2 + x^2 + x$ (ii) $2 - x^2 + x^3$ (iii) $\frac{\pi}{2}x^2 + x$ (iv) $\sqrt{2}x - 1$

3. Give one example each of a binomial of degree 35, and of a monomial of degree 100.

4. Write the degree of each of the following polynomials:

(i) $5x^3 + 4x^2 + 7x$ (ii) $4 - y^2$

(iii) $5t - \sqrt{7}$ (iv) 3

5. Classify the following as linear, quadratic and cubic polynomials:

(i) $x^2 + x$ (ii) $x - x^3$ (iii) $y + y^2 + 4$ (iv) $1 + x$

(v) $3t$ (vi) r^2 (vii) $7x^3$

4.3 Zeroes of a Polynomial

Consider the polynomial $p(x) = 5x^3 - 2x^2 + 3x - 2$.

If we replace x by 1 everywhere in $p(x)$, we get

$$p(1) = 5 \times (1)^3 - 2 \times (1)^2 + 3 \times (1) - 2$$
$$= 5 - 2 + 3 - 2$$
$$= 4$$

So, we say that the value of $p(x)$ at $x = 1$ is 4.

Similarly, $p(0) = 5(0)^3 - 2(0)^2 + 3(0) - 2$

$$= -2$$

Can you find $p(-1)$?

Example 2 : Find the value of each of the following polynomials at the indicated value of variables:

(i) $p(x) = 5x^2 - 3x + 7$ at $x = 1$.

(ii) $q(y) = 3y^3 - 4y + \sqrt{11}$ at $y = 2$.

(iii) $p(t) = 4t^4 + 5t^3 - t^2 + 6$ at $t = a$.

Solution : (i) $p(x) = 5x^2 - 3x + 7$

The value of the polynomial $p(x)$ at $x = 1$ is given by

$$p(1) = 5(1)^2 - 3(1) + 7$$
$$= 5 - 3 + 7 = 9$$

(ii) $q(y) = 3y^3 - 4y + \sqrt{11}$

The value of the polynomial $q(y)$ at $y = 2$ is given by

$$q(2) = 3(2)^3 - 4(2) + \sqrt{11} = 24 - 8 + \sqrt{11} = 16 + \sqrt{11}$$

(iii) $p(t) = 4t^4 + 5t^3 - t^2 + 6$

The value of the polynomial $p(t)$ at $t = a$ is given by

$$p(a) = 4a^4 + 5a^3 - a^2 + 6$$

Now, consider the polynomial $p(x) = x - 1$.

What is $p(1)$? Note that : $p(1) = 1 - 1 = 0$.

As $p(1) = 0$, we say that 1 is a *zero* of the polynomial $p(x)$.

Similarly, you can check that 2 is a *zero* of $q(x)$, where $q(x) = x - 2$.

In general, we say that a *zero* of a polynomial $p(x)$ is a number c such that $p(c) = 0$.

You must have observed that the zero of the polynomial $x - 1$ is obtained by equating it to 0, i.e., $x - 1 = 0$, which gives $x = 1$. We say $p(x) = 0$ is a polynomial equation and 1 is the *root of the polynomial* equation $p(x) = 0$. So we say 1 is the zero of the polynomial $x - 1$, or a *root* of the polynomial equation $x - 1 = 0$.

Now, consider the constant polynomial 5. Can you tell what its zero is? It has no zero because replacing x by any number in $5x^0$ still gives us 5. In fact, *a non-zero constant polynomial has no zero*. What about the zeroes of the zero polynomial? By convention, *every real number is a zero of the zero polynomial*.

Example 3 : Check whether –2 and 2 are zeroes of the polynomial $x + 2$.

Solution : Let $p(x) = x + 2$.

Then $p(2) = 2 + 2 = 4$, $p(-2) = -2 + 2 = 0$

Therefore, –2 is a zero of the polynomial $x + 2$, but 2 is not.

Example 4 : Find a zero of the polynomial $p(x) = 2x + 1$.

Solution : Finding a zero of $p(x)$, is the same as solving the equation

$$p(x) = 0$$

Now, $2x + 1 = 0$ gives us $x = -\frac{1}{2}$

So, $-\frac{1}{2}$ is a zero of the polynomial $2x + 1$.

