CHAPTER

3

Three-Dimensional Geometry

- Direction Cosines and Direction Ratios
- Equation of Straight Line Passing through a Given Point and Parallel to a Given Vector
- Equation of Line Passing through Two Given Points
- Angle between Two Lines
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DIRECTION COSINES AND DIRECTION RATIOS

From Chapter 1, recall that if a directed line L, passing through the origin, makes angles α, β and γ with the x-, y- and z-axes, respectively, called direction angles, then the cosines of these angles, namely, $\cos\alpha$, $\cos\beta$ and $\cos\gamma$, are called the direction cosines of the directed line L.

If we reverse the direction of L, the direction angles are replaced by their supplements, i.e., $\pi-\alpha$, $\pi-\beta$ and $\pi-\gamma$. Thus, the signs of the direction cosines are reversed.

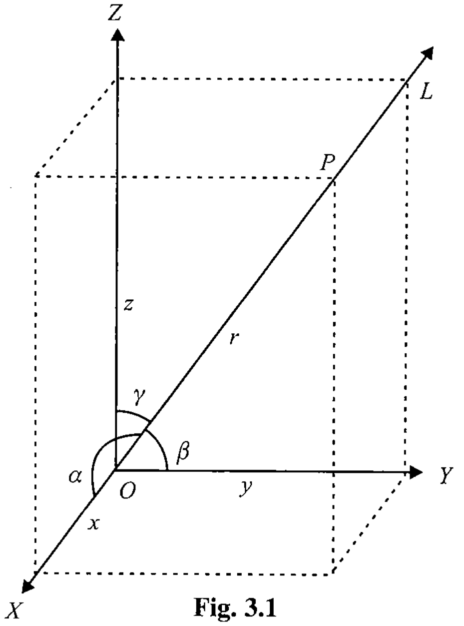


Fig. 3.1

Note that a given line in space can be extended in two opposite directions, and so it has two sets of direction cosines. In order to have a unique set of direction cosines for a given line in space, we must take the given line as a directed line. These unique direction cosines are denoted by l, m and n.

If the given line in space does not pass through the origin, then in order to find its direction cosines, we draw a line through the origin and parallel to the given line. Now take one of the directed lines from the origin and find its direction cosines as two parallel lines have same set of direction cosines.

Any three numbers which are proportional to the direction cosines of a line are called the *direction ratios* of the line. If l, m and n are direction cosines and a, b and c are the direction ratios of a line, then $a=\lambda l$, $b=\lambda m$ and $c=\lambda n$ for any non-zero $\lambda \in R$.

Notes:

1. *Direction cosines of the x-axis are* (1, 0, 0).

   Direction cosines of the y-axis are (0, 1, 0).

   Direction cosines of the z-axis are (0, 0, 1).

2. *Let OP be any line passing through the origin O which has direction cosines* $\cos\alpha$, $\cos\beta$ *and* $\cos\gamma$, *i.e.,* (l, m, n) *where distance* $OP = r \Rightarrow$ *Coordinates of P are* $(r\cos\alpha, r\cos\beta, r\cos\gamma)$.

3. *If* l, m *and* n *are the direction cosines of a vector, then* $l^2+m^2+n^2=1$.

4. $\vec{r} = |\vec{r}|\,(l\hat{i}+m\hat{j}+n\hat{k})$ *and* $\hat{r} = l\hat{i}+m\hat{j}+n\hat{k}$.

Direction Ratios

Let l, m and n be the direction cosines of a vector $\vec{r}$ and a, b and c be three numbers such that a, b, c are proportional to l, m and n. Therefore,

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k \text{ or } (l, m, n) = (ka, kb, kc)$$

Hence, a, b and c are direction ratios.

For example, if $(1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$ are direction cosines of a vector $\vec{r}$, then its direction ratios are $(1, -1, 1)$ or $(-1, 1, -1)$ or $(2, -2, 2)$ or $(\lambda, -\lambda, \lambda)$ ...

It is evident from the above definition that to obtain the direction ratios of a vector from its direction cosines, we just multiply them by a common number.

"That shows there can be an infinite number of direction ratios for a given vector, but the direction cosines are unique."

Direction ratios of a line joining two points

For points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$,

Vector $\overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$,

Then the direction ratios of PQ are $< (x_2 - x_1), (y_2 - y_1), (z_2 - z_1) >$.

To obtain direction cosines from direction ratios

Let a, b and c b e the direction ratios of a vector $\vec{r}$ having direction cosines l, m and n.

Then, $l = \lambda a$, $m = \lambda b$, $n = \lambda c$ (by definition)

$\therefore \quad l^2 + m^2 + n^2 = 1$

$\Rightarrow \quad a^2\lambda^2 + b^2\lambda^2 + c^2\lambda^2 = 1$

$\Rightarrow \quad \lambda = \pm\dfrac{1}{\sqrt{a^2+b^2+c^2}}$

$\Rightarrow \quad l = \pm\dfrac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\dfrac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm\dfrac{c}{\sqrt{a^2+b^2+c^2}}$

Example:

Let the direction ratios of a line be 3, 1 and -2.

Direction cosines are

$$\left(\frac{3}{\sqrt{3^2+1^2+(-2)^2}}, \frac{1}{\sqrt{3^2+1^2+(-2)^2}}, \frac{-2}{\sqrt{3^2+1^2+(-2)^2}}\right) \Rightarrow \left(\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}\right)$$

Notes:

1. *If* $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ *is a vector having direction cosines l, m and n, then* $l = \dfrac{a}{|\vec{r}|}, m = \dfrac{b}{|\vec{r}|}, n = \dfrac{c}{|\vec{r}|}$.

2. *Direction cosines of parallel vectors:*

   Let $\vec{a}$ *and* $\vec{b}$ *be two parallel vectors. Then* $\vec{b} = \lambda\vec{a}$ *for some* λ.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, *then* $\vec{b} = \lambda\vec{a} \Rightarrow \vec{b} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

This shows that $\vec{b}$ *has direction ratios* λa_1, λa_2 *and* λa_3, *i.e.,* a_1, a_2 *and* a_3 *because* $\lambda a_1 : \lambda a_2 : \lambda a_3 = a_1 : a_2 : a_3$. *Thus,* $\vec{a}$ *and* $\vec{b}$ *have equal direction ratios and hence equal direction cosines too.*

3. *If the direction ratios of* $\vec{r}$ *are a, b and c* $\Rightarrow \vec{r} = \dfrac{|\vec{r}|}{\sqrt{a^2+b^2+c^2}}(a\hat{i} + b\hat{j} + c\hat{k})$.
4. *Projections of* $\vec{r}$ *on the coordinate axes are:* $l|\vec{r}|$, $m|\vec{r}|$ and $n|\vec{r}|$.
5. *The projection of a segment joining points* $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$ on *a line with direction cosines l, m* and *n* is $(x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$.
6. *If* l_1, m_1, n_1 *and* l_2, m_2, n_2 *are the direction cosines of two concurrent lines, then the direction cosines of the lines bisecting the angles between them are proportional to* $l_1 \pm l_2$, $m_1 \pm m_2$ and $n_1 \pm n_2$.
7. *Acute angle* θ *between the two lines having direction cosines* l_1, m_1, n_1 and l_2, m_2, n_2 *is given by*

   $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$, $\sin\theta = \sqrt{(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2}$
8. *If* a_1, b_1, c_1 *and* a_2, b_2, c_2 *be the direction ratios of two lines, then the acute angle* θ *between them is given by*

   $$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}},$$

   $$\sin\theta = \frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$
9. *Two lines having direction cosines* l_1, m_1, n_1 *and* l_2, m_2, n_2 *are*
   - **a.** *perpendicular if and only if* $l_1l_2 + m_1m_2 + n_1n_2 = 0$.
   - **b.** *parallel if and only if* $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$
10. *Two lines having direction ratios* a_1, b_1, c_1 and a_2, b_2, c_2 *are*
    - **a.** *perpendicular if and only if* $a_1a_2 + b_1b_2 + c_1c_2 = 0$
    - **b.** *parallel if and only if* $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

Direction ratio of line along the bisector of two given lines

If l_1, m_1 and n_1 and l_2, m_2 and n_2 are the direction cosines of the two lines inclined to each other at an angle θ, then the direction cosines of the

a. internal bisector of the angle between these lines are $\dfrac{l_1 + l_2}{2\cos(\theta/2)}$, $\dfrac{m_1 + m_2}{2\cos(\theta/2)}$ and $\dfrac{n_1 + n_2}{2\cos(\theta/2)}$, and

b. external bisector of the angle between the lines are $\dfrac{l_1 - l_2}{2\sin(\theta/2)}$, $\dfrac{m_1 - m_2}{2\sin(\theta/2)}$ and $\dfrac{n_1 - n_2}{2\sin(\theta/2)}$.

Proof:

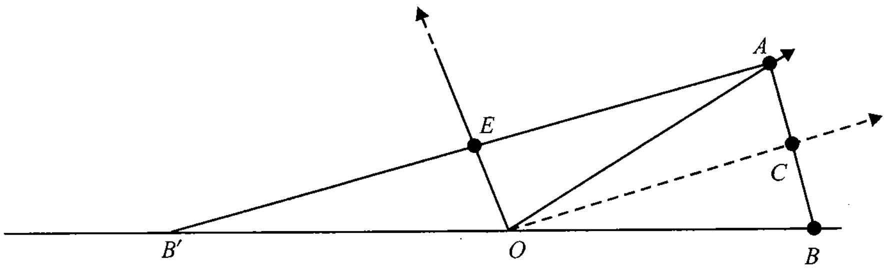


Fig. 3.2

Let OA and OB be two lines with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2. Let $OA = OB = 1$. Then the coordinates of A and B are (l_1, m_1, n_1) and (l_2, m_2, n_2), respectively. Let OC be the bisector of $\angle AOB$. Then C is the midpoint of AB and so its coordinates are

$$\left(\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}, \frac{n_1+n_2}{2}\right)$$

Therefore, the direction ratios of OC are $\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}$ and $\frac{n_1+n_2}{2}$.

We have $OC = \sqrt{\left(\frac{l_1+l_2}{2}\right)^2 + \left(\frac{m_1+m_2}{2}\right)^2 + \left(\frac{n_1+n_2}{2}\right)^2}$

$$= \frac{1}{2}\sqrt{(l_1^2+m_1^2+n_1^2)+(l_2^2+m_2^2+n_2^2)+2(l_1l_2+m_1m_2+n_1n_2)}$$

$$= \frac{1}{2}\sqrt{2+2\cos\theta} \quad (\because \cos\theta = l_1l_2+m_1m_2+n_1n_2)$$

$$= \frac{1}{2}\sqrt{2(1+\cos\theta)} = \cos\left(\frac{\theta}{2}\right)$$

Therefore, the direction cosines of $\overrightarrow{OC}$ are $\frac{l_1+l_2}{2(OC)}, \frac{m_1+m_2}{2(OC)}, \frac{n_1+n_2}{2(OC)}$

or $\frac{l_1+l_2}{2\cos(\theta/2)}, \frac{m_1+m_2}{2\cos(\theta/2)}, \frac{n_1+n_2}{2\cos(\theta/2)}$

In Fig. 3.2, OE is the external bisector.

The coordinates of E are $\frac{l_1-l_2}{2}, \frac{m_1-m_2}{2}$ and $\frac{n_1-n_2}{2}$.

Therefore, direction ratios of OE are $\frac{l_1-l_2}{2}, \frac{m_1-m_2}{2}$ and $\frac{n_1-n_2}{2}$.

Also, $OE = \frac{1}{2}\sqrt{2-2\cos\theta}$

$$= \frac{1}{2}\sqrt{2(1-\cos\theta)}$$

$$= \sin(\theta/2)$$

Therefore, the direction cosines of $\overrightarrow{OE}$ are $\frac{l_1-l_2}{2\sin(\theta/2)}, \frac{m_1-m_2}{2\sin(\theta/2)}$ and $\frac{n_1-n_2}{2\sin(\theta/2)}$.

Example 3.1 **If α, β and γ are the angles which a directed line makes with the positive directions of the co-ordinates axes, then find the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$.**

Sol. The direction cosines of the line are $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.

Since $l^2 + m^2 + n^2 = 1$, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow 1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$

$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Example 3.2 **A line OP through origin O is inclined at 30° and 45° to OX and OY, respectively. Then find the angle at which it is inclined to OZ.**

Sol. Let l, m and n be the direction cosines of the given vector. Then $l^2 + m^2 + n^2 = 1$.

If $l = \cos 30° = \sqrt{3}/2$, $m = \cos 45° = 1/\sqrt{2}$, then $\frac{3}{4} + \frac{1}{2} + n^2 = 1$.

$\Rightarrow n^2 = -1/4$, which is not possible. So, such a line cannot exist.

Example 3.3 ***ABC* is a triangle and $A = (2, 3, 5)$, $B = (-1, 3, 2)$ and $C = (\lambda, 5, \mu)$. If the median through A is equally inclined to the axes, then find the value of λ and μ.**

Sol.

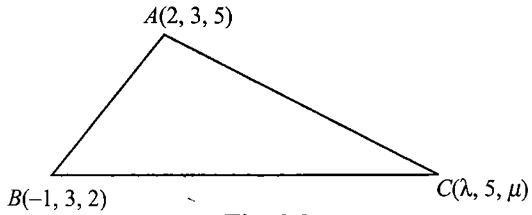


Fig. 3.3

Midpoint of BC is $\left(\frac{\lambda-1}{2}, 4, \frac{2+\mu}{2}\right)$

Direction ratios of the median through A are $\frac{\lambda-1}{2} - 2$, $4 - 3$ and $\frac{2+\mu}{2} - 5$, i.e., $\frac{\lambda-5}{2}$, 1 and $\frac{\mu-8}{2}$.

The median is equally inclined to the axes; so the direction ratios must be equal. Therefore,

$$\frac{\lambda-5}{2} = 1 = \frac{\mu-8}{2} \Rightarrow \lambda = 7, \mu = 10$$

Example 3.4 **A line passes through the points $(6, -7, -1)$ and $(2, -3, 1)$. Find the direction cosines of the line if the line makes an acute angle with the positive direction of the x-axis.**

Sol. Let l, m and n be the direction cosines of the given line. As it makes an acute angle with the x-axis, $l > 0$. The line passes through $(6, -7, -1)$ and $(2, -3, 1)$; therefore, its direction ratios are $(6 - 2, -7 + 3, -1 - 1)$ or $(4, -4, -2)$. Hence the direction cosines of the given line are 2/3, –2/3 and –1/3.

Example 3.5 **Find the ratio in which the y-z plane divides the join of the points $(-2, 4, 7)$ and $(3, -5, 8)$**

Sol. Let the y-z plane divide the join of $P(-2, 4, 7)$ and $Q(3, -5, 8)$ in the ratio $\lambda : 1$.

$\left(\frac{3\lambda-2}{\lambda+1}, \frac{-5\lambda+4}{\lambda+1}, \frac{8\lambda+7}{\lambda+1}\right)$ is in the y-z plane. Then its x-coordinate is zero.

$\frac{3\lambda-2}{\lambda+1} = 0$ or $3\lambda - 2 = 0$

$\therefore \lambda = 2/3$

Example 3.6 **If $A(3, 2, -4)$, $B(5, 4, -6)$ and $C(9, 8, -10)$ are three collinear points, then find the ratio in which point C divides AB.**

Sol. Let C divide AB in the ratio $\lambda : 1$. Then

$$C \equiv \left(\frac{5\lambda+3}{\lambda+1}, \frac{4\lambda+2}{\lambda+1}, \frac{-6\lambda-4}{\lambda+1}\right) = (9, 8 -10)$$

Comparing, $5\lambda + 3 = 9\lambda + 9$ or $4\lambda = -6$

$\therefore\ \lambda = -3/2$

Also, from $4\lambda + 2 = 8\lambda + 8$ and $-6\lambda - 4 = -10\lambda - 10$, we get the same value of λ.

Example 3.7 **If the sum of the squares of the distance of a point from the three coordinate axes is 36, then find its distance from the origin.**

Sol. Let $P(x, y, z)$ be the point. Now under the given condition,

$$[\sqrt{x^2 + y^2}]^2 + [\sqrt{y^2 + z^2}]^2 + [\sqrt{z^2 + x^2}]^2 = 36$$

$\Rightarrow x^2 + y^2 + z^2 = 18$

Then distance from the origin to point (x, y, z) is

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{18} = 3\sqrt{2}$$

Example 3.8 **A line makes angles α, β, γ and δ with the diagonals of a cube; show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 4/3$.**

Sol. The four diagonals of a cube are AL, BM, CN and OP.

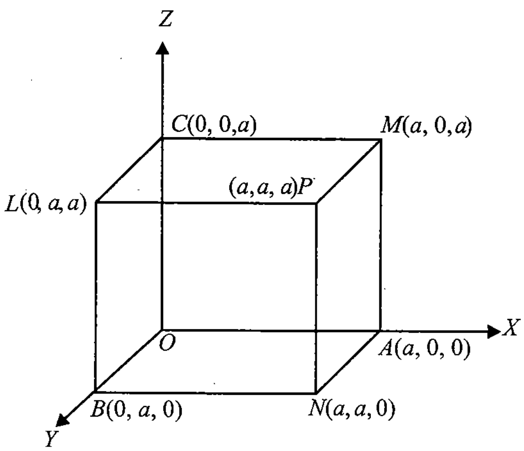


Fig. 3.4

Direction cosines of OP are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

Direction cosines of AL are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

Direction cosines of BM are $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

Direction cosines of CN are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{-1}{\sqrt{3}}$.

Let l, m and n be the direction cosines of a line which is inclined at angles α, β, γ and δ, respectively, to the four diagonals; then

$$\cos\alpha = l\cdot\frac{1}{\sqrt{3}} + m\cdot\frac{1}{\sqrt{3}} + n\cdot\frac{1}{\sqrt{3}}$$

$$= \frac{l+m+n}{\sqrt{3}}$$

Similarly, $\cos\beta = \frac{-l+m+n}{\sqrt{3}}$

$$\cos\gamma = \frac{l-m+n}{\sqrt{3}}$$

$$\cos\delta = \frac{l+m-n}{\sqrt{3}}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{1}{3}[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2]$$

$$= \frac{1}{3}\cdot 4(l^2+m^2+n^2) = \frac{4}{3}$$

Example 3.9 **Find the angle between the lines whose direction cosines are given by $l + m + n = 0$ and $2l^2 + 2m^2 - n^2 = 0$.**

Sol. $l^2 + m^2 + n^2 = 1$ (i)

$l + m + n = 0$ (ii)

$2l^2 + 2m^2 - n^2 = 0$ (iii)

$2(1 - n^2) = n^2 \Rightarrow 3n^2 = 2 \Rightarrow n = \pm\sqrt{2/3}$ (iv)

$2(l^2 + m^2) = n^2 = (-(l+m))^2 \Rightarrow l = m$ (v)

$l + m = \pm\sqrt{2/3} \Rightarrow 2l = \pm\sqrt{2/3}$

$l = \pm 1/\sqrt{6}, m = \pm 1/\sqrt{6}$

Direction cosines are

$$\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right) \text{and} \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}\right)$$

or

$$\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right) \text{and} \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\sqrt{\frac{2}{3}}\right)$$

The angle between these lines in both the cases is $\cos^{-1}\left(-\frac{1}{3}\right)$.

Example 3.10 **A mirror and a source of light are situated at the origin O and at a point on OX, respectively. A ray of light from the source strikes the mirror and is reflected. If the direction ratios of the normal to the plane are 1, –1, 1, then find the DCs of the reflected ray.**

Sol. Let the source of light be situated at $A(a, 0, 0)$, where $a \neq 0$.
Let OA be the incident ray and OB the reflected ray.
ON is the normal to the mirror at O. Therefore,
$\angle AON = \angle NOB = \theta/2$ (say)
Direction ratios of OA are a, 0 and 0 and so its direction cosines are 1, 0 and 0.
Direction ratios of ON are $1/\sqrt{3}, -1/\sqrt{3}$ and $1/\sqrt{3}$. Therefore,
$\angle AON = \angle NOB = (\theta/2)$ (say)
$\cos(\theta/2) = 1/\sqrt{3}$
Let l, m and n be the direction cosines of the reflected ray OB.

$$\frac{l+1}{2\cos(\theta/2)} = \frac{1}{\sqrt{3}}, \frac{m+0}{2\cos(\theta/2)} = -\frac{1}{\sqrt{3}} \text{ and } \frac{n+0}{2\cos(\theta/2)} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow l = \frac{2}{3} - 1, m = \frac{-2}{3}, n = \frac{2}{3}$$

$$\Rightarrow l = -\frac{1}{3}, m = -\frac{2}{3}, n = \frac{2}{3}$$

Concept Application Exercise 3.1

1. If the x-coordinate of a point P on the join of $Q(2, 2, 1)$ and $R(5, 1, -2)$ is 4, then find its z-coordinate.
2. Find the distance of the point $P(a, b, c)$ from the x-axis.
3. If $\vec{r}$ is a vector of magnitude 21 and has direction ratios 2, –3 and 6, then find $\vec{r}$.
4. If $P(x, y, z)$ is a point on the line segment joining $Q(2, 2, 4)$ and $R(3, 5, 6)$ such that the projections of $\overrightarrow{OP}$ on the axes are 13/5, 19/5 and 26/5, respectively, then find the ratio in which P divides QR.
5. If O is the origin, $OP = 3$ with direction ratios –1, 2 and –2, then find the coordinates of P.
6. A line makes angles α, β and γ with the coordinate axes. If $\alpha + \beta = 90°$, then find γ.
7. The line joining the points (–2, 1, –8) and (a, b, c) is parallel to the line whose direction ratios are 6, 2 and 3. Find the values of a, b and c.
8. If a line makes angles α, β and γ with three-dimensional coordinate axes, respectively, then find the value of $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$.
9. A parallelopiped is formed by planes drawn through the points $P(6, 8, 10)$ and $Q(3, 4, 8)$ parallel to the coordinate planes. Find the length of edges and diagonal of the parallolepiped.
10. Find the angle between any two diagonals of a cube.
11. Direction ratios of two lines are a, b, c and $1/bc$, $1/ca$, $1/ab$. Then the lines are_________.
12. Find the angle between the lines whose direction cosines are connected by the relations $l + m + n = 0$ and $2lm + 2nl - mn = 0$.

EQUATION OF STRAIGHT LINE PASSING THROUGH A GIVEN POINT AND PARALLEL TO A GIVEN VECTOR

Vector Form

Line Passing through Point A($\vec{a}$) and Parallel to Vector $\vec{b}$

Let A be the given point and let AP be the given line through A.

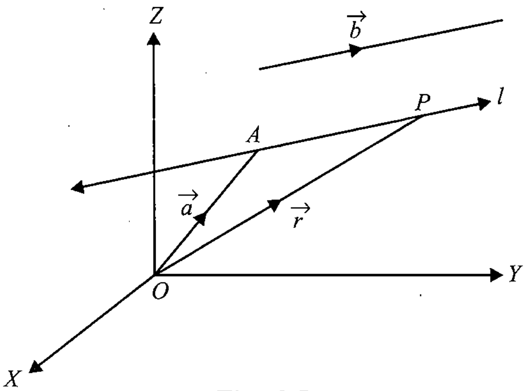


Fig. 3.5

Let $\vec{b}$ be any vector parallel to the given line.
Position vector of point A is $\vec{a}$.
Let P be any point on line AP, and let its position vector be $\vec{r}$.

Then, we have $\vec{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \vec{a} + \lambda\vec{b}$ (where, $\overrightarrow{AP} = \lambda\vec{b}$).

Hence, the vector equation of straight line; $\vec{r} = \vec{a} + \lambda\vec{b}$. (i)

Here, $\vec{r}$ is the position vector of any point $P(x, y, z)$ on the line. So $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

In particular, the equation of straight line through origin and parallel to $\vec{b}$ is $\vec{r} = \lambda\vec{b}$.

Cartesian Form

Let the coordinates of the given point A be (x_1, y_1, z_1) and the direction ratios of the line be a, b and c. Consider the coordinates of any point P be (x, y, z). Then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k};\ \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} \text{ and } \vec{b} = a\hat{i} + b\hat{j} + c\hat{k}.$$

Substituting these values in (i) and equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$x = x_1 + \lambda a;\ y = y_1 + \lambda b;\ z = z_1 + \lambda c$$

These are parametric equations of the line.

Eliminating the parameter λ, we get $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$.

Notes:

1. *Here any point on the line is* $(x, y, z) \equiv (x_1 + \lambda a, y_1 + \lambda b, z_1 + \lambda c)$ *(λ being a parameter).*
2. *Since the x-, y- and z-axes pass through the origin and have direction cosines* (1, 0, 0), (0, 1, 0) *and* (0, 0, 1), *their equations are*

Equation of x-axis : $\frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0}$ *or* $y = 0, z = 0$

Equation of y-axis : $\frac{x-0}{0} = \frac{y-0}{1} = \frac{z-0}{0}$ *or* $x = 0, z = 0$

Equation of z-axis : $\frac{x-0}{0} = \frac{y-0}{0} = \frac{z-0}{1}$ *or* $x = 0, y = 0$

EQUATION OF LINE PASSING THROUGH TWO GIVEN POINTS

Vector Form

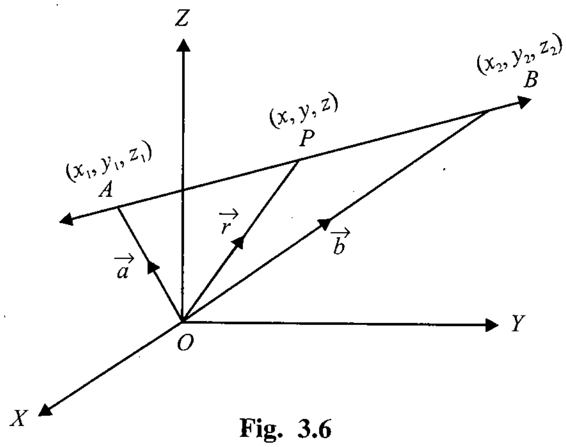


Fig. 3.6

From the figure, $\overrightarrow{OP} = \vec{r}$, $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

Since $\overrightarrow{AP}$ is collinear with $\overrightarrow{AB}$, $\overrightarrow{AP} = \lambda \overrightarrow{AB}$ for some scalar λ.

$\Rightarrow \overrightarrow{OP} - \overrightarrow{OA} = \lambda(\overrightarrow{OB} - \overrightarrow{OA})$

$\Rightarrow \vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$

$\Rightarrow \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$ (i)

Therefore, the equation of a straight line passing through $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

Cartesian Form

We have $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$.

Substituting these values in (i), we get

$$x\hat{i}+y\hat{j}+z\hat{k} = x_1\hat{i}+y_1\hat{j}+z_1\hat{k}+\lambda[(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}]$$

Equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$x=x_1+\lambda(x_2-x_1);\ y=y_1+\lambda(y_2-y_1);\ z=z_1+\lambda(z_2-z_1)$$

On eliminating λ, we obtain $\dfrac{x-x_1}{x_2-x_1}=\dfrac{y-y_1}{y_2-y_1}=\dfrac{z-z_1}{z_2-z_1}=\lambda$

which is the equation of the line in Cartesian form.

Example 3.11 **The Cartesian equation of a line is $\dfrac{x-3}{2}=\dfrac{y+1}{-2}=\dfrac{z-3}{5}$. Find the vector equation of the line.**

Sol. The given line is $\dfrac{x-3}{2}=\dfrac{y+1}{-2}=\dfrac{z-3}{5}$.

Note that it passes through (3, –1, 3) and is parallel to the line whose direction ratios are 2, –2 and 5. Therefore, its vector equation is $\vec{r}=3\hat{i}-\hat{j}+3\hat{k}+\lambda(2\hat{i}-2\hat{j}+5\hat{k})$, where λ is a parameter.

Example 3.12 **The Cartesian equations of a line are $6x-2=3y+1=2z-2$. Find its direction ratios and also find a vector equation of the line.**

Sol. The given line is $6x-2=3y+1=2z-2$ (i)

To put it in the symmetrical form, we must make the coefficients of x, y and z as 1. To do this, we divide each of the expressions in (i) by 6 and obtain $\dfrac{x-(1/3)}{1}=\dfrac{y+(1/3)}{2}=\dfrac{z-1}{3}$.

This shows that the given line passes through (1/3, –1/3, 1) and is parallel to the line whose direction ratios are 1, 2 and 3.

Therefore, its vector equation is $\vec{r}=\frac{1}{3}\hat{i}-\frac{1}{3}\hat{j}+\hat{k}+\lambda(\hat{i}+2\hat{j}+3\hat{k})$.

Example 3.13 **A line passes through the point with position vector $2\hat{i}-3\hat{j}+4\hat{k}$ and is in the direction of $3\hat{i}+4\hat{j}-5\hat{k}$. Find the equations of the line in vector and Cartesian forms.**

Sol. Since the line passes through $2\hat{i}-3\hat{j}+4\hat{k}$ and has direction of $3\hat{i}+4\hat{j}-5\hat{k}$, its vector equation is

$\vec{r}=\hat{a}+\lambda\hat{b} \Rightarrow \vec{r}=2\hat{i}-3\hat{j}+4\hat{k}+\lambda(3\hat{i}+4\hat{j}-5\hat{k})$, where λ is a parameter. (i)

Cartesian equivalent of (i) is $\dfrac{x-2}{3}=\dfrac{y+3}{4}=\dfrac{z-4}{-5}$

Example 3.14 **Find the vector equation of line passing through $A(3, 4, -7)$ and $B(1, -1, 6)$. Also find its cartesian equations.**

Sol. Since the line passes through $A(3\hat{i}+4\hat{j}-7\hat{k})$ and $B(\hat{i}-\hat{j}+6\hat{k})$, its vector equation is

$$\vec{r}=3\hat{i}+4\hat{j}-7\hat{k}+\lambda[(\hat{i}-\hat{j}+6\hat{k})-(3\hat{i}+4\hat{j}-7\hat{k})]$$

or $\vec{r}=3\hat{i}+4\hat{j}-7\hat{k}-\lambda(2\hat{i}+5\hat{j}-13\hat{k})$ (i)

where λ is a parameter.

The Cartesian equivalent of (i) is $\dfrac{x-3}{2}=\dfrac{y-4}{5}=\dfrac{z+7}{-13}$.

Example 3.15 **Find the vector equation of a line passing through (2, –1, 1) and parallel to the line whose equation is $\frac{x-3}{2}=\frac{y+1}{7}=\frac{z-2}{-3}$.**

Sol. Since the required line is parallel to $\frac{x-3}{2}=\frac{y+1}{7}=\frac{z-2}{-3}$, it follows that the required line passing through $A(2\hat{i}-\hat{j}+\hat{k})$ has the direction of $2\hat{i}+7\hat{j}-3\hat{k}$. Hence, the vector equation of the required line is $\vec{r}=2\hat{i}-\hat{j}+\hat{k}+\lambda(2\hat{i}+7\hat{j}-3\hat{k})$ where λ is a parameter.

Example 3.16 **Find the equation of a line which passes through the point (2, 3, 4) and which has equal intercepts on the axes.**

Sol. Since line has equal intercepts on axes, it is equally inclined to axes.

$\Rightarrow$ line is along the vector $a(\hat{i}+\hat{j}+\hat{k})$

$\Rightarrow$ Equation of line is $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{1}$

Example 3.17 **Find the points where line $\frac{x-1}{2}=\frac{y+2}{-1}=\frac{z}{1}$ intersects *xy*, *yz* and *zx* planes.**

Sol. Line meets *xy*-plane where $z=0$

Hence from the given equation of line, $\frac{x-1}{2}=\frac{y+2}{-1}=\frac{0}{1}$

$\Rightarrow$ $x=1$ and $y=-2$.

$\Rightarrow$ Line meets *xy*-plane at $(1, -2, 0)$.

Line meets *yz*-plane where $x=0$

Hence from the given equation of line, $\frac{0-1}{2}=\frac{y+2}{-1}=\frac{z}{1}$

$\Rightarrow$ $z=\frac{-1}{2}$ and $y=-\frac{3}{2}$

$\Rightarrow$ Line meets *yz*-plane at $\left(0, -\frac{3}{2}, \frac{-1}{2}\right)$

Line meets *zx*-plane where $y=0$

Hence from the given equation of line $\frac{x-1}{2}=\frac{0+2}{-1}=\frac{z}{1}$

$\Rightarrow$ $z=-2, x=-3$

$\Rightarrow$ Line meets *zx*-plane at $(-3, 0, -2)$

Example 3.18 **Find the equation of line $x+y-z-3=0=2x+3y+z+4$ in symmetric form. Find the direction ratios of the line.**

Sol. In the section of planes we will see that equation of the form $ax+by+cz+d=0$ is the equation of the plane in the space.

Now equation of line in the form $x+y-z-3=0=2x+3y+z+4$ means set of those points in space which are common to the planes $x+y-z-3=0$ and $2x+3y+z+4=0$, which lie on the line of intersection of planes.

For example, equation of x-axis is $y = z = 0$ where xy-plane ($z = 0$) and xz-plane ($y = 0$) intersect.
Now to get the equation of line in symmetric form , in above equations, first of all we eliminate any one of the variables, say z.
Then adding $x + y - z - 3 = 0$ and $2x + 3y + z + 4 = 0$,
$3x + 4y + 1 = 0$ or $3x = -4y - 1 = \lambda$ (say)

$$\Rightarrow \quad x = \frac{\lambda}{3},\ y = \frac{\lambda + 1}{-4}$$

Putting these values in $x + y - z - 3 = 0$, we have $\frac{\lambda}{3} + \frac{\lambda + 1}{-4} - z - 3 = 0$

$\Rightarrow \quad \lambda = 39 + 12z$

Comparing values of λ, we have equation of line as

$$3x = -4y - 1 = 12z + 39$$

$$\text{or} \quad \frac{3x}{12} = \frac{-4y - 1}{12} = \frac{12z + 39}{12} \quad \text{or} \quad \frac{x}{4} = \frac{y + \frac{1}{4}}{-3} = \frac{z + \frac{13}{4}}{1}$$

Hence the line is passing through point $\left(0, -\frac{1}{4}, -\frac{13}{4}\right)$ and having direction ratios $4, -3, 1$.

If we eliminate x or y first we will get equation of line having same direction ratio but with different point on the line.

Example 3.19 **Find the equation of a line which passes through point $A(1, 0, -1)$ and is perpendicular to the straight lines $\vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \lambda(2\hat{i} + 7\hat{j} - 3\hat{k})$ and $\vec{r} = 3\hat{i} - \hat{j} + 3\hat{k} + \lambda(2\hat{i} - 2\hat{j} + 5\hat{k})$.**

Sol. Since the line to be determined is perpendicular to the given two straight lines, it is directed towards vector

$$(2\hat{i} + 7\hat{j} - 3\hat{k}) \times (2\hat{i} - 2\hat{j} + 5\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 7 & -3 \\ 2 & -2 & 5 \end{vmatrix} = 29\hat{i} - 16\hat{j} - 18\hat{k}$$

Hence, the equation of the line passing through point $A(1, 0, -1)$ and along vector $29\hat{i} - 16\hat{j} - 18\hat{k}$ is $\frac{x - 1}{29} = \frac{y}{-16} = \frac{z + 1}{-18}$.

Example 3.20 **Find the coordinates of a point on the line $\frac{x - 1}{2} = \frac{y + 1}{-3} = z$ at a distance $4\sqrt{14}$ from the point $(1, -1, 0)$.**

Sol. Any point on the given line is $(2r + 1, -3r - 1, r)$, its distance from $(1, -1, 0)$

$\Rightarrow (2r)^2 + (-3r)^2 + r^2 = \left(4\sqrt{14}\right)^2$

$\Rightarrow r = \pm 4$

$\Rightarrow$ Coordinates are $(9, -13, 4)$ and $(-7, 11, -4)$ and the point nearer to the origin is $(-7, 11, -4)$.

ANGLE BETWEEN TWO LINES

Let the given lines be

$$\left.\begin{aligned}\vec{r} &= \vec{a}+\lambda\vec{b} \quad &\text{(i)}\\ \vec{r} &= \vec{a}'+\lambda\vec{b}' \quad &\text{(ii)}\end{aligned}\right\}\rightarrow \text{Vector form}$$

$$\left.\begin{aligned}\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3}\\ \frac{x-a_1'}{b_1'}=\frac{y-a_2'}{b_2'}=\frac{z-a_3'}{b_3'}\end{aligned}\right\}\rightarrow \text{Cartesian form}$$

Clearly (i) and (ii) are straight lines in the directions of $\vec{b}$ and $\vec{b}'$, respectively.
Let θ be the angle between the straight lines (i) and (ii).
Then θ is the angle between vectors $\vec{b}$ and $\vec{b}'$. Therefore,

$$\cos\theta = \frac{\vec{b}\cdot\vec{b}'}{|\vec{b}||\vec{b}'|}$$

$$\vec{b} = b\hat{i}+b_2\hat{j}+b_3\hat{k},\ \vec{b}' = b_1'\hat{i}+b_2'\hat{j}+b_3'\hat{k}$$

$$\therefore\ \vec{b}\cdot\vec{b}' = b_1b_1'+b_2b_2'+b_3b_3'$$

and $|\vec{b}| = \sqrt{b_1^2+b_2^2+b_3^3}$, $|\vec{b}'| = \sqrt{b_1'^2+b_2'^2+b_3'^2}$

$$\Rightarrow \cos\theta = \frac{b_1b_1'+b_2b_2'+b_3b_3'}{\sqrt{b_1^2+b_2^2+b_3^3}\sqrt{b_1'^2+b_2'^2+b_3'^2}}$$

Notes:

1. *If the lines are perpendicular, then* $\vec{b}\cdot\vec{b}'=0 \Rightarrow b_1b_1'+b_2b_2'+b_3b_3'=0$.
2. *If the lines are parallel, then* $\vec{b}=\lambda\vec{b}'$ *for some scalar* $\lambda \Rightarrow \frac{b_1}{b_2'}=\frac{b_2}{b_2'}=\frac{b_3}{b_3'}$.

Example 3.21 **Find the angle between each of the following pairs of lines:**

i. $\vec{r}=3\hat{i}+2\hat{j}-4\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k}); \vec{r}=5\hat{i}-2\hat{k}+\mu(3\hat{i}+2\hat{j}+6\hat{k})$, **where λ and μ are parameters.**

ii. $\frac{x+4}{3}=\frac{y-1}{5}=\frac{z+3}{4};\ \frac{x+1}{1}=\frac{y-4}{1}=\frac{z-5}{2}$

Sol. **i.** Lines are along vectors, $\vec{b}_1 = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} + 2\hat{j} + 6\hat{k}$

If θ is the angle between the two given lines, then

$$\cos\theta = \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|} = \frac{(1)(3)+(2)(2)+(2)(6)}{\sqrt{1^2+2^2+2^2}\sqrt{3^2+2^2+6^2}} = \frac{19}{(3)(7)} = \frac{19}{21} \Rightarrow \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

ii. Lines are along vectors $\vec{b}_1 = 3\hat{i} + 5\hat{j} + 4\hat{k}$ and $\vec{b}_2 = \hat{i} + \hat{j} + 2\hat{k}$

If θ is the angle between the two given lines, then

$$\cos\theta = \frac{(3)(1)+(5)(1)+(4)(2)}{\sqrt{3^2+5^2+4^2}\sqrt{1^2+1^2+2^2}} = \frac{3+5+8}{\sqrt{9+25+16}\sqrt{1+1+4}}$$

$$= \frac{16}{5\sqrt{2}\sqrt{6}} = \frac{16}{5\sqrt{2}\sqrt{2}\sqrt{3}} = \frac{8\sqrt{3}}{15} \Rightarrow \theta = \cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$$

Example 3.22 **Find the condition if lines $x = ay + b, z = cy + d$ and $x = a'y + b', z = c'y + d'$ are perpendicular.**

Sol. The equations of straight lines can be rewritten as

$$x = ay + b,\ z = cy + d \Rightarrow \frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c}$$

and $x = a'y + b',\ z = c'y + d' \Rightarrow \dfrac{x-b'}{a'} = \dfrac{y-0}{1} = \dfrac{z-d'}{c'}$

The above lines are perpendicular if $aa' + 1\cdot 1 + c\cdot c' = 0$.

PERPENDICULAR DISTANCE OF A POINT FROM A LINE

Foot of Perpendicular from a Point on the Given Line

Cartesian form

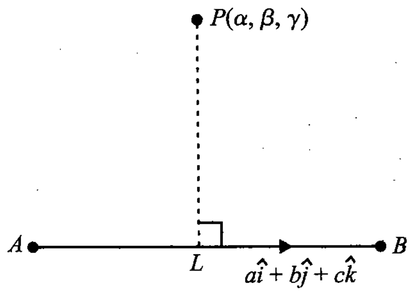


Fig. 3.7

Here, the equation of line AB is $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}$.

Let L be the foot of the perpendicular drawn from $P(\alpha, \beta, \gamma)$ on the line $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}$.

Let the coordiantes of L be $(x_1+a\lambda, y_1+b\lambda, z_1+c\lambda)$.

Then the direction ratios of PL are $(x_1+a\lambda-\alpha, y_1+b\lambda-\beta, z_1+c\lambda-\gamma)$.

Direction ratios of AB are (a, b, c).

Since PL is perpendicular to AB,

$$a(x_1+a\lambda-\alpha)+b(y_1+b\lambda-\beta)+c(z_1+c\lambda-\gamma)=0$$

$$\lambda=\frac{a(\alpha-x_1)+b(\beta-y_1)+c(\gamma-z_1)}{a^2+b^2+c^2}$$

Putting the value of λ in $(x_1+a\lambda, y_1+b\lambda, z_1+c\lambda)$, we get the foot of the perpendicular. Now we can get distance PL using distance formula.

Vector form

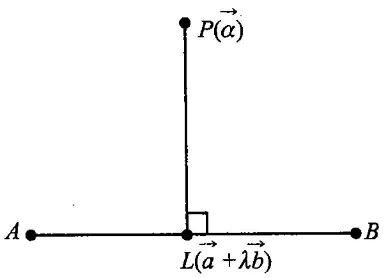


Fig. 3.8

Let L be the foot of the perpendicular drawn from $P(\vec{\alpha})$ on the line $\vec{r}=\vec{a}+\lambda\vec{b}$.

Since $\vec{r}$ denotes the position vector of any point on the line $\vec{r}=\vec{a}+\lambda\vec{b}$, the position vector of L will be $(\vec{a}+\lambda\vec{b})$

Directions ratios of $PL=\vec{a}-\vec{\alpha}+\lambda\vec{b}$

Since $\overrightarrow{PL}$ is perpendicular to $\vec{b}$,

$$(\vec{a}-\vec{\alpha}+\lambda\vec{b})\cdot\vec{b}=0$$

$$\Rightarrow (\vec{a}-\vec{\alpha})\cdot\vec{b}+\lambda\vec{b}\cdot\vec{b}=0$$

$$\Rightarrow \lambda=\frac{-(\vec{a}-\vec{\alpha})\cdot\vec{b}}{|\vec{b}|^2}$$

$\Rightarrow$ Position vector of L is $\vec{a}-\left(\dfrac{(\vec{a}-\vec{\alpha})\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$, which is the foot of the perpendicular.

Image of a Point in the Given Line

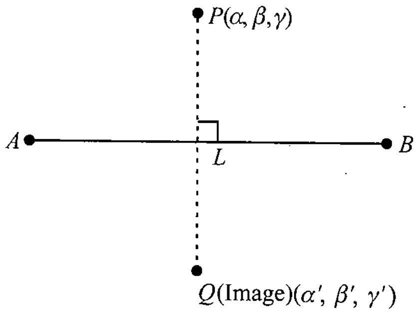


Fig. 3.9

Since L (foot of perpendicular) is the midpoint of P and Q (image of a point P in the line), we can get Q if L is found out.

Example 3.23 **Find the coordinates of the foot of the perpendicular drawn from point $A(1, 0, 3)$ to the join of points $B(4, 7, 1)$ and $C(3, 5, 3)$.**

Sol. Let D be the foot of the perpendicular and let it divide BC in the ratio $\lambda : 1$. Then the coordinates of D are $\frac{3\lambda+4}{\lambda+1}, \frac{5\lambda+7}{\lambda+1}$ and $\frac{3\lambda+1}{\lambda+1}$.

Now, $\overrightarrow{AD} \perp \overrightarrow{BC} \Rightarrow \overrightarrow{AD} \cdot \overrightarrow{BC} = 0$

$\Rightarrow (2\lambda + 3) + 2(5\lambda + 7) + 4 = 0$

$\Rightarrow \lambda = -\frac{7}{4}$

$\Rightarrow$ Coordinates of D are $\frac{5}{3}, \frac{7}{3}$ and $\frac{17}{3}$

Example 3.24 **Find the length of the perpendicular drawn from point $(2, 3, 4)$ to line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$.**

Sol. Let P be the foot of the perpendicular from $A(2, 3, 4)$ to the given line l whose equation is

$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$ or $\frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3} = k$ (say). Therefore, (i)

$x = 4 - 2k, y = 6k, z = 1 - 3k$

As P lies on (i), coordinates of P are $(4 - 2k, 6k, 1 - 3k)$ for some value of k.

The direction ratios of AP are

$(4 - 2k - 2, 6k - 3, 1 - 3k - 4)$ or $(2 - 2k, 6k - 3, -3 - 3k)$.

Also, the direction ratios of l are -2, 6 and -3.

Since $AP \perp l$,

$\Rightarrow -2(2 - 2k) + 6(6k - 3) - 3(-3 - 3k) = 0$

$\Rightarrow -4 + 4k + 36k - 18 + 9 + 9k = 0$ or $49k - 13 = 0$ or $k = 13/49$

We have $AP^2 = (4-2k-2)^2 + (6k-3)^2 + (1-3k-4)^2$

$= (2-2k)^2 + (6k-3)^2 + (-3-3k)^2$

$= 4 - 8k + 4k^2 + 36k^2 - 36k + 9 + 9 + 18k + 9k^2$

$= 22 - 26k + 49k^2$

$$= 22 - 26\left(\frac{13}{49}\right) + 49\left(\frac{13}{49}\right)^2$$

$$= \frac{22\times 49 - 26\times 13 + 13^2}{49} = \frac{909}{49}$$

$$AP = \frac{3}{7}\sqrt{101}$$

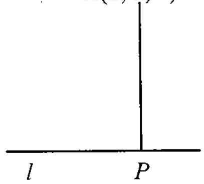


Fig. 3.10

SHORTEST DISTANCE BETWEEN TWO LINES

If two lines in space intersect at a point, then the shortest distance between them is zero. Also, if two lines in space are parallel, then the shortest distance between them will be the perpendicular distance, i.e., the length of the perpendicular drawn from any point on one line onto the other line. Further, in a space, there are lines which are neither intersecting nor parallel. In fact, such pair of lines are *non-coplanar* and are called *skew lines.*

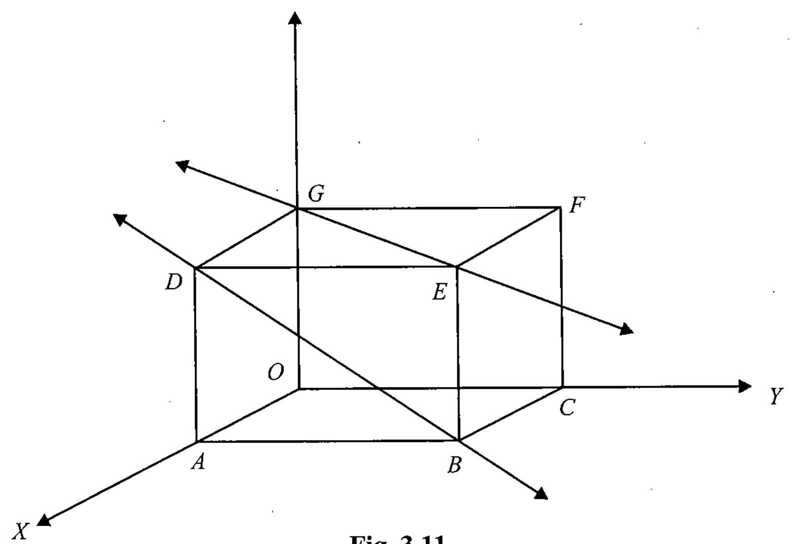


Fig. 3.11

Line GE goes diagonally across the ceiling and line DB passes through one corner of the ceiling directly above A and goes diagonally down the wall. These lines are skew because they are not parallel and also never meet.

By the shortest distance between two lines, we mean the join of a point in one line with one point on the other line so that the length of the segment so obtained is the smallest.

For skew lines, the line of the shortest distance will be perpendicular to both the lines.

Shortest Distance between Two Non-Coplanar Lines

Vector form

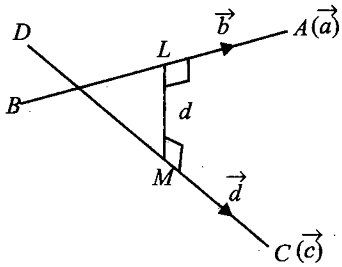


Fig. 3.12

Let the given lines be $\vec{r} = \vec{a} + t\,\vec{b}$ and $\vec{r} = \vec{c} + t_1\vec{d}$.

If two lines AB and CD do not intersect, there is always a line intersecting both the lines perpendicularly. The intercept on this line made by AB and CD is called the shortest distance between lines AB and CD. In Fig. 3.12, the shortest distance $= LM$, where $\angle ALM = \angle CML = 90°$. In the figure, the shortest distance $LM = |$projection of $\overrightarrow{AC}$ along $\overrightarrow{ML}|$

$$= \left|\overrightarrow{AC}\cdot\frac{\overrightarrow{ML}}{|\overrightarrow{ML}|}\right| = \frac{|(\vec{a}-\vec{c})\cdot\overrightarrow{LM}|}{|\overrightarrow{LM}|}$$

Now $\overrightarrow{LM}$ is perpendicular to both $\vec{b}$ and $\vec{d}$.

$$\Rightarrow \overrightarrow{LM} = \vec{b}\times\vec{d}$$

$$= \frac{|(\vec{a}-\vec{c})\cdot(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}$$

$$= \frac{|[\vec{b}\ \vec{d}\ \vec{a}-\vec{c}]|}{|\vec{b}\times\vec{d}|}$$

Cartesian form

Let the two skew lines be $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$ and $\frac{x-c_1}{d_1} = \frac{y-c_2}{d_2} = \frac{z-c_3}{d_3}$

$$\text{Then the shortest distance} = \frac{\left\|\begin{matrix} c_1-a_1 & c_2-a_2 & c_3-a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{matrix}\right\|}{\left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{matrix}\right\|}$$

Condition for Lines to Intersect

Two lines $\vec{r} = \vec{a} + t\vec{b}$ and $\vec{r} = \vec{c} + t_1\vec{d}$ are intersecting if

$$\left| \frac{(\vec{a} - \vec{c}) \cdot (\vec{b} - \vec{d})}{\vec{b} \times \vec{d}} \right| = 0$$

$$\Rightarrow \begin{vmatrix} c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

Distance Between Two Parallel Lines

If two lines l_1 and l_2 are parallel, then they are coplanar. Let the lines be given by

$$\vec{r} = \vec{a}_1 + \lambda\vec{b} \quad \text{(i)}$$

$$\vec{r} = \vec{a}_2 + \mu\vec{b} \quad \text{(ii)}$$

where $\vec{a}_1$ is the position vector of a point S on l_1 and $\vec{a}_2$ is the position vector of a point T on l_2.

