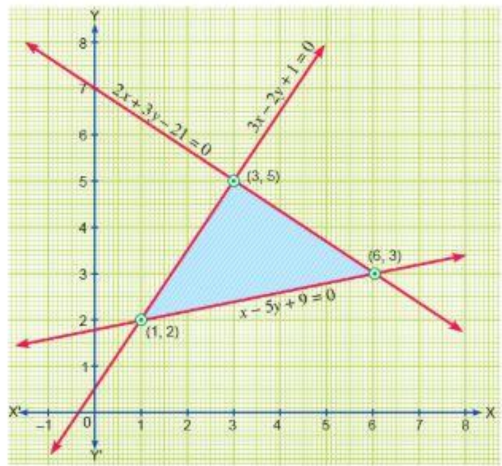
Long Answer Question-II (PYQ)

[6 marks]

Q.1. Using the method of integration, find the area of the region bounded by the lines $3x - 2y + 1 = 0$, $2x + 3y - 21 = 0$ and $x - 5y + 9 = 0$.

Ans.



Given lines are

$$3x - 2y + 1 = 0 \quad ...(i)$$

$$2x + 3y - 21 = 0 \quad ...(ii)$$

$$x - 5y + 9 = 0 \quad ...(iii)$$

For intersection of (*i*) and (*ii*)

Applying $(i) \times 3 + (ii) \times 2$, we get

$$9x - 6y + 3 + 4x + 6y - 42 = 0$$

$$\Rightarrow 13x - 39 = 0 \Rightarrow x = 3$$

Putting it in (*i*), we get

$$9 - 2y + 1 = 0$$

$$\Rightarrow 2x = 10 \Rightarrow y = 5$$

Intersection point of (*i*) and (*ii*) is (3, 5)

For intersection of (*ii*) and (*iii*)

Applying $(ii) - (iii) \times 2$, we get

$$2x + 3y - 21 - 2x + 10y - 18 = 0$$

$$\Rightarrow 13y - 39 = 0 \Rightarrow y = 3$$

Putting $y = 3$ in (*ii*), we get

$2x + 9 - 21 = 0$

$\Rightarrow 2x - 12 = 0 \Rightarrow x = 6$

Intersection point of (ii) and (iii) is $(6, 3)$

For intersection of (i) and (iii)

Applying $(i) - (iii) \times 3$, we get

$3x - 2y + 1 - 3x + 15y - 27 = 0$

$\Rightarrow 13y - 26 = 0 \quad \Rightarrow y = 2$

Putting $y = 2$ in (i), we get

$3x - 4 + 1 = 0 \quad \Rightarrow x = 1$

Intersection point of (i) and (iii) is $(1, 2)$.

With the help of point of intersection we draw the graph of lines (i), (ii) and (iii)

Shaded region is required region.

$$\therefore \text{Area of required region} = \int_1^3 \frac{3x+1}{2} dx + \int_3^6 \frac{-2x+21}{3} dx - \int_1^6 \frac{x+9}{5} dx$$

$$= \frac{3}{2}\int_1^3 x\,dx + \frac{1}{2}\int_1^3 dx - \frac{2}{3}\int_3^6 x\,dx + 7\int_3^6 dx - \frac{1}{5}\int_1^6 x\,dx - \frac{9}{5}\int_1^6 dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2}\right]_1^3 + \frac{1}{2}[x]_1^3 - \frac{2}{3}\left[\frac{x^2}{2}\right]_3^6 + 7[x]_3^6 - \frac{1}{5}\left[\frac{x^2}{2}\right]_1^6 - \frac{9}{5}[x]_1^6$$

$$= \frac{3}{4}(9-1) + \frac{1}{2}(3-1) - \frac{2}{6}(36-9) + 7(6-3) - \frac{1}{10}(36-1) - \frac{9}{5}(6-1)$$

$$= 6 + 1 - 9 + 21 - \frac{7}{2} - 9 = 10 - \frac{7}{2} = \frac{20-7}{2} = \frac{13}{2} sq \text{ sq units.}$$

Q.2. Find the area of the region {(*x*, *y*) : $x^2 + y^2 \leq 4$, $x + y \geq 2$}.

Ans.

Let $R = \{(x, y) : x^2 + y^2 \leq 4, x + y \geq 2\}$

$\Rightarrow \quad R = \{(x, y) : x^2 + y^2 \leq 4\} \cap \{(x, y) : x + y \geq 2\}$

i.e., $\quad R = R^1 \cap R^2$, where

$R_1 = \{(x, y) : x^2 + y^2 \leq 4\}$ and $R_2 = \{(x, y) : x + y \geq 2\}$

For region R_1

Obviously $x^2 + y^2 = 4$ is a circle having centre at $(0,0)$ and radius 2.

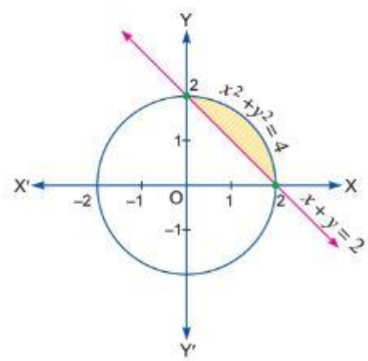
Since $(0,0)$ satisfy $x^2 + y^2 \leq 4$. Therefore region R_1 is the region lying interior of circle $x^2 + y^2 = 4$

For region R

x	0	2
y	2	0

$x + y = 2$ is a straight line passing through $(0, 2)$ and $(2, 0)$. Since $(0, 0)$ does not satisfy $x + y \geq 2$ therefore R_2 is that region which does not contain origin $(0, 0)$ *i.e.,* above the line $x + y = 2$

Hence, shaded region is required area.



Now, area of required region

$$= \int_0^2 \sqrt{4 - x^2}\,dx - \int_0^2 (2 - x)dx$$

$$= \left[\frac{1}{2}x\sqrt{4 - x^2} + \frac{1}{2}4\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 - 2[x]_0^2 + \left[\frac{x^2}{2}\right]_0^2$$

$$= [2\sin^{-1}1 - 0] - 2[2 - 0] + \left[\frac{4}{2} - 0\right]$$

$$= 2 \times \frac{\pi}{2} - 4 + 2 = (\pi - 2) \text{ sq units}$$

Q.3. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.

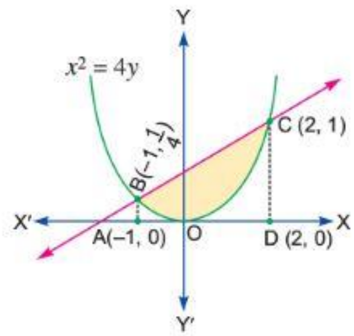
Ans.

