EXERCISE 7.6

Integrate the following:

QNo. 1 $x \sin x$.

Sol. $\int \underset{I}{x} \underset{II}{\sin x}\, dx = x \int \sin x\, dx - \int \left(\frac{d}{dx}(x) \int \sin x\, dx \right) dx + C$

$= x(-\cos x) - \int 1(-\cos x)\, dx + C$

$= -x\cos x + \sin x + C$

QNo 2. $x \sin 3x$.

Sol. $\int \underset{1}{x} \underset{2}{\sin 3x}\, dx = x \int \sin 3x - \int \left(\frac{d}{dx}(x) \int \sin 3x \right) dx + C$

$= x\left(\frac{-\cos 3x}{3}\right) - \int 1 \times \left(\frac{-\cos 3x}{3}\right) dx + C$

$= -\frac{x \cos 3x}{3} + \frac{\sin 3x}{9} + C$

QNo 3 $x^2 e^x$

Sol. $\int \underset{1}{x^2} \underset{2}{e^x}\, dx = x^2 \int e^x dx - \int \left(\left(\frac{d}{dx}(x^2)\right) \int e^x dx \right) dx + C$

$= x^2 e^x - \int 2x e^x dx + C$

Again Integrating taking x as first fn.

$= x^2 e^x - 2\left[x \int e^x - \int \left(\frac{d}{dx}(x) \int e^x dx \right) dx \right] + C$

$= x^2 e^x - 2x e^x + 2 \int e^x dx + C$

$= x^2 e^x - 2x e^x + 2e^x + C = (x^2 - 2x + 2)e^x + C.$

QNo 4. $x \log x$.

Sol. Integrate by parts taking $\log x$ as first fn.

$\int x \log x\, dx = \log x \left(\frac{x^2}{2}\right) - \int \frac{1}{x} \cdot \frac{x^2}{2} dx.$

$= \frac{x^2}{2} \log x - \frac{1}{2} \int x\, dx + C$

$= \frac{x^2}{2} \log x - \frac{1}{2} \frac{x^2}{2} + C = \frac{x^2}{2} \log x - \frac{1}{4} x^2 + C$

QNo. 5. $x \log 2x$.

$\int x \log 2x\, dx = \log 2x \int x\, dx - \int \left(\frac{d}{dx}(\log 2x) \int x\, dx \right) dx.$

$= \log 2x \frac{x^2}{2} \quad \int \frac{1}{2x} \cdot 2 \cdot \frac{x^2}{2} dx + C$

$= \frac{x^2}{2} \log 2x - \frac{1}{2} \int x\, dx + C$

$= \frac{x^2}{2} \log 2x - \frac{1}{2} \cdot \frac{x^2}{2} + C = \frac{x^2}{2} \log 2x - \frac{x^2}{4} + C$

Q No 6. $x^2 \log x$

$$\int \underbrace{x^2}_{2} \underbrace{\log x}_{1}\, dx = \log x \int x^2 dx - \int\left(\frac{d}{dx}(\log x)\int x^2 dx\right)dx$$

$$= \log x \cdot \frac{x^3}{3} - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx = \frac{x^3}{3}\log x - \frac{1}{3}\int x^2 dx + c$$

$$= \frac{x^3}{3}\log x - \frac{1}{3}\cdot\frac{x^3}{3} + c = \frac{x^3}{3}\left(\log x - \frac{1}{3}\right) + c$$

Q No 7 $x \sin^{-1} x$.