As l_1 and l_2 are coplanar, if the foot of the perpendicular from T on line l_1 is P, then the distance between the lines l_1 and $l_2 = |TP|$.

Let θ be the angle between vectors $\overrightarrow{ST}$ and $\vec{b}$.

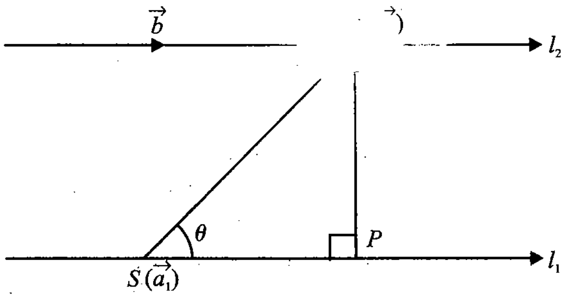


Fig. 3.13

Then $\vec{b} \times \overrightarrow{ST} = (|\vec{b}||\overrightarrow{ST}|\sin\theta)\hat{n}$ (iii)

where $\hat{n}$ is the unit vector perpendicular to the plane of the lines l_1 and l_2.

But $\overrightarrow{ST} = \vec{a}_2 - \vec{a}_1$

Therefore, from (iii), we get

$\vec{b} \times (\vec{a}_2 - \vec{a}_1) = |\vec{b}|\, PT\, \hat{n}$ (since $PT = ST\sin\theta$)

i.e., $|\vec{b} \times (\vec{a}_2 - \vec{a}_1)| = |\vec{b}|\, PT \cdot 1$ (as $|\hat{n}| = 1$)

Hence, the distance between the given parallel lines is

$$d = |\overrightarrow{PT}| = \left|\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{|\vec{b}|}\right|$$

Example 3.25 **Find the shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$. Also obtain the equation of the line of the shortest distance.**

Sol. (i) The two given lines are $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=r_1$ (say) (i)

and $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}=r_2$ (say) (ii)

Any point on (i) is given by $P(2r_1+1, 3r_1+2, 4r_1+3)$

And any point on (ii) is given by $Q(3r_2+2, 4r_2+4, 5r_2+5)$

Direction ratios of PQ are given by $3r_2-2r_1+1$, $4r_2-3r_1+2$ and $5r_2-4r_1+2$

Since PQ is perpendicular to (i), we get

$2(3r_2-2r_1+1)+3(4r_2-3r_1+2)+4(5r_2-4r_1+2)=0$

or $38r_2-29r_1+16=0$ (iii)

Also PQ is perpendicular to (ii), we get

$3(3r_2-2r_1+1)+4(4r_2-3r_1+2)+5(5r_2-4r_1+2)=0$

or $50r_2-38r_1+21=0$ (iv)

Solving (iii) and (iv), we obtain $r_2=-(1/6)$, $r_1=(1/3)$.

Therefore, coordinates of P and Q are $\left(\frac{5}{3}, 3, \frac{13}{3}\right)$ and $\left(\frac{3}{2}, \frac{10}{3}, \frac{25}{6}\right)$, respectively.

Thus, $PQ^2=\left(\frac{3}{2}-\frac{5}{3}\right)^2+\left(\frac{10}{3}-3\right)^2+\left(\frac{25}{6}-\frac{13}{3}\right)^2=\left(-\frac{1}{6}\right)^2+\left(\frac{1}{3}\right)^2+\left(-\frac{1}{6}\right)^2=\frac{1}{6}$

$\Rightarrow \quad PQ=\frac{1}{\sqrt{6}}$

The equation of the line of the shortest distance is given by

$$\frac{x-(5/3)}{(3/2)-(5/3)}=\frac{y-3}{(10/3)-3}=\frac{z-(13/3)}{(25/6)-(13/3)}$$

$$\frac{x-(5/3)}{-(1/6)}=\frac{y-3}{(1/3)}=\frac{z-(13/3)}{-(1/6)}$$

$$\frac{x-(5/3)}{1}=\frac{y-3}{-2}=\frac{z-(13/3)}{1}$$

Alternative method for finding the shortest distance:

Line (i) is passing through the point $(x_1, y_1, z_1)\equiv(1, 2, 3)$ and is parallel to vector

$a_1\hat{i} + b_1\hat{j} + c_1\hat{k} \equiv 2\hat{i} + 3\hat{j} + 4\hat{k}$.

Line (ii) is passing through the point $(x_2, y_2, z_2) \equiv (2, 4, 5)$ and is parallel to the vector

$a_2\hat{i} + b_2\hat{j} + c_2\hat{k} \equiv 3\hat{i} + 4\hat{j} + 5\hat{k}$.

Hence the shortest distance between the lines using the formula

$$\frac{\left|\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}\right|}{\left|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}\right|} \text{ is } \frac{\left|\begin{vmatrix} 2-1 & 4-2 & 5-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}\right|}{\left|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}\right|} = \frac{1}{\sqrt{6}}$$

Example 3.26 **Determine whether the following pair of lines intersect or not.**

i. $\vec{r} = \hat{i} - \hat{j} + \lambda(2\hat{i} + \hat{k}); \vec{r} = 2\hat{i} - \hat{j} + \mu(\hat{i} + \hat{j} - \hat{k})$

ii. $\vec{r} = \hat{i} + \hat{j} - \hat{k} + \lambda(3\hat{i} - \hat{j}); \vec{r} = 4\hat{i} - \hat{k} + \mu(2\hat{i} + 3\hat{k})$

Sol. **i.** Here $\vec{a_1} = \hat{i} - \hat{j}$, $\vec{a_2} = 2\hat{i} - \hat{j}$, $\vec{b_1} = 2\hat{i} + \hat{k}$ and $\vec{b_2} = \hat{i} + \hat{j} - \hat{k}$

$$\text{Now } [\vec{a_2} - \vec{a_1}\ \vec{b_1}\ \vec{b_2}] = \begin{vmatrix} 2-1 & -1+1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1 \neq 0$$

Thus, the two given lines do not intersect.

ii. Here $\vec{a_1} = \hat{i} + \hat{j} - \hat{k}$, $\vec{a_2} = 4\hat{i} - \hat{k}$, $\vec{b_1} = 3\hat{i} - \hat{j}$ and $\vec{b_2} = 2\hat{i} + 3\hat{k}$

$$\Rightarrow [\vec{a_2} - \vec{a_1}\ \vec{b_1}\ \vec{b_2}] = \begin{vmatrix} 4-1 & 0-1 & -1+1 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = 0$$

Thus, the two given lines intersect. Let us obtain the point of intersection of the two given lines.
For some values of λ and μ, the two values of $\vec{r}$ must coincide.

Thus, $\hat{i} + \hat{j} - \hat{k} + \lambda(3\hat{i} - \hat{j}) = 4\hat{i} - \hat{k} + \mu(2\hat{i} + 3\hat{k})$

$\Rightarrow (3+2\mu-3\lambda)\hat{i}+(\lambda-1)\hat{j}+3\mu\hat{k}=0$

$\Rightarrow 3+2\mu-3\lambda=0, \lambda-1=0, 3\mu=0$

Solving, we obtain $\lambda = 1$ and $\mu = 0$

Therefore, the point of intersection is $\vec{r}=4\hat{i}-\hat{k}$ (by putting $\mu = 0$ in the second equation).

Example 3.27 **Find the shortest distance between lines $\vec{r}=(\hat{i}+2\hat{j}+\hat{k}) + \lambda(2\hat{i}+\hat{j}+2\hat{k})$ and $\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu(2\hat{i}+\hat{j}+2\hat{k})$.**

Sol. Here lines (i) ane (ii) are passing through the points $\vec{a}_1=\hat{i}+2\hat{j}+\hat{k}$ and $\vec{a}_2=2\hat{i}-\hat{j}-\hat{k}$, respectively, and are parallel to the vector $\vec{b}=2\hat{i}+\hat{j}+2\hat{k}$.

Hence, the distance between the lines using the formula

$$\frac{|\vec{b}\times(\vec{a}_2-\vec{a}_1)|}{|\vec{b}|}=\frac{\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & 2\\ 1 & -3 & -2\end{vmatrix}}{3}=\frac{|4\hat{i}-6\hat{j}-7\hat{k}|}{3}=\frac{\sqrt{16+36+49}}{3}=\frac{\sqrt{101}}{3}$$

Example 3.28 **If the straight lines $x = -1 + s$, $y = 3 - \lambda s$, $z = 1 + \lambda s$ and $x = \frac{t}{2}$, $y = 1 + t$, $z = 2 - t$, with parameters s and t, respectively, are coplanar, then find λ.**

Sol. The given lines $\frac{x+1}{1}=\frac{y-3}{-\lambda}=\frac{z-1}{\lambda}=s$ and

$\frac{x-0}{1/2}=\frac{y-1}{1}=\frac{y-2}{-1}=t$ are coplanar if $\begin{vmatrix}0+1 & 1-3 & 2-1\\ 1 & -\lambda & \lambda\\ 1/2 & 1 & -1\end{vmatrix}=0$

$$\begin{vmatrix}1 & -2 & 1\\ 1 & -\lambda & \lambda\\ 1/2 & 1 & -1\end{vmatrix}=0$$

$$\Rightarrow 1(\lambda-\lambda)+2\left(-1-\frac{\lambda}{2}\right)+1\left(1+\frac{\lambda}{2}\right)=0$$

$$\Rightarrow \lambda=-2$$

Example 3.29 **Find the equation of a line which passes through the point (1, 1, 1) and intersects the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4}$.**

Sol. Any line passing through the point (1, 1, 1) is $\frac{x-1}{a}=\frac{y-1}{b}=\frac{z-1}{c}$ (i)

This line intersects the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$.

If $a:b:c \neq 2:3:4$ and $\begin{vmatrix} 1-1 & 2-1 & 3-1 \\ a & b & c \\ 2 & 3 & 4 \end{vmatrix} = 0$

$\Rightarrow \quad a - 2b + c = 0$ (ii)

Again, line (i) intersects line $\frac{x-(-2)}{1} = \frac{y-3}{2} = \frac{z-(-1)}{4}$.

If $a:b:c \neq 1:2:4$ and $\begin{vmatrix} -2-1 & 3-1 & -1-1 \\ a & b & c \\ 1 & 2 & 4 \end{vmatrix} = 0$

$\Rightarrow 6a + 5b - 4c = 0$ (iii)

From (ii) and (iii) by cross multiplication, we have $\frac{a}{8-5} = \frac{b}{6+4} = \frac{c}{5+12}$

$\Rightarrow \frac{a}{3} = \frac{b}{10} = \frac{c}{17}$

So, the required line is $\frac{x-1}{3} = \frac{y-1}{10} = \frac{z-1}{17}$

Concept Application Exercise 3.2

1. Find the point where line which passes through point (1, 2, 3) and is parallel to line $\vec{r} = \hat{i} - \hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 3\hat{k})$ meets the xy-plane.
2. Find the equation of the line passing through the points (1, 2, 3) and (– 1, 0, 4).
3. Find the equation of the line passing through the point (2, –1, –1) and parallel to the line $-6x - 2 = 3y + 1 = 2z - 2$.
4. Find the equation of the line passing through the point (–1, 2, 3) and perpendicular to the lines $\frac{x}{2} = \frac{y-1}{-3} = \frac{z+2}{-2}$ and $\frac{x+3}{-1} = \frac{y+3}{2} = \frac{z-1}{3}$.
5. Find the equation of the line passing through the intersection of $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ and also through the point (2, 1, – 2).
6. The straight line $\frac{x-3}{3} = \frac{y-2}{1} = \frac{z-1}{0}$ is
   a. parallel to the x-axis **b.** parallel to the y-axis
   c. parallel to the z-axis **d.** perpendicular to the z-axis
7. Find the angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$.
8. If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5}$ are at right angle, then find the value of k.

9. The equations of motion of a rocket are $x = 2t$, $y = -4t$ and $z = 4t$, where time t is given in seconds, and the coordinates of a moving point in kilometres. What is the path of the rocket? At what distance will be the rocket from the starting point $O(0, 0, 0)$ in 10?

10. Find the length of the perpendicular drawn from the point (5, 4, −1) to the line $\vec{r} = \hat{i} + \lambda(2\hat{i} + 9\hat{j} + 5\hat{k})$, where λ is a parameter.

11. Find the image of point (1, 2, 3) in the line $\dfrac{x-6}{3} = \dfrac{y-7}{2} = \dfrac{z-7}{-2}$.

12. Find the shortest distance between the lines $\vec{r} = (1-\lambda)\hat{i} + (\lambda-2)\hat{j} + (3-2\lambda)\hat{k}$ and $r = (\mu+1)\hat{i} + (2\mu-1)\hat{j} - (2\mu+1)\hat{k}$.

13. If the lines $\dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-1}{4}$ and $\dfrac{x-3}{1} = \dfrac{y-k}{2} = \dfrac{z}{1}$ intersect, then find the value of k.

PLANE

A plane is a surface such that if any two points are taken on it, the line segment joining them lies completely on the surface.

A plane is determined uniquely if:

i. The normal to the plane and its distance from the origin is given, i.e., the equation of a plane in normal form.

ii. It passes through a point and is perpendicular to a given direction.

iii. It passes through three given non-collinear points.

Equation of a Plane in Normal Form

Consider a plane whose perpendicular distance from the origin is d ($d \neq 0$). If $\overrightarrow{ON}$ is the normal from the origin to the plane, and $\hat{n}$ is the unit normal vector along $\overrightarrow{ON}$, then $\overrightarrow{ON} = d\hat{n}$. Let P be any point on the plane. Therefore, $\overrightarrow{NP}$ is perpendicular to $\overrightarrow{ON}$.

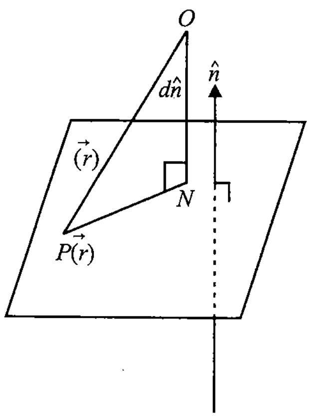


Fig. 3.14

Therefore, $\overrightarrow{NP} \cdot \overrightarrow{ON} = 0$ (i)

Let $\vec{r}$ be the position vector of the point P. Then $\overrightarrow{NP} = \vec{r} - d\hat{n}$ (as $\overrightarrow{ON} + \overrightarrow{NP} = \overrightarrow{OP}$)

Therefore, (i) becomes

$(\vec{r} - d\hat{n})\cdot d\hat{n} = 0$

$\Rightarrow (\vec{r} - d\hat{n})\cdot \hat{n} = 0 (d \neq 0)$

$\Rightarrow (\vec{r}\cdot\hat{n}) - d\hat{n}\cdot\hat{n} = 0$

$\Rightarrow \vec{r}\cdot\hat{n} = d (\text{as } \hat{n}\cdot\hat{n} = 1)$ (ii)

This is the vector form of the equation of the plane.

Cartesian form

Equation (ii) gives the vector equation of a plane, where $\hat{n}$ is the unit vector normal to the plane. Let $P(x, y, z)$ be any point on the plane. Then

$\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Let l, m and n be the direction cosines of $\hat{n}$.

Then $\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$

Therefore, (ii) gives

$(x\hat{i} + y\hat{j} + z\hat{k})\cdot(l\hat{i} + m\hat{j} + n\hat{k}) = d$

or $lx + my + nz = d$ (iii)

This is the Cartesian equation of the plane in the normal form.

Note: *Equation (iii) shows that if $\vec{r}\cdot(a\hat{i} + b\hat{j} + c\hat{k}) = d$ is the vector equation of a plane, then $ax + by + cz = d$ is the Cartesian equation of the plane, where a, b and c are the direction ratios of the normal to the plane.*

Example 3.30 **Find the equation of plane which is at a distance $\frac{4}{\sqrt{14}}$ from the origin and is normal to vector $2\hat{i} + \hat{j} - 3\hat{k}$.**

Sol. Here $\vec{n} = 2\hat{i} + \hat{j} - 3\hat{k}$. Then $\frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} + \hat{j} - 3\hat{k}}{\sqrt{2^2 + 1^2 + (-3)^2}} = \frac{2\hat{i} + \hat{j} - 3\hat{k}}{\sqrt{14}}$

Hence required equation of plane is $\vec{r}\cdot\frac{1}{\sqrt{14}}(2\hat{i} + \hat{j} - 3\hat{k}) = \pm\frac{1}{\sqrt{14}}$

or $\vec{r}\cdot(2\hat{i} + \hat{j} - 3\hat{k}) = \pm 1$ (vector form)

or $2x + y - 3z = \pm 1$ (cartesian form)

Example 3.31 **Find the unit vector perpendicular to the plane $\vec{r}\cdot(2\hat{i} + \hat{j} + 2\hat{k}) = 5$.**

Sol. Vector normal to the plane is $\vec{n} = 2\hat{i} + \hat{j} + 2\hat{k}$

Hence unit vector perpendicular to the plane is $\frac{\vec{n}}{|\vec{n}|}=\frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{2^2+1^2+2^2}}=\frac{1}{3}(2\hat{i}+\hat{j}+2\hat{k})$

Example 3.32 **Find the distance of the plane $2x-y-2z-9=0$ from the origin.**

Sol. The plane can be put in vector form as $\vec{r}\cdot(2\hat{i}-\hat{j}-2\hat{k})=9$ where $\vec{r}=2\hat{i}-\hat{j}-2\hat{k}$.

Here $\vec{n}=2\hat{i}-\hat{j}-2\hat{k}$

$$\Rightarrow \quad \frac{\vec{n}}{|\vec{n}|}=\frac{2\hat{i}-\hat{j}-2\hat{k}}{3}$$

Dividing equation throughout by 3, we have equation of plane in normal form as

$\vec{r}\cdot\frac{(2\hat{i}-\hat{j}-2\hat{k})}{3}=3$, in which 3 is the distance of the plane from the origin.

Example 3.33 **Find the vector equation of a line passing through $3\hat{i}-5\hat{j}+7\hat{k}$ and perpendicular to the plane $3x-4y+5z=8$.**

Sol. The given plane $3x-4y+5z=8$ or $(3\hat{i}-4\hat{j}+5\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})=8$.

This shows that $\vec{d}=3\hat{i}-4\hat{j}+5\hat{k}$ is normal to the given plane.

Therefore, the required line is parallel to $3\hat{i}-4\hat{j}+5\hat{k}$.

Since the required line passes through $3\hat{i}-5\hat{j}+7\hat{k}$, its equation is given by

$\vec{r}=3\hat{i}-5\hat{j}+7\hat{k}+\lambda(3\hat{i}-4\hat{j}+5\hat{k})$, where λ is a parameter.

Vector Equation of a Plane Passing through a Given Point and Normal to a Given Vector

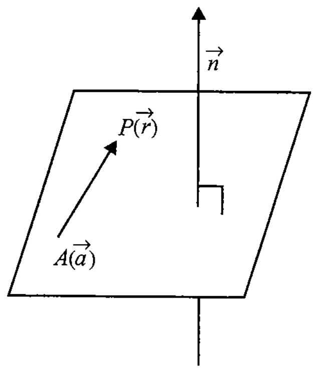


Fig. 3.15

Suppose the plane passes through a point having position vector $\vec{a}$ and is normal to vector $\vec{n}$.

Then for any position of point $P(\vec{r})$ on the plane, $\overrightarrow{AP} \perp \vec{n}$

$\Rightarrow \overrightarrow{AP} \cdot \vec{n} = 0$

$\Rightarrow (\vec{r} - \vec{a}) \cdot \vec{n} = 0 \quad (\because \overrightarrow{AP} = \vec{r} - \vec{a})$

Hence the required equation of the plane is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$.

Note:

The above equation can be written as $\vec{r} \cdot \vec{n} = d$, *where* $d = \vec{a} \cdot \vec{n}$ *(known as scalar product form of plane).*

The equation $\vec{r} \cdot \vec{n} = d$ *is in normal form if* $\vec{n}$ *is a unit vector and d is the distance of the plane from the origin. If* $\vec{n}$ *is not a unit vector, then we reduce the equation* $\vec{r} \cdot \vec{n} = d$ *to the normal form by dividing both sides by* $|\vec{n}|$; we get $\frac{\vec{r} \cdot \vec{n}}{|\vec{n}|} = \frac{d}{|\vec{n}|} \Rightarrow \vec{r} \cdot \hat{n} = \frac{d}{|n|} = p$ (distance from the origin).

Cartesian form

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$, then

$$(\vec{r} - \vec{a}) = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$$

Then equation of the plane can be written as

$$((x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}) \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = 0$$

$$\Rightarrow a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Thus, the coefficients of x, y and z in the Cartesian equation of a plane are the direction ratios of the normal to the plane.

Example 3.34 **Find the equation of the plane passing through the point (2, 3, 1) having (5, 3, 2) as the direction ratios of the normal to the plane.**

Sol. The equation of the plane passing through (x_1, y_1, z_1) and perpendicular to the line with direction ratios a, b and c is given by $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

Now, since the plane passes through (2, 3, 1) and is perpendicular to the line having direction ratios (5, 3, 2), the equation of the plane is given by $5(x - 2) + 3(y - 3) + 2(z - 1) = 0$ or $5x + 3y + 2z = 21$.

Example 3.35 **The foot of the perpendicular drawn from the origin to a plane is (12, – 4, 3). Find the equation of the plane.**

Sol. Since $P(12, -4, 3)$ is the foot of the perpendicular from the origin to the plane, OP is normal to the plane. Thus, the direction ratios of normal to the plane are 12, – 4 and 3.

Now, since the plane passes through (12, –4, 3), its equation is given by

$12(x - 12) - 4(y + 4) + 3(z - 3) = 0$

or $12x - 4y + 3z - 169 = 0$.

Example 3.36 **Find the equation of the plane such that image of point (1, 2, 3) in it is (−1, 0, 1).**

Sol. Since the image of A (1, 2, 3) in the plane is B (−1, 0, 1), the plane passes through the midpoint (0, 1, 2) of AB and is normal to the vector $\overrightarrow{AB} \equiv -2\hat{i} - 2\hat{j} - 2\hat{k}$

Hence, the equation of the plane is $-2(x-0) - 2(y-1) - 2(z-2) = 0$ or $x + y + z = 3$

Equation of a Plane Passing through Three Given Points

Cartesian form

Let the plane be passing through points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$.

Let $P(x, y, z)$ be any point on the plane.

Then vectors $\overrightarrow{PA}$, $\overrightarrow{BA}$ and $\overrightarrow{CA}$ are coplanar.

$[\overrightarrow{PA}\ \overrightarrow{BA}\ \overrightarrow{CA}] = 0$

$$\Rightarrow \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0, \text{ which is the required equation of the plane}$$

Vector form

Vector form of the equation of the plane passing through three points A, B and C having position vectors $\vec{a}, \vec{b}$ and $\vec{c}$, respectively.

Let $\vec{r}$ be the position vector of any point P in the plane.

Hence vectors $\overrightarrow{AP} = \vec{r} - \vec{a}$ $\overrightarrow{AB} = \vec{b} - \vec{a}$ and $\overrightarrow{AC} = \vec{c} - \vec{a}$ are coplanar.

Hence, $(\vec{r} - \vec{a}) \cdot \{(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\} = 0$

$\Rightarrow (\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}) = 0$

$\Rightarrow (\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}) = 0$

$\Rightarrow \vec{r} \cdot (\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}) = \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{a} \times \vec{b}) + \vec{a} \cdot (\vec{c} \times \vec{a})$

$\Rightarrow [\vec{r}\ \vec{b}\ \vec{c}] + [\vec{r}\ \vec{a}\ \vec{b}] + [\vec{r}\ \vec{c}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]$

which is the required equation of the plane.

Notes:

1. *If p is the length of perpendicular from the origin on this plane, then* $p = [\vec{a}\ \vec{b}\ \vec{c}]/n$, *where* $n = |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$.

2. *Four points $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar if $\vec{d}$ lies on the plane containing $\vec{a}, \vec{b}$ and $\vec{c}$.*

$$or\ \vec{d}\cdot[\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]$$

$$or\ [\vec{d}\ \vec{a}\ \vec{b}]+[\vec{d}\ \vec{b}\ \vec{c}]+[\vec{d}\ \vec{c}\ \vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}].$$

Example 3.37 **Find the equation of the plane passing through A (2, 2, –1), B (3, 4, 2) and C (7, 0, 6). Also find a unit vector perpendicular to this plane.**

Sol. Here $(x_1, y_1, z_1) \equiv (2, 2, -1)$, $(x_2, y_2, z_2) \equiv (3, 4, 2)$ and $(x_3, y_3, z_3) \equiv (7, 0, 6)$

Then the equation of the plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0 \text{ or } \begin{vmatrix} x-2 & y-2 & z-(-1) \\ 3-2 & 4-2 & 2-(-1) \\ 7-2 & 0-2 & 6-(-1) \end{vmatrix}=0$$

or

$5x + 2y - 3z = 17$

A normal vector to this plane is $\vec{d}=5\hat{i}+2\hat{j}-3\hat{k}$ (i)

Therefore, a unit vector normal to (i) is given by

$$\hat{n}=\frac{\vec{d}}{|\vec{d}|}=\frac{5\hat{i}+2\hat{j}-3\hat{k}}{\sqrt{25+4+9}}=\frac{1}{\sqrt{38}}(5\hat{i}+2\hat{j}-3\hat{k})$$

Example 3.38 **Show that the line of intersection of the planes $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=0$ and $\vec{r}\cdot(3\hat{i}+2\hat{j}+\hat{k})=0$ is equally inclined to $\hat{i}$ and $\hat{k}$. Also find the angle it makes with $\hat{j}$.**

Sol. The line of intersection of the two planes will be perpendicular to the normals to the planes. Hence it is parallel to the vector $(\hat{i}+2\hat{j}+3\hat{k})\times(3\hat{i}+2\hat{j}+\hat{k})=(-4\hat{i}+8\hat{j}-4\hat{k})$.

Now, $(-4\hat{i}+8\hat{j}-4\hat{k}).\hat{i}=-4$ and $(-4\hat{i}+8\hat{j}-4\hat{k})\cdot\hat{k}=-4$

Hence the line is equally inclined to $\hat{i}$ and $\hat{k}$.

$$\text{Also, } \frac{(-4\hat{i}+8\hat{j}-4\hat{k})}{\sqrt{16+64+16}}.\hat{j}=\frac{8}{\sqrt{96}}=\sqrt{\frac{2}{3}}$$

If θ is the required angle, then $\cos\theta=\sqrt{\frac{2}{3}} \Rightarrow \theta=\cos^{-1}\sqrt{\frac{2}{3}}$

Equation of the Plane that Passes through Point A with Position Vector $\vec{a}$ and is Parallel to Given Vectors $\vec{b}$ and $\vec{c}$

Vector form

Let $\vec{r}$ be the position vector of any point P in the plane. Then

$$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=\vec{r}-\vec{a}$$

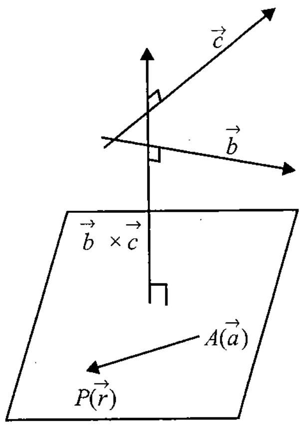


Fig. 3.16

Since vectors $\vec{r} - \vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar,

$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$

$\Rightarrow \vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c}) \Rightarrow [\vec{r}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}]$

which is the required equation of the plane.

Cartesian form

From $(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$, we have $[\vec{r} - \vec{a}\ \vec{b}\ \vec{c}]$

$$\Rightarrow \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$, which is the required equation of the plane,

where $\vec{b} = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}$ and $\vec{c} = x_3 \hat{i} + y_3 \hat{j} + z_3 \hat{k}$.

Example 3.39 **Find the vector equation of the following planes in cartesian form:**
$\vec{r} = \hat{i} - \hat{j} + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k})$.

Sol. The equation of the plane is $\vec{r} = \hat{i} - \hat{j} + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k})$.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Hence, the equation is $(x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - \hat{j}) = \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k})$

Thus vectors $(x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - \hat{j}), \hat{i} + \hat{j} + \hat{k}, \hat{i} - 2\hat{j} + 3\hat{k}$ are coplanar.

Therefore, the equation of the plane is $\begin{vmatrix} x - 1 & y - (-1) & z - 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 0$ or $5x - 2y - 3z - 7 = 0$

Equation of a Plane Passing through a Given Point and Line

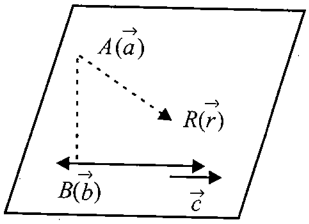


Fig. 3.17

Let the plane pass through a given point $A\ (\vec{a})$ and line $\vec{r} = \vec{b} + \lambda\, \vec{c}$.

For any position of point $R\ (\vec{r})$ on the plane, vectors $\overrightarrow{AB}$, $\overrightarrow{RA}$ and $\vec{c}$ are coplanar. Then $[\vec{r} - \vec{a}\ \ \vec{b} - \vec{a}\ \ \vec{c}] = 0$, which is required equation of the plane.

Example 3.40 **Prove that the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 3$ contains the line $\vec{r} = \hat{i} + \hat{j} + \lambda\,(2\hat{i} + \hat{j} + 4\hat{k})$.**

Sol. To show that $\vec{r} = \hat{i} + \hat{j} + \lambda\,(2\hat{i} + \hat{j} + 4\hat{k})$ (i)

lies in the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 3$, (ii)

we must show that each point of (i) lies in (ii). In other words, we must show that $\vec{r}$ in (i) satisfies (ii) for every value of λ.

We have $[\hat{i} + \hat{j} + \lambda\,(2\hat{i} + \hat{j} + 4\hat{k})] \cdot (\hat{i} + 2\hat{j} - \hat{k})$

$$= (\hat{i} + \hat{j}) \cdot (\hat{i} + 2\hat{j} - \hat{k}) + \lambda\,(2\hat{i} + \hat{j} + 4\hat{k}) \cdot (\hat{i} + 2\hat{j} - \hat{k})$$

$$= (1)\,(1) + (1)\,(2) + \lambda\,[(2)\,(1) + (1)\,(2) + 4\,(-1)] = 3 + \lambda\,(0) = 3$$

Hence line (i) lies in plane (ii).

Example 3.41 **Find the equation of the plane which is parallel to the lines $\vec{r} = \hat{i} + \hat{j} + \lambda\,(2\hat{i} + \hat{j} + 4\hat{k})$ and $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z+2}{1}$ and is passing through the point (0, 1, –1).**

Sol. The plane is parallel to the given lines, which are directed along vectors $\vec{a} = 2\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = -3\hat{i} + 2\hat{j} + 1\hat{k}$.

Then the plane is normal to vector $\vec{a} \times \vec{b} \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 4 \\ -3 & 2 & 1 \end{vmatrix} = -7\hat{i} - 14\hat{j} + 7\hat{k}$

Also, the plane passes through the point (0, 1, –1).

Therefore, the equation of the plane is $-7\,(x-0) - 14\,(y-1) + 7\,(z+1) = 0$ or $7x + 14y - 7z = 21$

Intercept Form of a Plane

Let O be the origin and let OX, OY and OZ be the coordinate axes.

Let the plane meet the coordinate axes at the points A, B and C, respectively, such that

$OA = a$, $OB = b$ and $OC = c$. Then, the coordinates of points are $A\,(a, 0, 0)$, $B\,(0, b, 0)$ and $C\,(0, 0, c)$.

Let the equation of the plane be $Ax + By + Cz + D = 0$ (i)

Since (i) passes through $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, we have

$$Aa + D = 0 \Rightarrow A = \frac{-D}{a},\ Bb + D = 0 \Rightarrow B = \frac{-D}{b},\ Cc + D = 0 \Rightarrow C = \frac{-D}{c}$$

Putting these values in (i), we get the required equation of the plane as

$$\frac{-D}{a}x - \frac{D}{b}y - \frac{D}{c}z = -D \Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Example 3.42 **If a plane meets the coordinate axes at A, B and C such that the centroid of the triangle is (1, 2, 4), then find the equation of the plane.**

Sol. Let the plane meet the coordinate axes at $A\,(a, 0, 0)$, $B\,(0, b, 0)$, and $C\,(0, 0, c)$. Then,
$a = 3, b = 6, c = 12$.
Hence, the equation of required plane is $\frac{x}{3} + \frac{y}{6} + \frac{z}{12} = 1$ or $4x + 2y + z = 12$

Equation of a Plane Passing through Two Parallel Lines

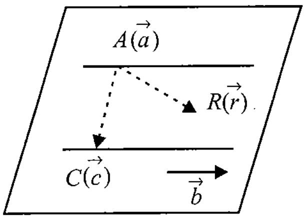


Fig. 3.18

Let the plane pass through parallel lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \mu\vec{b}$.

As shown in the diagram, for any position of R in the plane, vectors $\overrightarrow{RA}$, $\overrightarrow{AC}$ and $\vec{b}$ are coplanar. Then $[\vec{r} - \vec{a}\ \ \vec{c} - \vec{a}\ \ \vec{b}] = 0$, which is the required equation of the plane.

Equation of a Plane Parallel to a Given Plane

The general equation of the plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$, where k is any scalar, as normal to both the planes is $a\hat{i} + b\hat{j} + c\hat{k}$.

Example 3.43 **Find the equation of the plane passing through (3, 4, –1), which is parallel to the plane $\vec{r}\cdot(2\hat{i} - 3\hat{j} + 5\hat{k}) + 7 = 0$.**

Sol. The equation of any plane which is parallel to $\vec{r}\cdot(2\hat{i} - 3\hat{j} + 5\hat{k}) + 7 = 0$ is

$\vec{r} \cdot (2\hat{i} - 3\hat{j} + 5\hat{k}) + \lambda = 0$ (i)

or $2x - 3y + 5z + \lambda = 0$

Further (i) will pass through (3, 4, –1) if (2) (3) + (–3) (4) + 5 (–1) + $\lambda = 0$ or $-11 + \lambda = 0 \Rightarrow \lambda = 11$

Thus equation of the required plane is $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 5\hat{k}) + 11 = 0$.

ANGLE BETWEEN TWO PLANES

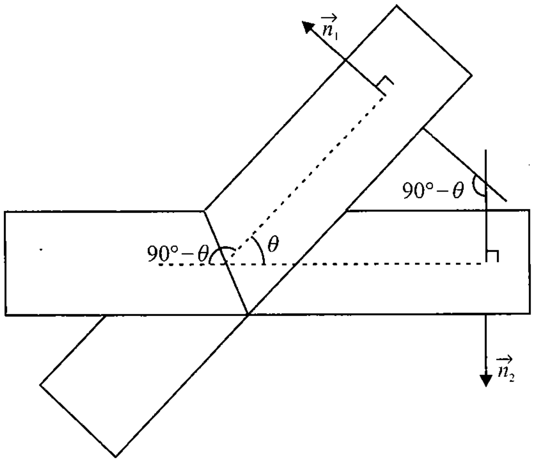


Fig. 3.19

The angle between two planes is defined as the angle between their normals.

Let θ be the angle between planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$

then $\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$

Condition for Perpendicularity

If the planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ are perpendicular, then $\vec{n}_1$ and $\vec{n}_2$ are perpendicular. Therefore,

$\vec{n}_1 \cdot \vec{n}_2 = 0$

Condition for Parallelism

If the planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ are parallel, there exists the scalar λ such that $\vec{n}_1 = \lambda \vec{n}_2$.

Cartesian form

If the planes are $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$

$$\Rightarrow \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Condition for parallelism: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \lambda$

Condition for perpendicularity: $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Example 3.44 **Find the angle between the planes $2x + y - 2x + 3 = 0$ and $\vec{r} \cdot (6\hat{i} + 3\hat{j} + 2\hat{k}) = 5$.**

Sol. Normals along the given planes are $2\hat{i} + \hat{j} - 2\hat{k}$ and $6\hat{i} + 3\hat{j} + 2\hat{k}$

Then angle between planes, $\theta = \cos^{-1}\frac{(2\hat{i} + \hat{j} - 2\hat{k}) \cdot (6\hat{i} + 3\hat{j} + 2\hat{k})}{\sqrt{(2)^2 + (1)^2 + (-2)^2}\sqrt{(6)^2 + (3)^2 + (2)^2}} = \cos^{-1}\frac{11}{21}$

Example 3.45 **Show that $ax + by + r = 0$, $by + cz + p = 0$ and $cz + ax + q = 0$ are perpendicular to x-y, y-z and z-x planes, respectively.**

Sol. The planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular to each other if and only if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

The equation of x-y, y-z and z-x planes are $z = 0$, $x = 0$ and $y = 0$, respectively.

Now we have to show that $z = 0$ is perpendicular to $ax + by + r = 0$.

It follows immediately, since $a(0) + b(0) + (0)(1) = 0$, other parts can be done similarly.

LINE OF INTERSECTION OF TWO PLANES

Let two non-parallel planes are $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$

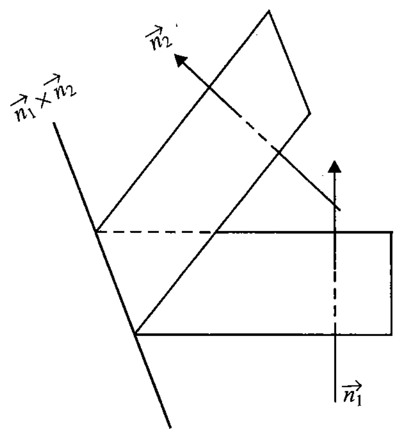


Fig. 3.20

Now line of intersection of planes is perpendicular to vectors $\vec{n}_1$ and $\vec{n}_2$.

$\therefore$ Line of intersection is parallel to vector $\vec{n}_1 \times \vec{n}_2$.

If we wish to find the equation of line of intersection of planes $a_1x + b_1y + c_1z - d_1 = 0$ and $a_2x + b_2y + c_2z - d_2 = 0$, then we find any point on the line by putting $z = 0$ (say), then we can find corresponding values of x and

y by solving equations $a_1x + b_1y - d_1 = 0$ and $a_2x + b_2y - d_2 = 0$. Thus by fixing the value of $z = \lambda$, we can find the corresponding value of x and y in terms of λ. After getting x, y and z in terms of λ, we can find the equation of line in symmetric form.

Example 3.46 **Reduce the equation of line $x - y + 2z = 5$ and $3x + y + z = 6$ in symmetrical form.**

or

Find the line of intersection of planes $x - y + 2z = 5$ and $3x + y + z = 6$.

Sol. Given $x - y + 2z = 5$, $3x + y + z = 6$.

Let $z = \lambda$.

Then $x - y = 5 - 2\lambda$ and $3x + y = 6 - \lambda$.

Solving these two equations, $4x = 11 - 3\lambda$ and $4y = 4x - 20 + 8\lambda = -9 + 5\lambda$.

The equation of the line is $\dfrac{4x-11}{-3} = \dfrac{4y+9}{5} = \dfrac{z-0}{1}$.

Example 3.47 **Find the equation of the plane passing through the points $(-1, 1, 1)$ and $(1, -1, 1)$ and perpendicular to the plane $x + 2y + 2z = 5$.**

Sol. The equation of any plane which passes through $(-1, 1, 1)$ is

$$a(x + 1) + b(y - 1) + c(z - 1) = 0 \qquad \text{(i)}$$

This plane will pass through $(1, -1, 1)$ if

$$2a - 2b = 0 \text{ or } a = b \qquad \text{(ii)}$$

Next, (i) will be perpendicular to $x + 2y + 2z = 5$ if

$$a + 2b + 2c = 0 \qquad \text{(iii)}$$

Using (ii), we can write (iii) as $a + 2a + 2c = 0$ or $c = -3a/2$.

Thus $a : b : c = a : a : \left(\dfrac{-3}{2}\right)a = 2 : 2 : -3$

Putting these values in (i), we get $2(x + 1) + 2(y - 1) - 3(z - 1) = 0$

or $2x + 2y - 3z + 3 = 0$, which is the equation of the required plane.

Alternative method:

The plane is passing through the points $A(-1, 1, 1)$ and $B(1, -1, 1)$.

Let any point on the plane be $P(x, y, z)$.

Then vector $\overrightarrow{AP} \times \overrightarrow{AB}$ is perpendicular to vector $\hat{i} + 2\hat{j} + 2\hat{k}$, which is normal to the plane $x + 2y + 2z = 5$.

Hence, the equation of the plane is $\begin{vmatrix} x-(-1) & y-1 & z-1 \\ 1-(-1) & -1-1 & 1-1 \\ 1 & 2 & 2 \end{vmatrix} = 0$ or $2x + 2y - 3z + 3 = 0$

Example 3.48 **Find the equation of the plane containing line $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z+2}{1}$ and point $(0, 7, -7)$.**

Sol. The equation of the plane containing line $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z+2}{1}$ is

$$a(x + 1) + b(y - 3) + c(z + 2) = 0, \qquad \text{(i)}$$

where $-3a + 2b + c = 0 \qquad \text{(ii)}$

This passes through (0, 7, – 7).

$\therefore a+4b-5c=0.$ (iii)

From (ii) and (iii), $\frac{a}{-14}=\frac{b}{-14}=\frac{c}{-14}$ or $\frac{a}{1}=\frac{b}{1}=\frac{c}{1}$

So, the required plane is $x+y+z=0$.

Example 3.49 **Find the distance of the point P (3, 8, 2) from the line $\frac{1}{2}(x-1)=\frac{1}{4}(y-3)=\frac{1}{3}(z-2)$ measured parallel to the plane $3x+2y-2z+15=0$.**

Sol.

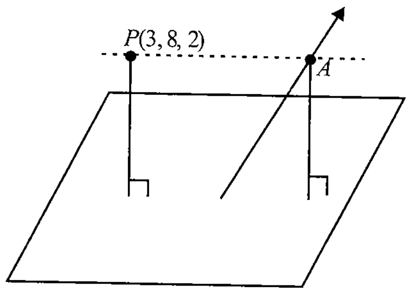


Fig. 3.21

Let the general point of the line be $A\ (2\lambda+1, 4\lambda+3, 3\lambda+2)$.

Let this point lie on the line such that AP is parallel to the plane

$\Rightarrow \overrightarrow{AP}\perp(3\hat{i}+2\hat{j}-2\hat{k})$

$\Rightarrow 3\cdot(2\lambda-2)+2(4\lambda-5)-2(3\lambda)=0$

$\Rightarrow \lambda=2$

Therefore, A is (5, 11, 8).

$PA=\sqrt{(5-3)^2+(11-8)^2+(8-2)^2}=\sqrt{4+9+36}=7$

Example 3.50 **Find the distance of the point (1, 0, –3) from the plane $x-y-z=9$ measured parallel to the line $\frac{x-2}{2}=\frac{y+2}{3}=\frac{z-6}{-6}$.**

Sol.

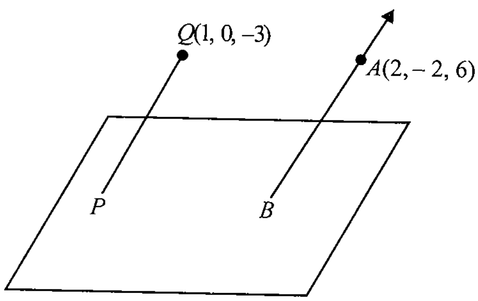


Fig. 3.22

The given plane is $x - y - z = 9$ (i)

The given line AB is $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$ (ii)

The equation of the line passing through $(1, 0, -3)$ and parallel to $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$ is

$$\frac{x-1}{2} = \frac{y-0}{3} = \frac{z+3}{-6} = r \quad \text{(iii)}$$

Coordinate of any point on (iii) may be given as $P(2r + 1, 3r, -6r - 3)$.

If P is the point of the intersection of (i) and (iii), then it must lie on (i). Therefore,

$$(2r + 1) - (3r) - (-6r - 3) = 9$$

$$2r + 1 - 3r + 6r + 3 = 9 \Rightarrow r = 1$$

Therefore, the coordinates of P are 3, 3, –9.

Distance between $Q(1, 0, -3)$ and $P(3, 3, -9) = \sqrt{(3-1)^2 + (3-0)^2 + (-9+3)^2}$

$$= \sqrt{4+9+36} = 7$$

ANGLE BETWEEN A LINE AND A PLANE

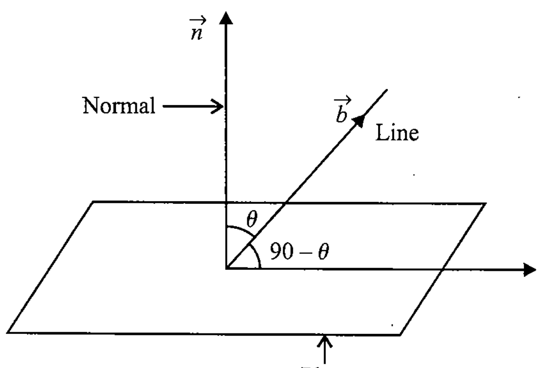


Fig. 3.23

The angle between a line and a plane is the complement of the angle between the line and the normal to the plane.

If the equation of the line is $\vec{r} = \vec{a} + \lambda \vec{b}$ and that of the plane is $\vec{r} \cdot \vec{n} = d$, then angle θ between the line and the normal to the plane is $\cos\theta = \left| \frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|} \right|$.

So the angle ϕ between the line and the plane is given by $90° - \theta$

$$\sin\phi = \left| \frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|} \right| \quad \text{or } \phi = \sin^{-1} \left| \frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|} \right|$$

Line $\vec{r}=\vec{a}+\lambda\vec{b}$ and plane $\vec{r}\cdot\vec{n}=d$ are perpendicular if $\vec{b}=\lambda\vec{n}$ or $\vec{b}\times\vec{n}=\vec{0}$ and parallel if $\vec{b}\perp\vec{n}$ or $\vec{b}\cdot\vec{n}=0$.

Example 3.51 **Find the angle between the line $\vec{r}=\hat{i}+2\hat{j}-\hat{k}+\lambda(\hat{i}-\hat{j}+\hat{k})$ and the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=4$.**

Sol. We know that if θ is the angle between the lines $\vec{r}=\vec{a}+\lambda\vec{b}$ and $\vec{r}\cdot\vec{n}=p$, then $\sin\theta=\left|\dfrac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$

Therefore, if θ is the angle between $\vec{r}=\hat{i}+2\hat{j}-\hat{k}+\lambda(\hat{i}-\hat{j}+\hat{k})$ and $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=4$, then

$$\sin\theta=\left|\frac{(\hat{i}-\hat{j}+\hat{k})\cdot(2\hat{i}-\hat{j}+\hat{k})}{|\hat{i}-\hat{j}+\hat{k}||2\hat{i}-\hat{j}+\hat{k}|}\right|$$

$$=\frac{2+1+1}{\sqrt{1+1+1}\sqrt{4+1+1}}$$

$$=\frac{4}{\sqrt{3}\sqrt{6}}=\frac{4}{3\sqrt{2}}$$

$$\Rightarrow \qquad \theta=\sin^{-1}\left(\frac{4}{3\sqrt{2}}\right)$$

EQUATION OF A PLANE PASSING THROUGH THE LINE OF INTERSECTION OF TWO PLANES

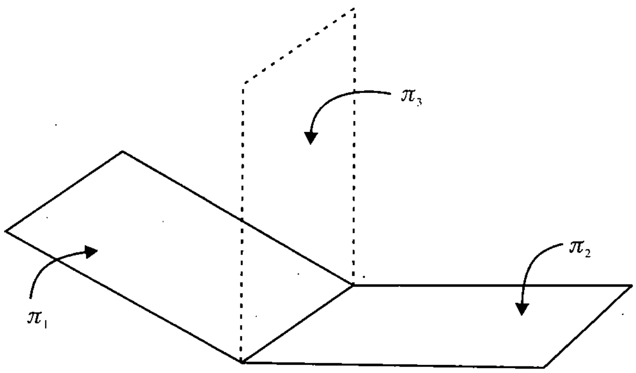


Fig. 3.24

Let π_1 and π_2 be two planes with equations $\vec{r}\cdot\hat{n}_1=d_1$ and $\vec{r}\cdot\hat{n}_2=d_2$, respectively. The position vector of any point on the line of intersection must satisfy both the equations.

If $\vec{t}$ is the position vector of a point on the line, then

$$\vec{t}\cdot\hat{n}_1=d_1 \text{ and } \vec{t}\cdot\hat{n}_2=d_2$$

Therefore, for all real values of λ, we have

$$\vec{t}\cdot(\hat{n}_1+\lambda\hat{n}_2)=d_1+\lambda d_2 \quad \text{(i)}$$

Since $\vec{t}$ is arbitrary, it satisfies for any point on the line.

Hence, the equation $\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$ represents a plane π_3 which is such that if any vector $\vec{r}$ satisfies both the equations π_1 and π_2, it also satisfies the equation π_3.

Cartesian Form

In Cartesian system, let $\vec{n}_1=A_1\hat{i}+B_1\hat{j}+C_1\hat{k}$, $\vec{n}_2=A_2\hat{i}+B_2\hat{j}+C_2\hat{k}$ and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$.

Then (i) becomes

$$x(A_1+\lambda A_2)+y(B_1+\lambda B_2)+z(C_1+\lambda C_2)=d_1+\lambda d_2$$

$$\text{or}\quad (A_1x+B_1y+C_1z-d_1)+\lambda(A_2x+B_2y+C_2z-d_2)=0 \quad \text{(ii)}$$

which is the required Cartesian form of the equation of the plane passing through the intersection of the given planes for each value of λ.

