Pair Of Linear Equations In Two Variables

- **Inconsistent, Consistent, and Dependent Pairs of Linear Equations**
  Let $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ be a system of linear equations.
  A pair of linear equations in two variables can be solved by

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1. **Graphical method**
2. **Algebraic method**

- **Algebraic method to solve linear equations**

  Case (i) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

  In this case, the given system is consistent.
  This implies that the system has a unique solution.

  Case (ii) $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

  In this case, the given system is inconsistent.
  This implies that the system has no solution.

  Case (iii) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

  In this case, the given system is dependent and consistent.
  This implies that the system has infinitely many solutions.

Example:

Find whether the following pairs of linear equations have unique solutions, no solutions, or infinitely many solutions?

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- $7x + 2y + 8 = 0$

  $14x + 4y + 16 = 0$
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- $2x + 3y - 10 = 0$

$5x - 2y - 6 = 0$

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- $3x - 8y + 12 = 0$

$6x - 16y + 14 = 0$

Solution:

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- $7x + 2y + 8 = 0$

$14x + 4y + 16 = 0$
Here, $a_1 = 7$, $b_1 = 2$, $c_1 = 8$
$a_2 = 14$, $b_2 = 4$, $c_2 = 16$
Now,

$$\frac{a_1}{a_2} = \frac{7}{14} = \frac{1}{2}, \frac{b_1}{b_2} = \frac{2}{4} = \frac{1}{2}, \frac{c_1}{c_2} = \frac{8}{16} = \frac{1}{2}$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Therefore, the given system has infinitely many solutions.

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- $2x + 3y - 10 = 0$

$5x - 2y - 6 = 0$
Here, $a_1 = 2$, $b_1 = 3$, $c_1 = -10$
$a_2 = 5$, $b_2 = -2$, $c_2 = -6$
Now,

$$\frac{a_1}{a_2} = \frac{2}{5}, \frac{b_1}{b_2} = \frac{3}{-2}$$

$$\therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

Therefore, the given system has a unique solution.

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- $3x - 8y + 12 = 0$

$6x – 16y + 14 = 0$
Here, $a_1 = 3$, $b_1 = –8$, $c_1 = 12$
$a_2 = 6$, $b_2 = – 16$, $c_2 = 14$
Now,

$$\frac{a_1}{a_2}=\frac{3}{6}=\frac{1}{2},\frac{b_1}{b_2}=\frac{-8}{-16}=\frac{1}{2},\frac{c_1}{c_2}=\frac{12}{14}=\frac{6}{7}$$

$$\therefore \frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$$

Therefore, the given system has no solutions.

- **Graphical representation of linear equations**
    - **Consistent system**
      A system of simultaneous linear equations is said to be consistent if it has at least one solution.
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- **Inconsistent system**
  A system of simultaneous linear equations is said to be inconsistent if it has no solution.

Nature of solution of simultaneous linear equations based on graph:

Case (i): The lines intersect at a point.
The point of intersection is the unique solution of the two equations. In this case, the pair of equations is **consistent**.

Case (ii): The lines coincide.
The pair of equations has infinitely many solutions – each point on the line is a solution. In this case, the pair of equations is dependent (which is **consistent**).

Case (iii): The lines are parallel.
The pair of equations has no solution. In this case, the pair of equations is **inconsistent**.

Example 1:
Show graphically that the system of equations $2x + 3y = 10$, $2x + 5y = 12$ is consistent.
Solution:
The given system of linear equations is
$2x + 3y = 10$ … (1)
$2x + 5y = 12$ … (2)