Given curves are

$$x^2 = 4y \qquad \ldots(i)$$

and $$x = 4y - 2 \qquad \ldots(ii)$$

Obviously, curve (i) is a parabola with vertex at $(0, 0)$ and axis along +ve direction of y-axis, while equation (ii) represents a straight line.



For points of intersection, we solve (i) and (ii) as

$$x^2 = x + 2$$

$\Rightarrow \quad x^2 - x - 2 = 0 \quad \Rightarrow \quad x^2 - 2x + x - 2 = 0$

$\Rightarrow \quad x(x-2) + 1(x-2) = 0 \quad \Rightarrow \quad (x-2)(x+1) = 0$

$\Rightarrow \quad x = -1, 2 \quad \Rightarrow \quad y = \frac{1}{4}, 1$

Hence, coordinates of intersection points of line and parabola are $\left(-1, \frac{1}{4}\right)$ and $(2, 1)$.

Shaded region is required region.

$\therefore$ Required area= area of trap. $ABCD$ – area of $ABOCDOA$

$$= \int_{-1}^{2}\left(\frac{x+2}{4}\right)dx - \int_{-1}^{2}\frac{x^2}{4}dx = \frac{1}{4}\left[\frac{(x+2)^2}{2}\right]_{-1}^{2} - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{8}[16-1] - \frac{1}{12}(8+1)$$
$$= \frac{15}{8} - \frac{9}{12} = \frac{27}{24} = \frac{9}{8} \text{ sq units.}$$

Q.3. Find the area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$.

Ans.

Given curves are

$$x^2 = 4y \qquad \ldots(i)$$

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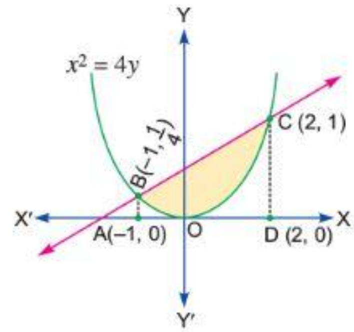
Obviously, curve (i) is a parabola with vertex at $(0, 0)$ and axis along +ve direction of y-axis, while equation (ii) represents a straight line.



For points of intersection, we solve (i) and (ii) as

$$x^2 = x + 2$$

$\Rightarrow \quad x^2 - x - 2 = 0 \quad \Rightarrow \quad x^2 - 2x + x - 2 = 0$

$\Rightarrow \quad x(x-2) + 1(x-2) = 0 \quad \Rightarrow \quad (x-2)(x+1) = 0$

$\Rightarrow \quad x = -1, 2 \quad \Rightarrow \quad y = \frac{1}{4}, 1$

Hence, coordinates of intersection points of line and parabola are $\left(-1, \frac{1}{4}\right)$ and $(2, 1)$.

Shaded region is required region.

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$$= \int_{-1}^{2}\left(\frac{x+2}{4}\right)dx - \int_{-1}^{2}\frac{x^2}{4}dx = \frac{1}{4}\left[\frac{(x+2)^2}{2}\right]_{-1}^{2} - \frac{1}{4}\left[\frac{x^3}{3}\right]_{-1}^{2}$$

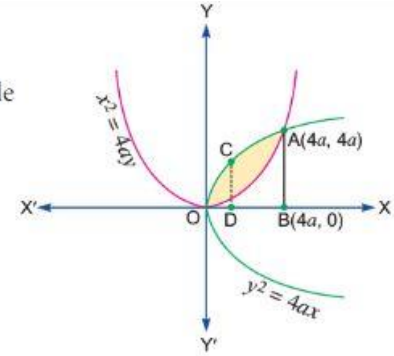
$$= \frac{1}{8}[16-1] - \frac{1}{12}(8+1)$$

$$= \frac{15}{8} - \frac{9}{12} = \frac{27}{24} = \frac{9}{8} \text{ sq units.}$$

Q.4. Find the area of the region included between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, where $a > 0$.

Ans.

Given parabolas are $y^2 = 4ax$ ...(i), $x^2 = 4ay$ ...(ii)



Obviously, curve (i) is right handed parabola having vertex at $(0, 0)$, while curve (ii) is upward parabola having vertex at $(0, 0)$.

Shaded region is required region.

For coordinate of intersection point A, (i) and (ii) are solved as

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$\Rightarrow \quad x^4 = 64a^3x$

$\Rightarrow \quad x(x^3 - 64a^3) = 0$

$\Rightarrow \quad x = 0 \quad \text{or} \quad x^3 - 64\,a^3 = 0$

$\Rightarrow \quad x = 4a \quad \text{and} \quad y = 4a$

Hence, coordinate of $A \equiv (4a, 4a)$

Therefore, area of required region = area of $OCABO$ – area of $ODABO$

$$= \int_0^{4a} \sqrt{4ax}\, dx - \int_0^{4a} \frac{x^2}{4a} dx \quad \Rightarrow \quad A = \left[\frac{4\sqrt{a}}{3} x^{3/2} - \frac{x^3}{12a}\right]_0^{4a}$$

$\Rightarrow \quad A = \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{(4a)^3}{12a}$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3} \text{ sq units.}$$

Q.5. Find the area of the region enclosed between the two circles: $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

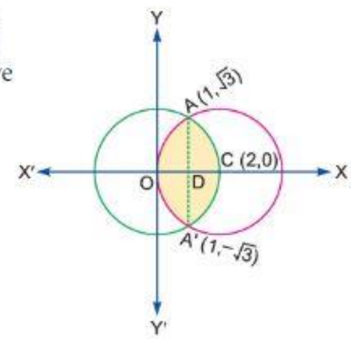
Ans.

Equations of the given circles are

$$x^2 + y^2 = 4 \qquad \text{...}(i)$$

and $$(x-2)^2 + y^2 = 4 \qquad \text{...}(ii)$$

Equation (i) is a circle with centre O at the origin and radius 2. Equation (ii) is a circle with centre $C\ (2, 0)$ and radius 2. Solving equations (i) and (ii), we have



$$(x-2)^2 + y^2 = x^2 + y^2$$

or $$x^2 - 4x + 4 + y^2 = x^2 + y^2$$

or $x = 1$ which gives $y = \pm\sqrt{3}$

Thus, the points of intersection of the given circles are $A\,(1, \sqrt{3})$ and $A'\,(1, -\sqrt{3})$ as shown in the fig.