Example 3.52 **Find the plane passing through the intersection of planes $\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})=1$ and $\vec{r}\cdot(\hat{i}-\hat{j})+4=0$ and perpendicular to $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=-8$.**

Sol. The equation of any plane through the line of intersection of the given planes is

$$\{\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})-1\}+\lambda\{\vec{r}\cdot(\hat{i}-\hat{j})+4\}=0$$

$$\vec{r}\cdot\{(2+\lambda)\hat{i}-(3+\lambda)\hat{j}+4\hat{k}\}=1-4\lambda \quad \text{(i)}$$

If it is perpendicular to $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})+8=0$, then

$$\{(2+\lambda)\hat{i}-(3+\lambda)\hat{j}+4\hat{k}\}\cdot(2\hat{i}-\hat{j}+\hat{k})=0$$

$$2(2+\lambda)+(3+\lambda)+4=0$$

$$\lambda=\frac{-11}{3}$$

Putting $\lambda=-11/3$ in (i), we obtain the equation of the required plane as $\vec{r}\cdot(-5\hat{i}+2\hat{j}+12\hat{k})=47$

Example 3.53 **Find the equation of a plane containing the line of intersection of the planes $x+y+z-6=0$ and $2x+3y+4z+5=0$ and passing through (1, 1, 1).**

Sol. The equation of a plane passing through the line of intersection of the given planes is

$$(x+y+z-6)+\lambda(2x+3y+4z+5)=0 \quad \text{(i)}$$

If it is passes through (1, 1, 1), $(1+1+1-6)+\lambda(2+3+4+5)=0$

$$\Rightarrow\lambda=\frac{3}{14}$$

Putting $\lambda=3/14$ in (i), we get

$$(x+y+z-6)+\frac{3}{14}(2x+3y+4z+5)=0$$

$$20x+23y+26z-69=0$$

Example 3.54 **The plane $ax+by=0$ is rotated through an angle α about its line of intersection with the plane $z=0$. Show that the equation to the plane in the new position is**

$ax+by\pm z\sqrt{a^2+b^2}\tan\alpha=0$.

Sol. Given planes are $ax+by=0$ (i)

and $z=0$ (ii)

Therefore, the equation of any plane passing through the line of intersection of planes (i) and (ii) may be taken as

$ax+by+kz=0$ (iii)

The direction cosines of a normal to the plane (iii) are

$$\frac{a}{\sqrt{a^2+b^2+k^2}},\ \frac{b}{\sqrt{a^2+b^2+k^2}} \text{ and } \frac{k}{\sqrt{a^2+b^2+k^2}}$$

The direction cosines of a normal to the plane (i) are

$$\frac{a}{\sqrt{a^2+b^2}}\ \frac{b}{\sqrt{a^2+b^2}} \text{ and } 0$$

Since the angle between the planes (i) and (iii) is α,

$$\cos\alpha=\frac{a\cdot a+b\cdot b+k\cdot 0}{\sqrt{a^2+b^2+k^2}\sqrt{a^2+b^2}}=\sqrt{\frac{a^2+b^2}{a^2+b^2+k^2}}$$

$$\Rightarrow k^2\cos^2\alpha=a^2(1-\cos^2\alpha)+b^2(1-\cos^2\alpha)$$

$$\Rightarrow k^2=\frac{(a^2+b^2)\sin^2\alpha}{\cos^2\alpha}\Rightarrow k=\pm\sqrt{a^2+b^2}\tan\alpha,$$

Putting this in (iii), we get the equation of the plane as $ax+by\pm z\sqrt{a^2+b^2}\tan\alpha=0$

DISTANCE OF A POINT FROM A PLANE

Vector Form

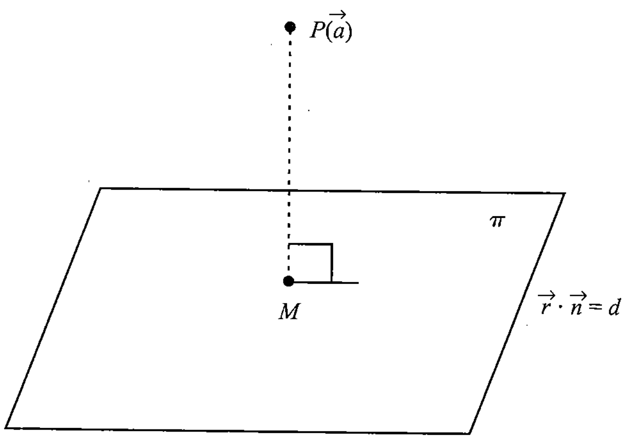


Fig. 3.25

Let π $(\vec{r}\cdot\vec{n}=d)$ be the given plane and P $(\vec{a})$ be the given point.

Let PM be the length of the perpendicular from P to the plane π.

Since line PM passes through $P(\vec{a})$ and is parallel to vector $\vec{n}$, which is normal to the plane π, the vector equation of line PM is: $\vec{r}=\vec{a}+\lambda\vec{n}$ (i)

Point M is the intersection of (i) and the given plane π. Therefore,

$$(\vec{a}+\lambda\vec{n})\cdot\vec{n}=d$$

$$\Rightarrow \vec{a}\cdot\vec{n}+\lambda\vec{n}\cdot\vec{n}=d$$

$$\Rightarrow \lambda=\frac{d-(\vec{a}\cdot\vec{n})}{|\vec{n}|^2}$$

Putting the value of λ in (i), we obtain the position vector of M given by $\vec{r}=\vec{a}+\left(\frac{d-\vec{a}\cdot\vec{n}}{|\vec{n}|^2}\right)\vec{n}$

$$\overrightarrow{PM} = \text{P.V. of } M - \text{P.V. of } P$$

$$=\vec{a}+\left(\frac{d-(\vec{a}\cdot\vec{n})}{|\vec{n}|^2}\right)\vec{n}-\vec{a}$$

$$=\left(\frac{d-(\vec{a}\cdot\vec{n})}{|\vec{n}|^2}\right)\vec{n}$$

$$\Rightarrow PM=|\overrightarrow{PM}|=\left|\frac{(d-\vec{a}\cdot\vec{n})\,\vec{n}}{|\vec{n}|^2}\right|=\frac{|d-(\vec{a}\cdot\vec{n})||\vec{n}|}{|\vec{n}|^2}=\frac{|d-(\vec{a}\cdot\vec{n})|}{|\vec{n}|},$$ which is the required length.

Cartesian Form

Let PM be the length of the perpendicular from a point P (x_1, y_1, z_1) to the plane $ax+by+cz+d=0$.

Then the equation of PM is $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=r$ (let) (i)

The coordinates of any point on this line are (x_1+ar, y_1+br, z_1+cr).

Thus the point coincides with M if and only if it lies on the plane.

i.e., $a(x_1+ar)+b(y_1+br)+c(z_1+cr)+d=0$

$$r=-\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2} \quad \text{(ii)}$$

Now, $PM=\sqrt{(x_1+ar-x_1)^2+(y_1+br-y_1)^2+(z_1+cr-z_1)^2}$

$$=\sqrt{(a^2+b^2+c^2)r^2}$$

$$=\sqrt{a^2+b^2+c^2}\;|r|$$

$$= \sqrt{a^2+b^2+c^2}\left|\frac{-(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}\right| \quad \text{from (ii)}$$

$$= \frac{|(ax_1+by_1+cz_1+d)|}{\sqrt{a^2+b^2+c^2}}$$

Also, if coordinates of M are (x_2, y_2, z_2), then

$$\frac{x_2-x_1}{a} = \frac{y_2-y_1}{b} = \frac{z_2-z_1}{c} = -\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2} \quad \text{(iii)}$$

Image of a Point in a Plane

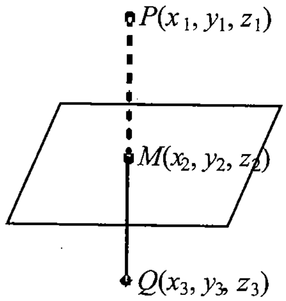


Fig. 3.26

Here Q is the image of P in the plane.

Therefore M is the midpoint of PQ.

Therefore from (iii)

$$\frac{\frac{x_3+x_1}{2}-x_1}{a} = \frac{\frac{y_3-y_1}{2}-y_1}{b} = \frac{\frac{z_3-z_1}{2}-z_1}{c}$$

$$= -\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$

or

$$\frac{x_3-x_1}{a} = \frac{y_3-y_1}{b} = \frac{z_3-z_1}{c} = \frac{-2(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$

DISTANCE BETWEEN PARALLEL PLANES

The distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is given by

$$d = \left|\frac{(d_2-d_1)}{\sqrt{a^2+b^2+c^2}}\right|$$

Proof:

Let $P(x_1, y_1, z_1)$ be point on plane $ax + by + cz + d_1 = 0$

then distance of this point from plane $ax + by + cz + d_2 = 0$ is

$$d = \frac{|ax_1 + by_1 + cz_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

also $ax_1 + by_1 + cz_1 + d_1 = 0$

$$\Rightarrow d = \frac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}}$$

Example 3.55 **Find the length and the foot of the perpendicular from the point (7, 14, 5) to the plane $2x + 4y - z = 2$.**

Sol. The required length $= \dfrac{2(7) + 4(14) - (5) - 2}{\sqrt{2^2 + 4^2 + 1^2}} = \dfrac{14 + 56 - 5 - 2}{\sqrt{4 + 16 + 1}} = \dfrac{63}{\sqrt{21}}$

Let the coordinates of the foot of the perpendicular from the point P (7, 14, 5) be $M(\alpha, \beta, \gamma)$.
Then the direction ratios of PM are $\alpha - 7$, $\beta - 14$ and $\gamma - 5$.
Therefore, the direction ratios of the normal to the plane are $\alpha - 7$, $\beta - 14$ and $\gamma - 5$.

But the direction ratios of normal to the given plane $2x + 4y - z = 2$ are 2, 4 and -1.

Hence, $\dfrac{\alpha - 7}{2} = \dfrac{\beta - 14}{4} = \dfrac{\gamma - 5}{-1} = k$

$\therefore\ \alpha = 2k + 7, \beta = 4k + 14$ and $\gamma = -k + 5$. (i)

Since α, β and γ lie on the plane $2x + 4y - z = 2$, $2\alpha + 4\beta - \gamma = 2$

$\Rightarrow\ 2(7 + 2k) + 4(14 + 4k) - (5 - k) = 2$

$\Rightarrow 14 + 4k + 56 + 16k - 5 + k = 2$

$\Rightarrow 21k = -63$

$\Rightarrow k = -3$

Now, putting $k = -3$ in (i), we get

$\alpha = 1, \beta = 2, \gamma = 8$

Hence the foot of the perpendicular is (1, 2, 8)

Example 3.56 **Find the distance between the parallel planes $x + 2y - 2z + 1 = 0$ and $2x + 4y - 4z + 5 = 0$.**

Sol. We know that the distance between parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

Therefore, the distance between $x + 2y - 2z + 1 = 0$ and $x + 2y - 2z + \frac{5}{2} = 0$ is

$$\frac{|(5/2) - 1|}{\sqrt{1 + 4 + 4}} = \frac{1}{2}$$

Example 3.57 **Find the image of the line $\frac{x-1}{9} = \frac{y-2}{-1} = \frac{z+3}{-3}$ in the plane $3x - 3y + 10z - 26 = 0$.**

Sol.

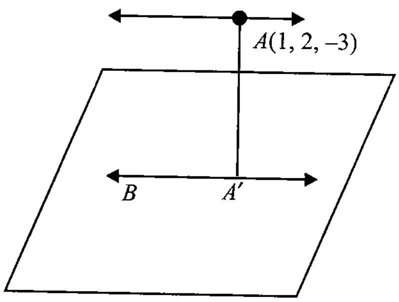


Fig. 3.27

$$\frac{x-1}{9} = \frac{y-2}{-1} = \frac{z+3}{-3} \quad \text{(i)}$$

$$3x - 3y + 10z - 26 = 0 \quad \text{(ii)}$$

The direction ratios of the line are 9, –1 and –3 and direction ratios of the normal to the given plane are 3, –3 and 10.

Since $9 \cdot 3 + (-1)(-3) + (-3)\,10 = 0$ and the point (1, 2, –3) of line (i) does not lie in plane (ii) for $3 \cdot 1 - 3 \cdot 2 + 10 \cdot (-3) - 26 \neq 0$, line (i) is parallel to plane (ii). Let A' be the image of point A (1, 2, –3) in plane (ii). Then the image of the line (i) in the plane (ii) is the line through A' and parallel to the line (i).

Let point A' be (p, q, r). Then

$$\frac{p-1}{3} = \frac{q-2}{-3} = \frac{r+3}{10} = -\frac{(3(1) - 3(2) + 10(-3) - 26)}{9+9+100} = \frac{1}{2}$$

The point is $A'(5/2, 1/2, 2)$

The equation of line BA' is $\frac{x-(5/2)}{9} = \frac{y-(1/2)}{-1} = \frac{z-2}{-3}$

EQUATION OF A PLANE BISECTING THE ANGLE BETWEEN TWO PLANES

Given planes are

$$a_1x + b_1y + c_1z + d_1 = 0 \quad \text{(i)}$$

$$a_2x + b_2y + c_2z + d_2 = 0 \quad \text{(ii)}$$

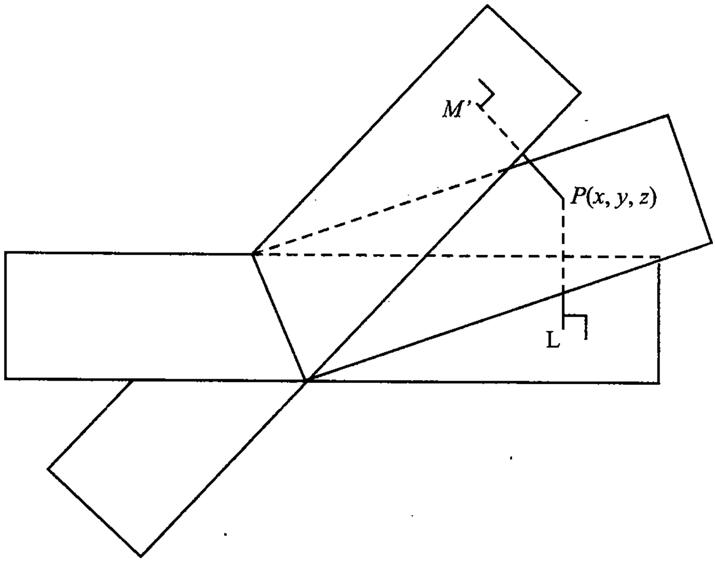


Fig. 3.28

Let $P(x, y, z)$ be a point on the plane bisecting the angle between (i) and (ii).

Let PL and PM be the length of the perpendiculars from P to planes (i) and (ii). Therefore,

$PL = PM$

$$\Rightarrow \left|\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}\right| = \left|\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

$$\Rightarrow \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

This is the equation of the plane bisecting the angles between planes (i) and (ii).

Vector form

The equation of the plane bisecting the angle between planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is

$$\left|\frac{\vec{r} \cdot \vec{n}_1 - d_1}{\vec{n}_1}\right| = \left|\frac{\vec{r} \cdot \vec{n}_2 - d_2}{\vec{n}_2}\right|$$

Bisector of the Angle Between the Two Planes Containing the Origin

Let the equation of the two planes be

$a_1x + b_1y + c_1z + d_1 = 0$ and (i)

$a_2x + b_2y + c_2z + d_2 = 0$ (ii)

where d_1 and d_2 are positive.

The equation of the bisector of the angle between the planes (i) and (ii) containing the origin is

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

Bisector of the Acute and Obtuse Angles Between Two Planes

Let the two planes be

$$a_1x+b_1y+c_1z+d_1=0 \quad \text{(i)}$$

$$a_2x+b_2y+c_2z+d_2=0 \quad \text{(ii)}$$

where $d_1, d_2>0$

i. If $a_1a_2+b_1b_2+c_1c_2>0$, the origin lies in the obtuse angle between the two planes and the equation of the bisector of the obtuse angle is $\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=-\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$.

ii. If $a_1a_2+b_1b_2+c_1c_2<0$, the origin lies in the acute angle between the two planes and the equation of the bisector of the acute angle between the two planes is $\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=-\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$.

Example 3.58 **Find the equations of the bisectors of the angles between the planes $2x-y+2z+3=0$ and $3x-2y+6z+8=0$ and specify the plane which bisects the acute angle and the plane which bisects the obtuse angle.**

Sol. The given planes are $2x-y+2z+3=0$ and $3x-2y+6z+8=0$, where $d_1, d_2>0$ and $a_1a_2+b_1b_2+c_1c_2=6+2+12>0$.

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=-\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \quad \text{(obtuse angle bisector)}$$

$$\frac{a_1x+b_1y+c_1z+d}{\sqrt{a_2^2+b_2^2+c_2^2}}=\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \quad \text{(acute angle bisector)}$$

i.e., $$\frac{2x-y+2z+3}{\sqrt{4+1+4}}=\pm\frac{3x-2y+6z+8}{\sqrt{9+4+36}}$$

$$\Rightarrow (14x-7y+14z+21)=\pm(9x-6y+18z+24)$$

Taking the positive sign on the right hand side, we get

$5x-y-4z-3=0$ (obtuse angle bisector)

Taking the negative sign on the right hand side, we get

$23x-13y+32z+45=0$ (acute angle bisector)

TWO SIDES OF A PLANE

Let $ax + by + cz + d = 0$ be the plane. Then the points (x_1, y_1, z_1) and (x_2, y_2, z_2) lie on the same side or the opposite sides as $\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} > 0$ or < 0, respectively.

Proof:

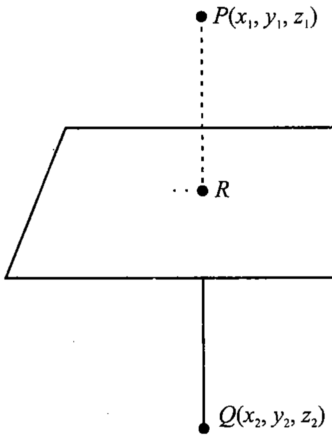


Fig. 3.29

Here the equation of the plane is $ax + by + cz + d = 0$. (i)

Let (i) divide the line segment joining P and Q at R internally in the ratio $m : n$.

Then $R\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}, \dfrac{mz_2 + nz_1}{m+n}\right)$

R lies on plane (i). Therefore,

$$a\left(\frac{mx_2 + nx_1}{m+n}\right) + b\left(\frac{my_2 + ny_1}{m+n}\right) + c\left(\frac{mz_2 + nz_1}{m+n}\right) + d = 0$$

$$a\,(mx_2 + nx_1) + b\,(my_2 + ny_1) + c\,(mz_2 + nz_1) + d\,(m + n) = 0$$

$$m\,(ax_2 + by_2 + cz_2 + d) + n\,(ax_1 + by_1 + cz_1 + d) = 0$$

$$\frac{m}{n} = -\frac{(ax_1 + by_1 + cz_1 + d)}{(ax_2 + by_2 + cz_2 + d)} \quad \text{(ii)}$$

Now, if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$

are of same sign $\dfrac{m}{n} < 0$ (external division)

are of opposite signs $\dfrac{m}{n} > 0$ (internal division)

If $\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} > 0$ (same side)

$\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} < 0$ (opposite side)

Concept Application Exercise 3.3

1. Find the angle between the line $\dfrac{x+1}{3} = \dfrac{y-1}{2} = \dfrac{z-1}{4}$ and the plane $2x + y - 3z + 4 = 0$.
2. Find the distance between the line $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z-2}{1}$ and the plane $x + y + z + 3 = 0$.
3. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\dfrac{x-2}{3} = \dfrac{y+1}{4} = \dfrac{z-2}{12}$ and plane $x - y + z = 5$.
4. Find the equation of a plane which passes through the point $(1, 2, 0)$ and which is perpendicular to the planes $x - y + z - 3 = 0$ and $2x + y - z + 4 = 0$.
5. Find the equation of the plane passing through the points $(1, 0, -1)$ and $(3, 2, 2)$ and parallel to the line $x - 1 = \dfrac{1-y}{2} = \dfrac{z-2}{3}$.
6. Find the equation of a plane containing the lines $\dfrac{x-5}{4} = \dfrac{y-7}{4} = \dfrac{z+3}{-5}$ and $\dfrac{x-8}{7} = \dfrac{y-4}{1} = \dfrac{z-5}{3}$.
7. Find the equation of the plane passing through the straight line $\dfrac{x-1}{2} = \dfrac{y+2}{-3} = \dfrac{z}{5}$ and perpendicular to the plane $x - y + z + 2 = 0$.
8. Find the equation of the plane perpendicular to the line $\dfrac{x-1}{2} = \dfrac{y-3}{-1} = \dfrac{z-4}{2}$ and passing through the origin.
9. Find the equation of the plane passing through the line $\dfrac{x-1}{5} = \dfrac{y+2}{6} = \dfrac{z-3}{4}$ and point $(4, 3, 7)$.
10. Find the angle between the line $\vec{r} = (\vec{i} + 2\vec{j} - \vec{k}) + \lambda(\vec{i} - \vec{j} + \vec{k})$ and the normal to the plane $\vec{r} \cdot (2\vec{i} - \vec{j} + \vec{k}) = 4$.
11. Find the equation of a plane which passes through the point $(1, 2, 3)$ and which is at the maximum distance from the point $(-1, 0, 2)$.
12. Find the direction ratios of orthogonal projection of line $\dfrac{x-1}{1} = \dfrac{y+1}{-2} = \dfrac{z-2}{3}$ in the plane $x - y + 2z - 3 = 0$. Also find the direction ratios of the image of the line in the plane.
13. Find the equation of a plane which is parallel to the plane $x - 2y + 2z = 5$ and whose distance from the point $(1, 2, 3)$ is 1.
14. Find the equation of a plane which passes through the point $(1, 2, 3)$ and which is equally inclined to the planes $x - 2y + 2z - 3 = 0$ and $8x - 4y + z - 7 = 0$.
15. Find the equation of the image of the plane $x - 2y + 2z - 3 = 0$ in the plane $x + y + z - 1 = 0$.

SPHERES

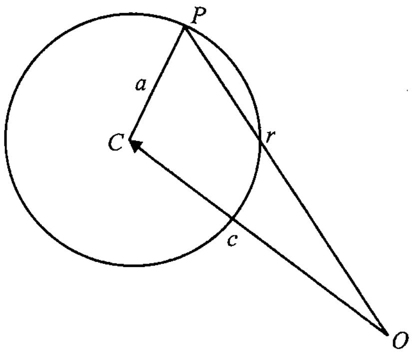


Fig. 3.30

A sphere is the locus of a point which moves in space in such a way that its distance from a fixed point always remains constant. The fixed point is called the centre and the constant distance is called the radius of the sphere.

Equation of a Sphere

Let $\vec{c}$ be the position vector of the centre C of the sphere and a be the radius of the sphere.

Let $\vec{r}$ be the position vector of any point P on the sphere.

Then $|\overrightarrow{CP}| = a$

But $\overrightarrow{CP} = \overrightarrow{OP} - \overrightarrow{OC} = \vec{r} - \vec{c}$

Thus, $|\vec{r} - \vec{c}| = a$

$\Rightarrow |\vec{r} - \vec{c}|^2 = a^2$

$\Rightarrow (\vec{r} - \vec{c}) \cdot (\vec{r} - \vec{c}) = a^2$

Cartesian form

If $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then $\vec{r} - \vec{c} = (x - c_1)\hat{i} + (y - c_2)\hat{j} + (z - c_3)\hat{k}$

Now, $|\vec{r} - \vec{c}| = \sqrt{(x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2}$

Therefore, the equation is $(x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2 = a^2$

$\Rightarrow \; x^2 + y^2 + z^2 - 2c_1x - 2c_2y - 2c_3z + c_1^2 + c_2^2 + c_3^2 - a^2 = 0$

We usually write this equation as $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (i)

Adding $u^2 + v^2 + w^2$ on both the sides of (i), we can write $(x + u)^2 + (y + v)^2 + (w + z)^2 = u^2 + v^2 + w^2 - d$.

This equation represents a sphere with centre at $(-u, -v, -w)$ and radius $\sqrt{u^2 + v^2 + w^2 - d}$. Note that we must have $u^2 + v^2 + w^2 - d \geq 0$.

Thus, (i) represents a sphere with centre at $(-u, -v, -w)$ and radius equal to $\sqrt{u^2 + v^2 + w^2 - d}$.

In particular, the equation of a sphere with centre at the origin is $|\vec{r}| = a$ or $x^2 + y^2 + z^2 = a^2$.

For a fixed sphere in space, we require four non-coplanar points which form a tetrahedron, or we can say that every tetrahedron has a unique circumscribed sphere.

Example 3.59 **Find the equation of a sphere whose centre is (3, 1, 2) and radius is 5.**

Sol. The equation of the sphere whose centre is (3, 1, 2) and radius is 5 is

$(x-3)^2 + (y-1)^2 + (z-2)^2 = 5^2$

$x^2 + y^2 + z^2 - 6x - 2y - 4z - 11 = 0$

Example 3.60 **Find the equation of the sphere passing through (0, 0, 0), (1, 0, 0), (0, 1, 0) and (0, 0, 1).**

Sol. Let the equation of the sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (i)

As (i) passes through (0, 0, 0), (1, 0, 0), (0, 1, 0) and (0, 0, 1), we must have $d = 0$, $1 + 2u + d = 0$
$1 + 2v + d = 0$ and $1 + 2w + d = 0$

Since $d = 0$, we get $2u = 2v = 2w = -1$

Thus, the equation of the required sphere is $x^2 + y^2 + z^2 - x - y - z = 0$

Example 3.61 **Find the equation of the sphere which passes through (1, 0, 0), (0,1, 0) and (0, 0, 1), and whose centre lies on the plane $3x - y + z = 2$.**

Sol. Let the equation of the required sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

As the sphere passes through (1, 0, 0), (0, 1, 0) and (0, 0, 1), we get

$1 + 2u + d = 0$, $1 + 2v + d = 0$ and $1 + 2w + d = 0$

$$\Rightarrow u = v = w = -\frac{d+1}{2}$$

Since the centre $(-u, -v, -w)$ lies on the plane $3x - y + z = 2$, we get $-3u + v - w = 2$

$$\Rightarrow \frac{3}{2}(d+1) = 2 \text{ or } d + 1 = \frac{4}{3} \text{ or } d = \frac{1}{3}$$

Thus, $u = v = w = -2/3$

Therefore, the equation of the required sphere is $x^2 + y^2 + z^2 - \left(\frac{2}{3}\right)x - \left(\frac{2}{3}\right)y - \left(\frac{2}{3}\right)z + \frac{1}{3} = 0$

or $3(x^2 + y^2 + z^2) - 2(x + y + z) + 1 = 0$

Example 3.62 **Find the equation of a sphere which passes through (1, 0, 0) (0, 1, 0) and (0, 0, 1), and has radius as small as possible.**

Sol. Let the equation of the required sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (i)

As the sphere passes through (1, 0, 0), (0, 1, 0) and (0, 0, 1), we get

$1 + 2u + d = 0$, $1 + 2v + d = 0$ and $1 + 2w + d = 0$

$\Rightarrow u = v = w = -\dfrac{1}{2}(d+1)$

If R is the radius of the sphere, then $R^2 = u^2 + v^2 + w^2 - d$

$$\Rightarrow R^2 = \frac{3}{4}(d+1)^2 - d$$

$$= \frac{3}{4}\left[d^2 + 2d + 1 - \frac{4}{3}d\right]$$

$$= \frac{3}{4}\left[d^2 + \frac{2}{3}d + 1\right]$$

$$= \frac{3}{4}\left[\left(d + \frac{1}{3}\right)^2 + 1 - \frac{1}{9}\right]$$

$$= \frac{3}{4}\left[\left(d + \frac{1}{3}\right)^2 + \frac{8}{9}\right]$$

The last equation shows that R^2 (and thus R) will be the least if and only if $d = -1/3$.

Therefore, $u = v = w = -\dfrac{1}{2}\left(1 - \dfrac{1}{3}\right) = -\dfrac{1}{3}$

Hence, the equation of the required sphere is $x^2 + y^2 + z^2 - \dfrac{2}{3}(x + y + z) - \dfrac{1}{3} = 0$

or $3(x^2 + y^2 + z^2) - 2(x + y + z) - 1 = 0$

Example 3.63 **Find the locus of a point which moves such that the sum of the squares of its distance from the points A (1, 2, 3), B (2, −3, 5) and C (0, 7, 4) is 120.**

Sol. Let $P(x, y, z)$ be any point on the locus. Then $PA^2 + PB^2 + PC^2 = 120$

$\Rightarrow (x-1)^2 + (y-2)^2 + (z-3)^2 + (x-2)^2 + (y+3)^2 + (z-5)^2 + (x-0)^2 + (y-7)^2 + (z-4)^2 = 120$

$3x^2 + 3y^2 + 3z^2 - 6x - 12y - 24z + 117 = 120$

$x^2 + y^2 + z^2 - 2x - 4y - 8z - 1 = 0$

This represents a sphere with centre at (1, 2, 4) and radius equal to $\sqrt{1^2 + 2^2 + 4^2 + 1} = \sqrt{22}$

Diameter Form of the Equation of a Sphere

Let AB be the diameter of a sphere whose centre is C. Let the vectors of the extremeties A and B of the diameter be $\vec{a}$ and $\vec{b}$, respectively. Let P be any point on this sphere. Suppose the position vector of P is $\vec{r}$. We know that the angle in a hemisphere is a right angle.

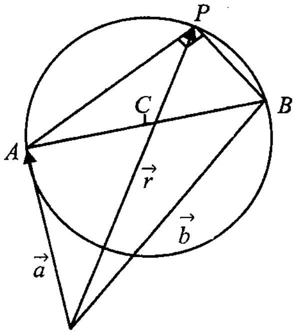


Fig. 3.31

Thus, $\angle APB = \pi/2$

$\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$ (i)

But $\overrightarrow{AP} = \vec{r} - \vec{a}$ and $\overrightarrow{BP} = \vec{r} - \vec{b}$

Thus, (i) can be written as $\left(\vec{r} - \vec{a}\right) \cdot \left(\vec{r} - \vec{b}\right) = 0$

This is the required equation of the sphere.

Cartesian form

$\vec{a} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}, \vec{b} = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\vec{r} - \vec{a} = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$

$\vec{r} - \vec{b} = (x - x_2)\hat{i} + (y - y_2)\hat{j} + (z - z_2)\hat{k}$

Thus, $(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0$ gives

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

Example 3.64 **Find the equation of the sphere described on the joint of points A and B having position vectors $2\hat{i} + 6\hat{j} - 7\hat{k}$ and $-2\hat{i} + 4\hat{j} - 3\hat{k}$, respectively, as the diameter. Find the centre and the radius of the sphere.**

Sol. If point P with positive vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is any point on the sphere, then

$\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$

$(x - 2)(x + 2) + (y - 6)(y - 4) + (z + 7)(z + 3) = 0$

$\Rightarrow (x^2 - 4) + (y^2 - 10y + 24) + (z^2 + 10z + 21) = 0$

$\Rightarrow x^2 + y^2 + z^2 - 10y + 10z + 41 = 0$

The centre of this sphere is $(0, 5, -5)$ and its radius is $\sqrt{5^2 + (-5)^2 - 41} = \sqrt{9} = 3$

Example 3.65 **Find the radius of the circular section in which the sphere $|\vec{r}| = 5$ is cut by the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k}) = 3\sqrt{3}$.**

Sol. Let A be the foot of the perpendicular from the centre O to the plane

$\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k}) - 3\sqrt{3} = 0$

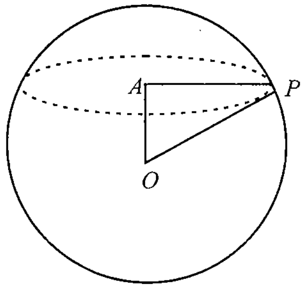


Fig. 3.32

Then $|OA| = \left|\dfrac{0\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)-3\sqrt{3}}{|\hat{i}+\hat{j}+\hat{k}|}\right| = \dfrac{3\sqrt{3}}{\sqrt{3}} = 3$ (Perpendicular distance of a point from the plane)

If P is any point on the circle, then P lies on the plane as well as on the sphere. Therefore,

OP = radius of the sphere = 5

Now $AP^2 = OP^2 - OA^2 = 5^2 - 3^2 = 16 \Rightarrow AP = 4$

Example 3.66 **Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$.**

Sol. The given plane will touch the given sphere if the perpendicular distance from the centre of the sphere to the plane is equal to the radius of the sphere. The centre of the given sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ is $(1, 2, -1)$ and its radius is $\sqrt{1^2 + 2^2 + (-1)^2 - (-3)} = 3$.

Length of the perpendicular from $(1, 2, -1)$ to the plane $2x - 2y + z + 12 = 0$ is

$$\left|\frac{2(1) - 2(2) + (-1) + 12}{\sqrt{2^2 + (-2)^2 + 1^2}}\right| = \frac{9}{3} = 3$$

Thus, the given plane touches the given sphere.

Example 3.67 **A variable plane passes through a fixed point (a, b, c) and cuts the coordinate axes at points A, B and C. Show that the locus of the centre of the sphere $OABC$ is $\dfrac{a}{x} + \dfrac{b}{y} + \dfrac{c}{z} = 2$.**

Sol. Let (α, β, γ) be any point on the locus. Then according to the given condition, (α, β, γ) is the centre of the sphere through the origin. Therefore, its equation is given by

$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = (0-\alpha)^2 + (0-\beta)^2 + (0-\gamma)^2$

$x^2 + y^2 + z^2 - 2\alpha x - 2\beta y - 2\gamma z = 0$

To obtain its point of intersection with the x-axis, we put $y = 0$ and $z = 0$, so that

$x^2 - 2\alpha x = 0 \Rightarrow x(x - 2\alpha) = 0 \Rightarrow x = 0$ or $x = 2\alpha$

Thus the plane meets x-axis at O (0, 0, 0) and A (2α, 0, 0). Similarly, it meets y-axis at O (0, 0, 0) and B (0, 2β, 0), and z-axis at O (0, 0, 0) and C (0, 0, 2γ).

The equation of the plane through A, B and C is

$$\frac{x}{2\alpha} + \frac{y}{2\beta} + \frac{z}{2\gamma} = 1 \quad \text{(intercept form)}$$

Since it passes through (a, b, c), we get

$$\frac{a}{2\alpha} + \frac{b}{2\beta} + \frac{c}{2\gamma} = 1 \text{ or } \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 2$$

Hence, locus of (α, β, γ) is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$

Example 3.68 **A sphere of constant radius k passes through the origin and meets the axes at A, B and C. Prove that the centroid of triangle ABC lies on the sphere $9(x^2 + y^2 + z^2) = 4k^2$.**

Sol. Let the equation of any sphere passing through the origin and having radius k be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0 \quad \text{(i)}$$

As the radius of the sphere is k, we get

$$u^2 + v^2 + w^2 = k^2 \quad \text{(ii)}$$

Note that (i) meets the x-axis at O (0, 0, 0) and A ($-2u$, 0, 0); y-axis at O (0, 0, 0) and B (0, $-2v$, 0); and z-axis at O (0, 0, 0) and C (0, 0, $-2w$).

Let the centroid of the triangle ABC be (α, β, γ). Then

$$\alpha = -\frac{2u}{3}, \beta = -\frac{2v}{3}, \gamma = -\frac{2w}{3} \quad \Rightarrow \quad u = -\frac{3\alpha}{2}, v = -\frac{3\beta}{2}, w = -\frac{3v}{2}$$

Putting this in (ii), we get

$$\left(\frac{-3}{2}\alpha\right)^2 + \left(\frac{-3}{2}\beta\right)^2 + \left(\frac{-3}{2}\gamma\right)^2 = k^2$$

$$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 = \frac{4}{9}k^2$$

This shows that the centroid of triangle ABC lies on $x^2 + y^2 + z^2 = \frac{4}{9}k^2$

Concept Application Exercise 3.4

1. Find the plane of the intersection of $x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$ and $4x^2 + 4y^2 + 4z^2 + 4x + 4y + 4z - 1 = 0$.
2. Find the radius of the circular section of the sphere $|\vec{r}| = 5$ by the plane $\vec{r}\cdot(\vec{i} + 2\vec{j} - \vec{k}) = 4\sqrt{3}$.
3. A point $P(x, y, z)$ is such that $3PA = 2PB$, where A and B are the points (1, 3, 4) and (1, −2, −1), respectively. Find the equation to the locus of the point P and verify that the locus is a sphere.
4. The extremities of a diameter of a sphere lie on the positive y- and positive z-axes at distances 2 and 4, respectively. Show that the sphere passes through the origin and find the radius of the sphere.
5. A plane passes through a fixed point (a, b, c). Show that the locus of the foot of the perpendicular to it from the origin is the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$.

Exercises

Subjective Type

Solutions on page 3.79

1. If variable lines in two adjacent positions have direction cosines l, m and n and $(l + \delta l)$, $(m + \delta m)$, $(n + \delta n)$, show that the small angle $\delta\theta$ between the two positions is given by $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.
2. Find the equation of the plane containing the line $\frac{y}{b} + \frac{z}{c} = 1$, $x = 0$, and parallel to the line $\frac{x}{a} - \frac{z}{c} = 1$, $y = 0$.
3. A variable plane passes through a fixed point (α, β, γ) and meets the axes at A, B and C. Show that the locus of the point of intersection of the planes through A, B and C parallel to the coordinate planes is $\alpha x^{-1} + \beta y^{-1} + \gamma z^{-1} = 1$.
4. Show that the straight lines whose direction cosines are given by the equations $al + bm + cn = 0$ and $ul^2 + vm^2 + wn^2 = 0$ are parallel or perpendicular as
$\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$ or $a^2(v + w) + b^2(w + u) + c^2(u + v) = 0$.
5. Find the perpendicular distance of a corner of a cube of unit side length from a diagonal not passing through it.
6. A point P moves on a plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. A plane through P and perpendicular to OP meets the coordinate axes at A, B and C. If the planes through A, B and C parallel to the planes $x = 0$, $y = 0$ and $z = 0$, respectively, intersect at Q, find the locus of Q.
7. If the planes $x - cy - bz = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$ pass through a straight line, then find the value of $a^2 + b^2 + c^2 + 2abc$.
8. P is a point and PM and PN are the perpendiculars from P to z-x and x-y planes. If OP makes angles θ, α, β and γ with the plane OMN and the x-y, y-z and z-x planes, respectively, then prove that $\operatorname{cosec}^2\theta = \operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma$.

9. A variable plane $lx + my + nz = p$ (where l, m, n are direction cosines of normal) intersects the coordinate axes at points A, B and C, respectively. Show that the foot of the normal on the plane from the origin is the orthocentre of triangle ABC and hence find the coordinates of the circumcentre of triangle ABC.

10. Let $x - y \sin \alpha - z \sin \beta = 0$, $x \sin \alpha + z \sin \gamma - y = 0$ and $x \sin \beta + y \sin \gamma - z = 0$ be the equations of the planes such that $\alpha + \beta + \gamma = \pi/2$ (where α, β and $\gamma \neq 0$). Then show that there is a common line of intersection of the three given planes.

11. Let a plane $ax + by + cz + 1 = 0$, where a, b and c are parameters, make an angle 60° with the line $x = y = z$, 45° with the line $x = y - z = 0$ and θ with the plane $x = 0$. The distance of the plane from point (2, 1, 1) is 3 units. Find the value of θ and the equation of the plane.

12. Prove that for all values of λ and μ, the planes $\frac{2x}{a} + \frac{y}{b} + \frac{2z}{c} - 1 + \lambda\left(\frac{x}{a} - \frac{2y}{b} - \frac{z}{c} - 2\right) = 0$ and $\frac{4x}{a} - \frac{3y}{b} - 5 + \mu\left(\frac{5y}{b} + \frac{4z}{c} + 3\right) = 0$ intersect on the same line.

13. OA, OB and OC, with O as the origin, are three mutually perpendicular lines whose direction cosines and l_r, m_r and n_r ($r = 1, 2$ and 3). If the projections of OA and OB on the plane $z = 0$ make angles ϕ_1 and ϕ_2, respectively, with the x-axis, prove that $\tan(\phi_1 - \phi_2) = \pm n_3 / n_1 n_2$.

14. O is the origin and lines OA, OB and OC have direction cosines l_r, m_r and n_r ($r = 1, 2$ and 3). If lines OA', OB' and OC' bisect angles BOC, COA and AOB, respectively, prove that planes AOA', BOB' and COC' pass through the line $\frac{x}{l_1 + l_2 + l_3} = \frac{y}{m_1 + m_2 + m_3} = \frac{z}{n_1 + n_2 + n_3}$.

15. If P be any point on the plane $lx + my + nz = p$ and Q be a point on the line OP such that $OP \cdot OQ = p^2$, then find the locus of the point Q.

16. If a variable plane forms a tetrahedron of constant volume $64k^3$ with the coordinate planes, find the locus of the centroid of the tetrahedron.

Objective Type

Solutions on page 3.89

Each question has four choices *a, b, c* and *d*, out of which *only one* answer is correct. Find the correct answer.

1. In a three-dimensional xyz space, the equation $x^2 - 5x + 6 = 0$ represents
   a. points **b.** planes
   c. curves **d.** pair of straight lines

2. The line $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-1}{-1}$ intersects the curve $xy = c^2$, $z = 0$ if c is equal to
   a. ± 1 **b.** $\pm 1/3$ **c.** $\pm \sqrt{5}$ **d.** none of these

3. Let the equations of a line and a plane be $\frac{x+3}{2}=\frac{y-4}{3}=\frac{z+5}{2}$ and $4x-2y-z=1$, respectively, then

a. the line is parallel to the plane
b. the line is perpendicular to the plane
c. the line lies in the plane
d. none of these

4. The length of the perpendicular from the origin to the plane passing through the point $\vec{a}$ and containing the line $\vec{r}=\vec{b}+\lambda\vec{c}$ is

a. $\dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}$
b. $\dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}|}$
c. $\dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}$
d. $\dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{c}\times\vec{a}+\vec{a}\times\vec{b}|}$

5. The distance of point $A(-2, 3, 1)$ from the line PQ through $P(-3, 5, 2)$, which makes equal angles with the axes is

a. $2/\sqrt{3}$ **b.** $\sqrt{14/3}$ **c.** $16/\sqrt{3}$ **d.** $5/\sqrt{3}$

6. The Cartesian equation of the plane $\vec{r}=(1+\lambda-\mu)\hat{i}+(2-\lambda)\hat{j}+(3-2\lambda+2\mu)\hat{k}$ is

a. $2x+y=5$ **b.** $2x-y=5$ **c.** $2x+z=5$ **d.** $2x-z=5$

7. A unit vector parallel to the intersection of the planes $\vec{r}\cdot\left(\hat{i}-\hat{j}+\hat{k}\right)=5$ and $\vec{r}\cdot\left(2\hat{i}+\hat{j}-3\hat{k}\right)=4$ is

a. $\dfrac{2\hat{i}+5\hat{j}-3\hat{k}}{\sqrt{38}}$ **b.** $\dfrac{2\hat{i}-5\hat{j}+3\hat{k}}{\sqrt{38}}$ **c.** $\dfrac{-2\hat{i}-5\hat{j}-3\hat{k}}{\sqrt{38}}$ **d.** $\dfrac{-2\hat{i}+5\hat{j}-3\hat{k}}{\sqrt{38}}$

8. Let L_1 be the line $\vec{r}_1=2\hat{i}+\hat{j}-\hat{k}+\lambda(\hat{i}+2\hat{k})$ and let L_2 be the line $\vec{r}_2=3\hat{i}+\hat{j}+\mu(\hat{i}+\hat{j}-\hat{k})$. Let π be the plane which contains the line L_1 and is parallel to L_2. The distance of the plane π from the origin is

a. $\sqrt{2/7}$ **b.** $1/7$ **c.** $\sqrt{6}$ **d.** none

9. For the line $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$, which one of the following is incorrect?

a. it lies in the plane $x-2y+z=0$
b. it is same as line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$
c. it passes through (2, 3, 5)
d. it is parallel to the plane $x-2y+z-6=0$

10. The value of m for which straight line $3x-2y+z+3=0=4x-3y+4z+1$ is parallel to the plane $2x-y+mz-2=0$ is

a. -2 **b.** 8 **c.** -18 **d.** 11

11. The intercept made by the plane $\vec{r}\cdot\vec{n}=q$ on the x-axis is

a. $\dfrac{q}{\hat{i}\cdot\vec{n}}$ **b.** $\dfrac{\hat{i}\cdot\vec{n}}{q}$ **c.** $\dfrac{\hat{i}\cdot\vec{n}}{q}$ **d.** $\dfrac{q}{|\vec{n}|}$

12. Equation of a line in the plane $\pi \equiv 2x - y + z - 4 = 0$ which is perpendicular to the line l whose equation is $\frac{x-2}{1} = \frac{y-2}{-1} = \frac{z-3}{-2}$ and which passes through the point of intersection of l and π is

a. $\frac{x-2}{1} = \frac{y-1}{5} = \frac{z-1}{-1}$

b. $\frac{x-1}{3} = \frac{y-3}{5} = \frac{z-5}{-1}$

c. $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z+1}{1}$

d. $\frac{x-2}{2} = \frac{y-1}{-1} = \frac{z-1}{1}$

13. If the foot of the perpendicular from the origin to a plane is $P(a, b, c)$, the equation of the plane is

a. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$

b. $ax + by + cz = 3$

c. $ax + by + cz = a^2 + b^2 + c^2$

d. $ax + by + cz = a + b + c$

14. The equation of a plane which passes through the point of intersection of lines $\frac{x-1}{3} = \frac{y-2}{1} = \frac{z-3}{2}$, and $\frac{x-3}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and at greatest distance from point (0, 0, 0) is

a. $4x + 3y + 5z = 25$ b. $4x + 3y = 5z = 50$ c. $3x + 4y + 5z = 49$ d. $x + 7y - 5z = 2$

15. Let $A(\vec{a})$ and $B(\vec{b})$ be points on two skew lines $\vec{r} = \vec{a} + \lambda\vec{p}$ and $\vec{r} = \vec{b} + u\vec{q}$ and the shortest distance between the skew lines is 1, where $\vec{p}$ and $\vec{q}$ are unit vectors forming adjacent sides of a parallelogram enclosing an area of $\frac{1}{2}$ units. If an angle between AB and the line of shortest distance is 60°, then $AB =$

a. $\frac{1}{2}$ b. 2 c. 1 d. $\lambda \in R - \{0\}$

16. Let $A(1, 1, 1)$, $B(2, 3, 5)$ and $C(-1, 0, 2)$ be three points, then equation of a plane parallel to the plane ABC which is at distance 2 is

a. $2x - 3y + z + 2\sqrt{14} = 0$

b. $2x - 3y + z - \sqrt{14} = 0$

c. $2x - 3y + z + 2 = 0$

d. $2x - 3y + z - 2 = 0$

17. The point on the line $\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z+5}{-2}$ at a distance of 6 from the point $(2, -3, -5)$ is

a. $(3, -5, -3)$ b. $(4, -7, -9)$ c. $(0, 2, -1)$ d. $(-3, 5, 3)$

18. The coordinates of the foot of the perpendicular drawn from the origin to the line joining the points $(-9, 4, 5)$ and $(10, 0, -1)$ will be

a. $(-3, 2, 1)$ b. $(1, 2, 2)$ c. $(4, 5, 3)$ d. none of these

19. If $P_1 : \vec{r}\cdot\vec{n_1} - d_1 = 0$, $P_2 : \vec{r}\cdot\vec{n_2} - d_2 = 0$ and $P_3 : \vec{r}\cdot\vec{n_3} - d_3 = 0$ are three planes and $\vec{n_1}, \vec{n_2}$ and $\vec{n_3}$ are three non-coplanar vectors, then three lines $P_1 = 0, P_2 = 0$; $P_2 = 0, P_3 = 0$ and $P_3 = 0, P_1 = 0$ are

a. parallel lines b. coplanar lines c. coincident lines d. concurrent lines

20. The length of projection of the line segment joining the points (1, 0, – 1) and (–1, 2, 2) on the plane $x + 3y - 5z = 6$, is equal to

a. 2 b. $\sqrt{\frac{271}{53}}$ c. $\sqrt{\frac{472}{31}}$ d. $\sqrt{\frac{474}{35}}$

21. The number of planes that are equidistant from four non-coplanar points is

a. 3 b. 4 c. 7 d. 9

22. In a three dimensional co-ordinate system, P, Q and R are images of a point A (a, b, c) in the x-y, y-z and z-x planes, respectively. If G is the centroid of triangle PQR, then area of triangle AOG is (O is the origin)

a. 0 b. $a^2 + b^2 + c^2$ c. $\frac{2}{3}(a^2 + b^2 + c^2)$ d. none of these

23. A plane passing through (1, 1, 1) cuts positive direction of co-ordinate axes at A, B and C, then the volume of tetrahedron $OABC$ satisfies

a. $V \le \frac{9}{2}$ b. $V \ge \frac{9}{2}$ c. $V = \frac{9}{2}$ d. none of these

24. If lines $x = y = z$ and $x = \frac{y}{2} = \frac{z}{3}$, and third line passing through (1, 1, 1) form a triangle of area $\sqrt{6}$ units, then point of intersection of third line with second line will be

a. $(1, 2, 3)$ b. $(2, 4, 6)$ c. $\left(\frac{4}{3}, \frac{8}{3}, \frac{12}{3}\right)$ d. none of these

25. The point of intersection of the line passing through (0, 0, 1) and intersecting the lines $x + 2y + z = 1$, $-x + y - 2z = 2$ and $x + y = 2$, $x + z = 2$ with xy plane is

a. $\left(\frac{5}{3}, -\frac{1}{3}, 0\right)$ b. $(1, 1, 0)$ c. $\left(\frac{2}{3}, -\frac{1}{3}, 0\right)$ d. $\left(-\frac{5}{3}, \frac{1}{3}, 0\right)$