$$\int \underbrace{x}_{2}\, \underbrace{\sin^{-1}x}_{1}\, dx = \sin^{-1}x \int x\, dx - \int\left(\frac{d}{dx}(\sin^{-1}x)\int x\, dx\right)dx$$

$$= \sin^{-1}x \cdot \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{1}{2}x^2\sin^{-1}x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$$= \frac{1}{2}x^2\sin^{-1}x + \frac{1}{2}\int \frac{-x^2+1-1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{1}{2}x^2\sin^{-1}x + \frac{1}{2}\int \sqrt{1-x^2}\,dx - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{1}{2}x^2\sin^{-1}x + \frac{1}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1^2}{2}\sin^{-1}\frac{x}{1}\right] - \frac{1}{2}\sin^{-1}x + c$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} + \frac{1}{4}\sin^{-1}x - \frac{1}{2}\sin^{-1}x + c$$

$$= \frac{1}{4}\left[\sin^{-1}x(2x^2-1) + x\sqrt{1-x^2}\right] + c$$

Q No. 8 $x \tan^{-1} x$

$$\int \underbrace{x}_{2}\, \underbrace{\tan^{-1}x}_{1}\, dx = \tan^{-1}x \cdot \int x\, dx - \int\left(\frac{d}{dx}(\tan^{-1}x)\int x\, dx\right)dx$$

$$= \tan^{-1}x \cdot \frac{x^2}{2} - \int \frac{1}{1+x^2}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int 1\cdot dx + \frac{1}{2}\int \frac{1}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + c$$

$$= \frac{1}{2}\left[(x^2+1)\tan^{-1}x - x\right] + c$$

Q No 9 $x\cos^{-1}x$

Sol. Same as Q No. 7.

Alternatively

Put $\cos^{-1}x = \theta \Rightarrow x = \cos\theta$

$\therefore dx = -\sin\theta \, d\theta$.

$$\therefore I = \int \cos\theta \cdot \theta(-\sin\theta)\,d\theta = -\frac{1}{2}\int \theta \cdot 2\sin\theta\cos\theta \cdot d\theta$$

$$= -\frac{1}{2}\int \underset{1}{\theta} \cdot \underbrace{\sin 2\theta}_{2}\, d\theta$$

$$= -\frac{1}{2}\left[\theta \cdot \int \sin 2\theta \cdot d\theta - \int\left(\frac{d}{d\theta}(\theta)\int \sin 2\theta \cdot d\theta\right)d\theta\right.$$

$$= -\frac{1}{2}\left[\theta\left(-\frac{\cos 2\theta}{2}\right) - \int 1\times\left(-\frac{\cos 2\theta}{2}\right)d\theta\right] + c$$

$$= \frac{\theta\cos 2\theta}{4} - \frac{1}{8}\sin 2\theta + c$$

$$= \frac{1}{4}\theta\left(2\cos^2\theta - 1\right) - \frac{1}{8}\left(2\sin\theta\cos\theta\right) + c.$$

$$= \frac{1}{4}\left(\cos^{-1}x\right)\left(2x^2-1\right) - \frac{1}{4}\sin\left(\cos^{-1}x\right)x + c.$$

$$= \left(\frac{2x^2-1}{4}\right)\cos^{-1}x - \frac{x}{4}\sqrt{1-x^2} + c \quad \left[\because \sin\left(\cos^{-1}x\right) = \sqrt{1-x^2}\right]$$

Q No 10 $(\sin^{-1}x)^2$.

Sol. Put $\sin^{-1}x = \theta \Rightarrow x = \sin\theta \Rightarrow dx = \cos\theta \, d\theta$

$$\therefore I = \int (\sin^{-1}x)^2 dx = \int \theta^2 \cos\theta \cdot d\theta$$

Integrating by parts taking θ^2 as first function.

$$I = \theta^2\int\cos\theta\, d\theta - \int\left(\frac{d}{d\theta}(\theta^2)\int\cos\theta\, d\theta\right)d\theta$$

$$= \theta^2\sin\theta - \int 2\theta\sin\theta\, d\theta$$

Again integrate by parts taking 'θ' as first fn.

$$= \theta^2\sin\theta - 2\left\{\theta(-\cos\theta) - \int 1(-\cos\theta)\,d\theta\right] + c$$

$$= \theta^2\sin\theta + 2\theta\cdot\cos\theta - 2\sin\theta + c$$

$$= (\sin^{-1}x)^2 x + 2\sin^{-1}x \cos(\sin^{-1}x) - 2x + c$$
$$= x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\,\sin^{-1}x. - 2x + c$$