Required area of the enclosed region $OACA'O$ between circles

$= 2$ [area of the region $ODCAO$]

$= 2$ [area of the region $ODAO$ + area of the region $DCAD$]

$$= 2\left[\int_0^1 y\ dx + \int_1^2 y\ dx\right]$$

$$= 2\left[\int_0^1 \sqrt{4-(x-2)^2}\,dx + \int_1^2 \sqrt{4-x^2}\,dx\right]$$

$$= 2\left[\frac{1}{2}(x-2)\sqrt{4-(x-2)^2} + \frac{1}{2}\times 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1 + 2\left[\frac{1}{2}x\sqrt{4-x^2} + \frac{1}{2}\times 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \left[(x-2)\sqrt{4-(x-2)^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1 + \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \left[\left(-\sqrt{3} + 4\sin^{-1}\left(\frac{-1}{2}\right)\right) - 4\sin^{-1}(-1)\right] + \left[4\sin^{-1}1 - \sqrt{3} - 4\sin^{-1}\frac{1}{2}\right]$$

$$= \left[\left(-\sqrt{3} - 4\times\frac{\pi}{6}\right) + 4\times\frac{\pi}{2}\right] + \left[4\times\frac{\pi}{2} - \sqrt{3} - 4\times\frac{\pi}{6}\right]$$

$$= \left(-\sqrt{3} - \frac{2\pi}{3} + 2\pi\right) + \left(2\pi - \sqrt{3} - \frac{2\pi}{3}\right) = \frac{8\pi}{3} - 2\sqrt{3} \text{ sq units.}$$

Q.6. Using integration find the area of the region

$\{(x, y) : 25x^2 + 9y^2 \leq 225$ and $5x + 3y \geq 15\}$.

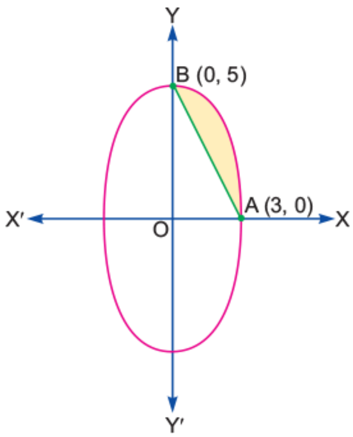
Ans.

The given region is $\{(x, y) : 25x^2 + 9y^2 \geq 225 \text{ and } 5x + 3y \geq 15\}$

Its corresponding equations are

$$25x^2 + 9y^2 = 225 \quad \Rightarrow \quad \frac{x^2}{9} + \frac{y^2}{25} = 1 \qquad \ldots(i)$$

and $\quad 5x + 3y = 15 \qquad \ldots(ii)$



Obviously; curve (i) is an ellipse with centre at $(0, 0)$ and major axis along y-axis while (ii) is at straight line. Intersection points of (i) and (ii) are $(0, 5)$ and $(3, 0)$.

Shaded region is required region and is given by

$$\int_0^3 5\sqrt{1-\frac{x^2}{9}}dx - \int_0^3 \frac{15-5x}{3}dx$$

$$= \frac{5}{3}\int_0^3 \sqrt{9-x^2} - 5\int_0^3 \text{dx} + \frac{5}{3}\int_0^3 x \ \text{dx}$$

$$= \left[\frac{5x}{6}\sqrt{9-x^2} + \frac{15}{2}\sin^{-1}\frac{x}{3}\right]_0^3 - 5/x/_0^3 + \left[\frac{5x^2}{6}\right]_0^3$$

$$= \frac{15}{2}.\frac{\pi}{2} - 15 + \frac{15}{2} = \frac{15}{2}\left(\frac{\pi}{2} - 1\right) \text{ sq units .}$$

Q.7. Find the area of the region in the first quadrant enclosed by the *x*-axis, the line *y* = *x* and the circle $x^2 + y^2 = 32$.

Ans.

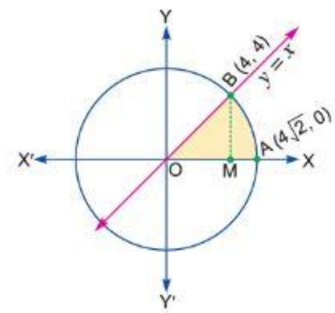
The given equations are

$$y = x \qquad ...(i)$$

and $$x^2 + y^2 = 32 \qquad ...(ii)$$

Solving (i) and (ii), we find that the line and the circle meet at $B(4, 4)$ in the first quadrant. Draw perpendicular BM to the x-axis.

Therefore, the required area = area of the region $OBMO$ + area of the region $BMAB$.



Now, the area of the region $OBMO$

$$= \int_0^4 y\,dx = \int_0^4 x\,dx = \frac{1}{2}[x^2]_0^4 = 8 \qquad ...(iii)$$

Again, the area of the region $BMAB$

$$= \int_4^{4\sqrt{2}} y\,dx = \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx = \left[\frac{1}{2}x\sqrt{32 - x^2} + \frac{1}{2} \times 32 \times \sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$$= \left(\tfrac{1}{2}4\sqrt{2} \times 0 + \tfrac{1}{2} \times 32 \times \sin^{-1}1\right) - \left(\tfrac{4}{2}\sqrt{32 - 16} + \tfrac{1}{2} \times 32 \times \sin^{-1}\tfrac{1}{\sqrt{2}}\right)$$

$$= 8\pi - (8 + 4\pi) = 4\pi - 8 \qquad ....(iv)$$

Adding (iii) and (iv), we get the required area = 4π sq units.

Q.8. Find the area enclosed by the parabola $y^2 = x$ and line $y + x = 2$.

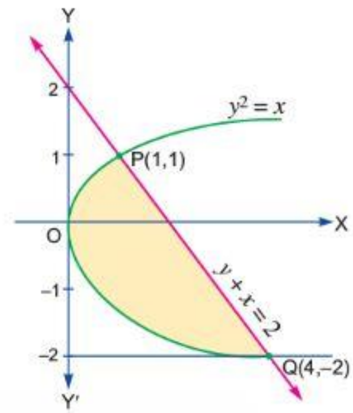
Ans.

To get the point of intersection, we have to solve the equation of line $y + x = 2$ and parabola $y^2 = x$.

On solving them we find the coordinates of points of intersection as $(4, -2)$ and $(1, 1)$. Drawing perpendiculars from these points on y-axis, we obtain the coordinate as $(0, 1)$ and $(0, -2)$.

Thus, required area $= \int_{-2}^{1}(2-y-y^2)dy = \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-2}^{1}$

$= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - \frac{4}{2} + \frac{8}{3}\right)$
$= 2 - \frac{5}{6} + 6 - \frac{8}{3} = 8 - \frac{21}{6} = \frac{27}{6}$
$= \frac{9}{2}$ sq units.



Q.9. Using integration, find the area of the region bounded by the parabola $y^2 = 4x$ and the circle $4x^2 + 4y^2 = 9$.

Ans.

Given curves are

$y^2 = 4x \ldots(i)$

$4x^2 + 4y^2 = 9 \ldots(ii)$

$\Rightarrow \quad x^2 + y^2 = \left(\frac{3}{2}\right)^2$

Obviously, curve (i) is right handed parabola having vertex at $(0, 0)$ and axis along +ve direction of x-axis while curve (ii) is a circle having centre at $(0, 0)$ and radius $\frac{3}{2}$.