26. Shortest distance between the lines $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{1}$ and $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{1}$ is equal to

a. $\sqrt{14}$ b. $\sqrt{7}$ c. $\sqrt{2}$ d. none of these

27. Distance of point P $(\vec{p})$ from the plane $\vec{r} \cdot \vec{n} = 0$ is

a. $|\vec{p} \cdot \vec{n}|$ b. $\frac{|\vec{p} \times \vec{n}|}{|\vec{n}|}$ c. $\frac{|\vec{p} \cdot \vec{n}|}{|\vec{n}|}$ d. none of these

28. The reflection of the point $\vec{a}$ in the plane $\vec{r} \cdot \vec{n} = q$ is

a. $\vec{a} + \frac{(\vec{q} - \vec{a} \cdot \vec{n})}{|\vec{n}|}$ b. $\vec{a} + 2\left(\frac{(\vec{q} - \vec{a} \cdot \vec{n})}{|\vec{n}|^2}\right)\vec{n}$

c. $\vec{a} + \frac{2(\vec{q} + \vec{a} \cdot \vec{n})}{|\vec{n}|}\vec{n}$ d. none of these

29. Line $\vec{r} = \vec{a} + \lambda\vec{b}$ will not meet the plane $\vec{r} \cdot \vec{n} = q$, if

a. $\vec{b} \cdot \vec{n} = 0, \vec{a} \cdot \vec{n} = q$ b. $\vec{b} \cdot \vec{n} \ne 0, \vec{a} \cdot \vec{n} \ne q$

c. $\vec{b} \cdot \vec{n} = 0, \vec{a} \cdot \vec{n} \ne q$ d. $\vec{b} \cdot \vec{n} \ne 0, \vec{a} \cdot \vec{n} = q$

30. If a line makes an angle of $\frac{\pi}{4}$ with the positive direction of each of x-axis and y-axis, then the angle that the line makes with the positive direction of the z-axis is

a. $\frac{\pi}{3}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{2}$ d. $\frac{\pi}{6}$

31. The ratio in which the plane $\vec{r}\cdot(\vec{i}-2\vec{j}+3\vec{k})=17$ divides the line joining the points $-2\vec{i}+4\vec{j}+7\vec{k}$ and $3\vec{i}-5\vec{j}+8\vec{k}$ is

a. 1 : 5 b. 1 : 10 c. 3 : 5 d. 3 :10

32. The image of the point $(-1, 3, 4)$ in the plane $x-2y=0$ is

a. $\left(-\frac{17}{3},-\frac{19}{3},4\right)$ b. $(15,11,4)$ c. $\left(-\frac{17}{3},-\frac{19}{3},1\right)$ d. $\left(\frac{9}{5},-\frac{13}{5},4\right)$

33. The distance between the line: $\vec{r}=2\hat{i}-2\hat{j}+3\hat{k}+\lambda(\hat{i}-\hat{j}+4\hat{k})$ and the plane $\vec{r}\cdot(\hat{i}+5\hat{j}+\hat{k})=5$ is

a. $\frac{10}{3\sqrt{3}}$ b. $\frac{10}{9}$ c. $\frac{10}{3}$ d. $\frac{3}{10}$

34. Let L be the line of intersection of the planes $2x+3y+z=1$ and $x+3y+2z=2$. If L makes an angle α with the positive x-axis, then $\cos\alpha$ equals

a. $\frac{1}{2}$ b. 1 c. $\frac{1}{\sqrt{2}}$ d. $\frac{1}{\sqrt{3}}$

35. The length of the perpendicular drawn from (1, 2, 3) to the line $\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}$ is

a. 4 b. 5 c. 6 d. 7

36. If angle θ between the line $\frac{x+1}{1}=\frac{y-1}{2}=\frac{z-2}{2}$ and the plane $2x-y+\sqrt{\lambda}\,z+4=0$ is such that $\sin\theta=\frac{1}{3}$, the value of λ is

a. $\frac{-3}{5}$ b. $\frac{5}{3}$ c. $\frac{-4}{3}$ d. $\frac{3}{4}$

37. The intersection of the spheres $x^2+y^2+z^2+7x-2y-z=13$ and $x^2+y^2+z^2-3x+3y+4z=8$ is the same as the intersection of one of the spheres and the plane

a. $x-y-z=1$ b. $x-2y-z=1$ c. $x-y-2z=1$ d. $2x-y-z=1$

38. A plane makes intercepts OA, OB and OC whose measurements are b and c on the OX, OY and OZ axes. The area of triangle ABC is

a. $\frac{1}{2}(ab+bc+ca)$ b. $\frac{1}{2}abc\,(a+b+c)$

c. $\frac{1}{2}(a^2b^2+b^2c^2+c^2a^2)^{1/2}$ d. $\frac{1}{2}(a+b+c)^2$

39. A line makes an angle θ with each of the x- and z-axes. If the angle β, which it makes with y-axis, is such that $\sin^2 \beta = 3 \sin^2 \theta$, then $\cos^2 \theta$ equals

a. $\frac{2}{3}$ **b.** $\frac{1}{5}$ **c.** $\frac{3}{5}$ **d.** $\frac{2}{5}$

40. The shortest distance from the plane $12x + y + 3z = 327$ to the sphere $x^2 + y^2 + z^2 + 4x - 2y - 6z = 155$ is

a. 39 **b.** 26 **c.** $41\frac{4}{13}$ **d.** 13

41. A tetrahedron has vertices $O(0, 0, 0)$, $A(1, 2, 1)$, $B(2, 1, 3)$ and $C(-1, 1, 2)$, then angle between faces OAB and ABC will be:

a. $\cos^{-1}\left(\frac{17}{31}\right)$ **b.** 30° **c.** 90° **d.** $\cos^{-1}\left(\frac{19}{35}\right)$

42. The radius of the circle in which the sphere $x^2 + y^2 + z^2 + 2z - 2y - 4z - 19 = 0$ is cut by the plane $x + 2y + 2z + 7 = 0$ is

a. 2 **b.** 3 **c.** 4 **d.** 1

43. The lines: $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar if:

a. $k = 1$ or -1 **b.** $k = 0$ or -3 **c.** $k = 3$ or -3 **d.** $k = 0$ or -1

44. The point of intersection of the lines $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1}$ and $\frac{x+3}{-36} = \frac{y-3}{2} = \frac{z-6}{4}$ is

a. $\left(21, \frac{5}{3}, \frac{10}{3}\right)$ **b.** $(2, 10, 4)$ **c.** $(-3, 3, 6)$ **d.** $(5, 7, -2)$

45. Two systems of rectangular axes have the same origin. If a plane cuts them at distance a, b, c and a', b', c' from the origin, then:

a. $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2} = 0$ **b.** $\frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{c^2} + \frac{1}{a'^2} - \frac{1}{b'^2} - \frac{1}{c'^2} = 0$

c. $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a'^2} - \frac{1}{b'^2} - \frac{1}{c'^2} = 0$ **d.** $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2} = 0$

46. The plane, which passes through the point (3, 2, 0) and the line $\frac{x-3}{1} = \frac{y-6}{5} = \frac{z-4}{4}$ is:

a. $x - y + z = 1$ **b.** $x + y + z = 5$ **c.** $x + 2y - z = 1$ **d.** $2x - y + z = 5$

47. The direction ratios of a normal to the plane through (1, 0, 0) and (0, 1, 0), which makes an angle of $\frac{\pi}{4}$ with the plane $x + y = 3$ are

a. $< 1, \sqrt{2}, 1 >$ **b.** $< 1, 1, \sqrt{2} >$ **c.** $< 1, 1, 2 >$ **d.** $< \sqrt{2}, 1, 1 >$

48. The centre of the circle given by: $\vec{r}\cdot(\hat{i}+2\hat{j}+2\hat{k})=15$ and $|\vec{r}-(\hat{j}+2\hat{k})|=4$ is

a. $(0,1,2)$ b. $(1,3,4)$ c. $(-1,3,4)$ d. none of these

49. The lines which intersect the skew lines $y=mx, z=c$; $y=-mx, z=-c$ and the x-axis lie on the surface

a. $cz=mxy$ b. $xy=cmz$ c. $cy=mxz$ d. none of these

50. Distance of the point $P(\vec{p})$ from the line $\vec{r}=\vec{a}+\lambda\vec{b}$ is

a. $\left|(\vec{a}-\vec{p})+\dfrac{((\vec{p}-\vec{a})\cdot\vec{b})\vec{b}}{|\vec{b}|^2}\right|$

b. $\left|(\vec{b}-\vec{p})+\dfrac{((\vec{p}-\vec{a})\cdot\vec{b})\vec{b}}{|\vec{b}|^2}\right|$

c. $\left|(\vec{a}-\vec{p})+\dfrac{((\vec{p}-\vec{b})\cdot\vec{b})\vec{b}}{|\vec{b}|^2}\right|$

d. none of these

51. From the point P(a, b, c), let perpendiculars PL and PM be drawn to YOZ and ZOX planes, respectively. Then the equation of the plane OLM is

a. $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=0$

b. $\dfrac{x}{a}+\dfrac{y}{b}-\dfrac{z}{c}=0$

c. $\dfrac{x}{a}-\dfrac{y}{b}-\dfrac{z}{c}=0$

d. $\dfrac{x}{a}-\dfrac{y}{b}+\dfrac{z}{c}=0$

52. The plane $\vec{r}\cdot\vec{n}=q$ will contain the line $\vec{r}=\vec{a}+\lambda\vec{b}$, if

a. $\vec{b}\cdot\vec{n}\neq 0, \vec{a}\cdot\vec{n}\neq q$

b. $\vec{b}\cdot\vec{n}=0, \vec{a}\cdot\vec{n}\neq q$

c. $\vec{b}\cdot\vec{n}=0, \vec{a}\cdot\vec{n}=q$

d. $\vec{b}\cdot\vec{n}\neq 0, \vec{a}\cdot\vec{n}=q$

53. The projection of point $P(\vec{p})$ on the plane $\vec{r}\cdot\vec{n}=q$ is $(\vec{s})$, then

a. $\vec{s}=\dfrac{(q-\vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$

b. $\vec{s}=\vec{p}+\dfrac{(q-\vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$

c. $\vec{s}=\vec{p}-\dfrac{(\vec{p}\cdot\vec{n}\cdot)\vec{n}}{|\vec{n}|^2}$

d. $\vec{s}=\vec{p}-\dfrac{(q-\vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$

54. The angle between $\hat{i}$ line of the intersection of the plane $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=0$ and $\vec{r}\cdot(3\hat{i}+3\hat{j}+\hat{k})=0$, is

a. $\cos^{-1}\left(\dfrac{1}{3}\right)$ b. $\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ c. $\cos^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$ d. none of these

55. The line $\dfrac{x+6}{5}=\dfrac{y+10}{3}=\dfrac{z+14}{8}$ is the hypotenuse of an isosceles right angled triangle whose opposite vertex is (7, 2, 4). Then which of the following is not the side of the triangle?

a. $\frac{x-7}{2}=\frac{y-2}{-3}=\frac{z-4}{6}$

b. $\frac{x-7}{3}=\frac{y-2}{6}=\frac{z-4}{2}$

c. $\frac{x-7}{3}=\frac{y-2}{5}=\frac{z-4}{-1}$

d. none of these

56. The equation of the plane which passes through the line of intersection of planes $\vec{r}\cdot\vec{n}_1=q_1$, $\vec{r}\cdot\vec{n}_2=q_2$ and is parallel to the line of intersection of planes $\vec{r}\cdot\vec{n}_3=q_3$ and $\vec{r}\cdot\vec{n}_4=q_4$, is

a. $[\vec{n}_2\ \vec{n}_3\ \vec{n}_4](\vec{r}\cdot\vec{n}_1-q_1)=[\vec{n}_1\ \vec{n}_3\ \vec{n}_4](\vec{r}\cdot\vec{n}_2-q_2)$

b. $[\vec{n}_1\ \vec{n}_2\ \vec{n}_3](\vec{r}\cdot\vec{n}_4-q_4)=[\vec{n}_4\ \vec{n}_3\ \vec{n}_1](\vec{r}\cdot\vec{n}_2-q_2)$

c. $[\vec{n}_4\ \vec{n}_3\ \vec{n}_1](\vec{r}\cdot\vec{n}_4-q_4)=[\vec{n}_1\ \vec{n}_2\ \vec{n}_3](\vec{r}\cdot\vec{n}_2-q_2)$

d. none of these

57. Consider triangle AOB in the x-y plane, where $A\equiv(1,0,0)$; $B\equiv(0,2,0)$; and $O\equiv(0,0,0)$. The new position of O, when triangle is rotated about side AB by 90° can be

a. $\left(\frac{4}{5},\frac{3}{5},\frac{2}{\sqrt{5}}\right)$ b. $\left(\frac{-3}{5},\frac{\sqrt{2}}{5},\frac{2}{\sqrt{5}}\right)$ c. $\left(\frac{4}{5},\frac{2}{5},\frac{2}{\sqrt{5}}\right)$ d. $\left(\frac{4}{5},\frac{2}{5},\frac{1}{\sqrt{5}}\right)$

58. Let $\vec{a}=\hat{i}+\hat{j}$ and $\vec{b}=2\hat{i}-\hat{k}$, then the point of intersection of the lines $\vec{r}\times\vec{a}=\vec{b}\times\vec{a}$ and $\vec{r}\times\vec{b}=\vec{a}\times\vec{b}$ is

a. (3, – 1, 1) b. (3, 1, – 1) c. (–3, 1, 1) d. (–3, – 1, – 1)

59. The coordinates of the point P on the line $\vec{r}=(\hat{i}+\hat{j}+\hat{k})+\lambda(-\hat{i}+\hat{j}-\hat{k})$ which is nearest to the origin is

a. $\left(\frac{2}{3},\frac{4}{3},\frac{2}{3}\right)$ b. $\left(-\frac{2}{3},-\frac{4}{3},\frac{2}{3}\right)$ c. $\left(\frac{2}{3},\frac{4}{3},-\frac{2}{3}\right)$ d. None of these

60. The ratio in which the line segment joining the points whose position vectors are $2\hat{i}-4\hat{j}-7\hat{k}$ and $-3\hat{i}+5\hat{j}-8\hat{k}$ is divided by the plane whose equation is $\hat{r}\cdot(\hat{i}-2\hat{j}+3\hat{k})=13$, is

a. 13 : 12 internally b. 12 : 25 externally c. 13 : 25 internally d. 37 : 25 internally

61. Which of the following are equations for the plane passing through the points P (1, 1, – 1), Q (3, 0, 2) and R (– 2, 1, 0)?

a. $(2\hat{i}-3\hat{j}+3\hat{k})\cdot((x+2)\hat{i}+(y-1)\hat{j}+z\hat{k})=0$

b. $x=3-t$, $y=-11t$, $z=2-3t$

c. $(x+2)+11(y-1)=3z$

d. $(2\hat{i}-\hat{j}+3\hat{k})\times(-3\hat{i}+\hat{k})\cdot((x+2)\hat{i}+(y-1)\hat{j}+z\hat{k})=0$

62. Given $\vec{\alpha}=3\hat{i}+\hat{j}+2\hat{k}$ and $\vec{\beta}=\hat{i}-2\hat{j}-4\hat{k}$ are the position vectors of the points A and B. Then the distance of the point $-\hat{i}+\hat{j}+\hat{k}$ from the plane passing through B and perpendicular to AB is

a. 5 b. 10 c. 15 d. 20

63. L_1 and L_2 are two lines whose vector equations are

$L_1 : \vec{r} = \lambda((\cos\theta + \sqrt{3})\hat{i} + (\sqrt{2}\sin\theta)\hat{j} + (\cos\theta - \sqrt{3})\hat{k})$

$L_2 : \vec{r} = \mu\left(a\hat{i} + b\hat{j} + c\hat{k}\right)$, where λ and μ are scalars and α is the acute angle between L_1 and L_2. If the angle 'α' is independent of θ, then the value of 'α' is

a. $\frac{\pi}{6}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{3}$ d. $\frac{\pi}{2}$

64. The shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is

a. $\sqrt{30}$ b. $2\sqrt{30}$ c. $5\sqrt{30}$ d. $3\sqrt{30}$

65. The line through $\hat{i} + 3\hat{j} + 2\hat{k}$ and $\perp$ to the line $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} + 6\hat{j} + \hat{k}) + \mu(\hat{i} + 2\hat{j} + 3\hat{k})$ is

a. $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(-\hat{i} + 5\hat{j} - 3\hat{k})$
b. $\vec{r} = \hat{i} + 3\hat{j} + 2\hat{k} + \lambda(\hat{i} - 5\hat{j} + 3\hat{k})$
c. $\vec{r} = \hat{i} + 3\hat{j} + 2\hat{k} + \lambda(\hat{i} + 5\hat{j} + 3\hat{k})$
d. $\vec{r} = \hat{i} + 3\hat{j} + 2\hat{k} + \lambda(-\hat{i} - 5\hat{j} - 3\hat{k})$

66. The equation of the plane passing through the lines $\frac{x-4}{1} = \frac{y-3}{1} = \frac{z-2}{2}$ and $\frac{x-3}{1} = \frac{y-2}{-4} = \frac{z}{5}$ is

a. $11x - y - 3z = 35$ b. $11x + y - 3z = 35$ c. $11x - y + 3z = 35$ d. none of these

67. The three planes $4y + 6z = 5$; $2x + 3y + 5z = 5$ and $6x + 5y + 9z = 10$

a. meet in a point
b. have a line in common
c. form a triangular prism
d. none of these

68. The equation of the plane through the line of intersection of the planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ and parallel to the line $y = 0$ and $z = 0$ is

a. $(ab' - a'b)x + (bc' - b'c)y + (ad' - a'd) = 0$
b. $(ab' - a'b)x + (bc' - b'c)y + (ad' - a'd)z = 0$
c. $(ab' - a'b)y + (ac' - a'c)z + (ad' - a'd) = 0$
d. none of these

69. Equation of the plane passing through the points (2, 2, 1) and (9, 3, 6), and $\perp$ to the plane $2x + 6y + 6z - 1 = 0$ is

a. $3x + 4y + 5z = 9$
b. $3x + 4y - 5z = 9$
c. $3x + 4y - 5z = 9$
d. none of the above

70. Value of λ such that the line $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{\lambda}$ is $\perp$ to normal to the plane $\vec{r} \cdot (2\vec{i} + 3\vec{j} + 4\vec{k}) = 0$ is

a. $-\frac{13}{4}$ b. $-\frac{17}{4}$ c. 4 d. none of these

71. The equation of the plane through the intersection of the planes $x + 2y + 3z - 4 = 0$ and $4x + 3y + 2z + 1 = 0$ and passing through the origin is

a. $17x + 14y + 11z = 0$
b. $7x + 4y + z = 0$
c. $x + 14y + 11z = 0$
d. $17x + y + z = 0$

72. The plane $4x + 7y + 4z + 81 = 0$ is rotated through a right angle about its line of intersection with the plane $5x + 3y + 10z = 25$. The equation of the plane in its new position is

a. $x - 4y + 6z = 106$
b. $x - 8y + 13z = 103$
c. $x - 4y + 6z = 110$
d. $x - 8y + 13z = 105$

73. The vector equation of the plane passing through the origin and the line of intersection of the planes $\vec{r} \cdot \vec{a} = \lambda$ and $\vec{r} \cdot \vec{b} = \mu$ is

a. $\vec{r} \cdot (\lambda \vec{a} - \mu \vec{b}) = 0$
b. $\vec{r} \cdot (\lambda \vec{b} - \mu \vec{a}) = 0$
c. $\vec{r} \cdot (\lambda \vec{a} + \mu \vec{b}) = 0$
d. $\vec{r} \cdot (\lambda \vec{b} + \mu \vec{a}) = 0$

74. The lines $\vec{r} = \vec{a} + \lambda (\vec{b} \times \vec{c})$ and $\vec{r} = \vec{b} + \mu (\vec{c} \times \vec{a})$ will intersect if

a. $\vec{a} \times \vec{c} = \vec{b} \times \vec{c}$
b. $\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c}$
c. $\vec{b} \times \vec{a} = \vec{c} \times \vec{a}$
d. none of these

75. The projection of the line $\frac{x+1}{-1} = \frac{y}{2} = \frac{z-1}{3}$ on the plane $x - 2y + z = 6$ is the line of intersection of this plane with the plane

a. $2x + y + 2 = 0$
b. $3x + y - z = 2$
c. $2x - 3y + 8z = 3$
d. none of these

76. The direction cosines of a line satisfy the relations $\lambda (l + m) = n$ and $mn + nl + lm = 0$. The value of λ, for which the two lines are perpendicular to each other, is

a. 1
b. 2
c. 1/2
d. none of these

77. The intercepts made on the axes by the plane which bisects the line joining the points (1, 2, 3) and (–3, 4, 5) at right angles are

a. $\left(-\frac{9}{2}, 9, 9\right)$
b. $\left(\frac{9}{2}, 9, 9\right)$
c. $\left(9, -\frac{9}{2}, 9\right)$
d. $\left(9, \frac{9}{2}, 9\right)$

78. The pair of lines whose direction cosines are given by the equations $3l + m + 5n = 0$ and $6mn - 2nl + 5lm = 0$, are

a. parallel
b. perpendicular
c. inclined at $\cos^{-1}\left(\frac{1}{6}\right)$
d. none of these

79. A sphere of constant radius $2k$ passes through the origin and meets the axes in A, B and C. The locus of a centroid of the tetrahedron $OABC$ is

a. $x^2 + y^2 + z^2 = 4k^2$
b. $x^2 + y^2 + z^2 = k^2$
c. $2(k^2 + y^2 + z)^2 = k^2$
d. none of these

80. A plane passes through a fixed point (a, b, c). The locus of the foot of the perpendicular to it from the origin is a sphere of radius

a. $\frac{1}{2}\sqrt{a^2 + b^2 + c^2}$
b. $\sqrt{a^2 + b^2 + c^2}$
c. $a^2 + b^2 + c^2$
d. $\frac{1}{2}(a^2 + b^2 + c^2)$

81. What is the nature of the intersection of the set of planes $x + ay + (b + c)z + d = 0$, $x + by + (c + a)z + d = 0$ and $x + cy + (a + b)z + d = 0$?

a. They meet at a point
b. They form a triangular prism
c. They pass through a line
d. They are at equal distance from the origin

82. Find the equation of a straight line in the plane $\vec{r} \cdot \vec{n} = d$ which is parallel to $\vec{r} = \vec{a} + \lambda \vec{b}$ and passes through the foot of the perpendicular drawn from point $P(\vec{a})$ to $\vec{r} \cdot \vec{n} = d$ (where $\vec{n} \cdot \vec{b} = 0$).

a. $\vec{r} = \vec{a} + \left(\dfrac{d - \vec{a} \cdot \vec{n}}{n^2}\right)\vec{n} + \lambda \vec{b}$
b. $\vec{r} = \vec{a} + \left(\dfrac{d - \vec{a} \cdot \vec{n}}{n}\right)\vec{n} + \lambda \vec{b}$
c. $\vec{r} = \vec{a} + \left(\dfrac{\vec{a} \cdot \vec{n} - d}{n^2}\right)\vec{n} + \lambda \vec{b}$
d. $\vec{r} = \vec{a} + \left(\dfrac{\vec{a} \cdot \vec{n} - d}{n}\right)\vec{n} + \lambda \vec{b}$

83. What is the equation of the plane which passes through the z-axis and is perpendicular to the line $\dfrac{x-a}{\cos\theta} = \dfrac{y+2}{\sin\theta} = \dfrac{z-3}{0}$?

a. $x + y \tan \theta = 0$
b. $y + x \tan\theta = 0$
c. $x \cos\theta - y \sin\theta = 0$
d. $x \sin\theta - y \cos\theta = 0$

84. A straight line L on the xy-plane bisects the angle between OX and OY. What are the direction cosines of L ?

a. $< (1/\sqrt{2}), (1/\sqrt{2}), 0 >$
b. $< (1/2), (\sqrt{3}/2), 0 >$
c. $< 0, 0, 1 >$
d. $< (2/3), (2/3), (1/3) >$

85. For what value(s) of a, will the two points $(1, a, 1)$ and $(-3, 0, a)$ lie on opposite sides of the plane $3x + 4y - 12z + 13 = 0$?

a. $a < -1$ or $a > 1/3$
b. $a = 0$ only
c. $0 < a < 1$
d. $-1 < a < 1$

86. If the plane $\dfrac{x}{2} + \dfrac{y}{3} + \dfrac{z}{6} = 1$ cuts the axes of coordinates at points A, B and C, then find the area of the triangle ABC.

a. 18 sq unit
b. 36 sq unit
c. $3\sqrt{14}$ sq unit
d. $2\sqrt{14}$ sq unit

Multiple Correct Answers Type

Solutions on page 3.111

Each question has four choices *a, b, c* and *d*, out of which *one or more* are correct.

1. Let PM be the perpendicular from the point $P(1, 2, 3)$ to the x-y plane. If $\overrightarrow{OP}$ makes an angle θ with the positive direction of the z-axis and $\overrightarrow{OM}$ makes an angle ϕ with the positive direction of x-axis, where O is the origin and θ and ϕ are acute angles, then

   a. $\cos\theta\cos\phi = 1/\sqrt{14}$ **b.** $\sin\theta\sin\phi = 2/\sqrt{14}$ **c.** $\tan\phi = 2$ **d.** $\tan\theta = \sqrt{5}/3$

2. Let P_1 denote the equation of a plane to which the vector $(\hat{i} + \hat{j})$ is normal and which contains the line whose equation is $\vec{r} = \hat{i} + \hat{j} + \hat{k} + \lambda(\hat{i} - \hat{j} - \hat{k})$ and P_2 denote the equation of the plane containing the line L and a point with position vector $\hat{j}$. Which of the following holds good?

   a. The equation of P_1 is $x + y = 2$.
   b. The equation of P_2 is $\vec{r}\cdot(\hat{i} - 2\hat{j} + \hat{k}) = 2$.
   c. The acute angle between P_1 and P_2 is $\cot^{-1}(\sqrt{3})$.
   d. The angle between the plane P_2 and the line L is $\tan^{-1}\sqrt{3}$.

3. If the planes $\vec{r}\cdot(\hat{i} + \hat{j} + \hat{k}) = q_1$, $\vec{r}\cdot(\hat{i} + 2a\hat{j} + \hat{k}) = q_2$ and $\vec{r}\cdot(a\hat{i} + a^2\hat{j} + \hat{k}) = q_3$ intersect in a line, then the value of a is

   a. 1 **b.** 1/2 **c.** 2 **d.** 0

4. A line with direction cosines proportional to 1, –5 and –2 meets lines $x = y + 5 = z + 1$ and $x + 5 = 3y = 2z$. The coordinates of each of the points of the intersection are given by

   a. $(2, -3, 1)$ **b.** $(1, 2, 3)$ **c.** $(0, 5/3, 5/2)$ **d.** $(3, -2, 2)$

5. Let $P = 0$ be the equation of a plane passing through the line of intersection of the planes $2x - y = 0$ and $3z - y = 0$ and perpendicular to the plane $4x + 5y - 3z = 8$. Then the points which lie on the plane $P = 0$ is/are

   a. (0, 9, 17) **b.** (1/7, 2, 1/9) **c.** (1, 3, – 4) **d.** (1/2, 1, 1/3)

6. The equation of the lines $x + y + z - 1 = 0$ and $4x + y - 2z + 2 = 0$ written in the symmetrical form is

   a. $\dfrac{x-1}{2} = \dfrac{y+2}{-1} = \dfrac{z-2}{2}$ **b.** $\dfrac{x+(1/2)}{1} = \dfrac{y-1}{-2} = \dfrac{z-(1/2)}{1}$

   c. $\dfrac{x}{1} = \dfrac{y}{-2} = \dfrac{z-1}{1}$ **d.** $\dfrac{x+1}{1} = \dfrac{y-2}{-2} = \dfrac{z-0}{1}$

7. Consider the planes $3x - 6y + 2z + 5 = 0$ and $4x - 12y + 3z = 3$. The plane $67x - 162y + 47z + 44 = 0$ bisects the angle between the given planes which

   a. contains the origin **b.** is acute **c.** is obtuse **d.** none of these

8. If the lines $\dfrac{x-2}{1} = \dfrac{y-3}{1} = \dfrac{z-4}{\lambda}$ and $\dfrac{x-1}{\lambda} = \dfrac{y-4}{2} = \dfrac{z-5}{1}$ intersect, then

   a. $\lambda = -1$ **b.** $\lambda = 2$ **c.** $\lambda = -3$ **d.** $\lambda = 0$

9. The equations of the plane which passes through (0, 0, 0) and which is equally inclined to the planes $x - y + z - 3 = 0$ and $x + y + z + 4 = 0$ is/are

a. $y = 0$ b. $x = 0$ c. $x + y = 0$ d. $x + z = 0$

10. The x-y plane is rotated about its line of intersection with the y-z plane by 45°, then the equation of the new plane is/are

a. $z + x = 0$ b. $z - y = 0$ c. $x + y + z = 0$ d. $z - x = 0$

11. The equation of the plane which is equally inclined to the lines $\frac{x-1}{2} = \frac{y}{-2} = \frac{z+2}{-1}$ and $\frac{x+3}{8} = \frac{y-4}{1} = \frac{z}{-4}$ and passing through the origin is/are

a. $14x - 5y - 7z = 0$ b. $2x + 7y - z = 0$ c. $3x - 4y - z = 0$ d. $x + 2y - 5z = 0$

12. Which of the following lines lie on the plane $x + 2y - z + 4 = 0$?

a. $\frac{x-1}{1} = \frac{y}{-1} = \frac{z-5}{-1}$

b. $x - y + z = 2x + y - z = 0$

c. $\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(3\hat{i} + \hat{j} + 5\hat{k})$

d. none of these

13. If the volume of tetrahedron $ABCD$ is 1 cubic units, where $A(0, 1, 2)$, $B(-1, 2, 1)$ and $C(1, 2, 1)$, then the locus of point D is

a. $x + y - z = 3$ b. $y + z = 6$ c. $y + z = 0$ d. $y + z = -3$

14. A rod of length 2 units whose one end is (1, 0, –1) and other end touches the plane $x - 2y + 2z + 4 = 0$, then

a. The rod sweeps the figure whose volume is π cubic units.

b. The area of the region which the rod traces on the plane is 2π.

c. The length of projection of the rod on the plane is $\sqrt{3}$ units.

d. The centre of the region which the rod traces on the plane is $\left(\frac{2}{3}, \frac{2}{3}, \frac{-5}{3}\right)$.

15. Consider a set of points R in the space which is at a distance of 2 units from the line $\frac{x}{1} = \frac{y-1}{-1} = \frac{z+2}{2}$ between the planes $x - y + 2z + 3 = 0$ and $x - y + 2z - 2 = 0$.

a. The volume of the bounded figure by points R and the planes is $(10/3\sqrt{3})\pi$ cube units.

b. The area of the curved surface formed by the set of points R is $(20\pi/\sqrt{6})$ sq. units.

c. The volume of the bounded figure by the set of points R and the planes is $(20\pi/\sqrt{6})$ cubic units.

d. The area of the curved surface formed by the set of points R is $(10/\sqrt{3})\pi$ sq. units.

16. The equation of a line passing through the point $\vec{a}$ parallel to the plane $\vec{r}.\vec{n} = q$ and perpendicular to the line $\vec{r} = \vec{b} + t\vec{c}$ is

a. $\vec{r} = \vec{a} + \lambda(\vec{n} \times \vec{c})$

b. $(\vec{r} - \vec{a})\times(\vec{n} \times \vec{c}) = 0$

c. $\vec{r} = \vec{b} + \lambda(\vec{n} \times \vec{c})$

d. none of these

Reasoning Type

Solutions on page 3.116

Each question has four choices *a*, *b*, *c* and *d*, out of which *only one* is correct. Each question contains Statement 1 and Statement 2.

- **a.** Both the statements are true, and Statement 2 is the correct explanation for Statement 1.
- **b.** Both the statements are true, but Statement 2 is not the correct explanation for Statement 1.
- **c.** Statement 1 is true and Statement 2 is false.
- **d.** Statment 1 is false and Statement 2 is true.

1. **Statement 1:** Lines $\vec{r} = \hat{i} - \hat{j} + \lambda(\hat{i} + \hat{j} - \hat{k})$ and $\vec{r} = 2\hat{i} - \hat{j} + \mu(\hat{i} + \hat{j} - \hat{k})$ do not intersect.

   Statement 2: Skew lines never intersect.

2. **Statement 1:** Lines $\vec{r} = \hat{i} + \hat{j} - \hat{k} + \lambda(3\hat{i} - \hat{j})$ and $\vec{r} = 4\hat{i} - \hat{k} + \mu(2\hat{i} + 3\hat{k})$ intersect.

   Statement 2: If $\vec{b} \times \vec{d} = \vec{0}$, then lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \lambda\vec{d}$ do not intersect.

3. The equation of two straight lines are $\frac{x-1}{2} = \frac{y+3}{1} = \frac{z-2}{-3}$ and $\frac{x-2}{1} = \frac{y-1}{-3} = \frac{z+3}{2}$.

   Statement 1: The given lines are coplanar.

   Statement 2: The equations $2x_1 - y_1 = 1, x_1 + 3y_1 = 4$ and $3x_1 + 2y_1 = 5$ are consistent.

4. **Statement 1:** A plane passes through the point $A(2, 1, -3)$. If distance of this plane from origin is maximum, then its equation is $2x + y - 3z = 14$.

   Statement 2: If the plane passing through the point $A(\vec{a})$ is at maximum distance from origin, then normal to the plane is vector $\vec{a}$.

5. **Statement 1:** Line $\frac{x-1}{1} = \frac{y-0}{2} = \frac{z+2}{-1}$ lies in the plane $2x - 3y - 4z - 10 = 0$.

   Statement 2: If line $\vec{r} = \vec{a} + \lambda\vec{b}$ lies in the plane $\vec{r} \cdot \vec{c} = n$ (where n is scalar), then $\vec{b} \cdot \vec{c} = 0$.

6. **Statement 1:** Let θ be the angle between the line $\frac{x-2}{2} = \frac{y-1}{-3} = \frac{z+2}{-2}$ and the plane $x + y - z = 5$. Then $\theta = \sin^{-1}(1/\sqrt{51})$.

   Statement 2: The angle between a straight line and a plane is the complement of the angle between the line and the normal to the plane.

7. **Statement 1:** Let $A(\vec{i} + \vec{j} + \vec{k})$ and $B(\vec{i} - \vec{j} + \vec{k})$ be two points. Then point $P(2\vec{i} + 3\vec{j} + \vec{k})$ lies exterior to the sphere with AB as its diameter.

   Statement 2: If A and B are any two points and P is a point in space such that $\overrightarrow{PA} \cdot \overrightarrow{PB} > 0$, then point P lies exterior to the sphere with AB as its diameter.

8. **Statement 1:** There exists a unique sphere which passes through the three non-collinear points and which has the least radius.

   Statement 2: The centre of such a sphere lies on the plane determined by the given three points.

9. **Statement 1:** There exist two points on the line $\frac{x-1}{1} = \frac{y}{-1} = \frac{z+2}{2}$ which are at a distance of 2 units from point $(1, 2, -4)$.

   Statement 2: Perpendicular distance of point $(1, 2, -4)$ from the line $\frac{x-1}{1} = \frac{y}{-1} = \frac{z+2}{2}$ is 1 unit.

10. **Statement 1:** The shortest distance between the lines $\frac{x}{-3}=\frac{y-1}{1}=\frac{z+1}{-1}$ and $\frac{x-2}{1}=\frac{y-3}{2}=\left(\frac{z+(13/7)}{-1}\right)$ is zero.

Statement 2: The given lines are perpendicular.

Linked Comprehension Type

Solutions on page 3.117

Based on each paragraph, three multiple-choice questions have to be answered. Each question has four choices *a, b, c* and *d*, out of which *only one* is correct.

For Problems 1–3

Given four points $A(2, 1, 0)$, $B(1, 0, 1)$, $C(3, 0, 1)$ and $D(0, 0, 2)$. Point D lies on a line L orthogonal to the plane determined by the points A, B and C.

1. The equation of the plane ABC is

   a. $x+y+z-3=0$ **b.** $y+z-1=0$ **c.** $x+z-1=0$ **d.** $2y+z-1=0$

2. The equation of the line L is

   a. $\vec{r}=2\hat{k}+\lambda(\hat{i}+\hat{k})$ **b.** $\vec{r}=2\hat{k}+\lambda(2\hat{j}+\hat{k})$ **c.** $\vec{r}=2\hat{k}+\lambda(\hat{j}+\hat{k})$ **d.** none

3. The perpendicular distance of D from the plane ABC is

   a. $\sqrt{2}$ **b.** 1/2 **c.** 2 **d.** $1/\sqrt{2}$

For Problems 4–6

A ray of light comes along the line $L=0$ and strikes the plane mirror kept along the plane $P=0$ at B. $A(2, 1, 6)$ is a point on the line $L=0$ whose image about $P=0$ is A'. It is given that $L=0$ is $\frac{x-2}{3}=\frac{y-1}{4}=\frac{z-6}{5}$ and $P=0$ is $x+y-2z=3$.

4. The coordinates of A' are

   a. $(6,5,2)$ **b.** $(6,5,-2)$ **c.** $(6,-5,2)$ **d.** none of these

5. The coordinates of B are

   a. $(5,10,6)$ **b.** $(10,15,11)$ **c.** $(-10,-15,-14)$ **d.** none of these

6. If $L_1=0$ is the reflected ray, then its equation is

   a. $\frac{x+10}{4}=\frac{y-5}{4}=\frac{z+2}{3}$ **b.** $\frac{x+10}{3}=\frac{y+15}{5}=\frac{z+14}{5}$

   c. $\frac{x+10}{4}=\frac{y+15}{5}=\frac{z+14}{3}$ **d.** none of these

For Problems 7–9

Consider three planes $2x + py + 6z = 8, x + 2y + qz = 5$ and $x + y + 3z = 4$.

7. Three planes intersect at a point if

a. $p = 2, q \neq 3$ **b.** $p \neq 2, q \neq 3$ **c.** $p \neq 2, q = 3$ **d.** $p = 2, q = 3$

8. Three planes do not have any common point of intersection if

a. $p = 2, q \neq 3$ **b.** $p \neq 2, q \neq 3$ **c.** $p \neq 2, q = 3$ **d.** $p = 2, q = 3$

9. The planes have infinite points common among them if

a. $p = 2, q \in 3$ **b.** $p \in 2, q \in 3$ **c.** $p \neq 2, q = 3$ **d.** $p = 2, q = 3$

For Problems 10–12

Consider a plane $x + y - z = 1$ and point $A(1, 2, -3)$. A line L has the equation $x = 1 + 3r,\ y = 2 - r$ and $z = 3 + 4r$.

10. The coordinate of a point B of line L such that AB is parallel to the plane is

a. (10, –1, 15) **b.** (–5, 4, –5) **c.** (4, 1, 7) **d.** (–8, 5, –9)

11. The equation of the plane containing line L and point A has the equation

a. $x - 3y + 5 = 0$ **b.** $x + 3y - 7 = 0$ **c.** $3x - y - 1 = 0$ **d.** $3x + y - 5 = 0$

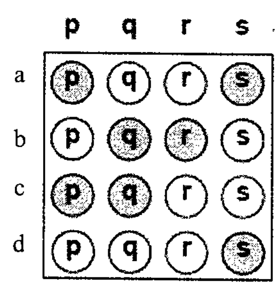
12. The distance between the points on the line which are at a distance of $4/\sqrt{3}$ from the plane is

a. $4\sqrt{26}$ **b.** 20 **c.** $10\sqrt{13}$ **d.** none of these

Matrix-Match Type

Solutions on page 3.120

Each question contains statements given in two columns which have to be matched. Statements (a, b, c, d) in Column I have to be matched with statements (p, q, r, s) in Column II. If the correct matches are a → p, s; b → q, r; c → p, q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:



1.

Column I	Column II
a. A vector perpendicular to the line $x = 2t + 1, y = t + 2$ and $z = -t - 3$	**p.** $7\hat{i} + 3\hat{j} + 5\hat{k}$
b. A vector parallel to the planes $x + y + z - 3 = 0$ and $2x - y + 3z = 0$	**q.** $4\hat{i} - \hat{j} - 3\hat{k}$
c. A vector along which the distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{-1}$ and $\vec{r} = (3\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + \hat{j} - 2\hat{k})$ is the shortest	**r.** $-11\hat{i} + 7\hat{j} + 5\hat{k}$
d. A vector normal to the plane $\vec{r} = -\hat{i} + 4\hat{j} - 6\hat{k} + \lambda(\hat{i} + 3\hat{j} - 2\hat{k}) + \mu(-\hat{i} + 2\hat{j} - 5\hat{k})$	**s.** $\hat{i} + 3\hat{j} + \hat{k}$

2.

Column I	Column II
a. Lines $\frac{x-1}{-2} = \frac{y+2}{3} = \frac{z}{-1}$ and $\vec{r} = (3\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + \hat{j} + \hat{k})$ are	**p.** intersecting
b. Lines $\frac{x+5}{1} = \frac{y-3}{7} = \frac{z+3}{3}$ and $x - y + 2z - 4 = 0 = 2x + y - 3z + 5 = 0$ are	**q.** perpendicular
c. Lines $(x = t - 3, y = -2t + 1, z = -3t - 2)$ and $\vec{r} = (t+1)\hat{i} + (2t+3)\hat{j} + (-t-9)\hat{k}$ are	**r.** parallel
d. Lines $\vec{r} = (\hat{i} + 3\hat{j} - \hat{k}) + t(2\hat{i} - \hat{j} - \hat{k})$ and $\vec{r} = (-\hat{i} - 2\hat{j} + 5\hat{k}) + s(\hat{i} - 2\hat{j} + \frac{3}{4}\hat{k})$ are	**s.** skew

3.

Column I	Column II
a. The coordinates of a point on the line $x = 4y + 5, z = 3y - 6$ at a distance 3 from the point $(5, 3, -6)$ is/are	**p.** $(-1, -2, 0)$
b. The plane containing the lines $\frac{x-2}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and parallel to $\hat{i} + 4\hat{j} + 7\hat{k}$ has the point	**q.** $(5, 0, -6)$
c. A line passes through two points $A(2, -3, -1)$ and $B(8, -1, 2)$. The coordinates of a point on this line nearer to the origin and at a distance of 14 units from A is/are	**r.** $(2, 5, 7)$
d. The coordinates of the foot of the perpendicular from the point $(3, -1, 11)$ on the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is/are	**s.** $(14, 1, 5)$

4.

	Column I	Column II
a.	The distance between the line $\vec{r} = (2\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ and plane $\vec{r} \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 5$	**p.** $\frac{25}{3\sqrt{14}}$
b.	Distance between parallel planes $\vec{r} \cdot (2\hat{i} - \hat{j} + 3\hat{k}) = 4$ and $\vec{r} \cdot (6\hat{i} - 3\hat{j} + 9\hat{k}) + 13 = 0$ is	**q.** 13/7
c.	The distance of a point (2, 5, – 3) from the plane $\vec{r} \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) = 4$ is	**r.** $\frac{10}{3\sqrt{3}}$
d.	The distance of the point (1, 0, – 3) from the plane $x - y - z - 9 = 0$ measured parallel to line $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$	**s.** 7

5.

	Column I	Column II
a.	Image of the point (3, 5, 7) in the plane $2x + y + z = -18$ is	**p.** $(-1, -1, -1)$
b.	The point of intersection of the line $\frac{x-2}{-3} = \frac{y-1}{-2} = \frac{z-3}{2}$ and the plane $2x + y - z = 3$ is	**q.** $(-21, -7, -5)$
c.	The foot of the perpendicular from the point (1, 1, 2) to the plane $2x - 2y + 4z + 5 = 0$ is	**r.** $\left(\frac{5}{2}, \frac{2}{3}, \frac{8}{3}\right)$
d.	The intersection point of the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ is	**s.** $\left(-\frac{1}{12}, \frac{25}{12}, \frac{-2}{12}\right)$

Integer Answer Type

Solutions on page 3.124

1. Find the number of spheres of radius r touching the coordinate axes.
2. Find the distance of the z-axis from the image of the point M (2, –3, 3) in the plane $x - 2y - z + 1 = 0$.
3. If the length of the projection of the line segment with points (1, 0, – 1) and (–1, 2, 2) to the plane $x + 3y - 5z = 6$ is d, then find the value of $[d/2]$ where [·] represent greatest integer function.
4. If the angle between the plane $x - 3y + 2z = 1$ and the line $\frac{x-1}{2} = \frac{y-1}{1} = \frac{z-1}{-3}$ is θ, then find the value of cosec θ.
5. Let A_1, A_2, A_3, A_4 be the areas of the triangular faces of a tetrahedron, and h_1, h_2, h_3, h_4 be the corresponding altitudes of the tetrahedron. If volume of tetrahedron is 1/6 cubic units, then find the minimum value of $(A_1 + A_2 + A_3 + A_4)(h_1 + h_2 + h_3 + h_4)$ (in cubic units).

6. Let the equation of the plane containing line $x-y-z-4=0=x+y+2z-4$ and parallel to the line of intersection of the planes $2x+3y+z=1$ and $x+3y+2z=2$ be $x+Ay+Bz+C=0$. Then find the value of $|A+B+C-4|$.
7. Let $P(a, b, c)$ be any point on the plane $3x+2y+z=7$, then find the least value of $2(a^2+b^2+c^2)$.
8. The plane denoted by $P_1 : 4x+7y+4z+81=0$ is rotated through a right angle about its line of intersection with the plane $P_2 : 5x+3y+10z=25$. If the plane in its new position be denoted by P, and the distance of this plane from the origin is d, then find the value of $[k/2]$ (where $[\cdot]$ represents greatest integer less than or equal to k).
9. The distance of the point $P(-2, 3, -4)$ from the line $\frac{x+2}{3}=\frac{2y+3}{4}=\frac{3z+4}{5}$ measured parallel to the plane $4x+12y-3z+1=0$ is d, then find the value of $(2d-8)$.
10. The position vectors of the four angular points of a tetrahedron $OABC$ are (0, 0, 0), (0, 0, 2), (0, 4, 0) and (6, 0, 0), respectively. A point P inside the tetrahedron is at the same distance 'r' from the four plane faces of the tetrahedron. Find the value of $9r$.

Archives

Solutions on page 3.127

Subjective Type

1. (i) Find the equation of the plane passing through the points (2, 1, 0), (5, 0, 1) and (4, 1, 1).

   (ii) If P is the point (2, 1, 6), then find the point Q such that PQ is perpendicular to the plane in (i) and the midpoint of PQ lies on it. **(IIT-JEE, 2003)**
2. Find the equation of a plane passing through (1, 1, 1) and parallel to the lines L_1 and L_2 having direction ratios (1, 0, –1) and (1, – 1, 0), respectively. Find the volume of tetrahedron formed by origin and the points where this plane intersects the coordinate axes. **(IIT-JEE, 2004)**
3. A parallelepiped S has base points A, B, C and D and upper face points A', B', C' and D'. The parallelepiped is compressed by upper face $A'B'C'D'$ to form a new parallelepiped T having upper face points A'', B'', C'' and D''. The volume of parallelepiped T is 90 percent of the volume of parallelepiped S. Prove that the locus of A'' is a plane. **(IIT-JEE, 2004)**
4. Find the equation of the plane containing the lines $2x-y+z-3=0$ and $3x+y=z=5$ and at a distance of $1/\sqrt{6}$ from the point $(2, 1, -1)$. **(IIT-JEE, 2005)**
5. A line with positive direction cosines passes through the point $P(2, -1, 2)$ and makes equal angles with the coordinate axes. The line meets the plane $2x+y+z=9$ at point Q. Find the length of the line segment PQ. **(IIT-JEE, 2009)**

Objective Type

Multiple choice questions with one correct answer

1. The value of k such that $\frac{x-4}{1}=\frac{y-2}{1}=\frac{z-k}{2}$ lies in the plane $2x-4y+z=7$, is

   a. 7 **b.** – 7 **c.** no real value **d.** 4

   (IIT-JEE, 2003)

2. If the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then the value of k is

a. 3/2 b. 9/2 c. −2/9 d. −3/2

(IIT-JEE, 2004)

3. A variable plane at a distance of 1 unit from the origin cuts the coordinate axes at A, B and C. If the centroid $D(x, y, z)$ of triangle ABC satisfies the relation $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=k$, then the value of k is

a. 3 b. 1 c. 1/3 d. 9

(IIT-JEE, 2005)

4. A plane which is perpendicular to two planes $2x-2y+z=0$ and $x-y+2z=4$ passes through $(1, -2, 1)$. The distance of the plane from the point $(1, 2, 2)$ is

a. 0 b. 1 c. $\sqrt{2}$ d. $2\sqrt{2}$

(IIT-JEE, 2006)

5. Let $P(3, 2, 6)$ be a point in space and Q be a point on line $\vec{r}=(\hat{i}-\hat{j}+2\hat{k})+\mu(-3\hat{i}+\hat{j}+5\hat{k})$. Then the value of μ for which the vector $\overrightarrow{PQ}$ is parallel to the plane $x-4y+3z=1$ is

a. 1/4 b. −1/4 c. 1/8 d. −1/8

(IIT-JEE, 2009)

6. Equation of the plane containing the straight line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$ is

a. $x+2y-2z=0$ b. $3x+2y-2z=0$ c. $x-2y+z=0$ d. $5x+2y-4z=0$

(IIT-JEE, 2010)

7. If the distance of the point $P(1, -2, 1)$ from the plane $x+2y-2z=\alpha$, where $\alpha>0$, is 5, then the foot of the perpendicular from P to the place is

a. $\left(\frac{8}{3}, \frac{4}{3}, -\frac{7}{3}\right)$ b. $\left(\frac{4}{3}, -\frac{4}{3}, \frac{1}{3}\right)$ c. $\left(\frac{1}{3}, \frac{2}{3}, \frac{10}{3}\right)$ d. $\left(\frac{2}{3}, -\frac{1}{3}, \frac{5}{2}\right)$

(IIT-JEE, 2010)

Assertion and reasoning type

Each question has four choices *a*, *b*, *c* and *d*, out of which *only one* is correct. Each question contains Statement 1 and Statement 2.

a. Both the statements are true, and Statement 2 is the correct explanation for Statement 1.
b. Both the statements are true, but Statement 2 is not the correct explanation for Statement 1.
c. Statement 1 is true and Statement 2 is false.
d. Statment 1 is false and Statement 2 is true.

1. Consider the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

   Statement 1: The parametric equations of the line of intersection of the given planes are $x = 3 + 14t, y = 2t, z = 15t$.