Q No 11. $\dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}$

Sol: Put $\cos^{-1}x = \theta \Rightarrow x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$.

$$\therefore I = \int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}\,dx = \int \frac{\cos\theta\cdot\theta\cdot(-\sin\theta)\,d\theta}{\sqrt{1-\cos^2\theta}} = \int \frac{-\theta\cdot\sin\theta\cos\theta\, d\theta}{\sin\theta}$$
$$= -\int \theta\cos\theta\, d\theta.$$

Integrating by parts taking θ as first function

$$= -\left\{\theta\cdot\sin\theta - \int 1\cdot\sin\theta\, d\theta\right\} + c.$$
$$= -\theta\cdot\sin\theta + \int \sin\theta\cdot d\theta + c.$$
$$= -\theta\,\sin\theta - \cos\theta + c.$$
$$= -\theta\sqrt{1-\cos^2\theta} - \cos\theta + C = -\cos^{-1}x\cdot\sqrt{1-x^2} - x + C.$$

Q No 12 $x\sec^2 x$

Sol: Integrating by parts taking x as first fn.

$$\int x\sec^2 x\,dx = x\cdot\int \sec^2 x\,dx - \int\left(\frac{d}{dx}(x)\int \sec^2 x\,dx\right)dx$$
$$= x\cdot\tan x - \int 1\cdot\tan x\,dx.$$
$$= x\tan x - (-\log|\cos x|) + c$$
$$= x\tan x + \log|\cos x| + c.$$

Q No 13 $\tan^{-1}x. = \tan^{-1}x \times 1$

Integrating by parts taking $\tan^{-1}x$ as first fn.

$$\int \tan^{-1}x \times 1\,dx = \tan^{-1}x\int 1\cdot dx - \int\left(\frac{d}{dx}(\tan^{-1}x)\int 1\cdot dx\right)dx$$
$$= \tan^{-1}x\cdot x - \int \frac{1}{1+x^2}\cdot x\,dx$$

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$

$$= x\tan^{-1}x - \frac{1}{2}\log|1+x^2| + c.$$

Q.No 14. $x(\log x)^2$.

Sol.- Integrating by parts taking $(\log x)^2$ as first function.

$$\int x(\log x)^2 dx = (\log x)^2\int x\,dx - \int\left(\frac{d}{dx}(\log x)^2\int x\,dx\right)dx.$$

$$= (\log x)^2\cdot\frac{x^2}{2} - \int \frac{2\log x}{x}\times\frac{x^2}{2}\,dx + c$$

$$= \frac{x^2}{2}(\log x)^2 - \int x\log x\,dx + c$$

$$= \frac{x^2}{2}(\log x)^2 - \left(\frac{x^2}{2}\log x - \frac{1}{4}x^2\right) + c \quad \left[\text{As in Q.No 4}\right]$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4} + c$$

Q.No 15. $(x^2+1)\log x$.

Sol. Integrating by parts taking $\log x$ as first function.

$$\int (x^2+1)\log x\,dx = \log x\int(x^2+1)dx - \int\left(\frac{d}{dx}(\log x)\int(x^2+1)dx\right)dx$$

$$= \log x\left(\frac{x^3}{3}+x\right) - \int\frac{1}{x}\left(\frac{x^3}{3}+x\right)dx + c$$

$$= \log x\left(\frac{x^3}{3}+x\right) - \frac{1}{3}\int x^2 dx - \int 1\,dx + c$$

$$= \left(\frac{x^3}{3}+x\right)\log x - \frac{1}{3}\cdot\frac{x^3}{3} - x + c$$

$$= \left(\frac{x^3}{3}+x\right)\log x - \frac{x^3}{9} - x + c$$

Q.No 16. $e^x(\sin x + \cos x)$

Sol. $$\int e^x(\sin x + \cos x)\,dx = \int e^x\left(\sin x + \frac{d}{dx}(\sin x)\right)dx$$