Now, if $p(x) = ax + b$, $a \neq 0$, is a linear polynomial, how can we find a zero of $p(x)$? Example 4 may have given you some idea. Finding a zero of the polynomial $p(x)$, amounts to solving the polynomial equation $p(x) = 0$.

Now, $p(x) = 0$ means $ax + b = 0, a \neq 0$

So, $ax = -b$

i.e., $x = -\frac{b}{a}$.

So, $x = -\frac{b}{a}$ is the only zero of $p(x)$, i.e., a *linear polynomial has one and only one zero*.

Now we can say that 1 is *the* zero of $x - 1$, and -2 is *the* zero of $x + 2$.

Example 5 : Verify whether 2 and 0 are zeroes of the polynomial $x^2 - 2x$.

Solution : Let $p(x) = x^2 - 2x$

Then $p(2) = 2^2 - 4 = 4 - 4 = 0$

and $p(0) = 0 - 0 = 0$

Hence, 2 and 0 are both zeroes of the polynomial $x^2 - 2x$.

Let us now list our observations:

(i) A zero of a polynomial need not be 0.

(ii) 0 may be a zero of a polynomial.

(iii) Every linear polynomial has one and only one zero.

(iv) A polynomial can have more than one zero.

EXERCISE 4.2

1. Find the value of the polynomial $5x - 4x^2 + 3$ at

(i) $x = 0$ (ii) $x = -1$ (iii) $x = 2$

2. Find $p(0)$, $p(1)$ and $p(2)$ for each of the following polynomials:

(i) $p(y) = y^2 - y + 1$ (ii) $p(t) = 2 + t + 2t^2 - t^3$

(iii) $p(x) = x^3$ (iv) $p(x) = (x - 1)(x + 1)$

3. Verify whether the following are zeroes of the polynomial, indicated against them.

(i) $p(x) = 3x + 1,\ x = -\frac{1}{3}$ (ii) $p(x) = 5x - \pi,\ x = \frac{4}{5}$

(iii) $p(x) = x^2 - 1,\ x = 1, -1$ (iv) $p(x) = (x + 1)(x - 2),\ x = -1, 2$

(v) $p(x) = x^2,\ x = 0$ (vi) $p(x) = lx + m,\ x = -\frac{m}{l}$

(vii) $p(x) = 3x^2 - 1,\ x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}$ (viii) $p(x) = 2x + 1,\ x = \frac{1}{2}$

4. Find the zero of the polynomial in each of the following cases:

(i) $p(x) = x + 5$ (ii) $p(x) = x - 5$ (iii) $p(x) = 2x + 5$

(iv) $p(x) = 3x - 2$ (v) $p(x) = 3x$ (vi) $p(x) = ax,\ a \neq 0$

(vii) $p(x) = cx + d,\ c \neq 0,\ c, d$ are real numbers.

4.4 Remainder Theorem

Let us consider two numbers 15 and 6. You know that when we divide 15 by 6, we get the quotient 2 and remainder 3. Do you remember how this fact is expressed? We write 15 as

$$15 = (6 \times 2) + 3$$

We observe that the *remainder* 3 is less than the *divisor* 6. Similarly, if we divide 12 by 6, we get

$$12 = (6 \times 2) + 0$$

What is the remainder here? Here the remainder is 0, and we say that 6 is a *factor* of 12 or 12 is a *multiple* of 6.

Now, the question is: can we divide one polynomial by another? To start with, let us try and do this when the divisor is a monomial. So, let us divide the polynomial $2x^3 + x^2 + x$ by the monomial x.

We have $$(2x^3 + x^2 + x) \div x = \frac{2x^3}{x} + \frac{x^2}{x} + \frac{x}{x}$$

$$= 2x^2 + x + 1$$

In fact, you may have noticed that x is common to each term of $2x^3 + x^2 + x$. So we can write $2x^3 + x^2 + x$ as $x(2x^2 + x + 1)$.

We say that x and $2x^2 + x + 1$ are *factors* of $2x^3 + x^2 + x$, and $2x^3 + x^2 + x$ is a *multiple* of x as well as a multiple of $2x^2 + x + 1$.