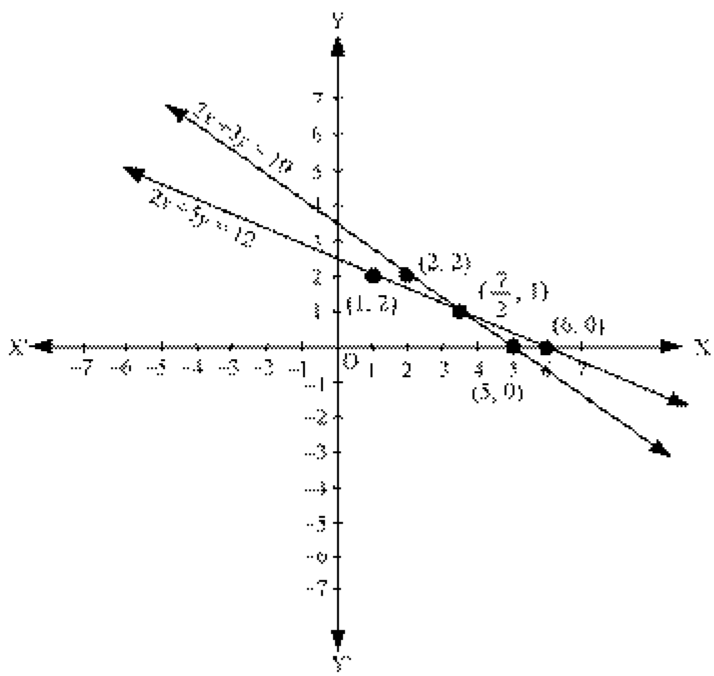
Table for equation (1)

x	5	2
y	0	2

Table for equations (2)

x	6	1
y	0	2

By plotting and joining the points (5, 0) and (2, 2), the line representing (1) can be obtained. Similarly, by plotting and joining the points (6, 0) and (1, 2), the line representing (2) can be obtained.



It is seen that the two lines intersect at point $\left(\frac{7}{2}, 1\right)$.

Therefore, the given system of equations is consistent and has a unique solution $\left(\frac{7}{2}, 1\right)$.

Example 2:

Show graphically that the system of equations $3x - 6y + 9 = 0$, $2x - 4y + 6 = 0$ has infinitely many solutions (that is, inconsistent).

Solution:

The given system of equations is

$3x - 6y + 9 = 0$ … (1)

$2x - 4y + 6 = 0$ … (2)

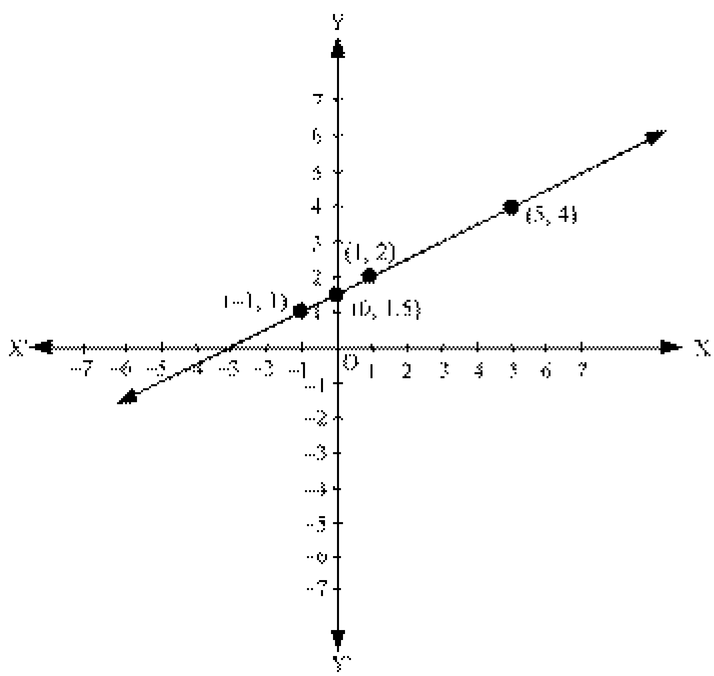
Table for equation (1)

x	–1	5
y	1	4

Table for equation (2)

x	0	1
y	1.5	2

By plotting and joining (–1, 1), (5, 4), the line representing equation (1) can be obtained.
By plotting and joining (0, 1.5), (1, 2), the line representing equation (2) can be obtained.



It is observed that the graphs of the two equations are coincident. Therefore, the given system of equations has infinitely many.

- **Expressing given situations mathematically**

Example:
Raj and Rohan each has certain number of chocolates. Raj says to Rohan, if you give me 6 of your chocolates, I will have twice the number of chocolates left with you. Rohan replies, if you give me 4 of your chocolates, I will have the same number of chocolates as left with you. Write this situation mathematically?