   Statement 2: The vector $14\hat{i} + 2\hat{j} + 15\hat{k}$ is parallel to the line of intersection of the given planes.

2. Consider three planes $P_1 : x - y + z = 1, P_2 : x + y - z = -1$ and $P_3 : x - 3y + 3z = 2$.

   Let L_1, L_2 and L_3 be the lines of intersection of the planes P_2 and P_3, P_3 and P_1, and P_1 and P_2, respectively.

   Statement 1: At least two of the lines L_1, L_2 and L_3 are non-parallel.

   Statement 2: The three planes do not have a common point.

(IIT-JEE, 2009)

Comprehension type

For Problems 1–3

Consider the lines $L_1 : \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}, L_2 : \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$ **(IIT-JEE, 2008)**

1. The unit vector perpendicular to both L_1 and L_2 is

   a. $\frac{-\hat{i}+7\hat{j}+7\hat{k}}{\sqrt{99}}$ **b.** $\frac{-\hat{i}-7\hat{j}+5\hat{k}}{5\sqrt{3}}$ **c.** $\frac{-\hat{i}+7\hat{j}+5\hat{k}}{5\sqrt{3}}$ **d.** $\frac{7\hat{i}-7\hat{j}-\hat{k}}{\sqrt{99}}$

2. The shortest distance between L_1 and L_2 is

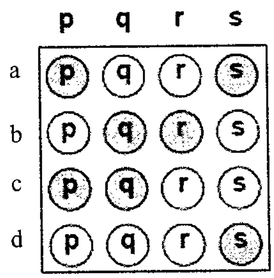
   a. 0 **b.** $\frac{17}{\sqrt{3}}$ **c.** $\frac{41}{5\sqrt{3}}$ **d.** $\frac{17}{5\sqrt{3}}$

3. The distance of the point (1, 1, 1) from the plane passing through the point (–1, –2, –1) and whose normal is perpendicular to both the lines L_1 and L_2 is

   a. $\frac{12}{\sqrt{65}}$ **b.** $\frac{14}{\sqrt{75}}$ **c.** $\frac{13}{\sqrt{75}}$ **d.** $\frac{13}{\sqrt{65}}$

Matrix-match type

Each question contains statements given in two columns which have to be matched. Statements (a, b, c, d) in Column I have to be matched with statements (p, q, r, s) in Column II. If the correct matches are a → p, s; b → q, r; c → p, q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:



1. Consider the linear equations $ax + by + cz = 0$, $bx + cy + az = 0$ and $cx + ay + bz = 0$.
Match the conditions/expressions in Column I with statements in Column II. **(IIT-JEE, 2007)**

Column I	Column II
a. $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	**p.** The equations represent planes meeting only at a single point.
b. $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	**q.** The equations represent the line $x = y = z$.
c. $a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	**r.** The equations represent identical planes.
d. $a + b + c = 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	**s.** The equations represent the whole of the three-dimensional space.

Integer Answer Type

1. If the distance between the plane $Ax - 2y + z = d$ and the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, then find the value of $|d|$.

(IIT-JEE, 2010)

ANSWERS AND SOLUTIONS

Subjective Type

1. Since l, m and n, and $(l + \delta m), (m + \delta m), (n + \delta n)$ are the direction cosines, we have

$l^2 + m^2 + n^2 = 1$ (i)

$(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1$

$\Rightarrow l^2 + m^2 + n^2 + 2l\,\delta l + 2m\,\delta m + 2n\,\delta n + (\delta l)^2 + (\delta m)^2 + (\delta n)^2 = 1$

$\Rightarrow 2(l\,\delta l + m\,\delta m + n\,\delta n) = -(\{\delta l)^2 + (\delta m)^2 + (\delta n)^2\}$ (ii)

Now it is given that $\delta\theta$ is the angle between two adjacent positions of the line. Therefore

$\cos\delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$ (iii)

Now $\cos\delta\theta = 1 - \frac{(\delta\theta)^2}{2!} + \frac{(\delta\theta)^2}{4!} - \cdots$

If $\delta\theta$ is small, then $\cos\delta\theta = 1 - \frac{(\delta\theta)^2}{2}$

Then from (iii), we have $1 - \frac{(\delta\theta)^2}{2} = (l^2 + m^2 + n^2) + (l\,\delta l + m\,\delta m + n\,\delta n)$

$\Rightarrow \quad 1 - \frac{(\delta\theta)^2}{2} = 1 - \frac{1}{2}\{(\delta l)^2 + (\delta m)^2 + (\delta n)^2\}$ (using (i) and (ii))

$\Rightarrow \quad (\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$

2. The equation of the first line may be written as $\frac{y}{b}-\frac{1}{2}=\frac{1}{2}-\frac{z}{c}, x=0$

or $\frac{x}{0}=\frac{y-\frac{1}{2}b}{b}=\frac{z-\frac{1}{2}c}{-c}$ (i)

Similarly, the equation of the second line may be written as

$\frac{x-\frac{1}{2}a}{a}=\frac{y}{0}=\frac{z+\frac{1}{2}c}{c}$ (ii)

The equation of any plane passing through line (i) is

$A(x)+B\left(y-\frac{1}{2}b\right)+C\left(z-\frac{1}{2}c\right)=0,$ (iii)

where $A\cdot 0+B\cdot b-C\cdot c=0$ (iv)

Now plane (iii) will be parallel to line (ii) if

$A\cdot a+B\cdot 0-C\cdot c=0$ (v)

Solving (iv) and (v), we have $\frac{A}{bc}=\frac{B}{-ca}=\frac{C}{-ab}$

Putting these values of A, B and C in (iii), the equation of the required plane is

$$bcx-ca\left(y-\frac{1}{2}b\right)-ab\left(z-\frac{1}{2}c\right)=0 \text{ or } \frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$$

3. Let the equation of the variable plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$ (i)

where a, b and c are the parameters.

Plane (i) passes through the point (α, β, γ). Therefore,

$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1$ (ii)

Plane (i) meets the coordinate axes at points A, B and C. The equations of the planes passing through A, B and C and parallel to the coordinate planes are, respectively,

$x=a, y=b, z=c$ (iii)

The locus of the point of intersection of these planes is obtained by eliminating the parameters a, b and c between (ii) and (iii). Putting the values of a, b and c from (iii) in (ii), the required locus is given by $\frac{\alpha}{x}+\frac{\beta}{y}+\frac{\gamma}{z}=1$ or $\alpha x^{-1}+\beta y^{-1}+\gamma z^{-1}=1$

4. Here, $l=-\frac{(bm+cn)}{a}$ and $ul^2+m^2v+wn^2=0$.

Eliminating l, we get

$$\frac{u(bm+cn)^2}{a^2}+vm^2+wn^2=0$$

$$u(b^2m^2+2bcmn+c^2n^2)+va^2m^2+wa^2n^2=0$$

$$(b^2u+a^2v)m^2+(2bcu)mn+(c^2u+a^2w)n^2=0$$

$\Rightarrow (b^2u+a^2v)\left(\frac{m}{n}\right)^2+(2bcu)\left(\frac{m}{n}\right)+(c^2u+a^2w)=0$, which is quadratic in (m/n) having roots m_1/n_1 and m_2/n_2

a. If the straight lines are parallel, the quadratic in m/n has equal roots, i.e., discriminant = 0

$\Rightarrow (2bcu)^2 - 4(b^2u + a^2v)(c^2u + a^2w) = 0$

$\Rightarrow b^2c^2u^2 = (b^2u + a^2v)(c^2u + a^2w)$

$\Rightarrow a^2vw + b^2uw + c^2uv = 0$

$\Rightarrow \frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$

b. If the straight lines are perpendicular,

$\Rightarrow \frac{m_1}{n_1}\cdot\frac{m_2}{n_2}=\frac{c^2u+a^2w}{b^2u+a^2v}$ (product of roots)

$$\Rightarrow \frac{m_1m_2}{c^2u+a^2w}=\frac{n_1n_2}{b^2u+a^2v} \quad \text{(i)}$$

Similarly, by eliminating n, we get

$$\frac{l_1l_2}{b^2w+c^2v}=\frac{m_1m_2}{c^2u+a^2w} \quad \text{(ii)}$$

From (i) and (ii)

$$\frac{l_1l_2}{b^2w+c^2v}=\frac{m_1m_2}{c^2u+a^2w}=\frac{n_1n_2}{b^2u+a^2v}=\lambda$$

Since they are perpendicular, $l_1l_2+m_1m_2+n_1n_2=0$

$\Rightarrow \lambda(b^2w+c^2v)+\lambda(c^2u+a^2w)+\lambda(b^2u+a^2v)=0$

$\Rightarrow a^2(v+w)+b^2(w+u)+c^2(u+v)=0$

5.

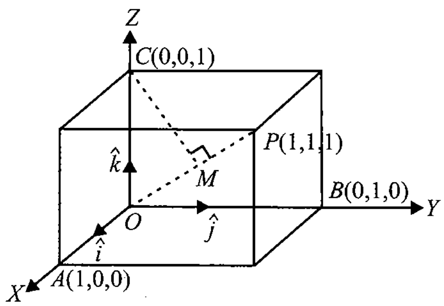


Fig. 3.33

Let the edges OA, OB and OC of the unit cube be along OX, OY and OZ, respectively.

Since $OA = OB = OC = 1$ unit, $\overrightarrow{OA}=\hat{i}$, $\overrightarrow{OB}=\hat{j}$ and $\overrightarrow{OC}=\hat{k}$

Let CM be perpendicular from the corner C on the diagonal OP. The vector equation of OP is

$\vec{r} = \lambda(\hat{i}+\hat{j}+\hat{k})$

OM = projection of $\overrightarrow{OC}$ on $\overrightarrow{OP} = \dfrac{\overrightarrow{OC}\cdot\overrightarrow{OP}}{|\overrightarrow{OP}|} = \hat{k}.\dfrac{(\hat{i}+\hat{j}+\hat{k})}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

Now $OC^2 = OM^2 + CM^2$

$\Rightarrow CM^2 = OC^2 - OM^2 = 1 - \dfrac{1}{3} = \dfrac{2}{3} \Rightarrow CM = \sqrt{\dfrac{2}{3}}$

6. The given plane is $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1.$ (i)

Let $P(h, k, l)$ be the point on the plane

$\dfrac{h}{a}+\dfrac{k}{b}+\dfrac{l}{c}=1$ (ii)

$\Rightarrow OP = \sqrt{h^2+k^2+l^2}$

Direction cosines of OP are $\dfrac{h}{\sqrt{h^2+k^2+l^2}}, \dfrac{k}{\sqrt{h^2+k^2+l^2}}$ and $\dfrac{l}{\sqrt{h^2+k^2+l^2}}$.

The equation of the plane through P and normal to OP is

$$\frac{hx}{\sqrt{h^2+k^2+l^2}}+\frac{ky}{\sqrt{h^2+k^2+l^2}}+\frac{lz}{\sqrt{h^2+k^2+l^2}} = \sqrt{h^2+k^2+l^2}$$

or $hx + ky + lz = h^2 + k^2 + l^2$

Therefore, $A \equiv \left(\dfrac{h^2+k^2+l^2}{h}, 0, 0\right), B \equiv \left(0, \dfrac{h^2+k^2+l^2}{k}, 0\right)$ and $C \equiv \left(0, 0, \dfrac{h^2+k^2+l^2}{l}\right)$

If $Q(\alpha, \beta, \gamma)$, then $\alpha = \dfrac{h^2+k^2+l^2}{h}, \beta = \dfrac{h^2+k^2+l^2}{k}$ and $\gamma = \dfrac{h^2+k^2+l^2}{l}$ (iii)

Now, $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}+\dfrac{1}{\gamma^2} = \dfrac{h^2+k^2+l^2}{(h^2+k^2+l^2)^2} = \dfrac{1}{h^2+k^2+l^2}$ (iv)

From (iii), $h = \dfrac{h^2+k^2+l^2}{\alpha} \Rightarrow \dfrac{h}{a} = \dfrac{h^2+k^2+l^2}{a\alpha}$

Similarly, $\dfrac{k}{b} = \dfrac{h^2+k^2+l^2}{b\beta}$ and $\dfrac{l}{c} = \dfrac{h^2+k^2+l^2}{c\gamma}$

$\dfrac{h^2+k^2+l^2}{a\alpha}+\dfrac{h^2+k^2+l^2}{b\beta}+\dfrac{h^2+k^2+l^2}{c\gamma} = \dfrac{h}{a}+\dfrac{k}{b}+\dfrac{l}{c}=1$ (from (ii)) or $\dfrac{1}{a\alpha}+\dfrac{1}{b\beta}+\dfrac{1}{c\gamma}=\dfrac{1}{h^2+k^2+l^2}=\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}+\dfrac{1}{\gamma^2}$ (from (iv))

The required equation of locus is $\dfrac{1}{ax}+\dfrac{1}{by}+\dfrac{1}{cz}=\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}$

7. The given planes are

$x - cy - bz = 0$ (i)

$cx - y + az = 0$ (ii)

$bx + ay - z = 0$ (iii)

The equation of the planes passing through the line of intersection of planes (i) and (ii) may be taken as $(x - cy - bz) + \lambda(cx - y + az) = 0$

or $x(1 + \lambda c) - y(c + \lambda) + z(-b + a\lambda) = 0$ (iv)

If planes (iii) and (iv) are the same, then Eqs. (iii) and (iv) will be identical.

$$\frac{1+c\lambda}{b}=\frac{-(c+\lambda)}{a}=\frac{-b+a\lambda}{-1}$$

$$\lambda=-\frac{(a+bc)}{(ac+b)} \text{ and } \lambda=-\frac{(ab+c)}{(1-a^2)}$$

$$\therefore \frac{-(a+bc)}{(ac+b)}=-\frac{(ab+c)}{(1-a^2)}$$

$a - a^3 + bc - a^2bc = a^2bc + ac^2 + ab^2 + bc$

$\Rightarrow 2a^2bc + ac^2 + ab^2 + a^3 - a = 0$

$\Rightarrow a(2abc + c^2 + b^2 + a^2 - 1) = 0$

$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$

Alternative method:

Since the planes pass through origin, the given planes have a common line of intersection if given system of equations has a non-trivial solution

$$\Rightarrow \begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix}=0$$

$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$

8. Let P be (x_1, y_1, z_1). Point M is $(x_1, 0, z_1)$ and N is $(x_1, y_1, 0)$.

So normal to plane OMN is $\overrightarrow{OM}\times\overrightarrow{ON}=\vec{x}$ (say). Therefore,

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & 0 & z_1 \\ x_1 & y_1 & 0 \end{vmatrix}=\hat{i}(-y_1 z_1)-\hat{j}(-x_1 z_1)+\hat{k}(x_1 y_1)$$

$$\sin\theta = \frac{-x_1y_1z + x_1y_1z + x_1y_1z_1}{\sqrt{x_1^2+y_1^2+z_1^2}\sqrt{\Sigma x_1^2 y_1^2}} \quad \left(\text{because } \sin\theta = \frac{\vec{n}\times\overrightarrow{OP}}{|\vec{n}||\overrightarrow{OP}|}\right)$$

$$\Rightarrow \operatorname{cosec}^2\theta = \frac{\Sigma x_1^2 \Sigma x_1^2 y_1^2}{(x_1y_1z_1)^2} = \frac{\Sigma x_1^2}{x_1^2} + \frac{\Sigma x_1^2}{y_1^2} + \frac{\Sigma x_1^2}{z_1^2}$$

Now, $\sin\alpha = \dfrac{\overrightarrow{OP}\cdot\hat{k}}{|\overrightarrow{OP}|} = \dfrac{z_1}{\sqrt{\Sigma x_1^2}}$, $\sin\beta = \dfrac{x_1}{\sqrt{\Sigma x_1^2}}$ and $\sin\gamma = \dfrac{y_1}{\sqrt{\Sigma x_1^2}}$

Now, $\operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma = \dfrac{\Sigma x_1^2}{x_1^2} + \dfrac{\Sigma x_1^2}{y_1^2} + \dfrac{\Sigma x_1^2}{z_1^2} = \operatorname{cosec}^2\theta$

Hence proved.

9. $\dfrac{x}{p/l} + \dfrac{y}{p/m} + \dfrac{z}{p/n} = 1$

The foot of normal on plane has coordinates $H(lp, mp, np)$.
Direction ratios of AH are $lp - (p/l)$, mp and np and direction ratios of BC are 0, $-p/m$, and p/n.

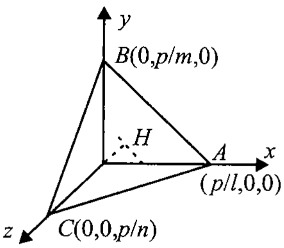


Fig. 3.34

$$\Rightarrow \left(lp - \frac{p}{l}\right)\cdot 0 + (mp)\left(-\frac{p}{m}\right) + (np)\left(\frac{p}{n}\right) = 0$$

Hence, AH is perpendicular to BC.
Similarly, BH is perpendicular to AC and CH is perpendicular to AB.
Hence, H is the orthocenter.
Moreover, in any triangle, G (centroid) divides OH in the ratio 1 : 2.
Hence,

$$G \equiv \left(\frac{p}{3l}, \frac{p}{3m}, \frac{p}{3n}\right)$$

$$H \equiv (lp, mp, np)$$

$$\Rightarrow O \equiv \left(\frac{p - l^2p}{2l}, \frac{p - m^2p}{2m}, \frac{p - n^2p}{2n}\right).$$

10. $x - y\sin\alpha - z\sin\beta = 0$ (i)

$x\sin\alpha + z\sin\gamma - y = 0$ (ii)

$x\sin\beta + y\sin\gamma - z = 0$ (iii)

These planes pass through origin. Let l, m and n be the direction cosines of the line of intersection of planes (i) and (ii). Then

$l\cdot 1 - m\sin\alpha - n\sin\beta = 0$

$l\sin\alpha - m\cdot 1 + n\sin\gamma = 0$

$$\Rightarrow \frac{l}{-\sin\gamma\sin\alpha - \sin\beta} = \frac{m}{-\sin\beta\sin\alpha - \sin\gamma} = \frac{n}{-1+\sin^2\alpha} \quad \text{(iv)}$$

If $\alpha + \beta + \gamma = \frac{\pi}{2} \Rightarrow \beta = \frac{\pi}{2} - (\alpha + \gamma)$

$$\sin\beta = \sin\left(\frac{\pi}{2} - (\alpha+\gamma)\right) = \cos(\alpha+\gamma)$$

$\sin\beta = \cos\alpha\cos\gamma - \sin\alpha\sin\gamma$

$\sin\beta + \sin\alpha\sin\gamma = \cos\alpha\cos\gamma$

Similarly, $\sin\gamma + \sin\beta\sin\alpha = \cos\alpha\cos\beta$

From equation (iv), we get $\dfrac{l}{\cos\alpha\cos\gamma} = \dfrac{m}{\cos\alpha\cos\beta} = \dfrac{n}{\cos^2\alpha}$

$$\frac{l}{\cos\gamma} = \frac{m}{\cos\beta} = \frac{n}{\cos\alpha} \quad \text{(v)}$$

The line of intersection of planes (i) and (ii) also passes through the origin. Then the equation of the line is

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n}$$

$$\Rightarrow \frac{x}{\cos\gamma} = \frac{y}{\cos\beta} = \frac{z}{\cos\alpha} \quad \text{(vi)}$$

If the line also lies on plane (iii), then the three planes will intersect on this straight line.

The angle between line and normal of plane (iii) should be $\pi/2$.

$\Rightarrow \cos\gamma\sin\beta + \cos\beta\sin\gamma + \cos\alpha(-1) = \sin(\beta+\gamma) - \cos\alpha$

$$= \sin\left(\frac{\pi}{2} - \alpha\right) - \cos\alpha = 0$$

Hence $\dfrac{x}{\cos\gamma} = \dfrac{y}{\cos\beta} = \dfrac{z}{\cos\alpha}$ is the common line of the intersection of the three given planes.

11. $ax + by + cz + 1 = 0$ (i)

It makes an angle 60° with the line $x = y = z$. So we get

$$\sin 60^\circ = \frac{a+b+c}{\sqrt{3}\,\Sigma a^2} \Rightarrow 3\sqrt{\Sigma a^2} = 2(a+b+c) \quad \text{(ii)}$$

Plane (i) makes an angle of 45° with the line $x = y - z = 0$ $\left(\text{or } \frac{x}{0} = \frac{y}{1} = \frac{z}{1}\right)$

$$\sin 45° = \frac{b+c}{\sqrt{2}\sqrt{\Sigma a^2}} \Rightarrow \sqrt{\Sigma a^2} = b+c \qquad \text{(iii)}$$

Plane (i) makes an angle θ with the plane $x = 0$. So we get

$$\cos\theta = \frac{a}{\sqrt{\Sigma a^2}} \qquad \text{(iv)}$$

From (ii) and (iii), we get

$$(\sqrt{\Sigma a^2}) = 2a$$

$$\Rightarrow \frac{a}{\sqrt{\Sigma a^2}} = \frac{1}{2}$$

From (iv), $\cos\theta = 1/2 \Rightarrow \theta = 60°$

Distance of plane (i) from the point (2, 1, 1) is 3 units.

$$\Rightarrow \frac{2a+b+c+1}{\sqrt{\Sigma a^2}} = \pm 3$$

$$\Rightarrow \pm 3\sqrt{\Sigma a^2} = 2a+b+c+1$$

Case I:

$$3\sqrt{\Sigma a^2} = 2a + b + c + 1 \qquad \text{(v)}$$

From (ii) and (v), we get

$$b + c - 1 = 0 \qquad \text{(vi)}$$

and from (iii) and (iv), we get

$$2a + b + c + 1 = 3(b + c) \qquad \text{(vii)}$$

From (vi) and (vii), we get

$$a = \frac{1}{2}, b = \frac{(2 \mp \sqrt{2})}{4} \text{ and } c = \frac{2 \pm \sqrt{2}}{4}.$$

Hence, the set of such planes is $2x + (2 \pm \sqrt{2})y + (2 \pm \sqrt{2})z + 4 = 0$.

Case II:

$$-3\sqrt{\Sigma a^2} = 2a+b+c+1$$

$$a = \frac{-1}{10}, b = \frac{-(2 \pm \sqrt{2})}{20} \text{ and } c = \frac{-(2 \mp \sqrt{2})}{20}$$

Hence, the other set of the planes is $2x + (2 \pm \sqrt{2})y + (2 \mp \sqrt{2})z - 20 = 0$.

12. Let the given planes intersect on the line with direction ratios l, m and n. In that case,

$$(2+\lambda)\frac{l}{a}+(1-2\lambda)\frac{m}{b}+(2-\lambda)\frac{n}{c}=0 \quad \text{(i)}$$

and $$\frac{4l}{a}-(3-5\mu)\frac{m}{b}+4\mu.\frac{n}{c}=0 \quad \text{(ii)}$$

Hence, $$\frac{l/a}{6-6\mu-3\lambda-3\lambda\mu}=\frac{m/b}{8-8\mu-4\lambda-4\lambda\mu}=\frac{n/c}{-10+10\mu+5\lambda+5\lambda\mu}$$

or $$\frac{l/a}{3(2-2\mu-\lambda-\lambda\mu)}=\frac{m/b}{4(2-2\mu-\lambda-\lambda\mu)}=\frac{n/c}{-5(2-2\mu-\lambda-\lambda\mu)}$$

or $$\frac{l/a}{3}=\frac{m/b}{4}=\frac{n/c}{-5} \quad (\text{provided } 2-2\mu-\lambda-\lambda\mu\neq 0)$$

which are independent of λ and μ. Hence a line with direction ratios $(3a, 4b, -5c)$ lies in both the planes.

For $2-2\mu-\lambda-\lambda\mu=0$ or $\lambda=\frac{2(1-\mu)}{1+\mu}$, planes (i) and (ii) coincide with each other. Hence, the two given families of planes intersect on the same line.

13. Let A_1 and B_1 be the projections of A and B on the plane $z=0$. Let OA, OB and OC be of the unit length each so that the coordinates of A, B and C are $A(l_1, m_1, n_1)$, $B(l_2, m_2, n_2)$ and $C(l_3, m_3, n_3)$. The coordinates of A_1 and B_1, therefore, are $A_1(l_1, m_1, 0)$ and $B_1(l_2, m_2, 0)$. Since OA_1 and OB_1 make angles ϕ_1 and ϕ_2, respectively, with the x-axis, the angle between OA_1 and OB_1 is $\phi_1 \sim \phi_2$. Hence

$$\cos(\phi_1-\phi_2)=\frac{l_1l_2+m_1m_2}{\sqrt{l_1^2+m_1^2}\sqrt{l_2^2+m_2^2}} \quad \text{(i)}$$

Also OA, OB and OC are mutually perpendicular so that

$l_1l_2+m_1m_2+n_1n_2=0$

and $l_1^2+m_1^2+n_1^2=1$

Eq. (i), therefore, yields

$$\cos(\phi_1-\phi_2)=\frac{-n_1n_2}{\sqrt{1-n_1^2}\sqrt{1-n_2^2}}$$

$$\Rightarrow \sec^2(\phi_1-\phi_2)=\frac{1-n_1^2-n_2^2+n_1^2n_2^2}{n_1^2n_2^2}=1+\frac{1-n_1^2-n_2^2}{n_1^2n_2^2}=1+\frac{n_3^2}{n_1^2n_2^2}$$

$$\Rightarrow \tan^2(\phi_1-\phi_2)=\frac{n_3^2}{n_1^2n_2^2}$$

$$\Rightarrow \tan(\phi_1-\phi_2)=\pm\frac{n_3}{n_1n_2}$$

14. If θ is the angle BCO, then the direction cosines of OA' (bisector of $\angle BOC$) are $\frac{l_2+l_3}{2\cos(\theta/2)}$, $\frac{m_2+m_3}{2\cos(\theta/2)}$ and $\frac{n_2+n_3}{2\cos(\theta/2)}$ or the direction ratios of OA' are l_2+l_3, m_2+m_3 and n_2+n_3.

Also, the direction cosines of OA are l_1, m_1 and n_1. Hence the equation of plane AOA' is

$$\begin{vmatrix} x & y & z \\ l_2+l_3 & m_2+m_3 & n_2+n_3 \\ l_1 & m_1 & n_1 \end{vmatrix}=0$$

Applying $R_2 \to R_2 + R_3$, we get the equation of plane AOA' as

$$\begin{vmatrix} x & y & z \\ l_1+l_2+l_3 & m_1+m_2+m_3 & n_1+n_2+n_3 \\ l_1 & m_1 & n_1 \end{vmatrix}=0$$

$\Rightarrow$ For all values of r, the point $((l_1+l_2+l_3)r, (m_1+m_2+m_3)r$ and $(n_1+n_2+n_3)r)$ lies on plane AOA'. Hence, the line $\dfrac{x}{l_1+l_2+l_3}=\dfrac{y}{m_1+m_2+m_3}=\dfrac{z}{n_1+n_2+n_3}=r$ lies on plane AOA'. Similarly, this line lies on planes BOB' and COC' also. Hence, all the three planes, AOA', BOB' and COC', pass through the line.

15. Let $P(\alpha, \beta, \gamma)$ and $Q(x_1, y_1, z_1)$ be the given points.

Direction ratios of OP are α, β and γ and those of OQ are x_1,y_1 and z_1.

Since O, Q and P are collinear, $\dfrac{\alpha}{x_1}=\dfrac{\beta}{y_1}=\dfrac{\gamma}{z_1}=k$ (say) (i)

As $P(\alpha, \beta, \gamma)$ lies on the plane $lx+my+nz=p$,

$l\alpha+m\beta+n\gamma=p$, or

$klx_1+kmy_1+knz_1=p$ (using (i)) (ii)

Since $OP \cdot OQ = p^2$,

$$\sqrt{\alpha^2+\beta^2+\gamma^2}\cdot\sqrt{x_1^2+y_1^2+z_1^2}=p^2$$

$$\Rightarrow \sqrt{k^2x_1^2+k^2y_1^2+k^2z_1^2}\cdot\sqrt{x_1^2+y_1^2+z_1^2}=p^2$$

$\Rightarrow k(x_1^2+y_1^2+z_1^2)=p^2$ (iii)

From (ii) and (iii), $\dfrac{lx_1+my_1+nz_1}{x_1^2+y_1^2+z_1^2}=\dfrac{1}{p}$ or $p(lx_1+my_1+nz_1)=(x_1^2+y_1^2+z_1^2)$

Hence, the locus of Q is $p(lx+my+nz)=(x^2+y^2+z^2)$

16. Let the variable plane intersect the coordinate axes at $A(a, 0, b)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Then the equation of the plane will be $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$ (i)

Let $P(\alpha, \beta, \gamma)$ be the centroid of tetrahedron $OABC$. Then,

$\alpha=\dfrac{a}{4}$, $\beta=\dfrac{b}{4}$ and $\gamma=\dfrac{c}{4}$, or $a=4\alpha$, $b=4\beta$ and $c=4\gamma$

$\Rightarrow$ Volume of tetrahedron = (Area of ΔAOB) OC

$$\Rightarrow 64k^3 = \frac{1}{3}\left(\frac{1}{2}ab\right)c = \frac{abc}{6} \Rightarrow 64\,k^3 = \frac{(4\alpha)(4\beta)(4\gamma)}{6} \Rightarrow k^3 = \frac{\alpha\beta\gamma}{6}$$

Therefore, the required locus of $P(\alpha, \beta, \gamma)$ is $xyz = 6k^3$

Objective Type

1. **b.** $x^2 - 5x + 6 = 0$
   $\Rightarrow x - 2 = 0, x - 3 = 0$
   which represents planes.
2. **c.** We have $z = 0$ for the point, where the line intersects the curve.

   Therefore, $\frac{x-2}{3} = \frac{y+1}{2} = \frac{0-1}{-1}$

   $\Rightarrow \frac{x-2}{3} = 1$ and $\frac{y+1}{2} = 1$

   $\Rightarrow x = 5$ and $y = 1$

   Putting these values in $xy = c^2$, we get

   $5 = c^2 \Rightarrow c = \pm\sqrt{5}$
3. **a.** $4(2) - 2(3) - 1(2) = 0$

   Also, point $(-3, 4, -5)$ does not lie on the plane.

   Therefore, the line is parallel to the plane.
4. **c.** The given plane passes through $\vec{a}$ and is parallel to the vectors $\vec{b} - \vec{a}$ and $\vec{c}$. So it is normal to $(\vec{b} - \vec{a}) \times \vec{c}$. Hence, its equation is

   $(\vec{r} - \vec{a}) \cdot ((\vec{b} - \vec{a}) \times \vec{c}) = 0$

   or $\vec{r} \cdot (\vec{b} \times \vec{c} + \vec{c} \times \vec{a}) = [\vec{a}\,\vec{b}\,\vec{c}]$

   The length of the perpendicular from the origin to this plane is

   $$\frac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{b} \times \vec{c} + \vec{c} \times \vec{a}|}$$
5. **b.** Here, $\alpha = \beta = \gamma$

   $\because \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

   $\therefore \cos\alpha = \frac{1}{\sqrt{3}}$

   DC's of PQ are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

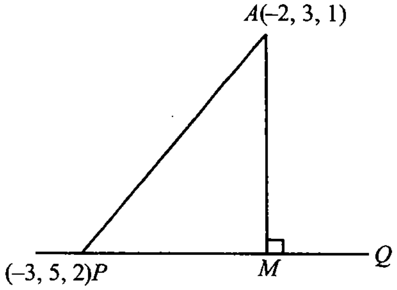


Fig. 3.35

PM = Projection of AP on PQ

$$= \left|(-2+3)\frac{1}{\sqrt{3}}+(3-5)\frac{1}{\sqrt{3}}+(1-2)\frac{1}{\sqrt{3}}\right| = \frac{2}{\sqrt{3}}$$

and $AP = \sqrt{(-2+3)^2+(3-5)^2+(1-2)^2} = \sqrt{6}$

$$AM = \sqrt{(AP)^2-(PM)^2} = \sqrt{6-\frac{4}{3}} = \sqrt{\frac{14}{3}}$$

6. **c.** Given plane is $\vec{r} = (1+\lambda-\mu)\hat{i}+(2-\lambda)\hat{j}+(3-2\lambda+2\mu)\hat{k}$

$\Rightarrow \quad \vec{r} = (\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-\hat{j}-2\hat{k})+\mu(-\hat{i}+2\hat{k})$

which is a plane passing through $\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$ and parallel to the vectors $\vec{b} = \hat{i}-\hat{j}-2\hat{k}$ and $\vec{c} = -\hat{i}+2\hat{k}$

Therefore, it is perpendicular to the vector $\vec{n} = \vec{b}\times\vec{c} = -2\hat{i}-\hat{k}$
Hence, equation of plane is $-2(x-1)+(0)(y-2)-(z-3)=0$ or $2x+z=5$

7. **c.** $\hat{a} = \pm\dfrac{\vec{n}_1\times\vec{n}_2}{|\vec{n}_1\times\vec{n}_2|} = \pm\dfrac{2\hat{i}+5\hat{j}+3\hat{k}}{\sqrt{38}}$ (where $\vec{n}_1$ and $\vec{n}_2$ are normal to the planes)

8. **a.** Equation of the plane containing L_1, $A(x-2)+B(y-1)+C(z+1)=0$
where $A+2C=0; A+B-C=0$
$\Rightarrow A=-2C,\ B=3C,\ C=C$
$\Rightarrow$ Plane is $-2(x-2)+3(y-1)+z+1=0$ or $2x-3y-z-2=0$

Hence, $p = \left|\dfrac{-2}{\sqrt{14}}\right| = \sqrt{\dfrac{2}{7}}$

9. **c.** (1, 2, 3) satisfies the plane $x - 2y + z = 0$ and also $(\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0$

Since the lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ both satisfy (0, 0, 0) and (1, 2, 3), both are same. Given line is obviously parallel to the plane $x - 2y + z = 6$.

10. **a.** Vector $\left((3\hat{i} - 2\hat{j} + \hat{k}) \times (4\hat{i} - 3\hat{j} + 4\hat{k})\right)$ is perpendicular to $2\hat{i} - \hat{j} + m\hat{k}$

$$\Rightarrow \begin{vmatrix} 3 & -2 & 1 \\ 4 & -3 & 4 \\ 2 & -1 & m \end{vmatrix} = 0 \qquad \Rightarrow m = -2$$

11. **a.** x intercept is say x_1

$\Rightarrow$ Plane passes through it

$\therefore\ x_1 \hat{i} \cdot \vec{n} = q \Rightarrow x_1 = \frac{q}{\hat{i} \cdot \vec{n}}$

12. **b.** Let direction ratios of the line be (a, b, c), then

$2a - b + c = 0$ and $a - b - 2c = 0$, i.e., $\frac{a}{3} = \frac{b}{5} = \frac{c}{-1}$

Therefore, direction ratios of the line are (3, 5, – 1).

Any point on the given line is $(2 + \lambda, 2 - \lambda, 3 - 2\lambda)$, it lies on the given plane π if

$2(2 + \lambda) - (2 - \lambda) + (3 - 2\lambda) = 4$

$\Rightarrow 4 + 2\lambda - 2 + \lambda + 3 - 2\lambda = 4 \Rightarrow \lambda = -1$

Therefore, the point of intersection of the line and the plane is (1, 3, 5).

Therefore, equation of the required line is

$$\frac{x-1}{3} = \frac{y-3}{5} = \frac{z-5}{-1}$$

13. **c.** Direction ratios of OP are (a, b, c)

Therefore, equation of the plane is

$a(x - a) + b(y - b) + c(z - c) = 0$

i.e. $xa + yb + zc = a^2 + b^2 + c^2$

14. **b.** Let a point $(3\lambda + 1, \lambda + 2, 2\lambda + 3)$ of the first line also lies on the second line

Then $\frac{3\lambda + 1 - 3}{1} = \frac{\lambda + 2 - 1}{2} = \frac{2\lambda + 3 - 2}{3} \Rightarrow \lambda = 1$

Hence, the point of intersection P of the two lines is (4, 3, 5).

Equation of plane perpendicular to OP, where O is (0, 0, 0) and passing through P is

$4x + 3y + 5z = 50$.

15. **b.** $1 = \left|(\vec{b} - \vec{a}) \cdot \frac{(\vec{p} \times \vec{q})}{|\vec{p} \times \vec{q}|}\right|$

$\Rightarrow |\vec{b} - \vec{a}| \cos 60^\circ = 1 \Rightarrow AB = 2$

16. **a.** $A(1, 1, 1), B(2, 3, 5), C(-1, 0, 2)$ direction ratios of AB are $<1, 2, 4>$.

Direction ratios of AC are $<-2, -1, 1>$.

Therefore, direction ratios of normal to plane ABC are $<2, -3, 1>$

As a result, equation of the plane ABC is $2x - 3y + z = 0$.

Let the equation of the required plane is $2x - 3y + z = k$, then $\left|\frac{k}{\sqrt{4+9+1}}\right| = 2$

$$k = \pm 2\sqrt{14}$$

Hence, equation of the required plane is $2x - 3y + z + 2\sqrt{14} = 0$

17. **b.** Direction cosines of the given line are $\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}$

Hence, the equation of line can be point in the form $\frac{x-2}{1/3} = \frac{y+3}{-2/3} = \frac{z+5}{-2/3} = r$

Therefore, any point on the line is $\left(2+\frac{r}{3}, -3-\frac{2r}{3}, -5-\frac{2r}{3}\right)$, where $r = \pm 6$.

Points are $(4, -7, -9)$ and $(0, 1, -1)$

18. **d.** Let AD be the perpendicular and D be the foot of the perpendicular which divides BC in the ratio $\lambda : 1$, then

$$D\left(\frac{10\lambda-9}{\lambda+1}, \frac{4}{\lambda+1}, \frac{-\lambda+5}{\lambda+1}\right). \quad \text{(i)}$$

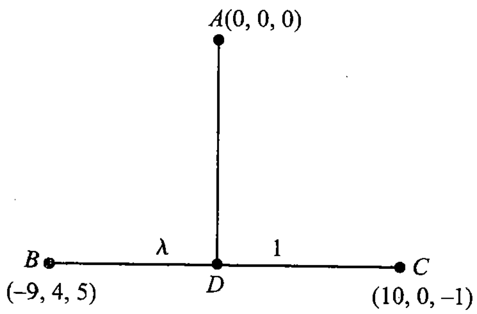


Fig. 3.36

The direction ratios of AD are $\frac{10\lambda-9}{\lambda+1}, \frac{4}{\lambda+1}$ and $\frac{-\lambda+5}{\lambda+1}$ and direction ratios of BC are 19, – 4 and – 6.

Since $AD \perp BC$, we get

$$19\left(\frac{10\lambda-9}{\lambda+1}\right) - 4\left(\frac{4}{\lambda+1}\right) - 6\left(\frac{-\lambda+5}{\lambda+1}\right) = 0$$

$$\Rightarrow \lambda = \frac{31}{28}$$

Hence, on putting the value of λ in (i), we get required foot of the perpendicular, i.e., $\left(\frac{58}{59}, \frac{112}{59}, \frac{109}{59}\right)$.

19. **d.** $P_1 = P_2 = 0$, $P_2 = P_3 = 0$ and $P_3 = P_1 = 0$ are lines of intersection of the three planes P_1, P_2 and P_3. As $\vec{n}_1, \vec{n}_2$ and $\vec{n}_3$ are non-coplanar, planes P_1, P_2 and P_3 will intersect at unique point. So the given lines will pass through a fixed point.

20. **d.** Let $A(1, 0, -1)$, $B(-1, 2, 2)$

Direction ratios of segment AB are $<2, -2, -3>$.

$$\cos\theta = \frac{|2\times 1 + 3(-2) - 5(-3)|}{\sqrt{1+9+25}\sqrt{4+4+9}} = \frac{11}{\sqrt{17}\sqrt{35}} = \frac{11}{\sqrt{595}}$$

Length of projection = $(AB)\sin\theta$

$$= \sqrt{(2)^2 + (2)^2 + (3)^2} \times \sqrt{1 - \frac{121}{595}}$$

$$= \sqrt{17}\,\frac{\sqrt{474}}{\sqrt{17}\sqrt{35}} = \sqrt{\frac{474}{35}} \text{ units}$$

21. **c.** Let the point be A, B, C and D.

The number of planes which have three points on one side and the fourth point on the other side is 4. The number of planes which have two points on each side of the plane is 3.

$\Rightarrow$ Number of planes is 7.

22. **a.** Point A is $(a, b, c) \Rightarrow$ Points P, Q, R are $(a, b, -c)$, $(-a, b, c)$ and $(a, -b, c)$, respectively

$\Rightarrow$ Centroid of triangle PQR is $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right) \Rightarrow G \equiv \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$

$\Rightarrow A, O, G$ are collinear $\Rightarrow$ area of triangle AOG is zero.

23. **b.** Let the equation of the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$\Rightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

$\Rightarrow$ Volume of tetrahedron $OABC = V = \frac{1}{6}(abc)$

Now $(abc)^{1/3} \geq \dfrac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \geq 3$ (G.M. $\geq$ H.M.)

$$\Rightarrow abc \geq 27 \Rightarrow V \geq \frac{9}{2}$$

24.

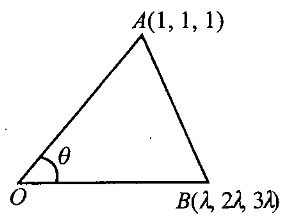


Fig. 3.37

b. Let any point on second line be $(\lambda, 2\lambda, 3\lambda)$

$$\cos\theta = \frac{6}{\sqrt{42}},\ \sin\theta = \frac{\sqrt{6}}{\sqrt{42}}$$

$$\Delta_{OAB} = \frac{1}{2}(OA)\ OB\sin\theta = \frac{1}{2}\sqrt{3}\ \lambda\ \sqrt{14} \times \frac{\sqrt{6}}{\sqrt{42}} = \sqrt{6}$$

$\Rightarrow \lambda = 2$

So B is $(2, 4, 6)$

25. **a.** Equation of line $x + 2y + z - 1 + \lambda(-x + y - 2z - 2) = 0$ (i)

$x + y - 2 + \mu(x + z - 2) = 0$ (ii)

$(0, 0, 1)$ lies on it $\Rightarrow \lambda = 0, \mu = -2$

For point of intersection, $z = 0$ and solve (i) and (ii).

26. **c.** Since the given lines are parallel.

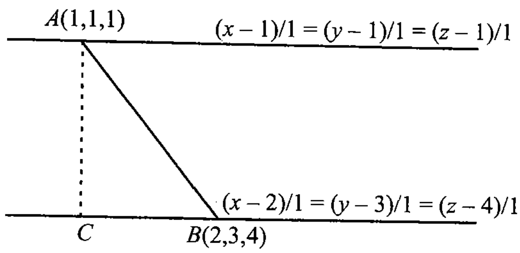


Fig. 3.38

From the figure, we get

$$BC = \frac{(2-1)1}{\sqrt{3}} + \frac{(3-1)1}{\sqrt{3}} + \frac{(4-1)1}{\sqrt{3}} = \frac{1+2+3}{\sqrt{3}} = 2\sqrt{3}$$

$$AB = \sqrt{1+4+9} = \sqrt{14}$$

Shortest distance $= AC = \sqrt{14-12} = \sqrt{2}$

27. **c.** Let $Q\ (\vec{q})$ be the foot of altitude drawn from 'P' to the plane $\vec{r}\cdot\vec{n} = 0$.

$\Rightarrow \vec{q} - \vec{p} = \lambda\vec{n} \Rightarrow \vec{q} = \vec{p} + \lambda\vec{n}$

Also $\vec{q}\cdot\vec{n} = 0 \Rightarrow (\vec{p} + \lambda\vec{n})\cdot\vec{n} = 0$

$$\Rightarrow \lambda = -\frac{\vec{p}\cdot\vec{n}}{|\vec{n}|^2} \Rightarrow \vec{q} - \vec{p} = -\frac{(\vec{p}\cdot\vec{n})}{|\vec{n}|^2}\vec{n}$$

Thus, required distance $= |\vec{q} - \vec{p}| = \dfrac{|\vec{p}\cdot\vec{n}|}{|\vec{n}|} = |\vec{p}\cdot\hat{n}|$

28. **b.** Given plane is $\vec{r} \cdot \vec{n} = q$ (i)

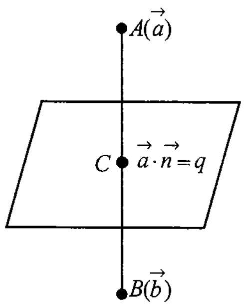


Fig. 3.39

Let the image of $A\ (\vec{a})$ in the plane be $B\ (\vec{b})$.

Equation of AC is $\vec{r} = \vec{a} + \lambda \vec{n}$ ($\because$ AC is normal to the plane) (ii)

Solving (i) and (ii), we get

$$(\vec{a} + \lambda \vec{n}) \cdot \vec{n} = q$$

$$\Rightarrow \lambda = \frac{q - \vec{a} \cdot \vec{n}}{|\vec{n}|^2}$$

$$\therefore\ \overrightarrow{OC} = \vec{a} + \frac{(q - \vec{a} \cdot \vec{n})}{|\vec{n}|^2} \cdot \vec{n}$$

But $\overrightarrow{OC} = \dfrac{\vec{a} + \vec{b}}{2}$

$$\therefore\ \vec{a} + \frac{(q - \vec{a} \cdot \vec{n})\vec{n}}{|\vec{n}|^2} = \frac{\vec{a} + \vec{b}}{2}$$

$$\Rightarrow \vec{b} = \vec{a} + 2\left(\frac{q - \vec{a} \cdot \vec{n}}{|\vec{n}|^2}\right)\vec{n}$$

29. **c.** We must have $\vec{b} \cdot \vec{n} = 0$ (because the line and the plane must be parallel) and $\vec{a} \cdot \vec{n} \neq q$ (as point $\vec{a}$ on the line should not lie on the plane).

30. **c.** Here $l = \cos \dfrac{\pi}{4}$, $m = \cos \dfrac{\pi}{4}$

Let the line make an angle 'γ' with z-axis

$\therefore\ l^2 + m^2 + n^2 = 1$

$$\Rightarrow \cos^2 \frac{\pi}{4} + \cos^2 \frac{\pi}{4} + \cos^2 \gamma = 1$$

$$\Rightarrow \frac{1}{2} + \frac{1}{2} + \cos^2 \gamma = 1$$

$$\Rightarrow 2\cos^2 \gamma = 0 \Rightarrow \cos \gamma = 0 \Rightarrow \gamma = \frac{\pi}{2}$$

31. **d.** Let the plane $\vec{r}\cdot(\vec{i}-2\vec{j}+3\vec{k})=17$ divide the line joining the points $-2\vec{i}+4\vec{j}+7\vec{k}$ and $3\vec{i}-5\vec{j}+8\vec{k}$ in the ratio $t:1$ at point P.

Therefore, point P is

$$\frac{3t-2}{t+1}\vec{i}+\frac{-5t+4}{t+1}\vec{j}+\frac{8t+7}{t+1}\vec{k}$$

This lies on the given plane

$$\therefore\ \frac{3t-2}{t+1}\cdot(1)+\frac{-5t+4}{t+1}(-2)+\frac{8t+7}{t+1}(3)=17$$

Solving, we get

$$t=\frac{3}{10}$$

32. **d.** Let $P(\alpha, \beta, \gamma)$ be the image of the point $Q(-1, 3, 4)$.

Midpoint of PQ lies on $x-2y=0$. Then,

$$\frac{\alpha-1}{2}-2\left(\frac{\beta+3}{2}\right)=0$$

$$\Rightarrow \alpha-1-2\beta-6=0 \Rightarrow \alpha-2\beta=7 \qquad \text{(i)}$$

Also PQ is perpendicular to the plane. Then,

$$\frac{\alpha+1}{1}=\frac{\beta-3}{-2}=\frac{\gamma-4}{0} \qquad \text{(ii)}$$

Solving (i) and (ii), we get

$$\alpha=\frac{9}{5}, \beta=-\frac{13}{5}, \gamma=4$$

Therefore, image is

$$\left(\frac{9}{5},-\frac{13}{5},4\right)$$

Alternative method:

For image,

$$\frac{\alpha-(-1)}{1}=\frac{\beta-3}{-2}=\frac{\gamma-4}{0}=\frac{-2(-1-2(3))}{(1)^2+(-2)^2}$$

$$\Rightarrow \quad \alpha=\frac{9}{5}, \beta=-\frac{13}{5}, \gamma=4$$

33. **a.** It is obvious that the given line and plane are parallel.

Given point on the line is $A(2, -2, 3)$.

$B(0, 0, 5)$ is a point on the plane

$\therefore \overrightarrow{AB} = (2-0)\hat{i} + (-2-0)\hat{j} + (3-5)\hat{k}$

Then distance of B from the plane = projection of $\overrightarrow{AB}$ on vector $\hat{i} + 5\hat{j} + \hat{k}$

$$p = \left| \frac{\left(2\hat{i} - 2\hat{j} - 2\hat{k}\right)\cdot\left(\hat{i} + 5\hat{j} + \hat{k}\right)}{\sqrt{1+25+1}} \right|$$

$$= \left| \frac{2-10-2}{\sqrt{27}} \right| = \frac{10}{3\sqrt{3}}$$

34. d. Since line of intersection is perpendicular to both the planes, direction ratios of the line of intersection

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{vmatrix} = 3\hat{i} - 3\hat{j} + 3\hat{k}.$$

Hence, $\cos\alpha = \dfrac{3}{\sqrt{9+9+9}} = \dfrac{1}{\sqrt{3}}$

35. d. Let P be the point (1, 2, 3) and PN be the length of the perpendicular from P on the given line.

Coordinates of point N are $(3\lambda + 6, 2\lambda + 7, -2\lambda + 7)$.

Now PN is perpendicular to the given line or vector $3\vec{i} + 2\vec{j} - 2\vec{k}$

$\Rightarrow 3(3\lambda+6-1) + 2(2\lambda+7-2) - 2(-2\lambda+7-3) = 0$

$\Rightarrow \lambda = -1$

Then, point N is (3, 5, 9)

$\Rightarrow PN = 7$

36. b. The line is $\dfrac{x+1}{1} = \dfrac{y-1}{2} = \dfrac{z-2}{2}$ and the plane is $2x - y + \sqrt{\lambda}\,z + 4 = 0$.