Consider another pair of polynomials $3x^2 + x + 1$ and x.

Here, $$(3x^2 + x + 1) \div x = (3x^2 \div x) + (x \div x) + (1 \div x).$$

We see that we cannot divide 1 by x to get a polynomial term. So in this case we stop here, and note that 1 is the remainder. Therefore, we have

$$3x^2 + x + 1 = \{x \times (3x + 1)\} + 1$$

In this case, $3x + 1$ is the quotient and 1 is the remainder. Do you think that x is a factor of $3x^2 + x + 1$? Since the remainder is not zero, it is not a factor.

Now let us consider an example to see how we can divide a polynomial by any non-zero polynomial.

Example 6 : Divide $p(x)$ by $g(x)$, where $p(x) = x + 3x^2 - 1$ and $g(x) = 1 + x$.

Solution : We carry out the process of division by means of the following steps:

Step 1 : We write the dividend $x + 3x^2 - 1$ and the divisor $1 + x$ in the standard form, i.e., after arranging the terms in the descending order of their degrees. So, the dividend is $3x^2 + x - 1$ and divisor is $x + 1$.

Step 2 : We divide the first term of the dividend by the first term of the divisor, i.e., we divide $3x^2$ by x, and get $3x$. This gives us the first term of the quotient.

$$\frac{3x^2}{x} = 3x = \text{first term of quotient}$$

Step 3 : We multiply the divisor by the first term of the quotient, and subtract this product from the dividend, i.e., we multiply $x + 1$ by $3x$ and subtract the product $3x^2 + 3x$ from the dividend $3x^2 + x - 1$. This gives us the remainder as $-2x - 1$.

$$\begin{array}{r|l} & 3x \\ \hline x+1 & 3x^2 + x - 1 \\ & 3x^2 + 3x \\ & - \quad - \\ \hline & -2x - 1 \end{array}$$

Step 4 : We treat the remainder $-2x - 1$ as the new dividend. The divisor remains the same. We repeat Step 2 to get the next term of the quotient, i.e., we divide the first term $-2x$ of the (new) dividend by the first term x of the divisor and obtain -2. Thus, -2 is the second term in the quotient.

$$\frac{-2x}{x} = -2$$
$= \text{second term of quotient}$

New Quotient $= 3x - 2$

Step 5 : We multiply the divisor by the second term of the quotient and subtract the product from the dividend. That is, we multiply $x + 1$ by -2 and subtract the product $-2x - 2$ from the dividend $-2x - 1$. This gives us 1 as the remainder.

$$\begin{array}{l|l} (x+1)(-2) & -2x-1 \\ = -2x-2 & -2x-2 \\ & \underline{+ \quad +} \\ & \quad +1 \end{array}$$

This process continues till the remainder is 0 or the degree of the new dividend is less than the degree of the divisor. At this stage, this new dividend becomes the remainder and the sum of the quotients gives us the whole quotient.

Step 6 : Thus, the quotient in full is $3x - 2$ and the remainder is 1.

Let us look at what we have done in the process above as a whole:

$$\begin{array}{r|l} & 3x - 2 \\ \hline x+1 & 3x^2 + x - 1 \\ & 3x^2 + 3x \\ & \underline{- \quad -} \\ & \quad -2x - 1 \\ & \quad -2x - 2 \\ & \quad \underline{+ \quad +} \\ & \qquad\quad 1 \end{array}$$

Notice that $3x^2 + x - 1 = (x + 1)(3x - 2) + 1$

i.e., **Dividend = (Divisor × Quotient) + Remainder**

In general, if $p(x)$ and $g(x)$ are two polynomials such that degree of $p(x) \geq$ degree of $g(x)$ and $g(x) \neq 0$, then we can find polynomials $q(x)$ and $r(x)$ such that:

$$p(x) = g(x)q(x) + r(x),$$

where $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$. Here we say that $p(x)$ divided by $g(x)$, gives $q(x)$ as quotient and $r(x)$ as remainder.

In the example above, the divisor was a linear polynomial. In such a situation, let us see if there is any link between the remainder and certain values of the dividend.