Shaded region is required bounded region, which is symmetrical about x-axis.

For coordinate of intersection points 'A' or 'B'.

$\Rightarrow 4x^2 + 16x - 9 = 0$

$\Rightarrow 4x^2 + 18x - 2x - 9 = 0$

$\Rightarrow 2x(2x+9) - 1(2x+9) = 0$

$\Rightarrow (2x-1)\ (2x+9) = 0$

$\Rightarrow \quad x = \frac{1}{2}$ or $x = -\frac{9}{2}$ (not possible)

i.e., $\quad x = \frac{1}{2} \qquad \therefore \quad y^2 = 4 \times \frac{1}{2} = 2$

Hence, area of required region $= 2$ [Area $OADO$ + Area $DACD$]

$$= 2\left[\int_0^{1/2} 2\sqrt{x}\,dx + \int_{1/2}^{3/2} \sqrt{\frac{9}{4} - x^2 dx}\right]$$

$$= 4.\tfrac{2}{3}[x^{3/2}]_0^{1/2} + \left[\tfrac{x}{2}\sqrt{\tfrac{9}{4} - x^2} + \tfrac{1}{2}.\tfrac{9}{4}\sin^{-1}\tfrac{2x}{3}\right]_{1/2}^{3/2}$$

$$= \tfrac{8}{3}\left(\tfrac{1}{2\sqrt{2}} - 0\right) + \left[\left\{\tfrac{9}{4}\sin^{-1}(1)\right\} - \left\{\tfrac{1}{\sqrt{2}} + \tfrac{9}{4}\sin^{-1}\left(\tfrac{1}{3}\right)\right\}\right]$$

$$= \tfrac{2\sqrt{2}}{3} + \left[\tfrac{9\pi}{8} - \tfrac{1}{\sqrt{2}} - \tfrac{9}{4}\sin^{-1}\left(\tfrac{1}{3}\right)\right]$$

$$= \tfrac{2\sqrt{2}}{3} - \tfrac{1}{\sqrt{2}} + \tfrac{9\pi}{8} - \tfrac{9}{4}\sin^{-1}\left(\tfrac{1}{3}\right) \text{ sq units.}$$

[**Note:** Equation of circle in standard form is

$(x - \alpha)^2 + (y - \beta)^2 = r^2$, where (α, β) is centre and r is radius.]

Q.10. Find the area of that part of the circle $x^2 + y^2 = 16$, which is exterior to the parabola $y^2 = 6x$.

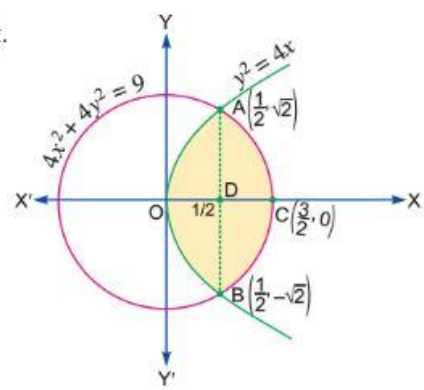
Ans.

Given curves are

$$x^2 + y^2 = 16 \qquad \ldots(i)$$

$$y^2 = 6x \qquad \ldots(ii)$$

Obviously curve (i) is a circle having centre at $(0, 0)$ and radius 4 unit. While curve (ii) is right handed parabola having vertex at $(0, 0)$ and axis along +ve direction of x-axis.



Required part in the shaded region.

Now, for intersection point of curve (i) & (ii)

$$x^2 + 6x = 16$$

$$\Rightarrow \qquad x^2 + 6x - 16 = 0$$

$\Rightarrow \quad x^2 + 8x - 2x - 16 = 0$

$\Rightarrow \quad x(x + 8) - 2(x + 8) = 0$

$\Rightarrow \quad (x + 8)(x - 2) = 0$

$\Rightarrow \quad x = -8$ or 2 $[\because x = -8$ is not possible as y^2 is +ve$]$

$\therefore \quad x = 2$

$\therefore \quad y = \pm\sqrt{12} = \pm 2\sqrt{3}$

Hence, coordinate of B is $(2, 2\sqrt{3})$.

Since, shaded region *i.e.*, required part is symmetrical about x-axis.

Therefore, area of required part

$= 2$ [Area $OBEDO$]

$= 2$ [Area $DEBFD$ – Area $OBFO$]

$= 2\left[\int_{-4}^{2} \sqrt{16 - x^2}\ \mathrm{dx} - \int_0^2 \sqrt{6x}\ \mathrm{dx}\right]$

$= 2\left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_{-4}^{2} - 2\sqrt{6}.\ \frac{2}{3}[x^{3/2}]_0^2$

$= \left[\left(2 \times \sqrt{12} + 16\sin^{-1}\frac{1}{2}\right) - (-4) \times 0 + 16\sin^{-1}(-1)\right] - \frac{4\sqrt{6}}{3}[2\sqrt{2} - 0]$

$= \left(\frac{8}{3} + 8\right)\pi + \left(4 - \frac{16}{3}\right)\sqrt{3}$

$= \left(\frac{32}{3}\pi - \frac{4}{3}\sqrt{3}\right)$ sq units.

Q.11. Using integration, find the area bounded by the lines $x + 2y = 2$, $y - x = 1$ and $2x + y = 7$.

Ans.

Given, $x + 2y = 2$ ...(*i*)

$y - x = 1$ ...(*ii*)

$2x + y = 7$ ...(*iii*)

On solving equation (*i*) and (*ii*), we get

Coordinate of $A \equiv (0, 1)$.

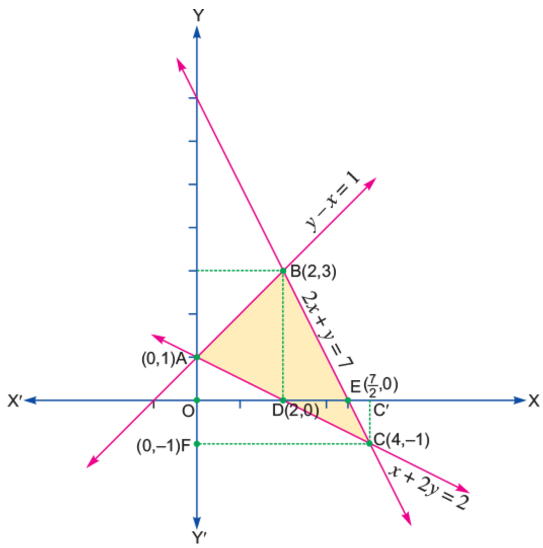
Similarly,

By solving (*ii*) and (*iii*), we get

Co-ordinate of $B \equiv (2, -3)$

and by solving (*i*) and (*iii*), we get

Co-ordinate of $C \equiv (4, -1)$



Shaded region is required region.