$$= e^x\sin x + c.$$

$$\because \int e^x(f(x) + f'(x))\,dx = e^x f(x) + c.$$

Q No 17. $\dfrac{x e^x}{(1+x)^2}$

Sol $I = \int \dfrac{x e^x}{(1+x)^2}\,dx = \int e^x \left(\dfrac{1+x-1}{(1+x)^2}\right) dx = \int e^x \left(\dfrac{1}{1+x} - \dfrac{1}{(1+x)^2}\right) dx.$

Now $\dfrac{d}{dx}\left(\dfrac{1}{1+x}\right) = \dfrac{d}{dx}(1+x)^{-1} = (-1)(1+x)^{-2}(0+1) = \dfrac{-1}{(1+x)^2}$

$\therefore\ I = \int e^x \left(\dfrac{1}{1+x} + \dfrac{d}{dx}\left(\dfrac{1}{1+x}\right)\right) dx.$

$= e^x \left(\dfrac{1}{1+x}\right) + c \qquad \left[\because \int e^x(f(x)+f'(x))\,dx = e^x f(x)+c\right.$

Q No 18. $e^x \left(\dfrac{1+\sin x}{1+\cos x}\right)$

Sol. $\int e^x \left(\dfrac{1+\sin x}{1+\cos x}\right) dx = \int e^x \left(\dfrac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right) dx$

$= \int e^x \left(\dfrac{1}{2\cos^2\frac{x}{2}} + \dfrac{\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2 x/2}\right) dx$

$= \int e^x \left(\dfrac{1}{2}\sec^2\dfrac{x}{2} + \tan\dfrac{x}{2}\right) dx = \int e^x \left(\tan\dfrac{x}{2} + \dfrac{d}{dx}\left(\tan\dfrac{x}{2}\right)\right) dx$

$= e^x \tan\dfrac{x}{2} + c$

Q No 19 $e^x\left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)$

Sol. $\int e^x \left(\dfrac{1}{x} - \dfrac{1}{x^2}\right) dx = \int e^x \left(\dfrac{1}{x} + \dfrac{d}{dx}\left(\dfrac{1}{x}\right)\right) dx.$

$= e^x \cdot \dfrac{1}{x} + c \qquad \left[\because \int e^x(f(x)+f'(x))\,dx = e^x f(x) + c\right.$

Q No 20. $\dfrac{(x-3)e^x}{(x-1)^3}$

Sol. $I = \int e^x \left(\dfrac{x-3}{(x-1)^3}\right) dx = \int e^x \dfrac{x-1-2}{(x-1)^3}\,dx$

$$= \int e^x \left(\frac{x-1}{(x-1)^3} - \frac{2}{(x-1)^3} \right) dx$$

$$= \int e^x \left(\frac{1}{(x-1)^2} + \frac{-2}{(x-1)^3} \right) dx$$

$$= \int e^x \left(\frac{1}{(x-1)^2} + \frac{d}{dx}\left(\frac{1}{(x-1)^2}\right) \right) dx \quad \left[\because \frac{d}{dx}\left(\frac{1}{(x-1)^2}\right) = \frac{d}{dx}(x-1)^{-2} = -2(x-1)^{-3}(1-0) = \frac{-2}{(x-1)^3} \right]$$

$$= \frac{e^x}{(x-1)^2} + c \quad \left[\because \int e^x (f(x) + f'(x)) dx = e^x f(x) + c \right.$$