In $p(x) = 3x^2 + x - 1$, if we replace x by -1, we have

$p(-1) = 3(-1)^2 + (-1) - 1 = 1$

So, the remainder obtained on dividing $p(x) = 3x^2 + x - 1$ by $x + 1$ is the same as the value of the polynomial $p(x)$ at the zero of the polynomial $x + 1$, i.e., -1.

Let us consider some more examples.

Example 7 : Divide the polynomial $3x^4 - 4x^3 - 3x - 1$ by $x - 1$.

Solution : By long division, we have:

$$
\begin{array}{r|l}
 & 3x^3 - x^2 - x - 4 \\
\hline
x - 1 & 3x^4 - 4x^3 - 3x - 1 \\
 & \underset{-}{}3x^4 \underset{+}{-} 3x^3 \\
\hline
 & \quad - x^3 \qquad - 3x - 1 \\
 & \quad \underset{+}{-} x^3 \underset{-}{+} x^2 \\
\hline
 & \qquad - x^2 - 3x - 1 \\
 & \qquad \underset{+}{-} x^2 \underset{-}{+} x \\
\hline
 & \qquad\quad - 4x - 1 \\
 & \qquad\quad \underset{+}{-} 4x \underset{-}{+} 4 \\
\hline
 & \qquad\qquad - 5
\end{array}
$$

Here, the remainder is -5. Now, the zero of $x - 1$ is 1. So, putting $x = 1$ in $p(x)$, we see that

$$
\begin{aligned}
p(1) &= 3(1)^4 - 4(1)^3 - 3(1) - 1 \\
&= 3 - 4 - 3 - 1 \\
&= -5, \text{ which is the remainder.}
\end{aligned}
$$

Example 8 : Find the remainder obtained on dividing $p(x) = x^3 + 1$ by $x + 1$.

Solution : By long division,

$$
\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
x + 1 & x^3 + 1 \\
 & \underset{-}{}x^3 \underset{-}{+} x^2 \\
\hline
 & \quad - x^2 \qquad + 1 \\
 & \quad \underset{+}{-} x^2 \underset{+}{-} x \\
\hline
 & \qquad\quad x + 1 \\
 & \qquad\quad \underset{-}{}x \underset{-}{+} 1 \\
\hline
 & \qquad\qquad 0
\end{array}
$$

So, we find that the remainder is 0.

Here $p(x) = x^3 + 1$, and the root of $x + 1 = 0$ is $x = -1$. We see that

$$p(-1) = (-1)^3 + 1$$
$$= -1 + 1$$
$$= 0,$$

which is equal to the remainder obtained by actual division.

Is it not a simple way to find the remainder obtained on dividing a polynomial by a *linear polynomial*? We shall now generalise this fact in the form of the following theorem. We shall also show you why the theorem is true, by giving you a proof of the theorem.

Remainder Theorem : *Let $p(x)$ be any polynomial of degree greater than or equal to one and let a be any real number. If $p(x)$ is divided by the linear polynomial $x - a$, then the remainder is $p(a)$.*

Proof : Let $p(x)$ be any polynomial with degree greater than or equal to 1. Suppose that when $p(x)$ is divided by $x - a$, the quotient is $q(x)$ and the remainder is $r(x)$, i.e.,

$$p(x) = (x - a)\, q(x) + r(x)$$

Since the degree of $x - a$ is 1 and the degree of $r(x)$ is less than the degree of $x - a$, the degree of $r(x) = 0$. This means that $r(x)$ is a constant, say r.

So, for every value of x, $r(x) = r$.

Therefore, $p(x) = (x - a)\, q(x) + r$

In particular, if $x = a$, this equation gives us

$$p(a) = (a - a)\, q(a) + r$$
$$= r,$$

which proves the theorem.

Let us use this result in another example.

Example 9 : Find the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$.

Solution : Here, $p(x) = x^4 + x^3 - 2x^2 + x + 1$, and the zero of $x - 1$ is 1.

So, $p(1) = (1)^4 + (1)^3 - 2(1)^2 + 1 + 1$

$= 2$

So, by the Remainder Theorem, 2 is the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$.