Solution:
Suppose Raj has x number of chocolates and Rohan has y number of chocolates.
According to the first condition, Rohan gives 6 chocolates to Raj so that Raj has twice the number of chocolates than what Rohan has.
$\Rightarrow x + 6 = 2(y - 6)$ … (1)
According to the second condition, Raj gives 4 chocolates to Rohan such that both have equal number of chocolates.
$\Rightarrow y + 4 = x - 4$ … (2)
Thus, the equation 1 and 2 are the algebraic representation of the given situation.

- **Solving given pairs of linear equations in two variables graphically:**

Example:
Solve the following system of linear equations graphically.
$x + y + 2 = 0$, $2x - 3y + 9 = 0$
Hence, find the area bounded by these two lines and the line $x = 0$
Solution:
The given equations are
$x + y + 2 = 0$ (1)
$2x - 3y + 9 = 0$ (2)

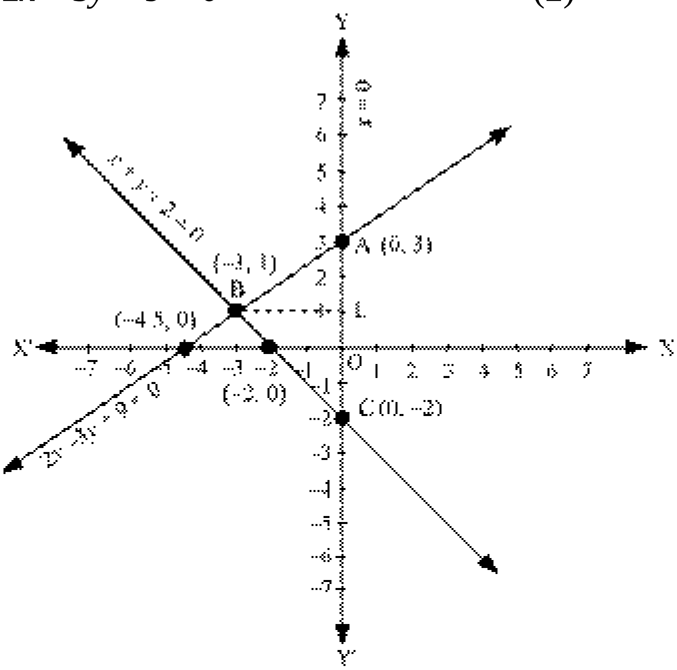


Table for the equations $x + y + 2 = 0$

x	0	–2
y	–2	0

Table for the equation $2x - 3y + 9 = 0$

x	0	–4.5

y	3	0

By plotting and joining the points (0, –2) and (–2, 0), the line representing equation (1) is obtained.
By plotting and joining the points (0, 3) and (–4.5, 0), the line representing equation (2) is obtained.
It is seen that the two lines intersect at point B (–3, 1).
Solution of the given system of equation is (–3, 1)
Area bound by the two lines and $x = 0$
= Area of ΔABC

$$= \frac{1}{2} \times AC \times BL = \frac{1}{2} \times 5 \times 3 \text{ square units} = 7.5 \text{ square units}$$

- **Substitution Method of Solving Pairs of Linear Equations**

In this method, we have **substituted** the value of one variable by expressing it in terms of the other variable to solve the pair of linear equations. That is why this method is known as the **substitution method.**

Example:

Solve the following system of equations by substitution method.
$x - 4y + 7 = 0$
$3x + 2y = 0$

Solution:

The given equations are
$x - 4y + 7 = 0$ … (1)
$3x + 2y = 0$ … (2)
From equation (2),
$3x = -2$

$$\Rightarrow x = -\frac{2}{3}y$$

Put $x = -\frac{2}{3}y$ in equation (1)

$$-\frac{2}{3}y - 4y + 7 = 0$$

$$\Rightarrow \frac{-2y - 12y}{3} = -7$$

$$\Rightarrow -14y = -21$$

$$\Rightarrow y = \frac{-21}{-14} = \frac{3}{2}$$

$$\therefore x = -\frac{2}{3}\left(\frac{3}{2}\right) = -1$$

Therefore, the required solution is $\left(-1, \frac{3}{2}\right)$.