$\therefore$ Area of required region

$$= \int_{-1}^{3} \frac{7-y}{2}\,dy - \int_{-1}^{1} (2-2y)\,dy - \int_{1}^{3} (y-1)\,dy$$

$$= \frac{1}{2}\left[7y - \frac{y^2}{2}\right]_{-1}^{3} - [2y - y^2]_{-1}^{1} - \left[\frac{y^2}{2} - y\right]_{1}^{3}$$

$$= \frac{1}{2}\left(21 - \frac{9}{2} + 7 + \frac{1}{2}\right) - (2 - 1 + 2 + 1) - \left(\frac{9}{2} - 3 - \frac{1}{2} + 1\right)$$

$= 12 - 4 - 2 = 6$ sq units

Q.12. Find the area of the region $\{(x, y) : y^2 \le 6ax$ and $x^2 + y^2 \le 16a^2\}$, using method of integration.

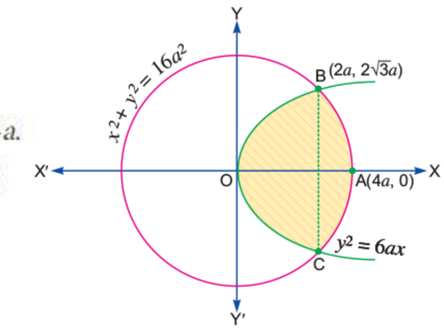
Ans.

Corresponding curves of given region

$\{(x, y): y^2 \le 6ax \text{ and } x^2 + y^2 \le 16a^2\}$ are

$$x^2 + y^2 = 16a^2 \qquad \ldots(i)$$

$$y^2 = 6ax \qquad \ldots(ii)$$



Obviously, curve (i) is a circle having centre $(0, 0)$ and radius $4a$. While curve (ii) is right handed parabola having vertex at $(0, 0)$ and axis along +ve direction of x-axis.

Obviously, shaded region $OCAB$ is area represented by

$$y^2 \le 6ax$$

and $\quad x^2 + y^2 \le 16a^2$

Now, intersection point of curve (i) and (ii)

$$x^2 + 6ax = 16a^2 \qquad \left[\text{Putting the value of } y^2 \text{ in } (i)\right]$$

$\Rightarrow \quad x^2 + 6ax - 16a^2 = 0 \qquad \Rightarrow \quad x^2 + 8ax - 2ax - 16a^2 = 0$

$\Rightarrow \quad x(x + 8a) - 2a(x + 8a) = 0 \quad \Rightarrow \quad (x + 8a)\,(x - 2a) = 0$

$\Rightarrow \quad x = 2a, -8a$

$\Rightarrow \quad x = 2a$ $\qquad$ [$\because x = -8a$ is not possible as y^2 is +ve]

$\therefore \quad y = 2\sqrt{3}a$

Since, shaded region is symmetrical about x-axis

$\therefore$ Required area= 2 [Area of $OABO$]

$$= 2\left[\int_0^{2a} \sqrt{6ax}\,dx + \int_{2a}^{4a} \sqrt{16a^2 - x^2}\,dx\right]$$

$$= 2\left[\sqrt{6a}\int_0^{2a} \sqrt{x}\,dx + \int_{2a}^{4a} \sqrt{(4a)^2 - x^2}\,dx\right]$$

$$= 2\sqrt{6a} \times \frac{2}{3}\left[x^{3/2}\right]_0^{2a} + 2\left[\frac{x}{2}\sqrt{16a^2 - x^2} + \frac{1}{2}16a^2\sin^{-1}\frac{x}{4a}\right]_{2a}^{4a}$$

$$= \frac{4\sqrt{6a}}{3}\left[(2a)^{3/2} - 0\right] + 2\left[(0 + 4a^2\pi) - \left(2\sqrt{3}a^2 + \frac{4a^2\pi}{3}\right)\right]$$

$$= \frac{8a^2\sqrt{12}}{3} + 8a^2\pi - 4\sqrt{3}a^2 - \frac{8a^2\pi}{3}$$

$$= \frac{16}{3}\sqrt{3}a^2 + \frac{16a^2\pi}{3} - 4\sqrt{3}a^2$$

$$= \frac{4\sqrt{3}}{3}a^2 + \frac{16a^2\pi}{3}$$

Q.13. Find the area of the region enclosed between the two circles $x^2 + y^2 = 9$ and $(x - 3)^2 + y^2 = 9$.

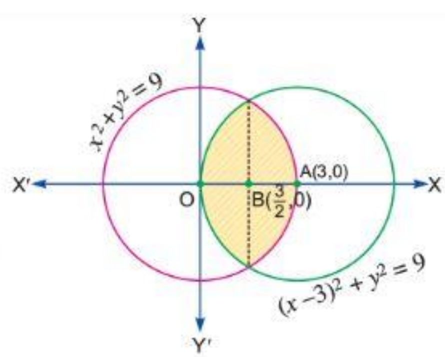
Ans.

The two circles are re-arranged and expressed as

$y^2 = 9 - x^2 \qquad \ldots (i)$

$y^2 = 9 - (x - 3)^2 \quad \ldots (ii)$

Obviously, circle (i) have centre $(0, 0)$ and radius 3 while (ii) have centre $(3, 0)$ and radius 3.



To find the point of intersection of the circles we equate y^2.

To find the point of intersection of the circles we equate y^2.

$\Rightarrow \quad 9 - x^2 = 9 - (x - 3)^2$

$\Rightarrow \quad 9 - x^2 = 9 - x^2 - 9 + 6x$

$\Rightarrow \quad x = \frac{3}{2}$

The circles are shown in the figure and the shaded area is the required area.

Now, area of shaded region

$$= 2\left[\int_0^{\frac{3}{2}} \sqrt{9-(x-3)^2}\,dx + \int_{\frac{3}{2}}^{3} \sqrt{9-x^2}\,dx\right]$$

$$= 2\left[\frac{x-3}{2}\sqrt{9-(x-3)^2} + \frac{9}{2}\sin^{-1}\frac{x-3}{3}\right]_0^{3/2} + 2\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_{\frac{3}{2}}^{3}$$

$$= 2\left[\frac{-3}{4}\sqrt{9-\frac{9}{4}} + \frac{9}{2}\sin^{-1}\left(-\frac{1}{2}\right) - \frac{9}{2}\sin^{-1}(-1)\right] + 2\left[\frac{9}{2}\sin^{-1}1 - \frac{3}{4}\sqrt{9-\frac{9}{4}} - \frac{9}{2}\sin^{-1}\frac{1}{2}\right]$$

$$= 2\left[\frac{-3}{4}\cdot\frac{3\sqrt{3}}{2} - \frac{9}{2}\cdot\frac{\pi}{6} + \frac{9}{2}\cdot\frac{\pi}{2}\right] + 2\left[\frac{9}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{3\sqrt{3}}{2} - \frac{9}{2}\cdot\frac{\pi}{6}\right]$$

$$= 2\left[-\frac{9\sqrt{3}}{8} - \frac{3\pi}{4} + \frac{9\pi}{4} + \frac{9\pi}{4} - \frac{9}{8}\sqrt{3} - \frac{3\pi}{4}\right]$$

$$= 2\left[\frac{-9\sqrt{3}}{4} - \frac{6\pi}{4} + \frac{18\pi}{4}\right]$$

$$= 2\left[-\frac{9\sqrt{3}}{4} + \frac{12\pi}{4}\right] = 6\pi - \frac{9\sqrt{3}}{2} \text{ sq units.}$$

Q.14. Find the area of the region included between the parabola $4y = 3x^2$ and the line $3x - 2y + 12 = 0$.