Example 10 : Check whether the polynomial $q(t) = 4t^3 + 4t^2 - t - 1$ is a multiple of $2t + 1$.

Solution : As you know, $q(t)$ will be a multiple of $2t + 1$ only, if $2t + 1$ divides $q(t)$ leaving remainder zero. Now, taking $2t + 1 = 0$, we have $t = -\frac{1}{2}\cdot$

Also, $$q\left(-\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right)^3 + 4\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) - 1 = -\frac{1}{2} + 1 + \frac{1}{2} - 1 = 0$$

So the remainder obtained on dividing $q(t)$ by $2t + 1$ is 0.

So, $2t + 1$ is a factor of the given polynomial $q(t)$, that is $q(t)$ is a multiple of $2t + 1$.

EXERCISE 2.3

1. Find the remainder when $x^3 + 3x^2 + 3x + 1$ is divided by

(i) $x + 1$ (ii) $x - \frac{1}{2}$ (iii) x (iv) $x + \pi$ (v) $5 + 2x$

2. Find the remainder when $x^3 - ax^2 + 6x - a$ is divided by $x - a$.

3. Check whether $7 + 3x$ is a factor of $3x^3 + 7x$.

4.5 Factorisation of Polynomials

Let us now look at the situation of Example 10 above more closely. It tells us that since the remainder, $q\left(-\frac{1}{2}\right) = 0$, $(2t + 1)$ is a factor of $q(t)$, i.e., $q(t) = (2t + 1)\, g(t)$ for some polynomial $g(t)$. This is a particular case of the following theorem.

Factor Theorem : If $p(x)$ is a polynomial of degree $n \geq 1$ and a is any real number, then (i) $x - a$ is a factor of $p(x)$, if $p(a) = 0$, and (ii) $p(a) = 0$, if $x - a$ is a factor of $p(x)$.

Proof: By the Remainder Theorem, $p(x)=(x - a)\, q(x) + p(a)$.

(i) If $p(a) = 0$, then $p(x) = (x - a)\, q(x)$, which shows that $x - a$ is a factor of $p(x)$.

(ii) Since $x - a$ is a factor of $p(x)$, $p(x) = (x - a)\, g(x)$ for same polynomial $g(x)$. In this case, $p(a) = (a - a)\, g(a) = 0$.

Example 11 : Examine whether $x + 2$ is a factor of $x^3 + 3x^2 + 5x + 6$ and of $2x + 4$.

Solution : The zero of $x + 2$ is -2. Let $p(x) = x^3 + 3x^2 + 5x + 6$ and $s(x) = 2x + 4$

Then, $p(-2) = (-2)^3 + 3(-2)^2 + 5(-2) + 6$

$= -8 + 12 - 10 + 6$

$= 0$

So, by the Factor Theorem, $x + 2$ is a factor of $x^3 + 3x^2 + 5x + 6$.

Again, $s(-2) = 2(-2) + 4 = 0$

So, $x + 2$ is a factor of $2x + 4$. In fact, you can check this without applying the Factor Theorem, since $2x + 4 = 2(x + 2)$.

Example 12 : Find the value of k, if $x - 1$ is a factor of $4x^3 + 3x^2 - 4x + k$.

Solution : As $x - 1$ is a factor of $p(x) = 4x^3 + 3x^2 - 4x + k$, $p(1) = 0$

Now, $p(1) = 4(1)^3 + 3(1)^2 - 4(1) + k$

So, $4 + 3 - 4 + k = 0$

i.e., $k = -3$

We will now use the Factor Theorem to factorise some polynomials of degree 2 and 3. You are already familiar with the factorisation of a quadratic polynomial like $x^2 + lx + m$. You had factorised it by splitting the middle term lx as $ax + bx$ so that $ab = m$. Then $x^2 + lx + m = (x + a)(x + b)$. We shall now try to factorise quadratic polynomials of the type $ax^2 + bx + c$, where $a \neq 0$ and a, b, c are constants.