If θ be the angle between the line and the plane, then $90° - \theta$ is the angle between the line and normal to the plane

$$\Rightarrow \cos(90° - \theta) = \frac{(1)(2) + (2)(-1) + (2)\left(\sqrt{\lambda}\right)}{\sqrt{1+4+4}\sqrt{4+1+\lambda}}$$

$$\Rightarrow \sin\theta = \frac{2-2+2\sqrt{\lambda}}{3\sqrt{5+\lambda}} \Rightarrow \frac{1}{3} = \frac{2\sqrt{\lambda}}{3\sqrt{5+\lambda}}$$

$\Rightarrow \sqrt{5+\lambda} = 2\sqrt{\lambda}$

$\Rightarrow 5 + \lambda = 4\lambda$

$\Rightarrow 3\lambda = 5$

$\Rightarrow \lambda = \dfrac{5}{3}$

37. **d.** The given spheres are

$x^2 + y^2 + z^2 + 7x - 2y - z - 13 = 0$ (i)

and $x^2 + y^2 + z^2 - 3x + 3y + 4z - 8 = 0$ (ii)

Subtracting (ii) from (i), we get

$10x - 5y - 5z - 5 = 0$

$\Rightarrow 2x - y - z = 1$

38. **c.** Plane meets axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Then area of ΔABC,

$$= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{2}|(-a\hat{i} + b\hat{j}) \times (-a\hat{i} + c\hat{k})|$$

$$= \frac{1}{2}\sqrt{(a^2b^2 + b^2c^2 + c^2a^2)}$$

39. **c.** Here $\sin^2\beta = 3\sin^2\theta$ (i)

By the question, $\cos^2\theta + \cos^2\theta + \cos^2\beta = 1$ (ii)

$\Rightarrow \cos^2\beta = 1 - 2\cos^2\theta$ (iii)

Adding (i) and (iii), we get

$1 = 1 + 3\sin^2\theta - 2\cos^2\theta$

$\Rightarrow 1 = 1 + 3(1 - \cos^2\theta) - 2\cos^2\theta$

$\Rightarrow 5\cos^2\theta = 3$

$\Rightarrow \cos^2\theta = \frac{3}{5}$

40. **d.** The given sphere is

$$x^2 + y^2 + z^2 + 4x - 2y - 6z - 155 = 0$$

Its centre is $(-2, 1, 3)$ and radius $= \sqrt{4 + 1 + 9 + 155} = \sqrt{169} = 13$

Therefore, distance of centre $(-2, 1, 3)$ from the plane $12x + 4y + 3z = 327$

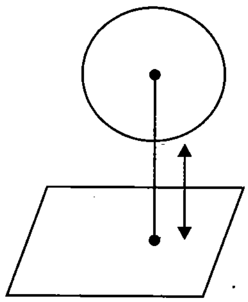


Fig. 3.40

$$= \frac{|12(-2) + 4(1) + 3(3) - 327|}{\sqrt{144 + 16 + 9}} = 26$$

Hence, the shortest distance is 13.

41. **d.** Vector perpendicular to the face OAB is

$$\overrightarrow{OA} \times \overrightarrow{OB} = (\hat{i} + 2\hat{j} + \hat{k}) \times (2\hat{i} + \hat{j} + 3\hat{k})$$
$$= 5\hat{i} - \hat{j} - 3\hat{k}$$

Vector perpendicular to face ABC is

$$\overrightarrow{AB} \times \overrightarrow{AC} = (\hat{i} - \hat{j} + 2\hat{k}) \times (-2\hat{i} - \hat{j} + \hat{k})$$
$$= \hat{i} - 5\hat{j} - 3\hat{k}$$

Since the angle between the face = angle between their normal, therefore

$$\cos\theta = \frac{5+5+9}{\sqrt{35}\sqrt{35}} = \frac{19}{35} \Rightarrow \theta = \cos^{-1}\left(\frac{19}{35}\right)$$

42. **b.** Center of the sphere is $(-1, 1, 2)$ and its radius = $\sqrt{1+1+4+19} = 5$

CL, perpendicular distance of C from plane, is $\left|\frac{-1+2+4+7}{\sqrt{1+4+4}}\right| = 4$

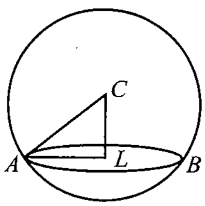


Fig. 3.41

Now $AL^2 = CA^2 - CL^2 = 25 - 16 = 9$

Hence, radius of the circle = $\sqrt{9} = 3$

43. **b.** The lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ (i)

and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ (ii)

are coplanar if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_1 & n_2 \end{vmatrix} = 0$

or $\begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$

$\Rightarrow k^2 + 3k = 0$

$\Rightarrow k = 0$ or -3

44. **a.** Given lines are

$$\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}=r_1 \quad \text{(say)}$$

and $\dfrac{x+3}{-36}=\dfrac{y-3}{2}=\dfrac{z-6}{4}=r_2 \quad \text{(say)}$

$\therefore x=3r_1+5=-36r_2-3,$

$y=-r_1+7=3+2r_2$

and $z=r_1-2=4r_2+6$

On solving, we get

$$x=21, y=\frac{5}{3}, z=\frac{10}{3}$$

45. **c.** The planes are $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$ and $\dfrac{x}{a'}+\dfrac{y}{b'}+\dfrac{z}{c'}=1$

Since the perpendicular distance of the origin on the planes is same, therefore

$$\left|\frac{-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|=\left|\frac{-1}{\sqrt{\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}}}\right|$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a'^2}-\frac{1}{b'^2}-\frac{1}{c'^2}=0$$

46. **a.** The required plane is $\begin{vmatrix} x-3 & y-6 & z-4 \\ 3-3 & 2-6 & 0-4 \\ 1 & 5 & 4 \end{vmatrix}=0$

$$\Rightarrow \begin{vmatrix} x-3 & y-z-2 & z-4 \\ 0 & 0 & -4 \\ 1 & 1 & 4 \end{vmatrix}=0 \quad (\text{Operating } C_2 \to C_2 - C_3)$$

$\Rightarrow 4(x-3-y+z+2)=0$

$\Rightarrow x-y+z=1$

47. **b.** Any plane through (1, 0, 0) is $a(x-1)+by+cz=0.$ (i)

It passes through (0, 1, 0).

$\therefore \quad a(0-1)+b(1)+c(0)=0 \Rightarrow -a+b=0$ (ii)

(i) makes an angle of $\dfrac{\pi}{4}$ with $x+y=3$, therefore

$$\cos\frac{\pi}{4}=\frac{a(1)+b(1)+c(0)}{\sqrt{a^2+b^2+c^2}\sqrt{1+1+0}}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{a+b}{\sqrt{2}\sqrt{a^2+b^2+c^2}}$$

$$\Rightarrow a+b=\sqrt{a^2+b^2+c^2}$$

Squaring, we get

$a^2+b^2+2ab=a^2+b^2+c^2$

$\Rightarrow 2ab=c^2 \Rightarrow 2a^2=c^2$ (using (ii))

$\Rightarrow c=\sqrt{2}\,a$

Hence, $a:b:c=a:a:\sqrt{2}\,a$

$=1:1:\sqrt{2}$

48. **b.** The equation of the line through the centre $\hat{j}+2\hat{k}$ and normal to the given plane is

$$\vec{r} = \hat{j}+2\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k}) \quad \text{(i)}$$

This meets the plane for which

$$[\hat{j}+2\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k})]\cdot(\hat{i}+2\hat{j}+2\hat{k})=15$$

$$\Rightarrow 6+9\lambda=15 \Rightarrow \lambda=1$$

Putting in (i), we get

$$\vec{r}=\hat{j}+2\hat{k}+(\hat{i}+2\hat{j}+2\hat{k}) = \hat{i}+3\hat{j}+4\hat{k}$$

Hence, centre is (1, 3, 4).

49. **c.** Equations of the planes through $y=mx$, $z=c$ and $y=-mx$, $z=-c$ are respectively,

$(y-mx)+\lambda_1(z-c)=0$ (i)

and $(y+mx)+\lambda_2(z+c)=0$ (ii)

It meets at x-axis, i.e., $y=0=z$.

$\therefore \lambda_2=\lambda_1$

From (i) and (ii), $\dfrac{y-mx}{z-c}=\dfrac{y+mx}{z+c}$

$\therefore cy=mzx$

50. **c.** Let $Q\,(\vec{q})$ be the foot of altitude drawn from

$P\,(\vec{p})$ to the line $\vec{r}=\vec{a}+\lambda\vec{b}$,

$$\Rightarrow (\vec{q}-\vec{p})\cdot\vec{b}=0 \text{ and } \vec{q}=\vec{a}+\lambda\vec{b}$$

$$\Rightarrow (\vec{a}+\lambda\vec{b}-\vec{p})\cdot\vec{b}=0$$

$$\Rightarrow (\vec{a}-\vec{p})\cdot\vec{b}+\lambda|\vec{b}|^2=0$$

$$\Rightarrow \lambda=\frac{(\vec{p}-\vec{a})\cdot\vec{b}}{|\vec{b}|^2}$$

$$\Rightarrow \vec{q} - \vec{p} = \vec{a} + \frac{((\vec{p} - \vec{a}) \cdot \vec{b})\vec{b}}{|\vec{b}|^2} - \vec{p}$$

$$\Rightarrow |\vec{q} - \vec{p}| = \left|(\vec{a} - \vec{p}) + \frac{((\vec{p} - \vec{a}) \cdot \vec{b})\vec{b}}{|\vec{b}|^2}\right|$$

51. **b.** Coordinates of L and M are $(0, b, c)$ and $(a, 0, c)$, respectively. Therefore, the equation of the plane passing through $(0, 0, 0)$, $(0, b, c)$ and $(a, 0, c)$ is

$$\begin{vmatrix} x-0 & y-0 & z-0 \\ 0 & b & c \\ a & 0 & c \end{vmatrix} = 0 \text{ or } \frac{x}{a} + \frac{y}{b} - \frac{z}{c} = 0$$

52. **c.** We must have $\vec{b} \cdot \vec{n} = 0$ and $\vec{a} \cdot \vec{n} = q$.

53. **b.** We have $\vec{s} - \vec{p} = \lambda \vec{n}$ and $\vec{s} \cdot \vec{n} = q$.

$$\Rightarrow (\lambda \vec{n} + \vec{p}) \cdot \vec{n} = q$$

$$\Rightarrow \lambda = \frac{q - \vec{p} \cdot \vec{n}}{|\vec{n}|^2}$$

$$\Rightarrow \vec{s} = \vec{p} + \frac{(q - \vec{p} \cdot \vec{n})\vec{n}}{|\vec{n}|^2}$$

54. **d.** Line of intersection of $\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 0$ and $\vec{r} \cdot (3\hat{i} + 3\hat{j} + \hat{k}) = 0$ will be parallel to $(3\hat{i} + 3\hat{j} + \hat{k}) \times (\hat{i} + 2\hat{j} + 3\hat{k})$, i.e., $7\hat{i} - 8\hat{j} + 3\hat{k}$.

If the required angle is θ, then

$$\cos\theta = \frac{7}{\sqrt{49 + 64 + 9}} = \frac{7}{\sqrt{122}}$$

55. **c.** Given one vertex $A(7, 2, 4)$ and line $\frac{x+6}{5} = \frac{y+10}{3} = \frac{z+14}{8}$

General point on above line $B \equiv (5\lambda - 6, 3\lambda - 10, 8\lambda - 14)$

Direction ratios of line AB are $<5\lambda - 13, 3\lambda - 12, 8\lambda - 18>$

Direction ratios of line BC are $<5, 3, 8>$

since angle between AB and BC is $\pi/4$

$$\cos\frac{\pi}{4} = \frac{(5\lambda - 3)5 + 3(3\lambda - 12) + 8(8\lambda - 18)}{\sqrt{5^2 + 3^2 + 8^2} \cdot \sqrt{(5\lambda - 13)^2 + (3\lambda - 12)^2 + (8\lambda - 18)^2}}$$

Squaring and solving, we have $\lambda = 3, 2$

Hence equation of lines are $\frac{x-7}{2} = \frac{y-2}{-3} = \frac{z-4}{6}$ and $\frac{x-7}{3} = \frac{y-2}{6} = \frac{z-4}{2}$

56. **a.** $\vec{r}\cdot\vec{n_1}+\lambda\vec{r}\cdot\vec{n_2}=q_1+\lambda q_2$ (i)

where λ is a parameter.

So, $\vec{n_1}+\lambda\vec{n_2}$ is normal to plane (i). Now, any plane parallel to the line of intersection of the planes $\vec{r}\cdot\vec{n_3}=q_3$ and $\vec{r}\cdot\vec{n_4}=q_4$ is of the form $\vec{r}\cdot(\vec{n_3}\times\vec{n_4})=d$. Hence we must have

$$[\vec{n_1}+\lambda\vec{n_2}]\cdot[\vec{n_3}\times\vec{n_4}]=0$$

$$\Rightarrow [\vec{n_1}\,\vec{n_3}\,\vec{n_4}]+\lambda[\vec{n_2}\,\vec{n_3}\,\vec{n_4}]=0$$

$$\Rightarrow \lambda=\frac{-[\vec{n_1}\,\vec{n_3}\,\vec{n_4}]}{[\vec{n_2}\,\vec{n_3}\,\vec{n_4}]}$$

$\Rightarrow$ On putting this value in Eq. (i), we have the equation of the required plane as

$$\vec{r}\cdot\vec{n_1}-q_1=\frac{[\vec{n_1}\,\vec{n_3}\,\vec{n_4}]}{[\vec{n_2}\,\vec{n_3}\,\vec{n_4}]}(\vec{r}\cdot\vec{n_2}-q_2)$$

$$\Rightarrow [\vec{n_2}\,\vec{n_3}\,\vec{n_4}](\vec{r}\cdot\vec{n_1}-q_1)=[\vec{n_1}\,\vec{n_3}\,\vec{n_4}]\,(\vec{r}\cdot\vec{n_2}-q_2)$$

57. **c.**

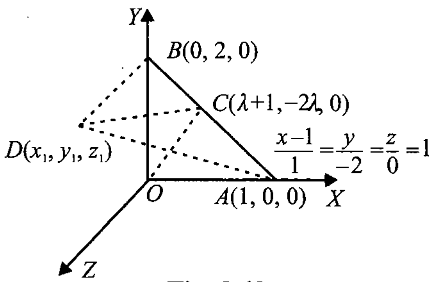


Fig. 3.42

Equation of line AB is $\dfrac{x-1}{1}=\dfrac{y}{-2}=\dfrac{z}{0}=\lambda$

Now $AB\perp OC \Rightarrow 1(\lambda+1)+(-2\lambda)(-2)=0 \Rightarrow 5\lambda=-1 \Rightarrow \lambda=-\dfrac{1}{5}$

$\Rightarrow C$ is $\left(\dfrac{4}{5},\dfrac{2}{5},0\right)$. Now

$x_1^2+(y_1-2)^2+z_1^2=4$ (i)

and $(x_1-1)^2+y_1^2+z_1^2=1$ (ii)

Now $OC\perp CD$

$$\Rightarrow \left(x_1-\frac{4}{5}\right)\frac{4}{5}+\left(y_1-\frac{2}{5}\right)\frac{2}{5}+(z_1-0)\,0=0 \quad \text{(iii)}$$

From (i) and (ii), we get

$-4y_1 + 2x_1 = 0 \Rightarrow x_1 = 2y_1$

From (iii), putting $x_1 = 2y_1 \Rightarrow 2y_1 = \frac{4}{5} \Rightarrow y_1 = \frac{2}{5} \Rightarrow x_1 = \frac{4}{5}$. Putting this value of x_1 and y_1 in (i), we get

$z_1 = \pm\frac{2}{\sqrt{5}}$

58. **b.** Let $\vec{r} \times \vec{a} = \vec{b} \times \vec{a}$

$\Rightarrow (\vec{r} - \vec{b}) \times \vec{a} = \vec{0} \Rightarrow \vec{r} = \vec{b} + t\vec{a}$

Similarly, other line $\vec{r} = \vec{a} + k\vec{b}$, where t and k are scalars.

Now $\vec{a} + k\vec{b} = \vec{b} + t\vec{a}$

$\Rightarrow t = 1, k = 1$

(equating the coefficients of $\vec{a}$ and $\vec{b}$)

$\therefore \vec{r} = \vec{a} + \vec{b} = \hat{i} + \hat{j} + 2\hat{i} - \hat{k} = 3\hat{i} + \hat{j} - \hat{k}$

i.e., (3, 1, – 1)

59. **a.** Let the point P be (x, y, z), then the vector $x\hat{i} + y\hat{j} + z\hat{k}$ will lie on the line

$\Rightarrow (x-1)\hat{i} + (y-1)\hat{j} + (z-1)\hat{k} = -\lambda\hat{i} + \lambda\hat{j} - \lambda\hat{k}$

$\Rightarrow x = 1 - \lambda, y = 1 + \lambda$ and $z = 1 - \lambda$

Now point P is nearest to the origin $\Rightarrow D = (1 - \lambda)^2 + (1 + \lambda)^2 + (1 - \lambda)^2$

$\Rightarrow \frac{dD}{d\lambda} = -4(1 - \lambda) + 2(1 + \lambda) = 0 \Rightarrow \lambda = \frac{1}{3}$

$\Rightarrow$ the point is $\left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right)$

60. **b.** Let P be the point and it divides the line segment in the ratio $\lambda : 1$. Then,

$$\overrightarrow{OP} = \vec{r} = \frac{-3\lambda + 2}{\lambda + 1}\hat{i} + \frac{5\lambda - 4}{\lambda + 1}\hat{j} + \frac{-8\lambda - 7}{\lambda + 1}\hat{k}$$

It satisfies $\vec{r} \cdot (\hat{i} - 2\hat{j} + 3\hat{k}) = 13$. So,

$$\frac{-3\lambda + 2}{\lambda + 1} - 2\frac{5\lambda - 4}{\lambda + 1} + 3\frac{-8\lambda - 7}{\lambda + 1} = 13$$

or $-3\lambda + 2 - 2(5\lambda - 4) + 3(-8\lambda - 7) = 13(\lambda + 1)$

or $-37\lambda - 11 = 13\lambda + 13$ or $50\lambda = -24$ or $\lambda = -\frac{12}{25}$

61. **d.** $\vec{V_1}, \vec{V_2}, \overrightarrow{PS}$ are in the same plane

$\therefore (2\hat{i} - \hat{j} + 3\hat{k}) \times (-3\hat{i} + \hat{k}) \cdot ((x + 2)\hat{i} + (y - 1)\hat{j} + z\hat{k}) = 0$

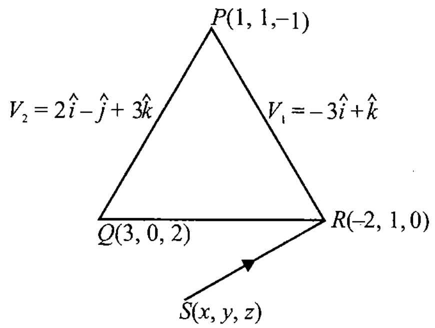


Fig. 3.43

62. **a.** $\overrightarrow{AB} = \vec{\beta} - \vec{\alpha} = -2\hat{i} - 3\hat{j} - 6\hat{k}$

Equation of the plane passing through B and perpendicular to AB is

$$(\vec{r} - \overrightarrow{OB}) \cdot \overrightarrow{AB} = 0$$

$$\vec{r} \cdot (2\hat{i} + 3\hat{j} + 6\hat{k}) + 28 = 0$$

Hence the required distance from $\vec{r} = -\hat{i} + \hat{j} + \hat{k}$

$$= \left| \frac{(-\hat{i} + \hat{j} + \hat{k}) \cdot (2\hat{i} + 3\hat{j} + 6\hat{k}) + 28}{|2\hat{i} + 3\hat{j} + 6\hat{k}|} \right| = \left| \frac{-2+3+6+28}{7} \right| = 5 \text{ units}$$

63. **a.** Both the lines pass through origin. Line L_1 is parallel to the vector $\vec{V_1}$

$$\vec{V_1} = (\cos\theta + \sqrt{3})\hat{i} + (\sqrt{2}\sin\theta)\hat{j} + (\cos\theta - \sqrt{3})\hat{k}$$

and L_2 is parallel to the vector $\vec{V_2}$

$$\vec{V_2} = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\therefore \cos\alpha = \frac{\vec{V_1} \cdot \vec{V_2}}{|\vec{V_1}||\vec{V_2}|}$$

$$= \frac{a(\cos\theta + \sqrt{3}) + (b\sqrt{2})\sin\theta + c(\cos\theta - \sqrt{3})}{\sqrt{a^2 + b^2 + c^2}\sqrt{(\cos\theta + \sqrt{3})^2 + 2\sin^2\theta + (\cos\theta - \sqrt{3})^2}}$$

$$= \frac{(a+c)\cos\theta + b\sqrt{2}\sin\theta + (a-c)\sqrt{3}}{\sqrt{a^2 + b^2 + c^2}\sqrt{2+6}}$$

For $\cos\alpha$ to be independent of θ, we get
$a + c = 0$ and $b = 0$

$$\therefore \cos\alpha = \frac{2a\sqrt{3}}{a\sqrt{2}\,2\sqrt{2}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \alpha = \frac{\pi}{6}.$$

64. **d.** Given lines are $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + l(3\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + m(-3\hat{i} + 2\hat{j} + 4\hat{k})$

Required shortest distance

$$= \frac{|(6\hat{i} + 15\hat{j} - 3\hat{k}) \cdot ((3\hat{i} - \hat{j} + \hat{k}) \times (-3\hat{i} + 2\hat{j} + 4\hat{k}))|}{|(3\hat{i} - \hat{j} + \hat{k}) \times (-3\hat{i} + 2\hat{j} + 4\hat{k})|}$$

$$= \frac{|(6\hat{i} + 15\hat{j} - 3\hat{k}) \cdot (-6\hat{i} - 15\hat{j} + 3\hat{k})|}{|-6\hat{i} - 15\hat{j} + 3\hat{k}|}$$

$$= \frac{36 + 225 + 9}{\sqrt{36 + 225 + 9}} = \frac{270}{\sqrt{270}} = \sqrt{270} = 3\sqrt{30}$$

65. **b.** The required line passes through the point $\hat{i} + 3\hat{j} + 2\hat{k}$ and is perpendicular to the lines $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} + 6\hat{j} + \hat{k}) + \mu(\hat{i} + 2\hat{j} + 3\hat{k})$; therefore it is parallel to the vector $\vec{b} = (2\hat{i} + \hat{j} + \hat{k}) \times (\hat{i} + 2\hat{j} + 3\hat{k}) = (\hat{i} - 5\hat{j} + 3\hat{k})$

Hence, the equation of the required line is

$$\vec{r} = (\hat{i} + 3\hat{j} + 2\hat{k}) + \lambda(\hat{i} - 5\hat{j} + 3\hat{k})$$

66. **d.** Here, the required plane is

$a(x - 4) + b(y - 3) + c(z - 2) = 0$

Also $a + b + 2c = 0$ and $a - 4b + 5c = 0$

Solving, we have

$$\frac{a}{5+8} = \frac{b}{2-5} = \frac{c}{-4-1} = k$$

$$\frac{a}{13} = \frac{b}{-3} = \frac{c}{-5} = k$$

Therefore, the required equation of plane is $-13x + 3y + 5z + 33 = 0$

67. **b.** Plane passing through the line of intersection of planes $4y + 6z = 5$ and $2x + 3y + 5z = 5$ is

$(4y + 6z - 5) + \lambda(2x + 3y + 5z - 5) = 0$, or

$2\lambda x + (3\lambda + 4)y + (5\lambda + 6)z - 5\lambda - 5 = 0$

Clearly, for $\lambda = -3$, we get the plane $6x + 5y + 9z = 10$.

Hence, the given three planes have common line of intersection.

68. **c.** The equation of a plane through the line of intersection of the planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ is

$(ax + by + cz + d) + \lambda(a'x + b'y + c'z + d') = 0$

or $x(a + \lambda a') + y(b + \lambda b') + z(c + \lambda c') + d + \lambda d' = 0$ (i)

This is parallel to x-axis, i.e., $y = 0, z = 0$. Therefore,

$1(a + \lambda a') + 0(b + \lambda b') + 0(c + \lambda c') = 0$

$\Rightarrow \lambda = -\dfrac{a}{a'}$

Putting the value of λ in (i), the required plane is $y(a'b - ab') + z(a'c - ac') + a'd - ad' = 0$

or $(ab' - a'b)y + (ac' - a'c)z + ad' - a'd = 0$

69. **b.** Any plane through (2, 2, 1) is

$a(x - 2) + b(y - 2) + c(z - 1) = 0$ (i)

It passes through (9, 3, 6) if $7a + b + 5c = 0$. (ii)

Also (i) is perpendicualr to $2x + 6y + 6z - 1 = 0$, we have

$2a + 6b + 6c = 0$

$\therefore a + 3b + 3c = 0$ (iii)

$\therefore \dfrac{a}{-12} = \dfrac{b}{-16} = \dfrac{c}{20}$ or $\dfrac{a}{3} = \dfrac{b}{4} = \dfrac{c}{-5}$ (from (ii) and (iii))

Therefore, the required plane is $3(x - 2) + 4(y - 2) - 5(z - 1) = 0$ or $3x + 4y - 5z - 9 = 0$.

70. **a.** Since line is parallel to the plane vector, $2\vec{i} + 3\vec{j} + \lambda\vec{k}$ is perpendicular to the normal to the plane $2\vec{i} + 3\vec{j} + 4\vec{k}$

$\Rightarrow 2 \times 2 + 3 \times 3 + 4\lambda = 0$

$\Rightarrow \lambda = -\dfrac{13}{4}$

71. **a.** Any plane through the given planes is $x + 2y + 3z - 4 + \lambda(4x + 3y + 2z + 1) = 0$

It passes through (0, 0, 0). Therefore,

$-4 + \lambda = 0$

$\therefore \lambda = 4$

Therefore, the required plane is $x + 2y + 3z + 4(4x + 3y + 2z) = 0$ or $17x + 14y + 11z = 0$.

72. **a.** The equation of the plane through the line of intersection of the planes $4x + 7y + 4z + 81 = 0$ and $5x + 3y + 10z = 25$ is $(4x + 7y + 4z + 81) + \lambda(5x + 3y + 10z - 25) = 0$

$\Rightarrow (4 + 5\lambda)x + (7 + 3\lambda)y + (4 + 10\lambda)z + 81 - 25\lambda = 0$ (i)

which is perpendicular to $4x + 7y + 4z + 81 = 0$

$\Rightarrow 4(4 + 5\lambda) + 7(7 + 3\lambda) + 4(4 + 10\lambda) = 0$

$\Rightarrow 81\lambda + 81 = 0$

$\Rightarrow \lambda = -1$

Hence the plane is $x - 4y + 6z = 106$

73. **b.** The equation of a plane through the line of intersection of the planes $\vec{r}\cdot\vec{a}=\lambda$ and $\vec{r}\cdot\vec{b}=\mu$ is

$(\vec{r}\cdot\vec{a}-\lambda)+k(\vec{r}\cdot\vec{b}-\mu)=0$ or $\vec{r}\cdot(\vec{a}+k\vec{b})=\lambda+k\mu$ (i)

This passes through the origin, therefore

$$\vec{0}(\vec{a}+k\vec{b})=\lambda+\mu k \Rightarrow k=\frac{-\lambda}{\mu}$$

Putting the value of k in (i), we get the equation of the required plane as

$$\vec{r}\cdot(\mu\vec{a}-\lambda\vec{b})=0 \Rightarrow \vec{r}\cdot(\lambda\vec{b}-\mu\vec{a})=0$$

74. **b.** The lines $\vec{r}=\vec{a}+\lambda(\vec{b}\times\vec{c})$ and $\vec{r}=\vec{b}+\mu(\vec{c}\times\vec{a})$ pass through points $\vec{a}$ and $\vec{b}$, respectively, and are parallel to the vectors $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$, respectively. Therefore, they intersect if $\vec{a}-\vec{b}, \vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ are coplanar and so

$$(\vec{a}-\vec{b})\cdot\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\}=0$$

$$\Rightarrow (\vec{a}-\vec{b})\cdot([\vec{b}\ \vec{c}\ \vec{a}]\vec{c}-[\vec{b}\ \vec{c}\ \vec{c}]\vec{a})=0$$

$$\Rightarrow ((\vec{a}-\vec{b})\cdot\vec{c})[\vec{b}\ \vec{c}\ \vec{a}]=0$$

$$\Rightarrow \vec{a}\cdot\vec{c}-\vec{b}\cdot\vec{c}=0 \Rightarrow \vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{c}$$

75. **a.** Equation of the plane through (–1, 0, 1) is

$a(x+1)+b(y-0)+c(z-1)=0$ (i)

which is parallel to the given line and perpendicular to the given plane

$-a+2b+3c=0$ (ii)

and $a-2b+c=0$ (iii)

From Eqs. (ii) and (iii), we get

$c=0, a=2b$

From Eq. (i), $2b(x+1)+by=0$

$\Rightarrow 2x+y+2=0$

76. **b.** Eliminating n, we get

$\lambda(l+m)^2+lm=0$

$$\Rightarrow \frac{\lambda l^2}{m^2}+(2\lambda+1)\frac{l}{m}+\lambda=0$$

$$\Rightarrow \frac{l_1l_2}{m_1m_2}=1 \quad \left(\text{product of roots } \frac{l_1}{m_1} \text{ and } \frac{l_2}{m_2}\right)$$

where l_1/m_1 and l_2/m_2 are the roots of this equation, further eliminating m, we get

$\lambda l^2-ln-n^2=0$

$$\Rightarrow \frac{l_1l_2}{n_1n_2}=-\frac{1}{\lambda}$$

Since the lines with direction cosines (l_1, m_1, n_1) and (l_2, m_2, n_2) are perpendicular, we have

$l_1l_2+m_1m_2+n_1n_2=0$

$\Rightarrow 1+1-\lambda=0$

$\Rightarrow \lambda=2$

77. **a.** Direction ratios of the line joining points $P(1, 2, 3)$ and $Q(-3, 4, 5)$ are $-4, 2, 2$ which are direction ratios of the normal to the plane.

Then, equation of plane is $-4x + 2y + 2z = k$.

Also this plane passes through the midpoint of PQ $(-1, 3, 4)$

$\Rightarrow -4(-1)+2(3)+2(4)=k$

$\Rightarrow k=18$

$\Rightarrow$ Equation of plane is $2x-y-z=-9$

Then, intercepts are $(-9/2)$, 9 and 9

78. **c.** $3l+m+5n=0$ (i)

$6mn-2nl+5ml=0$ (ii)

Substituting the value of n from Eq. (i) in Eq. (ii), we get

$6l^2+9lm-6m^2=0$

$$\Rightarrow 6\left(\frac{l}{m}\right)^2+9\left(\frac{l}{m}\right)-6=0$$

$$\therefore \frac{l_1}{m_1}=\frac{1}{2} \text{ and } \frac{l_2}{m_2}=-2$$

From Eq. (i), we get

$$\frac{l_1}{n_1}=-1 \text{ and } \frac{l_2}{n_2}=-2$$

$$\therefore \frac{l_1}{1}=\frac{m_1}{2}=\frac{n_1}{-1}=\sqrt{\frac{l_1^2+m_1^2+n_1^2}{1+4+1}}=\frac{1}{\sqrt{6}}$$

$$\text{and } \frac{l_2}{2}=\frac{m_2}{-1}=\frac{n_2}{-1}=\frac{\sqrt{l_2^2+m_2^2+n_2^2}}{\sqrt{4+1+1}}=\frac{1}{\sqrt{6}}$$

If θ be the angle between the lines, then

$$\cos\theta=\left(\frac{1}{\sqrt{6}}\right)\left(\frac{2}{\sqrt{6}}\right)+\left(\frac{2}{\sqrt{6}}\right)\left(-\frac{1}{\sqrt{6}}\right)+\left(-\frac{1}{\sqrt{6}}\right)\left(-\frac{1}{\sqrt{6}}\right)=\frac{1}{6}$$

$$\therefore \quad \theta=\cos^{-1}\left(\frac{1}{6}\right)$$

79. **b.** Let the equation of the sphere be $x^2+y^2+z^2-ax-by-cz=0$. This meets the axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Let (α, β, γ) be the coordinates of the centroid of the tetrahedron $OABC$. Then

$$\frac{a}{4}=\alpha, \frac{b}{4}=\beta, \frac{c}{4}=\gamma$$

$\Rightarrow a=4\alpha, b=4\beta, c=4\gamma$

Now, radius of the sphere = $2k$

$$\Rightarrow \frac{1}{2}\sqrt{a^2+b^2+c^2} = 2k \Rightarrow a^2+b^2+c^2 = 16k^2$$

$$\Rightarrow 16(\alpha^2+\beta^2+\gamma^2) = 16k^2$$

Hence, the locus of (α, β, γ) is $(x^2+y^2+z^2) = k^2$

80. **a.** Let the foot of the perpendicular from the origin on the given plane be $P(\alpha, \beta, \gamma)$. Since the plane passes through $A(a, b, c)$,

$$AP \perp OP \Rightarrow \overrightarrow{AP} \cdot \overrightarrow{OP} = 0$$

$$\Rightarrow [(\alpha - a)\hat{i} + (\beta - b)\hat{j} + (\gamma - c)\hat{k}] \cdot (\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}) = 0$$

$$\Rightarrow \alpha(\alpha - a) + \beta(\beta - b) + \gamma(\gamma - c) = 0$$

Hence, the locus of (α, β, γ) is

$$x(x-a) + y(y-b) + z(z-c) = 0$$

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

which is a sphere of radius $\frac{1}{2}\sqrt{a^2+b^2+c^2}$

81. **c.** $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} = 0$

82. **a.** Foot of the perpendicular from point $A(\vec{a})$ on the plane $\vec{r} \cdot \vec{n} = d$ is $\vec{a} + \frac{\left(d - \vec{a} \cdot \vec{n}\right)}{|\vec{n}|^2}\vec{n}$

Therefore, equation of the line parallel to $\vec{r} = \vec{a} + \lambda\vec{b}$ in the plane $\vec{r} . \vec{n}$ = d is given by

$$\vec{r} = \vec{a} + \frac{\left(d - \vec{a} . \vec{n}\right)}{|\vec{n}|^2}\vec{n} + \lambda\vec{b}$$

83. **a.** The plane is perpendicular to the line $\frac{x-a}{\cos\theta} = \frac{y+2}{\sin\theta} = \frac{z-3}{0}$.

Hence , the direction ratios of the normal of the plane are $\cos\theta$, $\sin\theta$ and 0. (i)

Now, the required plane passes through the z-axis. Hence the point (0, 0, 0) lies on the plane.

From Eqs. (i) and (ii), we get equation of the plane as

$$\cos\theta(x-0) + \sin\theta(y-0) + 0(z-0) = 0$$

$$\cos\theta x + \sin\theta y = 0$$

$$x + y\tan\theta = 0$$

84. **a.** The given line makes angles of $\pi/4$, $\pi/4$ and $\pi/2$ with the x-, y- and z-axes, respectively.

$\Rightarrow$ Direction cosines of the given line are

$\cos(\pi/4), \cos(\pi/4)$ and $\cos(\pi/2)$, or $(1/\sqrt{2}), (1/\sqrt{2})$ and 0.

85. **a.** We must have $(3 + 4a - 12 + 13)(-9 - 12a + 13) < 0$.

$\Rightarrow (a + 1)(12a - 4) > 0$

$\Rightarrow a < -1$ or $a > 1/3$

86. **c.** Plane meets axes at $A(2, 0, 0)$, $B(0, 3, 0)$ and $C(0, 0, 6)$.
Then area of ΔABC is

$$\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{2}|(-2\hat{i} + 3\hat{j}) \times (-2\hat{i} + 6\hat{j})|$$

$$= 3\sqrt{14} \text{ sq units}$$

Multiple Correct Answers Type

1. **b., c., d.**
If P be (x, y, z), then from the figure,
$x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$ and $z = r \cos \theta$
$1 = r \sin \theta \cos \phi$, $2 = r \sin \theta \sin \phi$ and $3 = r \cos \theta$

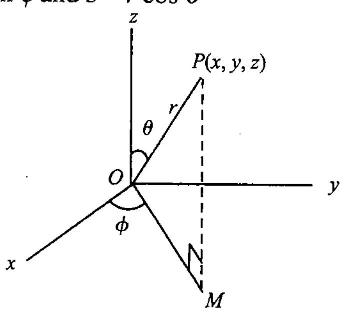


Fig. 3.44

$\Rightarrow 1^2 + 2^2 + 3^2 = r^2 \Rightarrow r = \pm\sqrt{14}$

$$\therefore \sin \theta \cos \phi = \frac{1}{\sqrt{14}}, \sin \theta \sin \phi = \frac{2}{\sqrt{14}} \text{ and } \cos \theta = \frac{3}{\sqrt{14}}$$

(neglecting negative sign as θ and ϕ are acute)

$$\frac{\sin\theta \sin\phi}{\sin\theta \cos\phi} = \frac{2}{1} \Rightarrow \tan \phi = 2$$

Also, $\tan \theta = \sqrt{5}/3$

2. a., c.

Plane P_1 contains the line $\vec{r} = \hat{i} + \hat{j} + \hat{k} + \lambda(\hat{i} - \hat{j} - \hat{k})$, hence contains the point $\hat{i} + \hat{j} + \hat{k}$ and is normal to vector $(\hat{i} + \hat{j})$.

Hence equation of plane is $(\vec{r} - (\hat{i} + \hat{j} + \hat{k}))\cdot(\hat{i} + \hat{j}) = 0$

or $x + y = 2$

Plane P_2 contains the line $\vec{r} = \hat{i} + \hat{j} + \hat{k} + \lambda(\hat{i} - \hat{j} - \hat{k})$ and point $\hat{j}$

Hence equation of plane is $\begin{vmatrix} x-0 & y-1 & z-0 \\ 1-0 & 1-1 & 1-0 \\ 1 & -1 & -1 \end{vmatrix} = 0$

or $x + 2y - z = 2$

If θ is the acute angle between P_1 and P_2, then

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \left|\frac{(\hat{i} + \hat{j})\cdot(\hat{i} + 2\hat{j} - \hat{k})}{\sqrt{2}\cdot\sqrt{6}}\right| = \frac{3}{\sqrt{2}\cdot\sqrt{6}} = \frac{\sqrt{3}}{2}$$

$$\theta = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$$

As L is contained in $P_2 \Rightarrow \theta = 0$

3. a., b. $\vec{r}\cdot\vec{n}_1 = q_1$ and $\vec{r}\cdot\vec{n}_2 = q_2$, $\vec{r}\cdot\vec{n}_3 = q_3$ intersect in a line if $[\vec{n}_1\ \vec{n}_2\ \vec{n}_3] = 0$. So,

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2a & 1 \\ a & a^2 & 1 \end{vmatrix} = 0$$

$\Rightarrow 2a - a^2 - 1 + a + a^2 - 2a^2 = 0$

$\Rightarrow 2a^2 - 3a + 1 = 0$

$\Rightarrow a = 1/2, 1$

4. a., b. Let the coordinates of the point(s) be a, b and c.

Therefore, the equation of the line passing through (a, b, c) and whose direction ratios are $1, -5$ and -2 is

$$\frac{x-a}{1} = \frac{y-b}{-5} = \frac{z-c}{-2} \qquad \text{(i)}$$

Line (i) intersects the line,

$$\frac{x}{1} = \frac{y+5}{1} = \frac{z+1}{1} \qquad \text{(ii)}$$

Therefore, these are coplanar.

$$\begin{vmatrix} 1 & -5 & -2 \\ 1 & 1 & 1 \\ a & b+5 & c+1 \end{vmatrix} = 0$$

or $a + b - 2c + 3 = 0$

Also, by using same procedure with the second equation, we get the condition

$11a + 15b - 32c + 55 = 0$

5. **a., d.** The equation of the plane passing through the intersection of the planes $2x - y = 0$ and $3z - y = 0$ is

$2x - y + \lambda(3z - y) = 0$ (i)

or $2x - y(\lambda + 1) + 3\lambda z = 0$

Plane (i) is perpendicular to $4x + 5y - 3z = 8$. Therefore,

$4 \times 2 - 5(\lambda + 1) - 9\lambda = 0$

$\Rightarrow 8 - 5\lambda - 5 - 9\lambda = 0$

$\Rightarrow 3 - 14\lambda = 0$

$\Rightarrow \lambda = 3/14$

$\therefore 2x - y + \frac{3}{14}(3z - y) = 0$

$28x - 17y + 9z = 0$

6. **b., c., d.**

$x + y + z - 1 = 0$

$4x + y - 2z + 2 = 0$

Therefore, the line is along the vector $(\hat{i} + \hat{j} + \hat{k}) \times (4\hat{i} + \hat{j} - 2\hat{k}) = 3\hat{i} - 6\hat{j} + 3\hat{k}$

Let $z = k$. Then $x = k - 1$ and $y = 2 - 2k$

Therefore, $(k - 1, 2 - 2k, k)$ is any point on the line.

Hence, $(-1, 2, 0)$, $(0, 0, 1)$ and $(-1/2, 1, 1/2)$ are the points on the line.

7. **a., b.**

$3x - 6y + 2z + 5 = 0$ (i)

$-4x + 12y - 3z + 3 = 0$ (ii)

Bisectors are $\frac{3x - 6y + 2z + 5}{\sqrt{9 + 36 + 4}} = \pm \frac{-4x + 12y - 3z + 3}{\sqrt{16 + 144 + 9}}$

The plane which bisects the angle between the planes that contains the origin.

$13(3x - 6y + 2z + 5) = 7(-4x + 12y - 3z + 3)$

$67x - 162y + 47z + 44 = 0$ (iii)

Further, $3 \times (-4) + (-6)(12) + 2 \times (-3) < 0$

Hence, the origin lies in the acute angle.

8. **a., d.** The given lines intersect if $\begin{vmatrix} 2-1 & 3-4 & 4-5 \\ 1 & 1 & \lambda \\ \lambda & 2 & 1 \end{vmatrix} = 0 \Rightarrow \lambda = 0, -1.$

9. **a., c.** The required plane is parallel to the bisector of the given planes.

Bisectors are $\frac{x-y+z-3}{\sqrt{3}}=\pm\frac{x+y+z+4}{\sqrt{3}}$

or $2y + 7 = 0$ and $2x + 2y + 1 = 0$. Hence, the planes are $y = 0$ and $x + y = 0$.

10. **a., d.**

The equation of a plane passing through the line of intersection of the x-y and y-z planes is $z + \lambda x = 0, \quad \lambda \in R$

This plane makes an angle 45° with the x-y plane ($z = 0$).

$\Rightarrow \cos 45° = \frac{1}{\sqrt{1}\sqrt{\lambda^2+1}}$

$\Rightarrow \lambda = \pm 1$

11. **a., b.** The plane is equally inclined to the lines. Hence, it is perpendicular to the angle bisector of the vectors $2\hat{i} - 2\hat{j} - \hat{k}$ and $8\hat{i} + \hat{j} - 4\hat{k}$.

Vector along the angle bisectors of the vectors are

$\frac{2\hat{i}-2\hat{j}-\hat{k}}{3} \pm \frac{8\hat{i}+\hat{j}-4\hat{k}}{9}$, or

$\frac{14\hat{i}-5\hat{j}-7\hat{k}}{9}$ and $\frac{-2\hat{i}-7\hat{j}+\hat{k}}{9}$.

Hence, the equations of the planes are $14x - 5y - 7z = 0$ or $2x + 7y - z = 0$

12. **a., c.**

For line $\frac{x-1}{1}=\frac{y}{-1}=\frac{z-5}{-1}$, point (1, 0, 5) lies on the plane. Also, the vector along the line $\hat{i} - \hat{j} - \hat{k}$ is perpendicular to the normal $\hat{i} + 2\hat{j} - \hat{k}$ to the plane. For line $\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(3\hat{i} + \hat{j} + 5\hat{k})$, point (2, −1, 4) lies on the plane and vector $3\hat{i} + \hat{j} + 5\hat{k}$ is perpendicular to the normal $\hat{i} + 2\hat{j} - \hat{k}$.

Line $x - y + z = 2x + y - z = 0$ passes through the origin, which is not on the given plane.

13. **b., c.**

Volume of tetrahedron $ABCD$ is $\frac{1}{6}|[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]| = 1$ cubic units.

$$\Rightarrow \begin{vmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ x-0 & y-1 & z-2 \end{vmatrix} = \pm 6$$

$\Rightarrow -2(y-1) - 2(z-2) = \pm 6$.

$\Rightarrow y - 1 + z - 2 = \pm 3$

$\Rightarrow y + z = 6$ or $y + z = 0$

14. a., c., d.

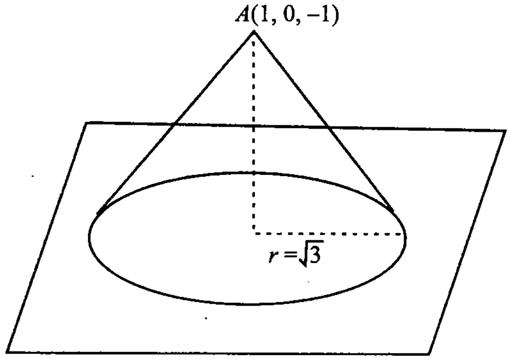


Fig. 3.45

The rod sweeps out the figure which is a cone.

The distance of point $A(1, 0, -1)$ from the plane is $\frac{|1-2+4|}{\sqrt{9}} = 1$ unit.

The slant height l of the cone is 2 units.

Then the radius of the base of the cone is $\sqrt{l^2 - 1} = \sqrt{4-1} = \sqrt{3}$.

Hence, the volume of the cone is $\frac{\pi}{3}(\sqrt{3})^2 \cdot 1 = \pi$ cubic units.

Area of the circle on the plane which the rod traces is 3π.

Also, the centre of the circle is $Q(x, y, z)$. Then $\frac{x-1}{1} = \frac{y-0}{-2} = \frac{z+1}{2} = \frac{-(1-0-2+4)}{1^2+(-2)^2+2^2}$, or

$Q(x, y, z) \equiv \left(\frac{2}{3}, \frac{2}{3}, \frac{-5}{3}\right)$.

15. b., c.

Distance between the planes is $h = 5/\sqrt{6}$.

Also the figure formed is cylinder, whose radius is $r = 2$ units.

Hence, the volume of the cylinder is $\pi r^2 h = \pi(2)^2 \cdot \frac{5}{\sqrt{6}} = \frac{20\pi}{\sqrt{6}}$ cubic units.

Also the curved surface area is $2\pi r h = 2\pi(2) \cdot \frac{5}{\sqrt{6}} = \frac{20\pi}{\sqrt{6}}$

16. **a., b.**

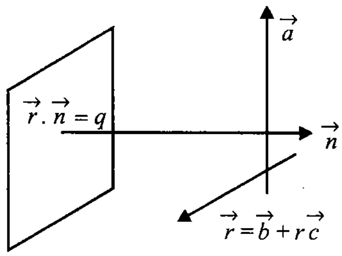


Fig. 3.46

Required line is parallel to $\vec{n}\times\vec{c}$

The equation of line is $\vec{r}=\vec{a}+\lambda(\vec{n}\times\vec{c})$

$\Rightarrow (\vec{r}-\vec{a})=\lambda(\vec{n}\times\vec{c})$

$\therefore (\vec{r}-\vec{a})\times(\vec{n}\times\vec{c})=0$

Reasoning Type

1. **b.** Given lines are parallel as both are directed along the same vector $(\hat{i}+\hat{j}-\hat{k})$; so they do not intersect. Also Statement 2 is correct by definition of skew lines, but skew lines are those which are neither parallel nor intersecting. Hence, both the statements are true, but Statement 2 is not the correct explanation for Statement 1.
2. **b.** For the given lines, let $\vec{a}_1=\hat{i}+\hat{j}-\hat{k}$, $\vec{a}_2=4\hat{i}-\hat{k}$, $\vec{b}_1=3\hat{i}-\hat{j}$ and $\vec{b}_2=2\hat{i}+3\hat{k}$. Therefore,

$$[\vec{a}_2-\vec{a}_1\ \vec{b}_1\ \vec{b}_2]=\begin{vmatrix}4-1 & 0-1 & -1+1\\ 3 & -1 & 0\\ 2 & 0 & 3\end{vmatrix}=\begin{vmatrix}3 & -1 & 0\\ 3 & -1 & 0\\ 2 & 0 & 3\end{vmatrix}=0$$

Hence, the lines are coplanar. Also vectors $\vec{b}_1$ and $\vec{b}_2$ along which the lines are directed are not collinear. Hence, the lines intersect. When $\vec{b}\times\vec{d}=\vec{0}$, vectors $\vec{b}$ and $\vec{d}$ are collinear; therefore, lines $\vec{r}=\vec{a}+\lambda\vec{b}$ and $\vec{r}=\vec{c}+\lambda\vec{d}$ are parallel and do not intersect. But this statement is not the correct explanation for Statement 1.

3. **a.** Any point on the first line is $(2x_1+1, x_1-3, -3x_1+2)$.
   Any point on the second line is $(y_1+2, -3y_1+1, 2y_1-3)$.
   If two lines are coplanar, then $2x_1-y_1=1$, $x_1+3y_1=4$ and $3x_1+2y_1=5$ are consistent.
4. **a.** The direction cosines of segment OA are $\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}$ and $\frac{-3}{\sqrt{14}}$.
   $OA=\sqrt{14}$
   This means OA will be normal to the plane and the equation of the plane is $2x+y-3z=14$.
5. **b.** Statement 2 is true as when the line lies in the plane, vector $\vec{b}$ along which the line is directed is perpendicular to the normal $\vec{c}$ of the plane, but it does not explain Statement 1 as for $\vec{b}\cdot\vec{c}=0$, the line

may be parallel to the plane. However, Statement 1 is correct as any point on the line $(t+1, 2t, -t-2)$ lies on the plane for $t \in R$.

6. **a.** $\sin\theta = \left|\dfrac{2-3+2}{\sqrt{4+9+4}\sqrt{3}}\right| = \dfrac{1}{\sqrt{51}}$

Therefore, Statement 1 is true and Statement 2 is also true by definition.

7. **a.** $\overrightarrow{PA}\cdot\overrightarrow{PB} = 9 > 0$. Therefore, P is exterior to the sphere. Statement 2 is also true (standard result).

8. **b.** Obviously the answer is (b).

9. **c.** Any point on the line $\dfrac{x-1}{1} = \dfrac{y}{-1} = \dfrac{z+2}{2}$ is $B(t+1, -t, 2t-2)$, $t \in R$.