Q No 21. $e^{2x} \sin x.$

Sol. $I = \int \underset{1}{e^{2x}} \underset{2}{\sin x}\, dx. = e^{2x} \int \sin x\, dx - \int \left(\frac{d}{dx}(e^{2x}) \int \sin x\, dx \right) dx$

$$= e^{2x}(-\cos x) - \int e^{2x} . 2(-\cos x)\, dx$$

$$= -e^{2x} \cos x + 2 \int e^{2x} \cos 2x\, dx.$$

$$= -e^{2x} \cos x + 2 \left[e^{2x} \int \cos x\, dx - \int \left(\frac{d}{dx}(e^{2x}) \int \cos x\, dx \right) dx \right.$$

$$= -e^{2x} \cos x + 2 \left[e^{2x} \sin x - \int e^{2x} . 2 \sin x . dx \right]$$

$$= -e^{2x} \cos x + 2 e^{2x} \sin x. - 4 \int e^{2x} \sin x\, dx + C$$

$$\therefore \quad I = -e^{2x} \cos x + 2 e^{2x} \sin x - 4I + C$$

$$\Rightarrow \quad 5I = -e^{2x} \cos x + 2 e^{2x} \sin x + c.$$

$$\therefore \quad I = \frac{e^{2x}}{5} \left[2 \sin x - \cos x \right] + c' \quad \text{where } c' = c/5 \text{ . is const.}$$

Q No 22 $\sin^{-1}\left(\frac{2x}{1+x^2} \right)$

Sol. $I = \int \sin^{-1}\left(\frac{2x}{1+x^2} \right) dx$

Put $x = \tan\theta \Rightarrow dx = \sec^2\theta \cdot d\theta$

$$\therefore \quad I = \int \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta} \right) . \sec^2\theta \cdot d\theta.$$

$$= \int \sin^{-1}(\sin 2\theta) \cdot \sec^2\theta \, d\theta = \int 2\theta \cdot \sec^2\theta \cdot d\theta$$

$$= 2\left[\theta \cdot \int \sec^2\theta \, d\theta - \int\left(\frac{d}{d\theta}(\theta) \int \sec^2\theta \, d\theta\right) d\theta\right]$$

$$= 2\left[\theta \cdot \tan\theta - \int \tan\theta \cdot d\theta\right] + c$$

$$= 2\left[\theta \cdot \tan\theta - \left(-\log|\cos\theta|\right)\right] + c$$

$$= 2\left[\theta \cdot \tan\theta + \log\left[\frac{1}{\sec\theta}\right]\right] + c$$

$$= 2\left[\theta \cdot \tan\theta + \log\left(\frac{1}{\sqrt{1+\tan^2\theta}}\right)\right] + c$$

$$= 2\left[\theta \cdot \tan\theta + \log\left(1+\tan^2\theta\right)^{-1/2}\right] + c$$

$$= 2\left[\theta \cdot \tan\theta - \frac{1}{2}\log\left(1+\tan^2\theta\right)\right] + c$$

$$= 2 \cdot \tan^{-1}x \cdot x - \log(1+x^2) + c$$

Choose the Correct answer in Q No 23 and 24.

Q No 23. $\int x^2 e^{x^3} dx$ equals

(A) $\frac{1}{3} e^{x^3} + c$ (B) $\frac{1}{3} e^{x^2} + c$ (C) $\frac{1}{2} e^{x^3} + c$ (D) $\frac{1}{2} e^{x^2} + c$

Sol. $I = \int x^2 e^{x^3} dx$ Put $x^3 = t \Rightarrow 3x^2 dx = dt$

$$\therefore I = \int e^t \frac{dt}{3} = \frac{1}{3}\int e^t dt = \frac{1}{3} e^t + c = \frac{1}{3} e^{x^3} + c$$

$\therefore$ Correct option is (A)

Q No 24. $\int e^x \sec x (1+\tan x)$ equals.

(A) $e^x \cos x + c$ (B) $e^x \sec x + c$ (C) $e^x \sin x + c$ (D) $e^x \tan x + c$

Sol. $\because \int e^x \sec x (1+\tan x) dx = \int e^x (\sec x + \sec x \cdot \tan x) dx$

$$= \int e^x \left(\sec x + \frac{d}{dx}(\sec x)\right) dx = e^x \sec x + c$$

$\therefore$ Correct option is B.