Factorisation of the polynomial $ax^2 + bx + c$ **by splitting the middle term** is as follows:

Let its factors be $(px + q)$ and $(rx + s)$. Then

$$ax^2 + bx + c = (px + q)(rx + s) = pr\,x^2 + (ps + qr)\,x + qs$$

Comparing the coefficients of x^2, we get $a = pr$.

Similarly, comparing the coefficients of x, we get $b = ps + qr$.

And, on comparing the constant terms, we get $c = qs$.

This shows us that b is the sum of two numbers ps and qr, whose product is $(ps)(qr) = (pr)(qs) = ac$.

Therefore, to factorise $ax^2 + bx + c$, we have to write b as the sum of two numbers whose product is ac. This will be clear from Example 13.

Example 13 : Factorise $6x^2 + 17x + 5$ by splitting the middle term, and by using the Factor Theorem.

Solution 1 : (By splitting method) **:** If we can find two numbers p and q such that $p + q = 17$ and $pq = 6 \times 5 = 30$, then we can get the factors.

So, let us look for the pairs of factors of 30. Some are 1 and 30, 2 and 15, 3 and 10, 5 and 6. Of these pairs, 2 and 15 will give us $p + q = 17$.

$$\begin{aligned} \text{So, } 6x^2 + 17x + 5 &= 6x^2 + (2 + 15)x + 5 \\ &= 6x^2 + 2x + 15x + 5 \\ &= 2x(3x + 1) + 5(3x + 1) \\ &= (3x + 1)(2x + 5) \end{aligned}$$

Solution 2 : (Using the Factor Theorem)

$6x^2 + 17x + 5 = 6\left(x^2 + \frac{17}{6}x + \frac{5}{6}\right) = 6\,p(x)$, say. If a and b are the zeroes of $p(x)$, then $6x^2 + 17x + 5 = 6(x - a)(x - b)$. So, $ab = \frac{5}{6}\cdot$ Let us look at some possibilities for a and b. They could be $\pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{5}{3}, \pm\frac{5}{2}, \pm 1$. Now, $p\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{17}{6}\left(\frac{1}{2}\right) + \frac{5}{6} \neq 0$. But $p\left(\frac{-1}{3}\right) = 0$. So, $\left(x + \frac{1}{3}\right)$ is a factor of $p(x)$. Similarly, by trial, you can find that $\left(x + \frac{5}{2}\right)$ is a factor of $p(x)$.

$$\begin{aligned} \text{Therefore,} \qquad 6x^2 + 17x + 5 &= 6\left(x + \frac{1}{3}\right)\left(x + \frac{5}{2}\right) \\ &= 6\left(\frac{3x + 1}{3}\right)\left(\frac{2x + 5}{2}\right) \\ &= (3x + 1)(2x + 5) \end{aligned}$$

For the example above, the use of the splitting method appears more efficient. However, let us consider another example.

Example 14 : Factorise $y^2 - 5y + 6$ by using the Factor Theorem.

Solution : Let $p(y) = y^2 - 5y + 6$. Now, if $p(y) = (y - a)(y - b)$, you know that the constant term will be ab. So, $ab = 6$. So, to look for the factors of $p(y)$, we look at the factors of 6.

The factors of 6 are 1, 2 and 3.

Now, $p(2) = 2^2 - (5 \times 2) + 6 = 0$

So, $y - 2$ is a factor of $p(y)$.

Also, $p(3) = 3^2 - (5 \times 3) + 6 = 0$

So, $y - 3$ is also a factor of $y^2 - 5y + 6$.

Therefore, $y^2 - 5y + 6 = (y - 2)(y - 3)$

Note that $y^2 - 5y + 6$ can also be factorised by splitting the middle term $-5y$.

Now, let us consider factorising cubic polynomials. Here, the splitting method will not be appropriate to start with. We need to find at least one factor first, as you will see in the following example.

Example 15 : Factorise $x^3 - 23x^2 + 142x - 120$.

Solution : Let $p(x) = x^3 - 23x^2 + 142x - 120$

We shall now look for all the factors of -120. Some of these are $\pm1, \pm2, \pm3,$

$\pm4, \pm5, \pm6, \pm8, \pm10, \pm12, \pm15, \pm20, \pm24, \pm30, \pm60$.

By trial, we find that $p(1) = 0$. So $x - 1$ is a factor of $p(x)$.