- **Elimination Method to Solve a Pair of Linear Equations**

  Example:

  Solve the following pair of linear equations by elimination method.

  $7x - 2y = 10$

  $5x + 3y = 6$

  Solution:

  $7x - 2y = 10$ … (1)

  $5x + 3y = 6$ … (2)

  Multiplying equation (1) by 5 and equation (2) by 7, we get

  $35x - 10y = 50$ … (3)

  $35x + 21y = 42$ … (4)

  Subtracting equation (4) from (3), we get

  $-31y = 8 \Rightarrow y = -\frac{8}{31}$

  Now, using equation (1):

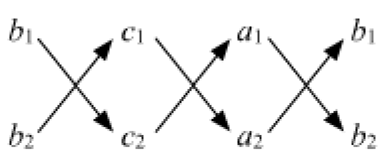
  $7x = 10 + 2y$

  $\Rightarrow x = \frac{1}{7}\left\{10 + 2 \times \frac{-8}{31}\right\} = \frac{42}{31}$.

  Required solution is $\left(\frac{42}{31}, -\frac{8}{31}\right)$.

- **Cross-Multiplication Method of Solving Pairs of Linear Equations**

  The solution of the system of linear equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ can be determined by the following diagram.



  That is,

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\Rightarrow x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \qquad (a_1b_2 - a_2b_1 \neq 0)$$

  Example:

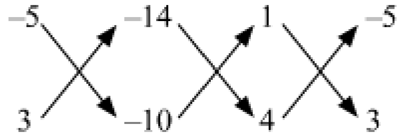
Solve the following pair of linear equations by the cross-multiplication method

$x - 5y = 14$, $4x + 3y = 10$

Solution:

$x - 5y - 14 = 0$

$4x + 3y - 10 = 0$



$$\frac{x}{(-5)\times(-10)-3\times(-14)}=\frac{y}{(-14)\times4-(-10)\times1}=\frac{1}{1\times3-4\times(-5)}$$

$$\Rightarrow\frac{x}{50+42}=\frac{y}{-56+10}=\frac{1}{3+20}$$

$$\Rightarrow\frac{x}{92}=\frac{y}{-46}=\frac{1}{23}$$

$$\Rightarrow x=\frac{92}{23}=4, y=-\frac{46}{23}=-2$$

∴ Required solution is (4, – 2).

Equations reducible to a pair of linear equations in two variables

Some pair of equations which are not linear can be reduced to linear form by suitable substitutions.

Example: Solve the following system of equations

$$\frac{2}{x-2}-\frac{1}{y-1}=1$$

$$\frac{5}{x-2}-\frac{6}{y-1}=20$$

Solution:

Let $\frac{1}{x-2}=u$ and $\frac{1}{y-1}=v$. Then, the given system of equations reduces to

$2u - v = 1$ … (1)

$5u - 6v = 20$ … (2)

Multiplying equation (1) by 6 and then subtracting from (2), we get

$$\begin{aligned} 5u - 6v &= 20 \\ 12u - 6v &= 6 \\ - \quad + \quad & \quad - \\ \hline -7u &= 14 \end{aligned}$$

$$\Rightarrow u = \frac{14}{-7} = -2$$

Equation (1) $\Rightarrow$ $v = 2u - 1$

$= 2(-2) - 1$

$= -4 - 1 = -5$

$$\therefore \frac{1}{x-2} = -2, \frac{1}{y-1} = -5$$

$$\Rightarrow \quad x - 2 = -\frac{1}{2} \quad \Rightarrow \quad y - 1 = -\frac{1}{5}$$

$$\Rightarrow \quad x = 2 - \frac{1}{2} = \frac{3}{2} \quad \Rightarrow \quad y = 1 - \frac{1}{5} = \frac{4}{5}$$

$\therefore \left(\frac{3}{2}, \frac{4}{5}\right)$ is the solution of the given system of equations.