Ans.

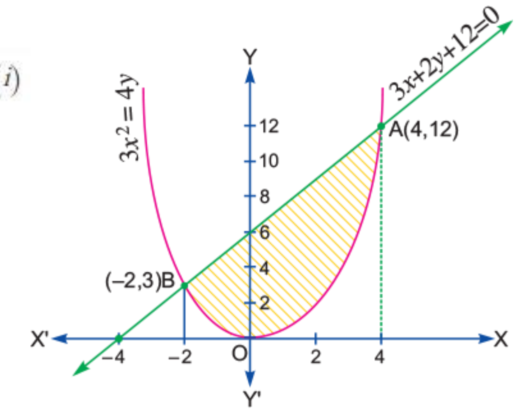
Given equation of parabola $4y = 3x^2 \Rightarrow y = \frac{3x^2}{4}$ ... (*i*)

and the line $3x - 2y + 12 = 0$

$\Rightarrow \frac{3x+12}{2} = y$ ... (*ii*)

The line intersect the parabola at $(-2, 3)$ and $(4, 12)$.

Hence, the required area will be the shaded region.



$$\text{Required area} = \int_{-2}^{4} \frac{3x+12}{2}dx - \int_{-2}^{4} \frac{3x^2}{4}dx$$

$$= \left[\frac{3}{4}x^2 + 6x - \frac{x^3}{4}\right]_{-2}^{4}$$

$= (12 + 24 - 16) - (3 - 12 + 2)$

$= 20 + 7$

$= 27$ square units.

Q.15. Using integration, find the area of the region

$\{(x, y)\} : 9x^2 + y^2 \le 36$ and $3x + y \ge 6$.

Ans.

Given region is $\{(x, y) : 9x^2 + y^2 \le 36 \text{ and } 3x + y \ge 6\}$

We draw the curves corresponding to equations

$9x^2 + y^2 = 36$ or

$\frac{x^2}{4} + \frac{y^2}{36} = 1$ and $3x + y = 6$

The curves intersect at $(2, 0)$ and $(0, 6)$

$\therefore$ Shaded area is enclosed by the two curves and is

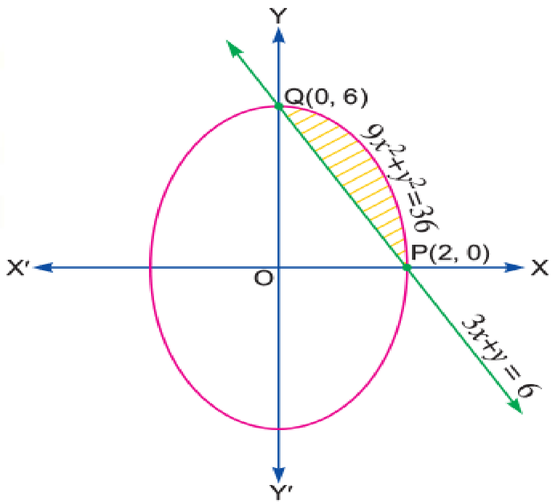
$$= \int_0^2 \sqrt{36\left(1-\frac{x^2}{4}\right)}\, dx - \int_0^2 (6-3x)dx$$

$$= 6\left[\frac{x}{4}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} - 2x + \frac{x^2}{2}\right]_0^2$$

$$= 6\ \left[\frac{2}{4}\sqrt{4-4} + \frac{4}{2}\sin^{-1}\frac{2}{2} - 4 + \frac{4}{2} - 0\right]$$

$$= 6\ \left[2\frac{\pi}{2} - 2\right]$$

= 6(p − 2) square units

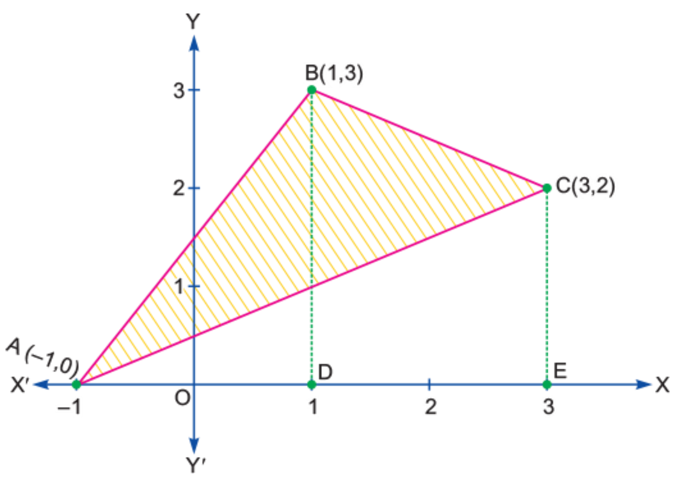


Q.16. Using integration, find the area of the triangle *ABC* with vertices as *A*(–1, 0), *B*(1, 3) and *C*(3, 2).

Ans.

We mark the points on the graph and get the triangle *ABC* as shown in the figure

Required area of triangle = area of D*ABD* + area of trapezium *EDEC* – area of D*ACE*



Equation of line $AB \Rightarrow y = \frac{3}{2}(x+1)$

Equation of line $BC \Rightarrow y = -\frac{x}{2} + \frac{7}{2}$

Equation of line $AC \Rightarrow y = \frac{x}{2} + \frac{1}{2}$

∴ Area of

$$\Delta ABC = \int_{-1}^{1}\left(\frac{3}{2}x+\frac{3}{2}\right)dx+\int_{1}^{3}\left(-\frac{x}{2}+\frac{7}{2}\right)dx-\int_{-1}^{3}\left(\frac{x}{2}+\frac{1}{2}\right)dx$$

$$=\left[\frac{3x^2}{4}+\frac{3}{2}x\right]_{-1}^{1}+\left[\frac{-x^2}{4}+\frac{7}{2}x\right]_{1}^{3}-\left[\frac{x^2}{4}+\frac{x}{2}\right]_{-1}^{3}$$

$$=\left(\frac{3}{4}+\frac{3}{2}-\frac{3}{4}+\frac{3}{2}\right)+\left(\frac{-9}{4}+\frac{21}{2}+\frac{1}{4}-\frac{7}{2}\right)-\left(\frac{9}{4}+\frac{3}{2}-\frac{1}{4}+\frac{1}{2}\right)$$

$$=3+\frac{-9+42+1-14}{4}-\left(\frac{9+6-1+2}{4}\right)=3+5-4=4 \text{ sq units}$$