Also, AB is perpendicular to the line, where A is $(1, 2, -4)$.

$\Rightarrow 1(t) - (-t-2) + 2(2t+2) = 0$

$\Rightarrow 6t + 6 = 0$

$\Rightarrow t = -1$

Point B is $(0, 1, -4)$

Hence, $AB = \sqrt{1+1+0} = \sqrt{2}$

10. **b.** Direction ratios of the given lines are $(-3, 1, -1)$ and $(1, 2, -1)$. Hence, the lines are perpendicular as $(-3)(1) + (1)(2) + (-1)(-1) = 0$.

Also lines are coplanar as $\begin{vmatrix} 0-2 & 1-3 & -1+(13/7) \\ -3 & 1 & -1 \\ 1 & 2 & -1 \end{vmatrix} = 0$

But Statement 2 is not enough reason for the shortest distance to be zero, as two skew lines can also be perpendicular.

Linked Comprehension Type

For Problems 1–3

1. b., 2. c., 3. d.

Sol.

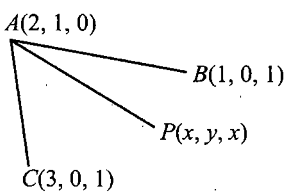


Fig. 3.47

$$\begin{vmatrix} x-2 & y-1 & z \\ 1-2 & 0-1 & 1-0 \\ 3-2 & 0-1 & 1-0 \end{vmatrix} = 0$$

$(x-2)[(-1)-(-1)]-(y-1)[(-1)-1]+z[1+1]=0$

$2(y-1)+2z=0$

$\Rightarrow y+z-1=0$

The vector normal to the plane is $\vec{r}=0\hat{i}+\hat{j}+\hat{k}$

The equation of the line through (0, 0, 2) and parallel to $\vec{n}$ is $\vec{r}=2\hat{k}+\lambda(\hat{j}+\hat{k})$

The perpendicular distance of $D(0, 0, 2)$ from plane ABC is $\left|\dfrac{2-1}{\sqrt{1^2+1^2}}\right| = \dfrac{1}{\sqrt{2}}$.

For Problems 4–6

4. b., 5. c., 6. c.

Sol.

4. **b.** Let $Q(x_2, y_2, z_2)$ be the image of $A(2, 1, 6)$ about mirror $x + y - 2z = 3$. Then,

$$\frac{x_2-2}{1}=\frac{y_2-1}{1}=\frac{z_2-6}{-2}=\frac{-2(2+1-12-3)}{1^2+1^2+2^2}=4$$

$\Rightarrow (x_2, y_2, z_2) \equiv (6, 5, -2)$.

5. **c.** Let $\dfrac{x-2}{3}=\dfrac{y-1}{4}=\dfrac{z-6}{5}=\lambda$

$x=2+3\lambda, y=1+4\lambda, z=6+5\lambda$ lies on plane $x+y-2z=3$

$\Rightarrow 2+3\lambda+1+4\lambda-2(6+5\lambda)=3$

$\Rightarrow 3+7\lambda-12-10\lambda=3 \Rightarrow -3\lambda=12 \Rightarrow \lambda=-4$

Point $B \equiv (-10, -15, -14)$

6. **c.** The equation of the reflected ray $L_1=0$ is the line joining $Q(x_2, y_2, z_2)$ and $B(-10, -15, -14)$.

$$\frac{x+10}{16}=\frac{y+15}{20}=\frac{z+14}{12}$$

or $\dfrac{x+10}{4}=\dfrac{y+15}{5}=\dfrac{z+14}{3}$

For Problems 7–9

7. b., 8. c., 9. b.

Sol.

The given system of equations is

$2x + py + 6z = 8$

$x+2y+qz=5$

$x+y+3z=4$

$$\Delta=\begin{vmatrix}2 & p & 6\\ 1 & 2 & q\\ 1 & 1 & 3\end{vmatrix}=(2-p)(3-q)$$

By Cramer's rule, if $\Delta \neq 0$, i.e., $p \neq 2$ and $q \neq 3$, the system has a unique solution.
If $p = 2$ or $q = 3$, $\Delta = 0$, then if $\Delta_x = \Delta_y = \Delta_z = 0$, the system has infinite solutions and if any one of Δ_x, Δ_y and $\Delta_z \neq 0$, the system has no solution.

$$\text{Now } \Delta_x = \begin{vmatrix} 8 & p & 6 \\ 5 & 2 & q \\ 4 & 1 & 3 \end{vmatrix}$$

$$= 30 - 8q - 15p + 4pq = (4q - 15) \cdot (p - 2)$$

$$\Delta_y = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & q \\ 1 & 4 & 3 \end{vmatrix}$$

$$= -8q + 8q = 0$$

$$\Delta_z = \begin{vmatrix} 2 & p & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix}$$

$$= p - 2$$

Thus, if $p = 2$, $\Delta_x = \Delta_y = \Delta_z = 0$ for all $q \in R$, the system has infinite solutions.
If $p \neq 2$, $q = 3$ and $\Delta_z \neq 0$, then the system has no solution.
Hence the system has (i) no solution if $p \neq 2$ and $q = 3$, (ii) a unique solution if $p \neq 2$ and $q \neq 3$ and (iii) infinite solutions if $p = 2$ and $q \in R$.

For Problems 10–12

10. d., 11. b., 12. d.

Sol.

10. **d.** The line $\dfrac{x-1}{3} = \dfrac{y-2}{-1} = \dfrac{z-3}{4} = r$

Any point say $B \equiv (3r + 1, 2 - r, 3 + 4r)$ (on the line L)

$\overrightarrow{AB} = 3r, -r, 4r + 6$

Hence,

$\overrightarrow{AB}$ is parallel to $x + y - z = 1$

$\Rightarrow 3r - r - 4r - 6 = 0$ or $r = -3$

B is $(-8, 5, -9)$

11. **b.** The equation of plane containing the line L is
$A(x-1) + B(y-2) + C(z-3) = 0$, where $3A - B + 4C = 0$ (i)
(i) also contains point $A(1, 2, -3)$.
Hence, $C = 0$ and $3A = B$.
The equation of plane $x - 1 + 3(y - 2) = 0$ or $x + 3y - 7 = 0$

12. **d.** The distance of point $(1 + 3r,\ 2 - r, 3 + 4r)$ from the plane is

$$\frac{|1+3r+2-r-3-4r-1|}{\sqrt{1+1+1}}=\frac{|2r+1|}{\sqrt{3}}=\frac{4}{\sqrt{3}}$$

$$\Rightarrow r=\frac{3}{2},-\frac{5}{2}$$

Hence, the points are $A\left(\frac{11}{2},\frac{1}{2},\frac{10}{2}\right)$ and $B\left(\frac{-13}{2},\frac{9}{2},\frac{-14}{2}\right)$

$$\Rightarrow AB=\sqrt{292}$$

Matrix-Match Type

1. **a → s; b → q; c → p; d → r**

a. Line $x = 2t + 1, y = t + 2, z = -t - 3$ or $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z+3}{-1}$, which is along the vector $2\hat{i}+\hat{j}-\hat{k}$.

Vector $\hat{i}+3\hat{j}+5\hat{k}$ is perpendicular to the line.

b. Normals to the planes $x + y + z - 3 = 0$ and $2x - y + 3z = 0$ are $\vec{n_1}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{n_2}=2\hat{i}-\hat{j}+3\hat{k}$

Then the vector along the line of intersection of planes is $\vec{n_1}\times\vec{n_2}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&1&1\\2&-1&3\end{vmatrix}=4\hat{i}-\hat{j}-3\hat{k}$

c. The shortest distance between the lines $\frac{x}{2}=\frac{y}{-3}=\frac{z}{-1}$ and $\vec{r}=(3\hat{i}-\hat{j}+\hat{k})+t(\hat{i}+\hat{j}-2\hat{k})$

occurs along the vector $(2\hat{i}-3\hat{j}-\hat{k})\times(\hat{i}+\hat{j}-2\hat{k})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&-3&-1\\1&1&-2\end{vmatrix}=7\hat{i}+3\hat{j}+5\hat{k}$

d. Normal to the plane $\vec{r}=-\hat{i}+4\hat{j}-6\hat{k}+\lambda(\hat{i}+3\hat{j}-2\hat{k})+\mu(-\hat{i}+2\hat{j}-5\hat{k})$ is $\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&3&-2\\-1&2&-5\end{vmatrix}$

$=-11\hat{i}+7\hat{j}+5\hat{k}$

2. **a → q, s; b → r; c → p, q; d → p**

a. Line $\frac{x-1}{-2}=\frac{y+2}{3}=\frac{z}{-1}$ is along the vector $\vec{a}=-2\hat{i}+3\hat{j}-\hat{k}$ and

line $\vec{r}=(3\hat{i}-\hat{j}+\hat{k})+t(\hat{i}+\hat{j}+\hat{k})$ is along the vector $\vec{b}=\hat{i}+\hat{j}+\hat{k}$. Here $\vec{a}\perp\vec{b}$.

Also $\begin{vmatrix} 3-1 & -1-(-2) & 1-0 \\ -2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$

b. The direction ratios of the line $x-y+2z-4=0=2x+y-3z+5=0$ are $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{vmatrix} = \hat{i}+7\hat{j}+3\hat{k}$.
Hence, the given two lines are parallel.

c. The given lines are ($x = t-3, y = -2t+1, z = -3t-2$) and $\vec{r} = (t+1)\hat{i} + (2t+3)\hat{j} + (-t-9)\hat{k}$, or

$$\frac{x+3}{1} = \frac{y-1}{-2} = \frac{z+2}{-3} \text{ and } \frac{x-1}{1} = \frac{y-3}{2} = \frac{z+9}{-1}.$$

The lines are perpendicular as $(1)(1) + (-2)(2) + (-3)(-1) = 0$.

Also $\begin{vmatrix} -3-1 & 1-3 & -2-(-9) \\ 1 & -2 & -3 \\ 1 & 2 & -1 \end{vmatrix} = 0$

Hence, the lines are intersecting.

d. The given lines are $\vec{r} = (\hat{i}+3\hat{j}-\hat{k}) + t(2\hat{i}-\hat{j}-\hat{k})$ and $\vec{r} = (-\hat{i}-2\hat{j}+5\hat{k}) + s(\hat{i}-2\hat{j}+\frac{3}{4}\hat{k})$.

$$\begin{vmatrix} 1-(-1) & 3-(-2) & -1-5 \\ 2 & -1 & -1 \\ 1 & -2 & 3/4 \end{vmatrix} = 0$$

Hence, the lines are coplanar and hence intersecting (as the lines are not parallel).

3. a → q; b → p; c → s; d → r

a. The given line is $x = 4y+5$, $z = 3y-6$, or

$$\frac{x-5}{4} = y, \quad \frac{z+6}{3} = y$$

or $\dfrac{x-5}{4} = \dfrac{y}{1} = \dfrac{z+6}{3} = \lambda$ (say)

Any point on the line is of the form $(4\lambda+5, \lambda, 3\lambda-6)$.
The distance between $(4\lambda+5, \lambda, 3\lambda-6)$ and $(5, 3, -6)$ is 3 units (given). Therefore
$(4\lambda+5-5)^2 + (\lambda-3)^2 + (3\lambda-6+6)^2 = 9$
$\Rightarrow 16\lambda^2 + \lambda^2 + 9 - 6\lambda + 9\lambda^2 = 9$
$\Rightarrow 26\lambda^2 - 6\lambda = 0$
$\Rightarrow \lambda = 0, 3/13$
The point is $(5, 0, -6)$

b. The equation of the plane containing the lines $\dfrac{x-2}{3} = \dfrac{y+3}{5} = \dfrac{z+5}{7}$ and

parallel to $\hat{i} + 4\hat{j} + 7\hat{k}$

$$\begin{vmatrix} x-2 & y+3 & z+5 \\ 1 & 4 & 7 \\ 3 & 5 & 7 \end{vmatrix} = 0$$

$\Rightarrow x-2y+z-3=0$

Point $(-1,-2,0)$ lies on this plane.

c. The line passing through points $A(2,-3,-1)$ and $B(8,-1,2)$ is $\dfrac{x-2}{8-2}=\dfrac{y+3}{-1+3}=\dfrac{z+1}{2+1}$ or

$\dfrac{x-2}{6}=\dfrac{y+3}{2}=\dfrac{z+1}{3}=\lambda$ (say).

Any point on this line is of the form $P(6\lambda+2, 2\lambda-3, 3\lambda-1)$, whose distance from point $A(2,-3,-1)$ is 14 units. Therefore,

$\Rightarrow\ PA = 14$

$\Rightarrow\ PA^2 = (14)^2$

$\Rightarrow\ (6\lambda)^2 + (2\lambda)^2 + (3\lambda)^2 = 196$

$\Rightarrow\ 49\lambda^2 = 196$

$\Rightarrow\ \lambda^2 = 4$

$\Rightarrow\ \lambda = \pm 2$

Therefore, the required points are $(14, 1, 5)$ and $(-10, -7-7)$. The point nearer to the origin is $(14, 1, 5)$.

d. Any point on line AB, $\dfrac{x}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}=\lambda$ is $M(2\lambda, 3\lambda+2, 4\lambda+3)$. Therefore the direction ratios of PM are $2\lambda-3$, $3\lambda+3$ and $4\lambda-8$.

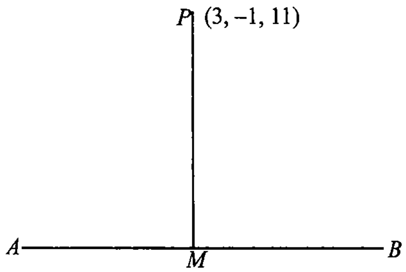


Fig. 3.48

But $PM \perp AB$

$\therefore\ 2(2\lambda-3) + 3(3\lambda+3) + 4(4\lambda-8) = 0$

$4\lambda - 6 + 9\lambda + 9 + 16\lambda - 32 = 0$

$29\lambda - 29 = 0$

$\lambda = 1$

Therefore, foot of the perpendicular is $M(2, 5, 7)$.

4. a → r; b → p; c → q; d → s

a. The given line and plane are $\vec{r} = (2\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ and $\vec{r}\cdot(\hat{i} + 5\hat{j} + \hat{k}) = 5$, respectively. Since $(\hat{i} - \hat{j} + 4\hat{k})\cdot(\hat{i} + 5\hat{j} + \hat{k}) = 0$, line and plane are parallel.

Hence, the required distance = distance of point (2, – 2, 3) from the plane $x + 5y + z - 5 = 0$,

which is $\dfrac{|2-10+3-5|}{\sqrt{1+25+1}} = \dfrac{10}{3\sqrt{3}}$

b. The distance between two parallel planes $\vec{r}\cdot(2i - j + 3k) = 4$ and $\vec{r}\cdot(6i - 3j + 9k) + 13 = 0$ is

$$d = \frac{|4-(-13/3)|}{\sqrt{(2)^2+(-1)^2+(3)^2}} = \frac{(25/3)}{\sqrt{14}} = \frac{25}{3\sqrt{14}}$$

c. The perpendicular distance of the point (2, 5, – 3) from the plane $\vec{r}\cdot(6i - 3j + 2k) = 4$ or $6x - 3y + 2z - 4 = 0$ is

$$d = \frac{|12-15-6-4|}{\sqrt{36+9+4}}$$

$$= 13/\sqrt{49} = 13/7$$

d. The equation of the line AB is

$$\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$$

The equation of line passing through (1, 0, – 3) and parallel to AB is

$$\frac{x-1}{2} = \frac{y}{3} = \frac{z+3}{-6} = r \text{ (say)}$$

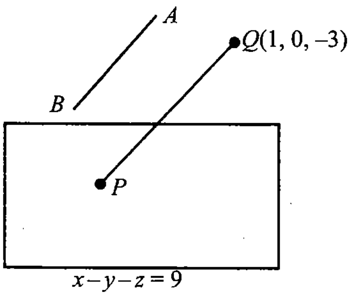


Fig. 3.49

The coordinates of any point on line $P(2r+1, 3r, -6r-3)$ which lie on plane

$(2r+1)-(3r)-(-6r-3) = 9$

$r = 1$

Point $P \equiv (3, 3, -9)$

Required distance $PQ = \sqrt{(3-1)^2+(3-0)^2+(-9+3)^2} = \sqrt{4+9+36} = 7$

5. **a → q; b → r; c → s; d → p**

a. If the required image is (x, y, z), then $\dfrac{x-3}{2} = \dfrac{y-5}{1} = \dfrac{z-7}{1} = -\dfrac{2(6+5+7+18)}{2^2+1^2+1^2} = -12$

or (–21, –7, –5).

b. Any point on the line $\frac{x-2}{-3}=\frac{y-1}{2}=\frac{z-3}{2}=\lambda$ is $(-3\lambda+2, 2\lambda+1, 2\lambda+3)$, which lies on plane $2x+y-z=3$. Therefore

$-6\lambda+4+2\lambda+1-2\lambda-3=3$

$-6\lambda=1$

$\lambda=-1/6$

Therefore, the point is $\left(\frac{5}{2}, \frac{2}{3}, \frac{8}{3}\right)$

c. If (x, y, z) is required foot of the perpendicular, then $\frac{x-1}{2}=\frac{y-1}{-2}=\frac{z-2}{4}=-\frac{(2-2+8+5)}{2^2+(-2)^2+4^2}$ or

$(x, y, z) \equiv \left(\frac{-1}{12}, \frac{25}{12}, \frac{-2}{12}\right)$

d. Any point on the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda$ is $P(2\lambda+1, 3\lambda+2, 4\lambda+3)$, which satisfies the line

$\frac{x-4}{5}=\frac{y-1}{2}=\frac{z}{1}$ or $\frac{2\lambda+1-4}{5}=\frac{3\lambda+2-1}{2}=\frac{4\lambda+3}{1}$

$\Rightarrow \lambda=-1$

The required point is $(-1, -1, -1)$

Integer Answer Type

1. **(8)** Obviously one in each octant.
2. **(1)** If image of point $(2, -3, 3)$ in the plane $x-2y-z+1=0$ is (a, b, c), then

$$\frac{a-2}{1}=\frac{b+3}{-2}=\frac{c-3}{-1}=\frac{-2(2-2(-3)-3+1)}{(1)^2+(-2)^2+(-1)^2}=-2$$

Hence the image is $(0, 1, 5)$

Obviously distance of image of the point from z-axis is 1.

3. **(3)** Let $A(1, 0, -1), B(-1, 2, 2)$

Direction ratios of AB are $(2, -2, -3)$

Let θ be the angle between the line and normal to plane, then

$$\cos\theta=\frac{|2.1+3(-2)-5(-3)|}{\sqrt{1+9+25}\sqrt{4+4+9}}=\frac{11}{\sqrt{17}\sqrt{35}}=\frac{11}{\sqrt{595}}$$

Length of projection

$=(AB)\sin\theta$

$$=\sqrt{(2)^2+(2)^2+(3)^2}\times\sqrt{1-\frac{121}{595}}$$

$$=\sqrt{\frac{474}{35}} \text{ units}$$

4. **(2)** Vector normal to the plane is $\vec{n} = \hat{i} - 3\hat{j} + 2\hat{k}$ and vector along the line is $\vec{v} = 2\hat{i} + \hat{j} - 3\hat{k}$

Now $\sin\theta = \dfrac{\vec{x}\cdot\vec{v}}{|\vec{x}|\,|\vec{v}|} = \left|\dfrac{2-3-6}{\sqrt{14}\,\sqrt{14}}\right| = \left|\dfrac{7}{14}\right|$

Hence $\operatorname{cosec}\theta = 2$

5. **(8)** Volume (V) $= \dfrac{1}{3} A_1 h_1 \quad \Rightarrow \quad h_1 = \dfrac{3V}{A_1}$

Similarly $h_2 = \dfrac{3V}{A_2}$, $h_3 = \dfrac{3V}{A_3}$ and $h_4 = \dfrac{3V}{A_4}$

So $(A_1 + A_2 + A_3 + A_4)(h_1 + h_2 + h_3 + h_4)$

$$= (A_1 + A_2 + A_3 + A_4)\left(\frac{3V}{A_1} + \frac{3V}{A_2} + \frac{3V}{A_3} + \frac{3V}{A_4}\right)$$

$$= 3V(A_1 + A_2 + A_3 + A_4)\left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4}\right)$$

Now using A.M.-H.M inequality in A_1, A_2, A_3, A_4, we get

$$\frac{A_1 + A_2 + A_3 + A_4}{4} \geq \frac{4}{\left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4}\right)}$$

$$\Rightarrow \quad (A_1 + A_2 + A_3 + A_4)\left(\frac{1}{A_1} + \frac{1}{A_2} + \frac{1}{A_3} + \frac{1}{A_4}\right) \geq 16$$

Hence the minimum value of $(A_1 + A_2 + A_3 + A_4)(h_1 + h_2 + h_3 + h_4) = 3V(16) = 48V = 48(1/6) = 8$

6. **(6)** A plane containing the line of intersection of the given planes is

$x - y - z - 4 + \lambda(x + y + 2z - 4) = 0$

i.e., $(\lambda + 1)x + (\lambda - 1)y + (2\lambda - 1)z - 4(\lambda + 1) = 0$

vector normal to it

$V = (\lambda + 1)\hat{i} + (\lambda - 1)\hat{j} + (2\lambda - 1)\hat{k}S$ (i)

Now the vector along the line of intersection of the planes

$2x + 3y + z - 1 = 0$ and $x + 3y + 2z - 2 = 0$ is given by

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{vmatrix} = 3(\hat{i} - \hat{j} + \hat{k})$$

As $\vec{n}$ is parallel to the plane (i), therefore

$\vec{n}\cdot\vec{V} = 0$

$(\lambda + 1) - (\lambda - 1) + (2\lambda - 1) = 0$

$2+2\lambda-1=0 \Rightarrow \lambda=\frac{-1}{2}$

Hence the required plane is

$\frac{x}{2}-\frac{3y}{2}-2z-2=0$

$x-3y-4z-4=0$

Hence $|A+B+C|=6$

7. (7) Clearly minimum value of $a^2+b^2+c^2$

$$=\left(\frac{|(3(0)+2(0)+(0)-7|}{\sqrt{(3)^2+(2)^2+(1)^2}}\right)^2=\frac{49}{14}=\frac{7}{2}\text{units}$$

8. (7) $4x+7y+4z+81=0$ (i)

$5x+3y+10z=25$ (ii)

Equation of plane passing through their line of intersection is

$(4x+7y+4z+81)+\lambda(5x+3y+10z-25)=0$

or $(4+5\lambda)x+(7+3\lambda)y+(4+10\lambda)z+81-25\lambda=0$ (iii)

plane (iii) $\perp$ to (i), so

$4(4+5\lambda)+7(7+3\lambda)+4(4+10\lambda)=0$

$\therefore \lambda=-1$

From (iii), equation of plane is $-x+4y-6z+106=0$. (iv)

Distance of (iv) from (0,0,0) $=\frac{106}{\sqrt{1+16+36}}=\frac{106}{\sqrt{53}}$

9. (9) Line through point $P(-2, 3, -4)$ and parallel to the given line $\frac{x+2}{3}=\frac{2y+3}{4}=\frac{3z+4}{5}$

is $\frac{x+2}{3}=\frac{y+\frac{3}{2}}{2}=\frac{z+\frac{4}{3}}{\frac{5}{3}}=\lambda$

Any point on this line is $Q\left[3\lambda-2, 2\lambda-\frac{3}{2}, \frac{5}{3}\lambda-\frac{4}{3}\right]$

Direction ratios of PQ are $\left[3\lambda, \frac{4\lambda-9}{2}, \frac{5\lambda+8}{3}\right]$

Now PQ is parallel to the given plane $4x+12y-3z+1=0$

$\Rightarrow$ line is perpendicular to the normal to the plane

$\Rightarrow 4(3\lambda)+12\left(\frac{4\lambda-9}{2}\right)-3\left(\frac{5\lambda+8}{3}\right)=0$

$\Rightarrow \lambda=2$

$\Rightarrow \quad Q\left(4, \frac{5}{2}, 2\right)$

$\Rightarrow \quad PQ = \sqrt{(6)^2 + \left(\frac{5}{2} - 3\right)^2 + (6)^2} = \frac{17}{2}$

10. **(6)** The given points are $O(0, 0, 0)$, $A(0, 0, 2)$, $B(0, 4, 0)$ and $C(6, 0, 0)$

Here three faces of tetrahedron are xy, yz, zx plane.

Since point P is equidistance from zx, xy and yz planes, its coordinates are $P(r, r, r)$

Equation of plane ABC is

$2x + 3y + 6z = 12$ (from intercept form)

P is also at distance r from plane ABC

$$\Rightarrow \quad \frac{|2r + 3r + 6r - 12|}{\sqrt{4 + 9 + 36}} = r$$

$\Rightarrow \quad |11r - 12| = 7r$

$\Rightarrow \quad 11r - 12 = \pm 7r$

$\Rightarrow \quad r = \frac{12}{18}, 3$

$\therefore \quad r = 2/3$ (as $r < 2$)

Archives

Subjective Type

1. (i) We know that equation of the plane passing through three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 2 & y - 1 & z - 0 \\ 5 - 2 & 0 - 1 & 1 - 0 \\ 4 - 2 & 1 - 1 & 1 - 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 2 & y - 1 & z \\ 3 & -1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow x + y - 2z = 3$

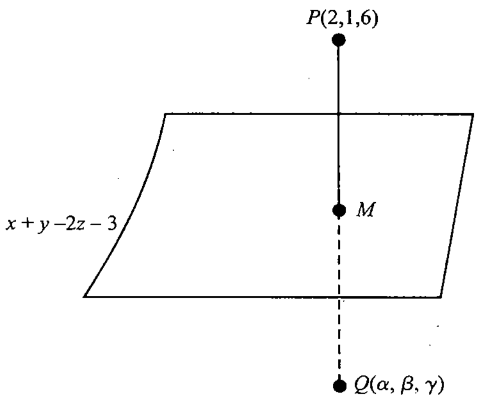


Fig. 3.50

According to the question, we have to find the image of $P(2, 1, 6)$ in the plane.

Let Q be (α, β, γ). Then $\frac{\alpha-2}{1}=\frac{\beta-1}{1}=\frac{\gamma-6}{-2}=\frac{-2(2+1-12-3)}{1^2+1^2+(-2)^2}=4$

$\Rightarrow Q(\alpha, \beta, \gamma) \equiv Q(6, 5, -2)$.

2. Since the plane is parallel to lines L_1 and L_2 with direction ratios as $(1, 0, -1)$ and $(1, -1, 0)$, a vector perpendicular to L_1 and L_2 will be parallel to the normal $\vec{n}$ to the plane. Therefore,

$$\vec{n}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix}=-\hat{i}-\hat{j}-\hat{k}$$

The equation of the plane passing through $(1, 1, 1)$ and having normal vector $\vec{n}=-\hat{i}-\hat{j}-\hat{k}$ is given by

$(\vec{r}-\vec{a})\cdot\vec{n}=0$

$\Rightarrow -1(x-1)-1(y-1)-1(z-1)=0$

$x+y+z=3$

$$\frac{x}{3}+\frac{y}{3}+\frac{z}{3}=1 \qquad \text{(i)}$$

The plane meets the axes at $A(3, 0, 0)$, $B(0, 3, 0)$ and $C(0, 0, 3)$ or $A(3\hat{i})$, $B(3\hat{j})$ and $C(3\hat{k})$.

Hence, the volume of tetrahedron $OABC=\frac{1}{6}[3\hat{i}\ 3\hat{j}\ 3\hat{k}]$

$$=\frac{27}{6}=\frac{9}{2}\text{ cubic units}$$

3. S is the parallelepiped with base point A, B, C and D and upper face points A', B', C' and D'. Let its volume be V_S. By compressing it by upper face A', B', C' and D', a new parallelepiped T is formed whose upper face points are now A'', B'', C'' and D''. Let its volume be V_T.

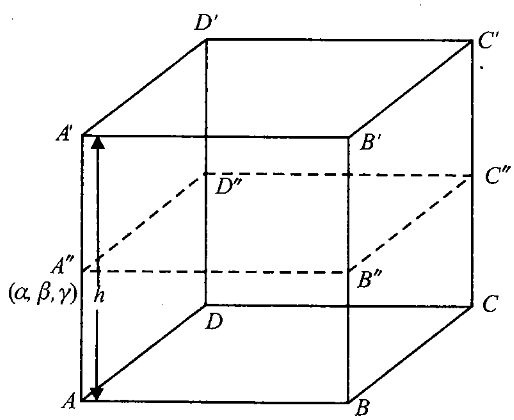


Fig. 3.51

Let h be the height of original parallelepiped S.
Then V_S = (area of $ABCD$) $\times h$ (i)
Let equation of plane $ABCD$ be $ax + by + cz + d = 0$ and $A''(\alpha, \beta, \gamma)$.
Then the height of the new parallelopiped T is the length of the perpendicular from A'' to $ABCD$,

i.e., $\dfrac{a\alpha + b\beta + c\gamma + d}{\sqrt{a^2 + b^2 + c^2}}$. Therefore

$$V_T = (\text{ar } ABCD) \times \frac{(a\alpha + b\beta + c\gamma + d)}{\sqrt{a^2 + b^2 + c^2}} \quad \text{(ii)}$$

But given that $V_T = \dfrac{90}{100} V_s$ (iii)

From (i), (ii) and (iii), we get

$$\frac{a\alpha + b\beta + c\gamma + d}{\sqrt{a^2 + b^2 + c^2}} = 0.9\,h$$

$$\Rightarrow a\alpha + b\beta + c\gamma + d - 0.9\,h\sqrt{a^2 + b^2 + c^2} = 0$$

Therefore, the locus of A'' (α, β, γ) is $ax + by + cz + d - 0.9h\sqrt{a^2 + b^2 + c^2} = 0$, which is a plane parallel to $ABCD$. Hence proved.

4. The given line is $2x - y + z - 3 = 0 = 3x + y + z - 5$, which is intersection of the following two planes:
$2x - y + z - 3 = 0$ (i)
$3x + y + z - 5 = 0$ (ii)
Any plane containing this line will be the plane passing through the intersection of planes (i) and (ii). Thus, the plane containing given line can be written as follows:
$(2x - y + z - 3) + \lambda(3x + y + z - 5) = 0$
$(3\lambda + 2)\,x + (\lambda - 1)\,y + (\lambda + 1)\,z + (-5\lambda - 3) = 0$
As its distance from the point $(2, 1, -1)$ is $1/\sqrt{6}$,

$$\left|\frac{(3\lambda+2)2+(\lambda-1)1+(\lambda+1)(-1)+(-5\lambda-3)}{\sqrt{(3\lambda+2)^2+(\lambda-1)^2+(\lambda+1)^2}}\right| = \frac{1}{\sqrt{6}}$$

$$\left|\frac{\lambda-1}{\sqrt{11\lambda^2+12\lambda+6}}\right| = \frac{1}{\sqrt{6}}$$

Squaring both sides, we get

$$\frac{(\lambda-1)^2}{11\lambda^2+12\lambda+6} = \frac{1}{6}$$

$$\Rightarrow 5\lambda^2 + 24\lambda = 0$$

$$\Rightarrow \lambda(5\lambda + 24) = 0$$

$$\Rightarrow \lambda = 0 \text{ or } -24/5$$

Therefore, the required equations of planes are $2x-y+z-3=0$ and

$$\left[3\left(\frac{-24}{5}\right)+2\right]x+\left[-\frac{24}{5}-1\right]y+\left[-\frac{24}{5}+1\right]z-5\left(\frac{-24}{5}\right)-3=0$$

or, $62x+29y+19z-105=0$

5. The direction cosines of the line are $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$.

Any point on the line at a distance t from $P(2, -1, 2)$ is $\left(2+\frac{t}{\sqrt{3}}, -1+\frac{t}{\sqrt{3}}, 2+\frac{t}{\sqrt{3}}\right)$, which lies on

$2x+y+z-9=0$

$\Rightarrow \quad t=\sqrt{3}$

Objective Type

Multiple choice questions with one correct answer

1. **a.** As the line $\frac{x-4}{1}=\frac{y-2}{1}=\frac{z-k}{2}$ lies in the plane $2x-4y+z=7$, the point $(4, 2, k)$ through which it passes must also lie on the given plane, and hence $2\times4-4\times2+k=7$ or $k=7$.

2. **b.** $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}=\lambda$

$\Rightarrow x=2\lambda+1, y=3\lambda-1$ and $z=4\lambda+1$

$\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}=\mu$

$\Rightarrow x=3+\mu, y=k+2\mu$ and $z=\mu$

Since the above lines intersect,

$2\lambda+1=3+\mu$ (i)

$3\lambda-1=2\mu+k$ (ii)

$\mu=4\lambda+1$ (iii)

Solving (i) and (iii) and putting the value of λ and μ in (ii), $k=9/2$

3. **d.** Let the equation of the variable plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, which meets the axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

The centroid of ΔABC is $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$ and it satisfies the relation $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=k$

$\Rightarrow \frac{9}{a^2}+\frac{9}{b^2}+\frac{9}{c^2}=k$

$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{k}{9}$ (i)

Also it is given that the distance of the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ from $(0, 0, 0)$ is 1 unit. Therefore,

$\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=1 \Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=1$ (ii)

From (i) and (ii), we get $k/9=1$, i.e. $k=9$

4. **d.** The equation of the plane passing through the point (1, – 2, 1) and perpendicular to the planes

$2x - 2y + z = 0$ and $x - y + 2z = 4$ is given by $\begin{vmatrix} x-1 & y+2 & z-1 \\ 2 & -2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 0$

$\Rightarrow x + y + 1 = 0$

Its distance from the point (1, 2, 2) is $\left|\frac{1+2+1}{\sqrt{2}}\right| = 2\sqrt{2}$

5. **a.** Any point on the line can be taken as

$Q = \{(1 - 3\mu), (\mu - 1), (5\mu + 2)\}$

$\overrightarrow{PQ} = \{-3\mu - 2, \mu - 3, 5\mu - 4)$

Now, $1(-3\mu - 2) - 4(\mu - 3) + 3(5\mu - 4) = 0$

$\Rightarrow -3\mu - 2 - 4\mu + 12 + 15\mu - 12 = 0$

$\Rightarrow 8\mu = 2 \Rightarrow \mu = 1/4$

6. **c.** Plane 1: $ax + by + cz = 0$ contains line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$

$\therefore \quad 2a + 3b + 4c = 0$ (i)

Plane 2: $a'x + b'y + c'z = 0$ is perpendicular to plane containing lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$

$\therefore \quad 3a' + 4b' + 2c' = 0$ and $4a' + 2b' + 3c' = 0$

$\Rightarrow \quad \frac{a'}{12-4} = \frac{b'}{8-9} = \frac{c'}{6-16}$

$\Rightarrow \quad 8a - b - 10c = 0$ (ii)

From (i) and (ii),

$\frac{a}{-30+4} = \frac{b}{32+20} = \frac{c}{-2-24}$

$\Rightarrow$ Equation of plane $x - 2y + z = 0$

7. **a.** Distance of point (1, – 2, 1) from plane $x + 2y - 2z = \alpha$ is 5 $\Rightarrow \alpha = 10$.

Equation of PQ, $\frac{x-1}{1} = \frac{y+2}{2} = \frac{z-1}{-2} = t$

$Q \equiv (t + 1, 2t - 2, -2t + 1)$ and $PQ = 5 \quad \Rightarrow \quad t = \frac{5+\alpha}{9} = \frac{5}{3} \Rightarrow Q \equiv \left(\frac{8}{3}, \frac{4}{3}, \frac{-7}{3}\right)$

Assertion and reasoning type

1. **d.** The line of intersection of the given plane is $3x - 6y - 2z - 15 = 0 = 2x + y - 2z - 5 = 0$
For $z = 0$, we obtain $x = 3$ and $y = -1$.
$\therefore$ Line passes through $(3, -1, 0)$
Also, the line is parallel to the cross product of normal to given planes, that is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\hat{i} + 2\hat{j} + 15\hat{k}$$

The equation of line is $\frac{x-3}{14} = \frac{y+1}{2} = \frac{z}{15} = t$, whose parametric form is
$x = 3 + 14t, y = -1 + 2t, z = 15t$
Therefore, Statement 1 is false.
However, Statement 2 is true.

2. **d.** The direction cosines of each of the lines L_1, L_2, L_3 are proportional to (0, 1, 1).

Comprehension type

For Problems 1–3

1. **b.** $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\hat{i} - 7\hat{j} + 5\hat{k}$

Hence, the unit vector will be $\frac{-\hat{i} - 7\hat{j} + 5\hat{k}}{5\sqrt{3}}$

2. **d.** Shortest distance = $\frac{(1+2)(-1) + (2-2)(-7) + (1+3)(5)}{5\sqrt{3}} = \frac{17}{5\sqrt{3}}$

3. **c.** The plane is given by $-(x + 1) - 7(y + 2) + 5(z + 1) = 0$
$\Rightarrow x + 7y - 5z + 10 = 0$

$\Rightarrow$ Distance = $\frac{1+7-5+10}{\sqrt{75}} = \frac{13}{\sqrt{75}}$

Matrix-match type

Sol. **a → r; b → q, c → p; d → s**
Here we have the determinant of the coefficient matrix of given equation as

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$= -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= -\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$

a. $a+b+c \neq 0$

and $a^2+b^2+c^2-ab-bc-ca=0$

$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0$

$\Rightarrow a=b=c$

Therefore, this equation represents identical planes.

b. $a+b+c=0$

and $a^2+b^2+c^2-ab-bc-ca \neq 0$

$\Rightarrow \Delta = 0$ and a, b and c are not all equal. Therefore, all equations are not identical but have infinite solutions. Hence,

$ax + by = (a + b)z$ (using $a + b + c = 0$)

and $bx + cy = (b + c)z$

$\Rightarrow (b^2 - ac)y = (b^2 - ac)z \Rightarrow y = z$

$\Rightarrow ax + by + cy = 0 \Rightarrow ax = ay$

$\Rightarrow x = y = z$

Therefore, the equations represent the line $x = y = z$.

c. $a+b+c \neq 0$ and $a^2+b^2+c^2-ab-bc-ca \neq 0$

$\Rightarrow \Delta \neq 0 \Rightarrow$ The equations have only trivial solution, i.e., $x = y = z = 0$.

Therefore, the equations represent the planes meeting at a single point, namely origin.

d. $a+b+c=0$ and $a^2+b^2+c^2-ab-bc-ca=0$

$\Rightarrow a=b=c$ and $\Delta = 0 \Rightarrow a=b=c=0$

$\Rightarrow$ All equations are satisfied by all x, y and z.

$\Rightarrow$ The equations represent the whole of the three-dimensional space.

Integer Answer Type

1. **(6)** Let normal to plane is $l\hat{i} + m\hat{j} + n\hat{k}$

$2l+3m+4n=0$

and $3l+4m+5n=0$

$\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$

Equation of plane will be

$a(x-1)+b(y-2)+c(z-3)=0$

$\Rightarrow -1(x-1)+2(y-2)-1(z-3)=0$

$\Rightarrow -x+1+2y-4-z+3=0$

$\Rightarrow -x+2y+z=0$

$\Rightarrow x-2y+z=0$

$\Rightarrow \frac{|d|}{\sqrt{6}} = \sqrt{6}$

$\Rightarrow d=6$

Appendix
Solutions to Concept Application Exercises

Chapter 1

Exercise 1.1

1. Since the diagonals of a rhombus bisect each other, $\overrightarrow{OA} = -\overrightarrow{OC}$ and $\overrightarrow{OB} = -\overrightarrow{OD}$ and so

   $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = \vec{0}$.

2. Let the position vectors of A, B and C be $\vec{a}$, $\vec{b}$ and $\vec{c}$, respectively. Then the position vectors of D, E and F are $(\vec{b}+\vec{c})/2$, $(\vec{c}+\vec{a})/2$ and $(\vec{a}+\vec{b})/2$, respectively. Therefore,

   $$\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF} = \left(\frac{\vec{b}+\vec{c}}{2}-\vec{a}\right)+\left(\frac{\vec{c}+\vec{a}}{2}-\vec{b}\right)+\left(\frac{\vec{a}+\vec{b}}{2}-\vec{c}\right)=\vec{0}$$

3. Since the diagonals of a parallelogram bisect each other, P is the middle point of AC and BD both. Therefore

   $\overrightarrow{OA} + \overrightarrow{OC} = 2\overrightarrow{OP}$ and $\overrightarrow{OB} + \overrightarrow{OD} = 2\overrightarrow{OP}$

4. F is the middle point of BD. Therefore

   $\overrightarrow{AB}+\overrightarrow{AD} = 2\overrightarrow{AF}$ (i)

   Similarly, $\overrightarrow{CB}+\overrightarrow{CD} = 2\overrightarrow{CE}$ (ii)

   Adding (i) and (ii), we get

   $$\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{CD} = 2(\overrightarrow{AF}+\overrightarrow{CF}) = -2(\overrightarrow{FA}+\overrightarrow{FC})$$

   $= -2(2(\overrightarrow{FE}))$ (because E is the midpoint of AC)

   $= 4\overrightarrow{EF}$

5. **b.** We have, $\overrightarrow{AO}+\overrightarrow{OB} = \overrightarrow{BO}+\overrightarrow{OC}$

   $\Rightarrow \quad \overrightarrow{AB} = \overrightarrow{BC}$

   Since the initial point of $\overrightarrow{BC}$ is the terminal point of $\overrightarrow{AB}$, A, B and C are collinear.

6. A vector along the internal bisector $= \dfrac{\vec{a}}{|\vec{a}|}+\dfrac{\vec{b}}{|\vec{b}|} = \dfrac{\hat{i}-2\hat{j}+2\hat{k}}{3}+\dfrac{2\hat{i}+\hat{j}+2\hat{k}}{3}$

   $$= \frac{1}{3}(3\hat{i}-\hat{j}+4\hat{k})$$

7.

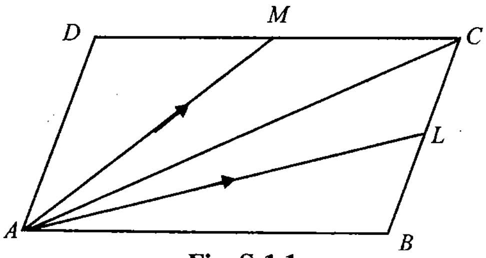


Fig. S-1.1

$$\overrightarrow{AL} = \overrightarrow{AB} + \overrightarrow{BL} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD}$$

$$\overrightarrow{AM} = \overrightarrow{AD} + \overrightarrow{DM} = \overrightarrow{AD} + \frac{1}{2}\overrightarrow{DC} = \overrightarrow{AD} + \frac{1}{2}\overrightarrow{AB}$$

Adding, $\overrightarrow{AL} + \overrightarrow{AM} + \frac{3}{2}(\overrightarrow{AB} + \overrightarrow{AD}) = \frac{3}{2}(\overrightarrow{AB} + \overrightarrow{BC}) = \frac{3}{2}\overrightarrow{AC}$

8. We know that the figure formed by the lines joining the midpoints of the sides of a quadrilateral is a parallelogram. Hence, $MPNQ$ is a parallelogram, whose diagonals are MN and PQ intersecting at E, which is the midpoint of both MN and PQ. For any origin O, we have $\overrightarrow{OA} + \overrightarrow{OB} = 2(\overrightarrow{OM})$ (as M is the midpoint of AB).

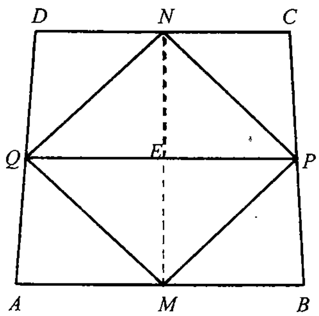


Fig. S-1.2

$$\overrightarrow{OC} + \overrightarrow{OB} = 2(\overrightarrow{ON}) \quad \text{(as } N \text{ is the midpoint of } BC)$$

$$\Rightarrow \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 2(\overrightarrow{OM} + \overrightarrow{ON})$$

$$= 2(2\overrightarrow{OE}) = 4\overrightarrow{OE}$$

where E is the midpoint of MN as it is the intersection of the diagonals of a parallelogram.

9. We have $\vec{a} = 3\hat{i} + 4\hat{j} - 2\hat{k}$. Therefore,

$$|\vec{a}| = \sqrt{9+16+4} = \sqrt{29}$$

Therefore, the unit vector parallel to $a = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{(29)}}(3\hat{i} + 4\hat{j} - 2\hat{k})$.

Now suppose $\vec{b}$ is the vector which when added to $\vec{a}$ gives the resultant $\hat{i}$.
Then $\vec{a} + \vec{b} = \hat{i}$ or $\vec{b} = \hat{i} - \vec{a} = \hat{i} - (3\hat{i} + 4\hat{j} - 2\hat{k})$. Therefore,
$\vec{b} = -2\hat{i} - 4\hat{j} + 2\hat{k}$

10. $|\overrightarrow{OA}| = |\overrightarrow{OB}| = \sqrt{14}$

ΔAOB is isosceles. Hence, the bisector of angle AOB will bisect the base AB.
Hence P is the midpoint (2, 2, – 2) of AB. Therefore,
$\overrightarrow{OP} = 2(\hat{i} + \hat{j} - \hat{k})$

11. $\vec{r_3} = p\vec{r_1} + q\vec{r_2}$

$$\Rightarrow \vec{r}_3 = \frac{p\vec{r_1} + (1-p)\vec{r_2}}{p + (1-p)}$$

r_3 divides $\vec{r_1}$ and $\vec{r_2}$ in the ratio $(1 - p): p$

Hence r_1, r_2 and r_3 are collinear.

Exercise 1.2

1. Since $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = \vec{0}$

$3\vec{a} + \vec{c} = 2\vec{b} + 2\vec{d}$

$$\Rightarrow \frac{3\vec{a} + \vec{c}}{4} = \frac{2\vec{b} + 2\vec{d}}{4} \Rightarrow \frac{3\vec{a} + \vec{c}}{3+1} = \frac{\vec{b} + \vec{d}}{2}$$

Therefore, P.V. of the point dividing AC in the ratio 1 : 3 is the same as the P.V. of midpoint of BD.

So AC and BD intersect at P, whose P.V. is $\frac{3\vec{a} + \vec{c}}{4}$ or $\frac{\vec{b} + \vec{d}}{2}$. Point P divides AC in the ratio 3 : 1 and BD in the ratio 1 : 1.

2. Consider $2\vec{a} - \vec{b} + 3\vec{c} = x(\vec{a} + \vec{b} - 2\vec{c}) + y(\vec{a} + \vec{b} - 3\vec{c})$

$$\Rightarrow \quad 2\vec{a} - \vec{b} + 3\vec{c} = (x + y)\vec{a} + (x + y)\vec{b} + (-2x - 3y)\vec{c}$$

$$x + y = 2 \qquad \text{(i)}$$

$$x + y = -1 \qquad \text{(ii)}$$

$$-2x - 3y = 3 \qquad \text{(iii)}$$

Multiplying (i) by 3 and adding it to (iii), we get

$$x = 9$$

From (i), $9 + y = 2 \Rightarrow y = -7$

Now putting $x = 9$ and $y = -7$ in (ii), we get

$$9 - 7 = -1$$

or $2 = -1$, which is not true.

Therefore, the given vectors are not coplanar.

Alternative method:

We have determinant of co-efficients as

$$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & -2 \\ 1 & 1 & -3 \end{vmatrix} = -3 \neq 0$$

Hence vectors are non-coplanar.

3. **(i)** Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}, \vec{b} = 2\vec{i} + 3\vec{j} - \vec{k}, \vec{c} = -\vec{i} - 2\vec{j} + 2\vec{k}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & -2 & 2 \end{vmatrix} = -1$$

Hence vectors are non-coplanar and linearly independent.

(ii) Let $\vec{a} = 3\vec{i} + \vec{j} - \vec{k}, \vec{b} = 2\vec{i} - \vec{j} + 7\vec{k}, \vec{c} = 7\vec{i} - \vec{j} + 13\vec{k}$

$$\begin{vmatrix} 3 & 1 & -1 \\ 2 & -1 & 7 \\ 7 & -1 & 13 \end{vmatrix} = 0$$

Hence vectors are coplanar and linearly dependent.

4. Putting the values of $\vec{A}$ and $\vec{B}$, and then equating the coefficients of $\vec{a}$ and $\vec{b}$ on both sides, we get

$3(p + 4q) = 2(-2p + q + 2)$

$3(2p + q + 1) = 2(2p - 3q - 1)$

$7p + 10q = 4$ and $2p + 9q = -5$

Solving, we get $p = 2$ and $q = -1$

5. Points $A(\ell_1 \vec{a} + m_1 \vec{b} + n_1 \vec{c}), B(\ell_2 \vec{a} + m_2\vec{b} + n_2 \vec{c}), C(\ell_3 \vec{a} + m_3 \vec{b} + n_3 \vec{c}), D(\ell_4 \vec{a} + m_4 \vec{b} + n_4 \vec{c})$ are coplanar.