Now we see that $x^3 - 23x^2 + 142x - 120 = x^3 - x^2 - 22x^2 + 22x + 120x - 120$

$= x^2(x - 1) - 22x(x - 1) + 120(x - 1)$ (Why?)

$= (x - 1)(x^2 - 22x + 120)$ [Taking $(x - 1)$ common]

We could have also got this by dividing $p(x)$ by $x - 1$.

Now $x^2 - 22x + 120$ can be factorised either by splitting the middle term or by using the Factor theorem. By splitting the middle term, we have:

$$x^2 - 22x + 120 = x^2 - 12x - 10x + 120$$

$$= x(x - 12) - 10(x - 12)$$

$$= (x - 12)(x - 10)$$

So, $$x^3 - 23x^2 - 142x - 120 = (x - 1)(x - 10)(x - 12)$$

EXERCISE 4.4

1. Determine which of the following polynomials has $(x + 1)$ a factor :

   (i) $x^3 + x^2 + x + 1$ (ii) $x^4 + x^3 + x^2 + x + 1$

   (iii) $x^4 + 3x^3 + 3x^2 + x + 1$ (iv) $x^3 - x^2 - \left(2 + \sqrt{2}\right)x + \sqrt{2}$

2. Use the Factor Theorem to determine whether $g(x)$ is a factor of $p(x)$ in each of the following cases:

(i) $p(x) = 2x^3 + x^2 - 2x - 1, g(x) = x + 1$

(ii) $p(x) = x^3 + 3x^2 + 3x + 1, g(x) = x + 2$

(iii) $p(x) = x^3 - 4x^2 + x + 6, g(x) = x - 3$

3. Find the value of k, if $x - 1$ is a factor of $p(x)$ in each of the following cases:

(i) $p(x) = x^2 + x + k$ (ii) $p(x) = 2x^2 + kx + \sqrt{2}$

(iii) $p(x) = kx^2 - \sqrt{2}x + 1$ (iv) $p(x) = kx^2 - 3x + k$

4. Factorise :

(i) $12x^2 - 7x + 1$ (ii) $2x^2 + 7x + 3$

(iii) $6x^2 + 5x - 6$ (iv) $3x^2 - x - 4$

5. Factorise :

(i) $x^3 - 2x^2 - x + 2$ (ii) $x^3 - 3x^2 - 9x - 5$

(iii) $x^3 + 13x^2 + 32x + 20$ (iv) $2y^3 + y^2 - 2y - 1$

4.6 Algebraic Identities

From your earlier classes, you may recall that an algebraic identity is an algebraic equation that is true for all values of the variables occurring in it. You have studied the following algebraic identities in earlier classes:

Identity I : $(x + y)^2 = x^2 + 2xy + y^2$

Identity II : $(x - y)^2 = x^2 - 2xy + y^2$

Identity III : $x^2 - y^2 = (x + y)(x - y)$

Identity IV : $(x + a)(x + b) = x^2 + (a + b)x + ab$

You must have also used some of these algebraic identities to factorise the algebraic expressions. You can also see their utility in computations.

Example 16 : Find the following products using appropriate identities:

(i) $(x + 3)(x + 3)$ (ii) $(x - 3)(x + 5)$

Solution : (i) Here we can use Identity I : $(x + y)^2 = x^2 + 2xy + y^2$. Putting $y = 3$ in it, we get

$$(x + 3)(x + 3) = (x + 3)^2 = x^2 + 2(x)(3) + (3)^2$$
$$= x^2 + 6x + 9$$

(ii) Using Identity IV above, i.e., $(x + a)(x + b) = x^2 + (a + b)x + ab$, we have

$$(x - 3)(x + 5) = x^2 + (-3 + 5)x + (-3)(5)$$
$$= x^2 + 2x - 15$$

Example 17 : Evaluate 105 × 106 without multiplying directly.

Solution :

$$105 \times 106 = (100 + 5) \times (100 + 6)$$
$$= (100)^2 + (5 + 6)\,(100) + (5 \times 6), \text{ using Identity IV}$$
$$= 10000 + 1100 + 30$$
$$= 11130$$

You have seen some uses of the identities listed above in finding the product of some given expressions. These identities are useful in factorisation of algebraic expressions also, as you can see in the following examples.