Q.17. Using integration, find the area of the following region:

$$\left\{(x,y):|x-1|\le y\le\sqrt{5-x^2}\right\}$$

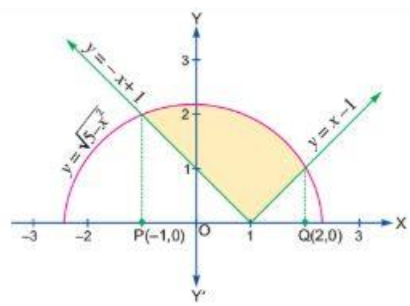
Ans.

We have provided

$$\left\{(x,y):|x-1|\le y\le\sqrt{5-x^2}\right\}$$

Equation of curve is $y=\sqrt{5-x^2}$ or $y^2+x^2=5$, which is a circle with centre at $(0, 0)$ and radius $\frac{5}{2}$.

Equation of line is $y=|x-1|$



Consider, $y = x - 1$ and $y = \sqrt{5 - x^2}$

Eliminating y, we get

$$x - 1 = \sqrt{5 - x^2}$$

$\Rightarrow \quad x^2 + 1 - 2x = 5 - x^2$

$\Rightarrow 2x^2 - 2x - 4 = 0$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow (x - 2)\ (x - 1) = 0$

$\Rightarrow \quad x = 2, -1$

The required area is

$$= \int_{-1}^{2} \sqrt{5 - x^2}\, dx - \int_{-1}^{1} (-x + 1) dx - \int_{1}^{2} (x - 1)\, dx$$

$$= \left[\frac{x}{2}\sqrt{5 - x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}\right]_{-1}^{2} - \left[-\frac{x^2}{2} + x\right]_{-1}^{1} - \left[\frac{x^2}{2} - x\right]_{1}^{2}$$

$$= \left(1 + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right) + 1 - \frac{5}{2}\sin^{-1}\left(-\frac{1}{\sqrt{5}}\right) - \left(\frac{-1}{2} + 1 + \frac{1}{2} + 1\right) - \left(2 - 2 - \frac{1}{2} + 1\right)$$

$$= \frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}}\right) + 2 - 2 - \frac{1}{2}$$

$$= \frac{5}{2}\sin^{-1}\left[\frac{2}{\sqrt{5}}\sqrt{1 - \frac{1}{5}} + \frac{1}{\sqrt{5}}\sqrt{1 - \frac{4}{5}}\right] - \frac{1}{2}$$

$$= \frac{5}{2}\left[\sin^{-1}\left(\frac{4}{5} + \frac{1}{5}\right)\right] - \frac{1}{2}$$

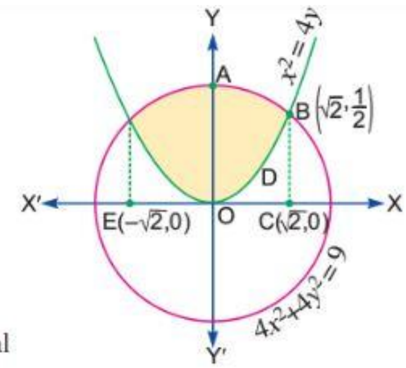
$$= \frac{5}{2}\sin^{-1}(1) - \frac{1}{2}$$

$$= \left(\frac{5\pi}{4} - \frac{1}{2}\right) \text{ sq units}.$$

Q.18. Find the area of the circle $4x^2 + 4y^2 = 9$, which is interior to the parabola $x^2 = 4y$.

Ans.

Equation of circle and parabola is



$4x^2 + 4y^2 = 9 \quad \Rightarrow \quad x^2 + y^2 = \frac{9}{4} \qquad \ldots(i)$

$x^2 = 4y \quad \Rightarrow \quad y = \frac{x^2}{4} \qquad \ldots(ii)$

Obviously given circle have centre at $(0, 0)$ and radius $\frac{3}{2}$, while given parabola is upward parabola having vertex at $(0, 0)$ and axis along +ve direction of *y*-axis. Shaded region is required region which is symmetrical along *y*-axis.

By putting value of equation (ii) in equation (i), we get

$$4x^2 + 4\left(\frac{x^2}{4}\right)^2 = 9$$

$$\Rightarrow x^4 + 16x^2 - 36 = 0$$

$$\Rightarrow (x^2 + 18)(x^2 - 2) = 0$$

$$\Rightarrow x^2 + 18 = 0, \; x^2 - 2 = 0$$

$$\Rightarrow x = -\sqrt{18}, \; x = \pm\sqrt{2}$$

$\Rightarrow x = \pm\sqrt{2}$ ($\because$ $x = -\sqrt{18}$ is not possible *i.e.*, intersecting points are $(-\sqrt{2}, 1/2)$, $(\sqrt{2}, 1/2)$)

$\therefore$ Required area = 2 [area of *OABCO* – area of *ODBCO*]

$$= 2\left[\int_0^{\sqrt{2}} \sqrt{\frac{9}{4} - x^2}\, dx - \int_0^{\sqrt{2}} \frac{x^2}{4} dx\right]$$

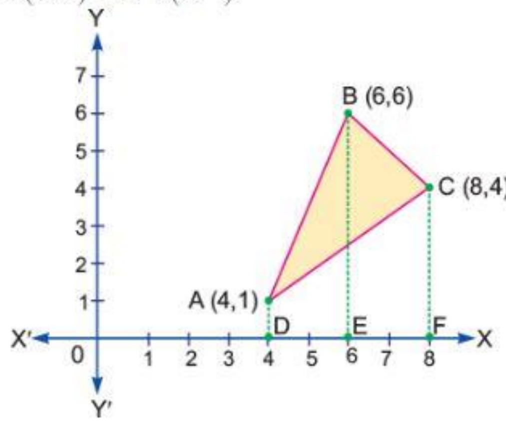
$$= 2\left\{\left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{9}{8}\sin^{-1}\frac{x}{3/2}\right]_0^{\sqrt{2}} - \left[\frac{x^3}{12}\right]_0^{\sqrt{2}}\right\}$$

$$= 2\left[\frac{\sqrt{2}}{4} + \frac{9}{8}\sin^{-1}\frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{6}\right] = \left(\frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}\right) \text{ sq units.}$$

Q.19. Using integration, find the area of the triangle *ABC*, co-ordinates of whose vertices are *A*(4, 1), *B*(6, 6) and *C*(8, 4)

Ans.