$\Rightarrow$ Vectors $\vec{AB} = (\ell_1 - \ell_2)\vec{a} + (m_1 - m_2)\vec{b} + (n_1 - n_2)\vec{c}$,

$\vec{AC} = (\ell_1 - \ell_3)\vec{a} + (m_1 - m_3)\vec{b} + (n_1 - n_3)\vec{c}$,

$\vec{AD} = (\ell_1 - \ell_4)\vec{a} + (m_1 - m_4)\vec{b} + (n_1 - n_4)\vec{c}$

are coplanar

$$\Rightarrow \quad \begin{vmatrix} \ell_1-\ell_2 & m_1-m_2 & n_1-n_2 \\ \ell_1-\ell_3 & m_1-m_3 & n_1-n_3 \\ \ell_1-\ell_3 & m_1-m_3 & n_1-n_4 \end{vmatrix} = 0$$

Now if $\begin{vmatrix} \ell_1 & \ell_2 & \ell_3 & \ell_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$

Then applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, $C_4 \to C_4 - C_1$, we have

$$\begin{vmatrix} \ell_1 & \ell_2-\ell_1 & \ell_3-\ell_1 & \ell_4-\ell_1 \\ m_1 & m_2-m_1 & m_3-m_1 & m_4-m_1 \\ n_1 & n_2-n_1 & n_3-n_1 & n_4-n_1 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} \ell_1-\ell_2 & m_1-m_2 & n_1-n_2 \\ \ell_1-\ell_3 & m_1-m_3 & n_1-n_3 \\ \ell_1-\ell_3 & m_1-m_3 & n_1-n_4 \end{vmatrix} = 0$$

6. Any vector $\vec{r}$ can be uniquely expressed as a linear combination of three non-coplanar vectors.

Let us choose that $7\vec{a} - 11\vec{b} + 15\vec{c} = x(\vec{a} - 2\vec{b} + 3\vec{c}) + y(2\vec{a} - 3\vec{b} + 4\vec{c}) + z(3\vec{a} - 4\vec{b} + 5\vec{c})$

Comparing the coefficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$ on both sides, we get

$x + 2y + 3z = 7$, $-2x - 3y - 4z = -11$, $3x + 4y + 5z = 15$

Eliminating x and then solving for y and z, we get $x = 1$, $y = 3$, $z = 0$

Chapter 2

Exercise 2.1

1. $|4\vec{a} + 3\vec{b}| = \sqrt{(4\vec{a} + 3\vec{b}) \cdot (4\vec{a} + 3\vec{b})}$

$$= \sqrt{16|\vec{a}|^2 + 9|\vec{b}|^2 + 24\vec{a}\cdot\vec{b}}$$

$$= \sqrt{144 + 144 + 24 \times 3 \times 4 \times \left(\frac{-1}{2}\right)}$$

$$= 12$$

2. It is given that vectors $\hat{i}-2x\hat{j}-3y\hat{k}$ and $\hat{i}+3x\hat{j}+2y\hat{k}$ are orthogonal. Therefore,

$$(\hat{i}-2x\hat{j}-3y\hat{k})\cdot(\hat{i}+3x\hat{j}+2y\hat{k})=0$$

$$\Rightarrow 1-6x^2-6y^2=0$$

$\Rightarrow 6x^2+6y^2=1$, which is a circle.

3. $|\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$

$$=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\vec{a}\cdot\vec{b}+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}$$

$$=1+4+4+0+0+0=9$$

$$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|=3$$

4. Given, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$$\vec{a}+\vec{b}=-\vec{c}$$

$$(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=(-\vec{c})\cdot(-\vec{c})$$

$$\Rightarrow a^2+b^2+2\,(\vec{a}\cdot\vec{b})=c^2$$

$$\Rightarrow 9+25+2\,(\vec{a}\cdot\vec{b})=49$$

$$\Rightarrow \vec{a}\cdot\vec{b}=15/2$$

$$\Rightarrow ab\cos\theta=15/2 \Rightarrow 3\cdot 5\cos\theta=15/2$$

$$\Rightarrow \cos\theta=1/2 \Rightarrow \theta=\pi/3$$

5. $|\vec{a}-\vec{b}|^2=(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$$=a^2+b^2-2(\vec{a}\cdot\vec{b})$$

$$=1+1-2\,(1\cdot 1\cdot\cos\theta)$$

$$=2\,(1-\cos\theta)$$

$$=2\left(1-\frac{1}{2}\right)=1$$

6. $\hat{n}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, where $a_1^2+a_2^2+a_3^2=1$

Given that $\vec{u}\cdot\hat{n}=0 \Rightarrow a_1+a_2=0$

Also, $\vec{v}\cdot\hat{n}=0 \Rightarrow a_1-a_2=0$

$a_1 = a_2 = 0$

$a_3 = 1$ or ∓ 1

$\hat{n} = \hat{k}$ or $-\hat{k}$

$|\vec{w}\cdot\hat{n}| = 3$

7. $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \vec{a} + \vec{b} + \vec{c}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{b}$ or $\overrightarrow{CA} = -(\vec{a} + \vec{b})$

$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \vec{b} + \vec{c}$

Therefore, $\overrightarrow{AB}\cdot\overrightarrow{CD} + \overrightarrow{BC}\cdot\overrightarrow{AD} + \overrightarrow{CA}\cdot\overrightarrow{BD}$

$= \vec{a}\cdot\vec{c} + \vec{b}\cdot(\vec{a} + \vec{b} + \vec{c}) + (-\vec{a} - \vec{b})\cdot(\vec{b} + \vec{c})$

$= \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} + \vec{b}\cdot\vec{c} - \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c} - \vec{b}\cdot\vec{b} - \vec{b}\cdot\vec{c} = 0$

8. $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \hat{i} - 2\hat{k}$

$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR} = 4\hat{i} - 4\hat{j} - \hat{k}$

Projection of $\overrightarrow{PQ}$ on $\overrightarrow{RS} = \dfrac{\overrightarrow{QP}\cdot\overrightarrow{RS}}{|\overrightarrow{RS}|} = \dfrac{6}{\sqrt{33}}$

9. $3\vec{p} + \vec{q}$ and $5\vec{p} - 3\vec{q}$ are perpendicular. Therefore,

$(3\vec{p} + \vec{q})\cdot(5\vec{p} - 3\vec{q}) = 0$

$15\vec{p}^2 - 3\vec{q}^2 = 4\vec{p}\cdot\vec{q}$ (i)

$2\vec{p} + \vec{q}$ and $4\vec{p} - 2\vec{q}$ are perpendicular. Therefore,

$(2\vec{p} + \vec{q})\cdot(4\vec{p} - 2\vec{q}) = 0$

$8\vec{p}^2 = 2\vec{q}^2$

$\vec{q}^2 = 4\vec{p}^2$ (ii)

Now, $\cos\theta = \dfrac{\vec{p}\cdot\vec{q}}{|\vec{p}||\vec{q}|}$

Substituting $\vec{q}^2 = 4\vec{p}^2$ in (i), $3\vec{p}^2 = 4\vec{p}\cdot\vec{q}$

$\therefore \cos\theta = \dfrac{3}{4}\cdot\dfrac{\vec{p}^2}{|\vec{p}|2|\vec{p}|} = \dfrac{3}{8}$

$\Rightarrow \theta = \cos^{-1}\dfrac{3}{8}$

10. $\vec{A}\cdot(\alpha\vec{A}+\vec{B})=\vec{B}\cdot(\alpha\vec{A}+\vec{B})$

$\Rightarrow \alpha+\vec{A}\cdot\vec{B}=\alpha\vec{A}\cdot\vec{B}+1$

$\Rightarrow (\vec{A}\cdot\vec{B})(1-\alpha)=(1-\alpha)$

Since $\vec{A}\cdot\vec{B}\neq 0 \Rightarrow \alpha=1$

11. $\vec{a}+\vec{b}+\vec{c}=\vec{x}$

Taking dot with $\vec{x}$ on both sides, we get

$\vec{x}\cdot\vec{a}+\vec{x}\cdot\vec{b}+\vec{x}\cdot\vec{c}=\vec{x}\cdot\vec{x}=|\vec{x}|^2=4$

$\Rightarrow 1+\frac{3}{2}+\vec{x}\cdot\vec{c}=4 \Rightarrow \vec{x}\cdot\vec{c}=\frac{3}{2}$

If θ be the angle between $\vec{c}$ and $\vec{x}$, then $|\vec{x}||\vec{c}|\cos\theta=3/2$

$\Rightarrow \cos\theta=3/4 \Rightarrow \theta=\cos^{-1}(3/4)$

12. Let θ be an angle between unit vectors $\vec{a}+\vec{b}$. Then

$\vec{a}\cdot\vec{b}=\cos\theta$

$|\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=2+2\cos\theta=4\cos^2\theta/2$

$\Rightarrow |\vec{a}+\vec{b}|=2\cos\frac{\theta}{2}$

Similarly, $|\vec{a}-\vec{b}|=2\sin\frac{\theta}{2}$

$\Rightarrow |\vec{a}+\vec{b}|+|\vec{a}-\vec{b}|=2\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)\leq 2\sqrt{2}$

13. Resultant force

$$\vec{F}=\vec{P}_1+\vec{P}_2+\vec{P}_3=2\hat{j}-\hat{k}$$

And displacement $=\vec{AB}$

$=$ P.V. of B $-$ P.V. of A

$=(6\hat{i}+\hat{j}-3\hat{k})-(4\hat{i}-3\hat{j}-2\hat{k})$

$=2\hat{i}+4\hat{j}-\hat{k}$

$\therefore$ Work done $=\vec{F}\cdot\vec{AB}$

$=(2\hat{j}-\hat{k})\cdot(2\hat{i}+4\hat{j}-\hat{k})$

$=9$ units

Exercise 2.2

1. $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & -5\\ m & n & 12\end{vmatrix}$

$=(36+5n)\,\hat{i}-(24+5m)\,\hat{j}+(2n-3m)\,\hat{k}=\vec{0}$

$m = \frac{-24}{5}, n = \frac{-36}{5}$

2. $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, but $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

$\Rightarrow \sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{4}{5} \Rightarrow \cos\theta = \frac{3}{5}$

Therefore, $\vec{a}\cdot\vec{b} = 2\times5\times\frac{3}{5} = 6$

3. Since $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} \neq \vec{0}$,

$\vec{a}\times\vec{b} - \vec{b}\times\vec{c} = \vec{0}$

$\Rightarrow \vec{a}\times\vec{b} + \vec{c}\times\vec{b} = \vec{0}$

$\Rightarrow (\vec{a}+\vec{c})\times\vec{b} = \vec{0}$

$\Rightarrow \vec{a}+\vec{c}$ is parallel to $\vec{b}$

$\vec{a}+\vec{c} = k\vec{b}$

4. $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ -1 & 2 & -4 \end{vmatrix} = -10\vec{i} + 9\vec{j} + 7\vec{k}$

$\vec{a}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4\vec{i} - 3\vec{j} - \vec{k}$

$\Rightarrow (\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c}) = -40 - 27 - 7 = -74$

5. Since $\vec{a}$, $\vec{c}$ and $\vec{b}$ form a right-handed system,

$\vec{c} = \vec{b}\times\vec{a}$

$= \hat{j}\times(x\hat{i} + y\hat{j} + z\hat{k})$

$= -x\hat{k} + z\hat{i} = z\hat{i} - x\hat{k}$

6. We have $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$. Therefore,

$\vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c} = 0 \Rightarrow \vec{a}\cdot(\vec{b} - \vec{c}) = 0$

Therefore, there are three possibilities: (i) $\vec{a} = \vec{0}$, (ii) $\vec{b} - \vec{c} = \vec{0}$ and (iii) $\vec{a}$ is perpendicular to $\vec{b} - \vec{c}$.

Again, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$. Therefore,

$$\vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{0}$$

$$\Rightarrow \vec{a} \times (\vec{b} - \vec{c}) = \vec{0}$$

Therefore, again there are three possibilities: (i) $\vec{a} = \vec{0}$, (ii) $\vec{b} - \vec{c} = \vec{0}$ and (iii) $\vec{a}$ is parallel to $\vec{b} - \vec{c}$.

Now $\vec{a}$ is given to be a non-zero vector. Therefore, we have the following possibilities left:

1. $\vec{b} - \vec{c} = \vec{0}$.
2. $\vec{a}$ is perpendicular to $\vec{b} - \vec{c}$ and $\vec{a}$ is parallel to $\vec{b} - \vec{c}$, which is absurd.

Therefore, the only possibility left is $\vec{b} - \vec{c} = \vec{0}$ or $\vec{b} = \vec{c}$.

7. $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}$

$$= \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b}$$

$$= \vec{0} + 2\vec{a} \times \vec{b} - \vec{0} = 2\vec{a} \times \vec{b}$$

Geometrically, the vector area of a parallelogram whose sides are along vectors $\vec{a}$ and $\vec{b}$ is $\vec{a} \times \vec{b}$. Also diagonals are along vectors $\vec{a} - \vec{b}$ and $\vec{a} + \vec{b}$ and the vector area in terms of diagonal vectors is $\frac{1}{2}[(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})]$.

8. $\vec{z} + \vec{z} \times \vec{x} = \vec{y} \Rightarrow |\vec{z} + \vec{z} \times \vec{x}|^2 = |\vec{y}|^2$

$\Rightarrow |\vec{z}|^2 + |\vec{z}|^2 |\vec{x}|^2 \sin^2\theta = 1$ (because $\vec{z} \cdot (\vec{z} \times \vec{x}) = 0$)

$$\Rightarrow |\vec{z}|^2 (1 + \sin^2\theta) = 1$$

$$\Rightarrow |\vec{z}| = \frac{1}{\sqrt{1 + \sin^2\theta}} = \frac{2}{\sqrt{7}}$$

$$\Rightarrow \sin\theta = \sqrt{3}/2$$

$$\Rightarrow \theta = \pi/3 = 60°$$

9. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$. Therefore,

$\vec{a} \cdot \hat{i} = a_1$, $\vec{a} \cdot \hat{j} = a_2$ and $\vec{a} \cdot \hat{k} = a_3$ and $\vec{a} \times \hat{i} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \hat{i} = -a_2\hat{k} + a_3\hat{j}$

Similarly, $\vec{a} \times \hat{j} = a_1\hat{k} - a_3\hat{i}$ and $\vec{a} \times \hat{k} = -a_1\hat{j} + a_2\hat{i}$

$$(\vec{a} \cdot \hat{i})(\vec{a} \times \hat{i}) + (\vec{a} \cdot \hat{j})(\vec{a} \times \hat{j}) + (\vec{a} \cdot \hat{k})(\vec{a} \times \hat{k}) = -a_1a_2\hat{k} + a_1a_3\hat{j} + a_1a_2\hat{k} - a_3a_2\hat{i} + a_3a_2\hat{i} - a_3a_1\hat{j}$$

$$= \vec{0}$$

10.

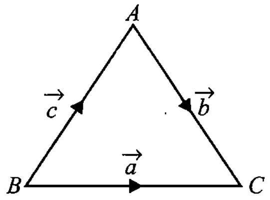


Fig. S-2.1

Clearly, $\vec{a}$, $\vec{b}$ and $\vec{c}$ represent the sides of a triangle.

$\Rightarrow \vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

$\Rightarrow \vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = 3\vec{a}\times\vec{b}$

$\Rightarrow 2\vec{b}\times\vec{a} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = 0 \Rightarrow \lambda = 2$

11.

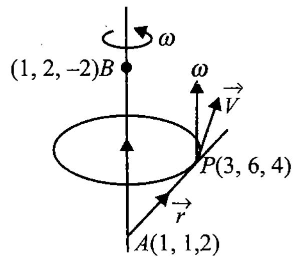


Fig. S-2.2

$\overrightarrow{OA} = \hat{i} + \hat{j} + 2\hat{k}$

$\overrightarrow{OB} = \hat{i} + 2\hat{j} - 2\hat{k}$

$\overrightarrow{AB} = \hat{j} - 4k \Rightarrow |\overrightarrow{AB}| = \sqrt{17}$

$\overrightarrow{AP} = (3\hat{i} + 6\hat{j} + 4\hat{k}) - (\hat{i} + \hat{j} + 2\hat{k})$

$= 2\hat{i} + 5\hat{j} + 2\hat{k}$

$\therefore \vec{\omega} = \frac{3}{\sqrt{17}}(\hat{j} - 4\hat{k})\,2$

$\vec{v} = \vec{\omega}\times\vec{r} = \frac{3}{\sqrt{17}}(\hat{j} - 4\hat{k})\times(2\hat{i} + 5\hat{j} + 2\hat{k})$

$= \frac{3}{\sqrt{17}}(22\hat{i} - 8\hat{j} - 2\hat{k})$

12. We have $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c} = 0$.

This implies that $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$.

Thus, $\vec{a}$ is a unit vector perpendicular to both $\vec{b}$ and $\vec{c}$.

Hence, $\vec{r} = \pm\dfrac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|} = \pm\dfrac{\vec{b}\times\vec{c}}{|\vec{b}||\vec{c}|\sin\pi/3} = \pm 2(\vec{b}\times\vec{c})$

13. Since $(\vec{a}\times\vec{b})^2 + (\vec{a}\cdot\vec{b})^2 = 144$, if the angle between $\vec{a}$ and $\vec{b}$ is θ, then

$|\vec{a}|^2|\vec{b}|^2 \sin^2\theta + |\vec{a}|^2|\vec{b}|^2 \cos^2\theta = 144$

$\Rightarrow |\vec{a}|^2|\vec{b}|^2 = 144$

$\Rightarrow |\vec{a}||\vec{b}| = 12$

$\Rightarrow 4|\vec{b}| = 12$

$\Rightarrow |\vec{b}| = 3$

14. We have, $|\vec{a}+\vec{b}| = \sqrt{3}$

$\Rightarrow |\vec{a}+\vec{b}|^2 = 3$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a}\cdot\vec{b}) = 3$

$\Rightarrow 1 + 1 + 2(\vec{a}\cdot\vec{b}) = 3$

$\Rightarrow \vec{a}\cdot\vec{b} = 1/2$ (i)

Now, $\vec{c} - \vec{a} - 2\vec{b} = 3(\vec{a}\times\vec{b})$

$\Rightarrow (\vec{c} - \vec{a} - 2\vec{b})\cdot\vec{b} = 3\{(\vec{a}\times\vec{b})\cdot\vec{b}\}$

$\Rightarrow \vec{c}\cdot\vec{b} - \vec{a}\cdot\vec{b} - 2(\vec{b}\cdot\vec{b}) = 0$ (because $[\vec{a}\ \vec{b}\ \vec{b}] = 0$)

$\Rightarrow \vec{c}\cdot\vec{b} - \dfrac{1}{2} - 2\times 1 = 0$ (Using (i))

$\Rightarrow \vec{c}\cdot\vec{b} = 5/2$

15. $\vec{F} = 3\hat{i} + 2\hat{j} - 4\hat{k}$

A is $(1, -1, 2)$, P is $(2, -1, 3)$

$\therefore \quad \vec{PA} = \text{P.V. of } A - \text{P.V. of } P$

$= (\hat{i} - \hat{j} + 2\hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k})$

$= -\hat{i} - \hat{k}$

Required vector moment $= \vec{PA} \times \vec{F}$

$$= (-\hat{i} - \hat{k}) \times (3\hat{i} + 2\hat{j} - 4\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix}$$

$$= 2\hat{i} - 7\hat{j} - 2\hat{k}$$

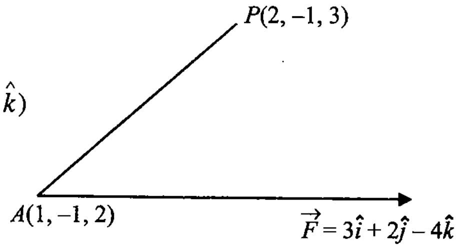


Fig. S-2.3

Exercise 2.3

1. Since $\vec{d}$ makes equal angles with the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

$$d = \frac{\mu(\vec{a} + \vec{b} + \vec{c})}{3} \quad \text{(i)}$$

($\vec{d}$ passes through the centroid of the triangle with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$)

Again $[\vec{a}\ \vec{b}\ \vec{c}]\vec{d} = [\vec{d}\ \vec{b}\ \vec{c}]\vec{a} + [\vec{d}\ \vec{c}\ \vec{a}]\vec{b} + [\vec{d}\ \vec{a}\ \vec{b}]\vec{c}$ (ii)

From (i) and (ii), we get $[\vec{d}\ \vec{b}\ \vec{c}] = [\vec{d}\ \vec{c}\ \vec{a}] = [\vec{d}\ \vec{a}\ \vec{b}]$

2. Let $\vec{l} = l_1\hat{i} + l_2\hat{j} + l_3\hat{k}$, $\vec{m} = m_1\hat{i} + m_2\hat{j} + m_3\hat{k}$, $\vec{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Therefore,

$\vec{l} \cdot \vec{a} = l_1a_1 + l_2a_2 + l_3a_3 = \sum l_1a_1$

Similarly, $\vec{l} \cdot \vec{b} = \sum l_1b_1$, etc.

Now, $[\vec{l}\ \vec{m}\ \vec{n}](\vec{a} \times \vec{b}) = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

$$= \begin{vmatrix} \sum l_1\hat{i} & \sum l_1a_1 & \sum l_1b_1 \\ \sum m_1\hat{i} & \sum m_1a_1 & \sum m_1b_1 \\ \sum n_1\hat{i} & \sum n_1a_1 & \sum n_1b_1 \end{vmatrix} = \begin{vmatrix} \vec{l} & \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} \\ \vec{m} & \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} \\ \vec{n} & \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} \end{vmatrix} = \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n} \end{vmatrix}$$

3. $\begin{vmatrix} 2 & 3 & 4 \\ 1 & \alpha & 2 \\ 1 & 2 & \alpha \end{vmatrix} = 15$

$\Rightarrow 2(\alpha^2 - 4) + 3(2 - \alpha) + 4(2 - \alpha) = 15$

$\Rightarrow 2\alpha^2 - 8 + 6 - 3\alpha + 8 - 4\alpha = 15$

$\Rightarrow 2\alpha^2 - 7\alpha - 9 = 0$

$\Rightarrow 2\alpha^2 - 9\alpha + 2\alpha - 9 = 0$

$\Rightarrow (\alpha + 1)(2\alpha - 9) = 0$

$\Rightarrow \alpha = -1, 9/2$

4. $\vec{a} \times \vec{b} = \vec{a} \times (\vec{a} \times \vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{a} - (\vec{a}\cdot\vec{a})\,\vec{c} = 2\vec{a} - 3\vec{c}$

But $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 3\hat{i} - 3\hat{k}$

Hence, $3\vec{c} = 2\vec{a} - (3\hat{i} - 3\hat{k}) = (2\hat{i} + 2\hat{j} + 2\hat{k}) - (3\hat{i} - 3\hat{k}) = -\hat{i} + 2\hat{j} + 5\hat{k}$

$\Rightarrow \vec{c} = \frac{1}{3}(-\hat{i} + 2\hat{j} + 5\hat{k})$

5. Since $\vec{x}$ is a non-zero vector, the given conditions will be satisfied if either (i) vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are zero or (ii) $\vec{x}$ is perpendicular to vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

In case (ii) $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar and so $[\vec{a}\,\vec{b}\,\vec{c}] = 0$.

6. $[\vec{a}\times\vec{b}\;\; \vec{b}\times\vec{c}\;\; \vec{c}\times\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$ (i)

Now let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Therefore,

$$[\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma a_1^2 & \Sigma a_1 b_1 & \Sigma a_1 c_1 \\ \Sigma b_1 a_1 & \Sigma b_1^2 & \Sigma b_1 c_1 \\ \Sigma c_1 a_1 & \Sigma c_1 b_1 & \Sigma c_1^2 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix} \quad \text{(ii)}$$

7. Here, $\vec{a} \times \vec{b} = \vec{c}$ (given)

$(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{c} \cdot \vec{c}$

$[\vec{a}\ \vec{b}\ \vec{c}] = |\vec{c}|^2$ (i)

Also, $\vec{b} \times \vec{c} = \vec{a}$ (given)

$\Rightarrow (\vec{b} \times \vec{c}) \cdot \vec{a} = \vec{a} \cdot \vec{a}$

$\Rightarrow [\vec{b}\ \vec{c}\ \vec{a}] = |\vec{a}|^2$

also $\vec{c} \times \vec{a} = \vec{b}$ (given) (ii)

$(\vec{c} \times \vec{a}) \cdot \vec{b} = \vec{b} \cdot \vec{b}$

$[\vec{c}\ \vec{a}\ \vec{b}] = |\vec{b}|^2$

Since $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$, (iii)

$|\vec{a}| = |\vec{b}| = |\vec{c}|$

8. $\vec{a} = \vec{p} + \vec{q}$

$\Rightarrow \vec{a} \times \vec{b} = \vec{p} \times \vec{b} + \vec{q} \times \vec{b}$

$\Rightarrow \vec{a} \times \vec{b} = \vec{q} \times \vec{b}$ $(\because \vec{p} \times \vec{b} = \vec{0})$

$\Rightarrow \vec{b} \times (\vec{a} \times \vec{b}) = \vec{b} \times (\vec{q} \times \vec{b})$

$= (\vec{b} \cdot \vec{b})\, \vec{q} - (\vec{b} \cdot \vec{q})\, \vec{b}$

$= (\vec{b} \cdot \vec{b})\, \vec{q}$ $(\because \vec{b} \cdot \vec{q} = 0)$

$$\Rightarrow \frac{\vec{b} \times (\vec{a} \times \vec{b})}{\vec{b} \cdot \vec{b}} = \vec{q}$$

9. $\vec{a} \cdot (\vec{b} \times \hat{i})\, \hat{i} = ((\vec{a} \times \vec{b}) \cdot \hat{i})\, \hat{i}$

If $\vec{a} \times \vec{b} = x\hat{i} + y\hat{j} + z\hat{k}$, then $(\vec{a} \times \vec{b}) \cdot \hat{i} = x$

Similarly, $(\vec{a} \cdot (\vec{b} \times \hat{j}))\, \hat{j} = y$ and $(\vec{a} \cdot (\vec{b} \times \hat{k}))\, \hat{k} = z$

$\Rightarrow (\vec{a} \cdot (\vec{b} \times \hat{i}))\, \hat{i} + (\vec{a} \cdot (\vec{b} \times \hat{j}))\, \hat{j} + (\vec{a} \cdot (\vec{b} \times \hat{k}))\, \hat{k} = x\hat{i} + y\hat{j} + z\hat{k} = \vec{a} \times \vec{b}$

$\left(\frac{3\lambda+2}{\lambda+1}\right)\hat{i}+\left(\frac{5\lambda+2}{\lambda+1}\right)\hat{j}+\left(\frac{6\lambda+4}{\lambda+1}\right)\hat{k}$. Therefore,

$$\frac{13}{5}\hat{i}+\frac{19}{5}\hat{j}+\frac{26}{5}\hat{k}=\left(\frac{3\lambda+2}{\lambda+1}\right)\hat{i}+\left(\frac{5\lambda+2}{\lambda+1}\right)\hat{j}+\left(\frac{6\lambda+4}{\lambda+1}\right)\hat{k}$$

Therefore, $\frac{3\lambda+2}{\lambda+1}=\frac{13}{5}$, $\frac{5\lambda+2}{\lambda+1}=\frac{19}{5}$ and $\frac{6\lambda+4}{\lambda+1}=\frac{26}{5}$

$\Rightarrow 2\lambda = 3 \Rightarrow \lambda = 3/2$

Hence, P divides QR in the ratio 3 : 2

5. The direction cosines of $\overrightarrow{OP}$ are $-\frac{1}{3}, \frac{2}{3}$ and $-\frac{2}{3}$.

Hence, $\overrightarrow{OP} = |\overrightarrow{OP}|\ (l\hat{i}+m\hat{j}+n\hat{k})$

$$= 3\left(-\frac{1}{3}\hat{i}+\frac{2}{3}\hat{j}-\frac{2}{3}\hat{k}\right)$$

$$= -1\hat{i}+2\hat{j}-2\hat{k}$$

So, the coordinates of P are –1, 2 and –2.

6. Here, $\cos^2\alpha + \cos^2(90-\alpha) + \cos^2\gamma = 1$

$\Rightarrow \cos^2\alpha + \sin^2\alpha + \cos^2\gamma = 1$

$\Rightarrow \cos^2\gamma + 1 = 1 \Rightarrow \gamma = 90°$

7. According to the question, $\frac{a+2}{6}=\frac{b-1}{2}=\frac{c+8}{3}=\lambda$

$\Rightarrow a = 6\lambda - 2, b = 2\lambda + 1, c = 3\lambda - 8$

8. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$

$= 2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1$

$= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3$

$= -1$

9. From the figure, it is clear that the length of the edges of the parallelopiped a, b, c is $x_2 - x_1, y_2 - y_1, z_2 - z_1$ or 6 –3, 8 –4 and 10 –8 or 3, 4 and 2. Therefore,

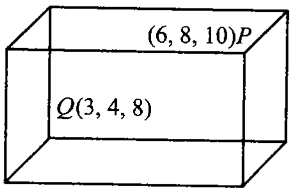


Fig. S-3.1

The length of the diagonal will be $\sqrt{a^2+b^2+c^2} = \sqrt{9+16+4} = \sqrt{39}$.

10.

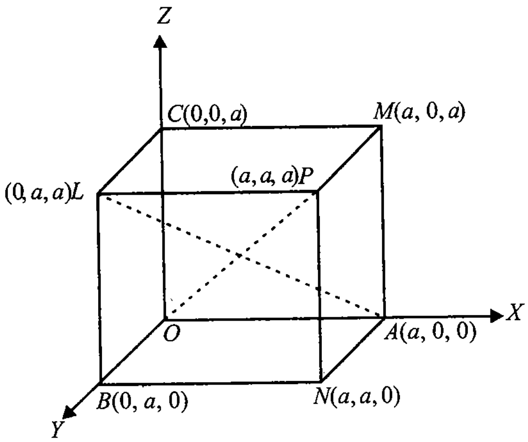


Fig. S-3.2

The direction ratios of OP are a, a and a or 1, 1 and 1 and those of AL are $-a, a$ and a, or -1, 1 and 1. Therefore,

$$\cos\theta = \frac{-1+1+1}{\sqrt{3}\cdot\sqrt{3}} = \frac{1}{3} \Rightarrow \theta = \cos^{-1}\frac{1}{3}$$

11. Since $\frac{a}{(1/bc)} = \frac{b}{(1/ca)} = \frac{c}{(1/ab)}$, hence lines are parallel.

12. Eliminating n, we have $(2l+m)(l-m)=0$.

When $2l+m=0$, then $\frac{l}{1} = \frac{m}{-2} = \frac{n}{-2}$.

When $l-m=0$, then $\frac{l}{1} = \frac{m}{1} = \frac{n}{-2}$. Therefore,

Direction ratios are 1, –2, 1 and 1, 1, and –2

$$\cos\theta = \frac{\sum l_1 l_2}{\sqrt{\left(\sum l_1^2\right)\left(\sum l_2^2\right)}} = -\frac{1}{2}$$

$\Rightarrow \theta = 120° = 2\pi/3$

Exercise 3.2

1. Line is passing through the point (1, 2, 3) and parallel to the line $\vec{r} = \hat{i} - \hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 3\hat{k})$ or parallel to the vector $\hat{i} - 2\hat{j} + 3\hat{k}$. Hence equation of line is

$$\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z-3}{3}$$

It meets xy-plane, where $z = 0$

Then from the equation of line, we have

$$\frac{x-1}{1}=\frac{y-2}{-2}=\frac{0-3}{3}$$

$\Rightarrow x=0, y=4.$

$\Rightarrow$ Line meets xy-plane at (0, 4, 0)

2. Since line is passing through the points $A(1, 2, 3)$ and $B(-1, 0, 4)$, it is along the vector $\vec{AB}=-2\hat{i}-2\hat{j}+\hat{k}$. Hence equation of line is

$$\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(-2\hat{i}-2\hat{j}+\hat{k}) \text{ or}$$

$$\vec{r}=-\hat{i}+4\hat{k}+\lambda(-2\hat{i}-2\hat{j}+\hat{k})$$

Or

$$\frac{x-1}{-2}=\frac{y-2}{-2}=\frac{z-3}{1} \quad \text{or} \quad \frac{x+1}{-2}=\frac{y-0}{-2}=\frac{z-4}{1}$$

3. The given line is $-6x-2=3y+1=2z-2$, or

$$\frac{x+(1/3)}{-1/6}=\frac{y+(1/3)}{1/3}=\frac{z-1}{1/2}$$

The direction ratios are $-\frac{1}{6}, \frac{1}{3}$ and $\frac{1}{2}$ or -1, 2 and 3.

The required equation is $\frac{x-2}{-1}=\frac{y+1}{2}=\frac{z+1}{3}$

4. The line through point $(-1, 2, 3)$ is perpendicular to the lines $\frac{x}{2}=\frac{y-1}{-3}=\frac{2+2}{-2}$ and $\frac{x+3}{-1}=\frac{y+3}{2}=\frac{z-1}{3}$. Therefore, the line is along the vector $\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -3 & -2\\ -1 & 2 & 3\end{vmatrix}$ or $-5\hat{i}-4\hat{j}+\hat{k}$.

Hence, equation of the line is $\frac{x+1}{5}=\frac{y-2}{4}=\frac{z-3}{-1}$.

5. Intersection point of the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ is $(-1,-1,-1)$ (on solving).

Therefore, the equation of the line passing through the points $(-1, -1, -1)$ and $(2, 1, -2)$ is $\frac{x+1}{3}=\frac{y+1}{2}=\frac{x+1}{-1}$.

6. The line is along the vector $3\hat{i}+\hat{j}$ which is perpendicular to the z-axis as $(3\hat{i}+\hat{j})\cdot\hat{k}=0$.

7. The lines are $\frac{x}{3}=\frac{y}{2}=\frac{z}{-6}$ and $\frac{x}{2}=\frac{y}{-12}=\frac{z}{-3}$.

Since $a_1a_2+b_1b_2+c_1c_2=6-24+18=0$,

$\theta=90°$

8. The lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Hence, $-3(3k) + 2k(1) + 2(-5) = 0 \Rightarrow k = -\frac{10}{7}$.

9. Eliminating t from the given equations, we get equation of the path.

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{4}$$

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{2}$$

Thus, the path of the rocket represents a straight line passing through the origin.

For $t = 10$ s, we have

$x = 20, y = -40$ and $z = 40$ and $|\vec{r}| = |\overrightarrow{OM}| = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{400 + 1600 + 1600} = 60$ km

10. Let P be the foot of the perpendicular from the point $A(5, 4, -1)$ to the line l whose equation is $\vec{r} = \hat{i} + \lambda(2\hat{i} + 9\hat{j} + 5\hat{k})$.

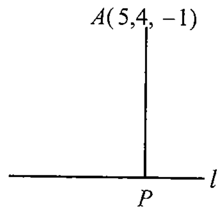


Fig. S-3.3

The coordinates of any point on the line are given by $x = 1 + 2\lambda$, $y = 9\lambda$ and $z = 5\lambda$.

The coordinates of P are given by $1 + 2\lambda$, 9λ and 5λ for some value of λ.

The direction ratios of AP are $1 + 2\lambda - 5$, $9\lambda - 4$ and $5\lambda - (-1)$ or $2\lambda - 4$, $9\lambda - 4$ and $5\lambda + 1$.

Also, the direction ratios of l are 2, 9 and 5.

Since $AP \perp l$, $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Rightarrow 2(2\lambda - 4) + 9(9\lambda - 4) + 5(5\lambda + 1) = 0 \Rightarrow 4\lambda - 8 + 81\lambda - 36 + 25\lambda + 5 = 0$

$\Rightarrow 110\lambda - 39 = 0 \Rightarrow \lambda = 39/110$

Now, $AP^2 = (1 + 2\lambda - 5)^2 + (9\lambda - 4)^2 + (5\lambda - (-1))^2 = (2\lambda - 4)^2 + (9\lambda - 4)^2 + (5\lambda + 1)^2$

$= 4\lambda^2 - 16\lambda + 16 + 81\lambda^2 - 72\lambda + 16 + 25\lambda^2 + 10\lambda + 1 = 110\lambda^2 - 78\lambda + 33$

$$= 110\left(\frac{39}{110}\right)^2 - 78\left(\frac{39}{110}\right) + 33 = \frac{39^2 - 78 \times 39 + 33 \times 110}{110} = \frac{2109}{110} \Rightarrow AP = \sqrt{\frac{2109}{110}}$$

11. Let the image of point $A(1, 2, 3)$ in the line l whose equation is

$\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2} = k$ (say) be A'. Then AA' is perpendicular to l and the point of intersection of AA' and l is the midpoint of AA'. Note that M is the foot of perpendicular from A to l. (i)

The coordinates of any point on the given line are of the form $(3k + 6, 2k + 7, -2k + 7)$. Therefore, the coordinates of M are $3k + 6$, $2k + 7$ and $-2k + 7$ for some value of k. The direction ratios of AM are $3k + 6 - 1, 2k + 7 - 2$ and $-2k + 7 - 3$ or $3k + 5, 2k + 5, -2k + 4$.

Also, the direction ratios of l are 3, 2 and -2.

Since $AM \perp l$, $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

$\Rightarrow 3(3k + 5) + 2(2k + 5) - 2(-2k + 4) = 0$

$\Rightarrow 17k + 17 = 0$ or $k = -1$

Thus, the coordinates of M are 3, 5 and 9.

Suppose coordinates of A' are x, y and z,

The coordinates of the midpoint of AA' are $\frac{x+1}{2}, \frac{y+2}{2}$ and $\frac{z+3}{2}$.

But the midpoint of AA' is (5, 3, 9). Therefore,

$$\frac{x+1}{2} = 3, \frac{y+2}{2} = 5 \text{ and } \frac{z+3}{2} = 9 \Rightarrow x = 5, y = 8, z = 15$$

Thus, the image of A in l is (5, 8, 15).

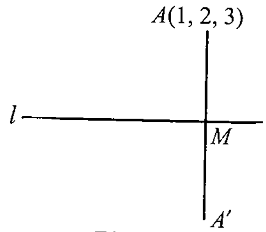


Fig. S-3.4

12. The lines are $\vec{r} = (1-\lambda)\hat{i} + (\lambda - 2)\hat{j} + (3 - 2\lambda)\hat{k}$ and $\vec{r} = (\mu + 1)\hat{i} + (2\mu - 1)\hat{j} - (2\mu + 1)\hat{k}$ or $\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} - 2\hat{j} - 2\hat{k})$ and $\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + \mu(\hat{i} + 2\hat{j} - 2\hat{k})$.

Line (i) passes through the point $(x_1, y_1, z_1) \equiv (1, -2, 3)$ and is parallel to the vector $a_1\hat{i} + b_1\hat{j} + c_1\hat{k} \equiv -\hat{i} - 2\hat{j} - 2\hat{k}$.

Line (ii) passes through the point $(x_2, y_2, z_2) \equiv (1, -1, -1)$ and is parallel to the vector $a_2\hat{i} + b_2\hat{j} + c_2\hat{k} \equiv \hat{i} + 2\hat{j} - 2\hat{k}$.

Hence, the shortest distance between the lines using the formula $\dfrac{\left|\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}\right|}{\left|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}\right|}$ is

$$\frac{\left|\begin{vmatrix} 1-1 & -1+2 & -1-3 \\ -1 & -2 & -2 \\ 1 & 2 & -2 \end{vmatrix}\right|}{\left|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & -2 \\ 1 & 2 & -2 \end{vmatrix}\right|} = \frac{4}{\sqrt{80}} = \frac{1}{\sqrt{5}}$$

13. $\dfrac{x-1}{2}=\dfrac{y+1}{3}=\dfrac{z-1}{4}=\lambda$

$\Rightarrow x=2\lambda+1, y=3\lambda-1$ and $z=4\lambda+1$.

$\dfrac{x-3}{1}=\dfrac{y-k}{2}=\dfrac{z}{1}=\mu$

$\Rightarrow x=3+\mu, y=k+2\mu$ and $z=\mu$.

Since the above lines intersect,

$2\lambda+1=3+\mu$ (i)

$3\lambda-1=2\mu+k$ (ii)

$\mu=4\lambda+1$ (iii)

Solving (i) and (iii) and putting the value of λ and μ in (ii), $k=9/2$.

Exercise 3.3

1. The angle between a line and a plane is complement of the angle between the line and the normal of the plane, i.e., 3, 2, 4 and normal 2, 1, –3. Therefore,

$\cos\theta=\dfrac{6+2-12}{\sqrt{29}\cdot\sqrt{14}}=-\dfrac{4}{\sqrt{406}}$

$\theta=\cos^{-1}(-4/\sqrt{406})$

$\phi=90°-\theta$

$\phi=90°-\cos^{-1}(-4/\sqrt{406})$

$\phi=\sin^{-1}(-4/\sqrt{406})$

2. The line is along the vector $\vec{a}=-3\hat{i}+2\hat{j}+\hat{k}$ and plane is normal to the vector $\vec{b}=\hat{i}+\hat{j}+\hat{k}$.

Since $\vec{a}\cdot\vec{b}=0$, the line is parallel to the plane.

Hence, the distance between the line and the plane is the distance of point (–1, 3, 2) from the plane,

$\dfrac{|-1+3+2+3|}{\sqrt{1+1+1}}=\dfrac{7}{\sqrt{3}}$

3. Any point on the line $\dfrac{x-2}{3}=\dfrac{y+1}{4}=\dfrac{z-2}{12}=\lambda$ is $(3\lambda+2, 4\lambda-1, 12\lambda+2)$.

This lies on $x-y+z=5$.

If $3\lambda+2-4\lambda+1+12\lambda+2=5\Rightarrow\lambda=0$, then the point is (2, –1, 2).

Its distance from (–1, –5, –10) is $\sqrt{(2+1)^2+(-1+5)^2+(2+10)^2}=\sqrt{9+16+144}=13$

4. Since the plane is perpendicular to the given two planes, it is parallel to the normals to the plane or the plane is perpendicular to the vector.

$$(\hat{i}-\hat{j}+\hat{k})\times(2\hat{i}+\hat{j}-\hat{k})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 1\\ 2 & 1 & -1\end{vmatrix}=3\hat{j}+3\hat{k}$$

Also the plane is passing through the point (1, 2, 0); hence the equation of the plane is
$0(x-1)+3(y-2)+3(z-0)=0$ or $y+z-2=0$.

5. The equation of any plane through the point (1, 0, –1) is

$A(x-1)+B(y-0)+C(z+1)=0$ (i)

Since it passes through the point (3, 2, 2), we get

$2A+2B+3C=0$ (ii)

Since plane (i) is parallel to the line $\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-2}{3}$, we have

$1A+(-2)B+3C=0$ (iii)

From (i) and (iii)

$A:B:C=4:-1:-2$

Substituting these values in (i), we get

$4(x-1)-1(y-0)-2(z+1)=0$, i.e., $4x-y-2z-6=0$

6. The required plane is

$$\begin{vmatrix} x-5 & y-7 & z+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}=0$$

$\Rightarrow 17(x-5)-(12+35)(y-7)+(4-28)(z+3)=0$

$\Rightarrow 17x-47y-24z+172=0$

7. Let the equation of a plane containing the line be $l(x-1)+m(y+2)+nz=0$
then $2l-3m+5n=0$ and $l-m+n=0$

$\therefore \frac{l}{2}=\frac{m}{3}=\frac{n}{1}$

$\therefore$ The plane is $2(x-1)+3(y+2)+z=0$

i.e., $2x+3y+z+4=0$

8. Any plane passing through the origin is $a(x-0)+b(y-0)+c(z-0)=0$

This is perpendicular to the given line. Therefore, the normal to the plane is parallel to the given line.

$\Rightarrow \frac{a}{2}=\frac{b}{-1}=\frac{c}{2}$

$\Rightarrow$ The required plane is $2(x-0)-1(y-0)+2(z-0)=0$

$\Rightarrow 2x-y+2z=0$

9. Any plane through $\frac{x-1}{5}=\frac{y+2}{6}=\frac{z-3}{4}$ is

$A(x-1)+B(y+2)+C(z-3)=0$, (i)

where $5A+6B+4C=0$ (ii)

Also, the plane passes through (4, 3, 7). Therefore,

$3A+5B+4C=0$ (iii)

By (ii) and (iii), $\frac{A}{4} = \frac{B}{-8} = \frac{C}{7}$

Therefore, the plane is $4(x-1) - 8(y+2) + 7(z-3) = 0$ or $4x - 8y + 7z = 41$.

10. The given line is $\vec{r} = (\vec{i} + 2\vec{j} - \vec{k}) + \lambda(\vec{i} - \vec{j} + \vec{k})$

Here, $\vec{b} = \vec{i} - \vec{j} + \vec{k}$ (type $\vec{r} = \vec{a} + \lambda\vec{b}$)

The given plane is $\vec{r} \cdot (2\vec{i} - \vec{j} + \vec{k}) = 4$ (type $\vec{r} \cdot \vec{n} = p$)

Here $\vec{n} = 2\vec{i} - \vec{j} + \vec{k}$

Now $\cos\theta = \frac{\vec{n} \cdot \vec{b}}{|\vec{n}||\vec{b}|}$ (If θ is the angle between the line and the normal to the plane)

$$= \frac{(2\vec{i} - \vec{j} + \vec{k}) \cdot (\vec{i} - \vec{j} + \vec{k})}{\sqrt{4+1+1}\sqrt{1+1+1}}$$

$$= \frac{2+1+1}{\sqrt{6}\sqrt{3}} = \frac{4}{\sqrt{2} \cdot 3} = \frac{2\sqrt{2}}{3}$$

$\therefore \quad \theta = \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

11. The plane passes through the point A (1, 2, 3) and is at the maximum distance from point B (–1, 0, 2); then the plane is perpendicular to line AB. Therefore, the direction ratios of the normal to the plane are 2, 2 and 1.

Hence, the equation of the plane is

$2(x-1) + 2(y-2) + 1(z-3) = 0$ or $2x + 2y + z = 9$

12. Any point on the line $\frac{x-1}{1} = \frac{y+1}{-2} = \frac{z-2}{3} = t$ is $(t+1, -2t-1, 3t+2)$, which lies on the given plane if $t + 1 + 2t + 1 + 6t + 4 - 3 = 0$ or $\Rightarrow t = -1/3$.

$\Rightarrow$ The point of intersection of the line and the plane is $P(2/3, -1/3, 1)$

Also, if the foot of the perpendicular from A (1, – 1, 2) on the plane is $Q(x, y, z)$, then

$$\frac{x-1}{1} = \frac{y+1}{-1} = \frac{z-2}{2} = -\frac{(1+1+4-3)}{1+1+4} = -\frac{1}{2}$$

Therefore, $Q(x, y, z)$ is $Q(1/2, -1/2, 1)$.

Hence, the direction ratios of PQ are $\frac{2}{3} - \frac{1}{2}, -\frac{1}{3} + \frac{1}{2}$ and $1 - 1$ or $\frac{1}{6}, \frac{1}{6}$ and 0.

If the image of point A (1, – 1, 2) in the plane is R, then Q is the midpoint of AR. Therefore, point R is (0, 0, 0).

Hence, the direction ratios of PR or the image of the line in the plane are 2/3, – 1/3 and 1.

13. The equation of the plane parallel to $x-2y+2z=5$ is $x-2y+2z+k=0$. (i)

Now, according to the equation,

$$\frac{1-4+6+k}{\sqrt{9}}=\pm 1$$

$k+3=\pm 3 \Rightarrow k=0$ or -6

The $x-2y+2z-6=0$ or $x-2y+2z=6$

14. Plane which is equally inclined to the given planes is parallel to the angle bisector of the given planes.

Now the angle bisector of the given planes is $\frac{x-2y+2z-3}{3}=\pm\frac{8x-4y+z-7}{9}$.

$5x+2y-5z+2=0$ and $11x-10y+7z-16=0$.

The equation of the required planes are $5x+2y-5z+p=0$ and $11x-10y+7z+q=0$.

Since both are passing through point (1, 2, 3), $p=6$ and $q=12$

The planes are $5x+2y-5z+6=0$ and $11x-10y+7z+12=0$

15. The image of the plane $x-2y+2z-3=0$ (i)

in the plane $x+y+z-1=0$ (ii)

passes through the line of intersection of the given planes

Therefore, the equation of such a plane is

$(x-2y+2z-3)+t(x+y+z-1)=0 \quad t\in R$

$(1+t)x+(-2+t)y+(2+t)z-3-t=0$ (iii)

Now plane (ii) makes the same angle with plane (i) and image plane (iii)

$$\Rightarrow \frac{1-2+2}{3\sqrt{3}}=\pm\frac{1+t-2+t+2+t}{\sqrt{3}\sqrt{(t+1)^2+(t-2)^2+(2+t)^2}}$$

$$\frac{1}{3}=\pm\frac{3t+1}{\sqrt{3t^2+2t+9}}$$

$3t^2+2t+9=9(9t^2+6t+1)$

$3t^2+2t+9=81t^2+54t+9$

$78t^2+52t=0$

$t=0$ or $t=-\frac{2}{3}$

For $t=0$, we get plane (i); hence for image plane, $t=-\frac{2}{3}$

The equation of the image plane is $3(x-2y+2z-3)-2(x+y+z-1)=0$

or, $x-8y+4z-7=0$

Exercise 3.4

1. The given spheres are

$x^2+y^2+z^2+2x+2y+2z+2=0$ and (i)

$x^2+y^2+z^2+x+y+z-(1/4)=0$ (ii)

The required plane is $(2x-x)+(2y-y)+(2z-z)+2+\frac{1}{4}=0$

or, $4x+4y+4z+9=0$

2. The radius of the sphere = 5

The given plane is $x + y - z = 4\sqrt{3}$

The length of the perpendiculor from the centre (0, 0, 0) of the sphere on the plane = $\frac{4\sqrt{3}}{\sqrt{1+1+1}} = 4$

Hence radius of the circular section = $\sqrt{25-16} = \sqrt{9} = 3$

3. Since $3PA = 2PB$, we get $9PA^2 = 4PB^2$

$9[(x-1)^2 + (y-3)^2 + (z-4)^2] = 4[x-1^2 + (y+2)^2 + (z+1)^2]$

$9[x^2 + y^2 + z^2 - 2x - 6y - 8z + 26] = 4[c^2 + y^2 + z^2 - 2x + 4y + 2z + 6]$

$5x^2 + 5y^2 - 10x - 70y - 80z + 210 = 0$

$x^2 + y^2 = z^2 - 2x - 14y - 16z + 42 = 0$

This represents a sphere with centre at (1, 7, 8) and radius equal to $\sqrt{1^2 + 7^2 + 8^2 - 42} = \sqrt{72} = 6\sqrt{2}$

4. We are given the extremities of the diameter as (0, 2, 0) and (0, 0, 4). Therefore, the equation of the sphere is $(x-0)(x-0) + (y-2)(y-0) + (z-0)(z-4) = 0$ or $x^2 + y^2 + z^2 - 2y - 4x = 0$. This sphere clearly passes through the origin.

5. Let (α, β, γ) be the foot of the perpendicular from the origin to a plane. Now this plane passes through (α, β, γ) and has direction ratios nomal to the plane as α, β and γ. Therefore, the equation of this plane is given by $\alpha(x-\alpha) + \beta(\gamma-\beta) + \gamma(z-\gamma) = 0$.

This plane will pass through (a, b, c) if $\alpha(a-\alpha) + \beta(b-\beta) + \gamma(c-\gamma) = 0$

$\Rightarrow a\alpha - \alpha + b\beta - \beta^2 + c\gamma - \gamma^2 = 0$

or, $\alpha^2 + \beta^2 + \gamma^2 - a\alpha - b\beta - c\gamma = 0$

Hence, the locus of (α, β, γ) is $x^2 + y^2 + z^2 - ax - by - cz = 0$