Example 18 : Factorise:

(i) $49a^2 + 70ab + 25b^2$ (ii) $\frac{25}{4}x^2 - \frac{y^2}{9}$

Solution : (i) Here you can see that

$$49a^2 = (7a)^2,\ 25b^2 = (5b)^2,\ 70ab = 2(7a)\,(5b)$$

Comparing the given expression with $x^2 + 2xy + y^2$, we observe that $x = 7a$ and $y = 5b$.

Using Identity I, we get

$$49a^2 + 70ab + 25b^2 = (7a + 5b)^2 = (7a + 5b)\,(7a + 5b)$$

(ii) We have $\frac{25}{4}x^2 - \frac{y^2}{9} = \left(\frac{5}{2}x\right)^2 - \left(\frac{y}{3}\right)^2$

Now comparing it with Identity III, we get

$$\frac{25}{4}x^2 - \frac{y^2}{9} = \left(\frac{5}{2}x\right)^2 - \left(\frac{y}{3}\right)^2$$
$$= \left(\frac{5}{2}x + \frac{y}{3}\right)\left(\frac{5}{2}x - \frac{y}{3}\right)$$

So far, all our identities involved products of binomials. Let us now extend the Identity I to a trinomial $x + y + z$. We shall compute $(x + y + z)^2$ by using Identity I.

Let $x + y = t$. Then,

$$(x + y + z)^2 = (t + z)^2$$
$$= t^2 + 2tz + t^2 \qquad \text{(Using Identity I)}$$
$$= (x + y)^2 + 2(x + y)z + z^2 \qquad \text{(Substituting the value of } t\text{)}$$

$= x^2 + 2xy + y^2 + 2xz + 2yz + z^2$ (Using Identity I)

$= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$ (Rearranging the terms)

So, we get the following identity:

Identity V : $\mathbf{(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx}$

Remark : We call the right hand side expression **the expanded form** of the left hand side expression. Note that the expansion of $(x + y + z)^2$ consists of three square terms and three product terms.

Example 19 : Write $(3a + 4b + 5c)^2$ in expanded form.

Solution : Comparing the given expression with $(x + y + z)^2$, we find that

$$x = 3a,\ y = 4b \text{ and } z = 5c.$$

Therefore, using Identity V, we have

$$(3a + 4b + 5c)^2 = (3a)^2 + (4b)^2 + (5c)^2 + 2(3a)(4b) + 2(4b)(5c) + 2(5c)(3a)$$

$$= 9a^2 + 16b^2 + 25c^2 + 24ab + 40bc + 30ac$$

Example 20 : Expand $(4a - 2b - 3c)^2$.

Solution : Using Identity V, we have

$$(4a - 2b - 3c)^2 = [4a + (-2b) + (-3c)]^2$$

$$= (4a)^2 + (-2b)^2 + (-3c)^2 + 2(4a)(-2b) + 2(-2b)(-3c) + 2(-3c)(4a)$$

$$= 16a^2 + 4b^2 + 9c^2 - 16ab + 12bc - 24ac$$

Example 21 : Factorise $4x^2 + y^2 + z^2 - 4xy - 2yz + 4xz$.

Solution : We have $4x^2 + y^2 + z^2 - 4xy - 2yz + 4xz = (2x)^2 + (-y)^2 + (z)^2 + 2(2x)(-y)$

$$+ 2(-y)(z) + 2(2x)(z)$$

$$= [2x + (-y) + z]^2 \quad \text{(Using Identity V)}$$

$$= (2x - y + z)^2 = (2x - y + z)(2x - y + z)$$

So far, we have dealt with identities involving second degree terms. Now let us extend Identity I to compute $(x + y)^3$. We have:

$$(x + y)^3 = (x + y)(x + y)^2$$

$$= (x + y)(x^2 + 2xy + y^2)$$

$$= x(x^2 + 2xy + y^2) + y(x^2 + 2xy + y^2)$$

$$= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

$$= x^3 + y^3 + 3xy(x + y)$$