Given triangle ABC, coordinates of whose vertices are $A(4, 1)$, $B(6, 6)$ and $C(8, 4)$.



Equation of AB is given by

$$y-6=\frac{6-1}{6-4}(x-6) \text{ or } y=\frac{5}{2}x-9$$

Equation of BC is given by

$$y-4=\frac{4-6}{8-6}(x-8) \text{ or } y=-x+12$$

Equation of AC is given by

$$y-4=\frac{4-1}{8-4}(x-8) \text{ or } y=\frac{3}{4}x-2$$

$\therefore$ Area of ΔABC = area of trap. $DABE$ + area of trap. $EBCF$ – area of trap. $DACF$

$$=\int_4^6\left(\frac{5}{2}x-9\right)dx+\int_6^8(-x+12)\,dx-\int_4^8\left(\frac{3}{4}x-2\right)dx$$

$$=\frac{5}{2}\left[\frac{x^2}{2}\right]_4^6-9[x]_4^6-\left[\frac{x^2}{2}\right]_6^8+12[x]_6^8-\frac{3}{4}\left[\frac{x^2}{2}\right]_4^8+2[x]_4^8$$

$$=\frac{5}{4}(36-16)-9(6-4)-\frac{1}{2}(64-36)+12(8-6)-\frac{3}{8}(64-16)$$

$$=\frac{5}{4}\times 20-18-\frac{28}{2}+24-\frac{3}{8}\times 48+8$$

$= 25 - 18 - 14 + 24 - 18 + 8 = 7$ sq units.

Q.20. Using integration, find the area of the region bounded by the lines $4x - y + 5 = 0$, $x + y - 5 = 0$ and $x - 4y + 5 = 0$.

Ans.

We have given

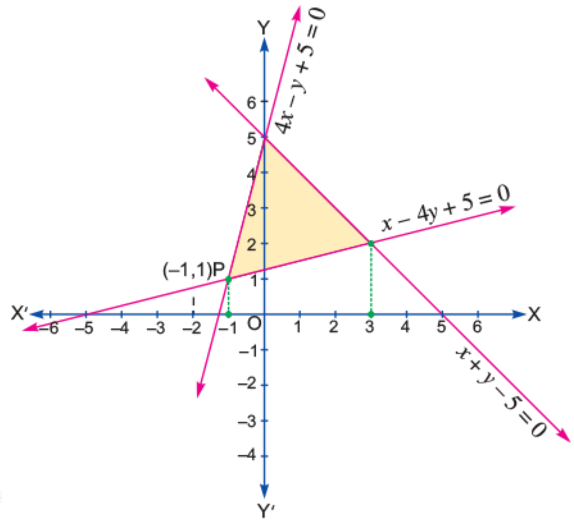
$$4x - y + 5 = 0 \qquad ...(i)$$

$$x + y - 5 = 0 \qquad ...(ii)$$

$$x - 4y + 5 = 0 \qquad ...(iii)$$

By solving equations (i) and (iii), we get $(-1, 1)$

and by solving (ii) and (iii), we get $(3, 2)$



$\therefore$ Area of region bounded by the lines is given by:

$$\int_{-1}^{0}\left\{(4x+5)-\left(\frac{x+5}{4}\right)\right\}dx+\int_{0}^{3}\left\{(5-x)-\left(\frac{x+5}{4}\right)\right\}dx$$

$$=\int_{-1}^{0}\left[\frac{15x}{4}+\frac{15}{4}\right]dx+\int_{0}^{3}\left[\frac{15}{4}-\frac{5x}{4}\right]dx$$

$$=\left[\frac{15x^2}{8}+\frac{15x}{4}\right]_{-1}^{0}+\left[\frac{15x}{4}-\frac{5x^2}{8}\right]_{0}^{3}$$

$$=0-\left(\frac{15}{8}-\frac{15}{4}\right)+\left(\frac{45}{4}-\frac{45}{8}\right)-0$$

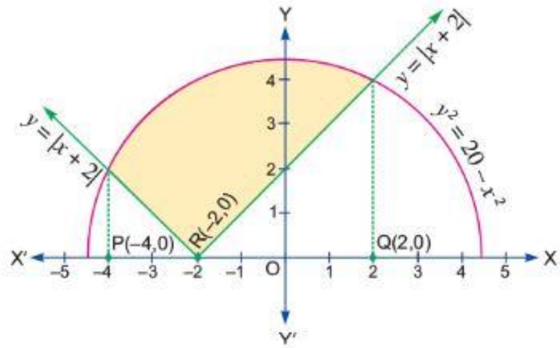
$$=\frac{15}{8}+\frac{45}{8}=\frac{15}{2}\text{ sq units.}$$

Q.21. Using integration, find the area of the following region:

$$\left\{(x,y) : |x+2| \leq y \leq \sqrt{20-x^2}\right\}$$

Ans.

Given region is $\left\{(x, y) : |x+2| \le y \le \sqrt{20-x^2}\right\}$



It consists of inequalities $y \ge |x+2|$ and $y \le \sqrt{20-x^2}$

Plotting these inequalities, we obtain the adjoining shaded region.

Solving $y = x + 2$ and $y^2 = 20 - x^2$

$\Rightarrow (x+2)^2 = 20 - x^2$

$\Rightarrow 2x^2 + 4x - 16 = 0$

or $(x+4)(x-2) = 0 \quad \Rightarrow x = -4, 2$

The required area $= \int_{-4}^{2} \sqrt{20-x^2}\, dx - \int_{-4}^{-2} -(x+2)\, dx - \int_{-2}^{2} (x+2)\, dx$

$= \left[\frac{x}{2}\sqrt{20-x^2} + \frac{20}{2}\sin^{-1}\frac{x}{\sqrt{20}}\right]_{-4}^{2} + \left[\frac{x^2}{2} + 2x\right]_{-4}^{-2} - \left[\frac{x^2}{2} + 2x\right]_{-2}^{2}$

$= 4 + 10\sin^{-1}\frac{1}{\sqrt{5}} + 4 + 10\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) + [2-4-8+8] - [2+4-2+4]$

$= 8 + 10\left(\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{2}{\sqrt{5}}\right) - 2 - 8 = -2 + 10\left(\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{2}{\sqrt{5}}\right)$

$= -2 + 10\sin^{-1}\left[\frac{1}{\sqrt{5}}\sqrt{1-\frac{4}{5}} + \frac{2}{\sqrt{5}}\sqrt{1-\frac{1}{5}}\right] = -2 + 10\sin^{-1}\left[\frac{1}{5} + \frac{4}{5}\right] = -2 + 10\sin^{-1} 1$

$= -2 + 10\frac{\pi}{2} = (5\pi - 2)$ sq units.

Q.22. Using integration find the area of the triangular region whose sides have equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

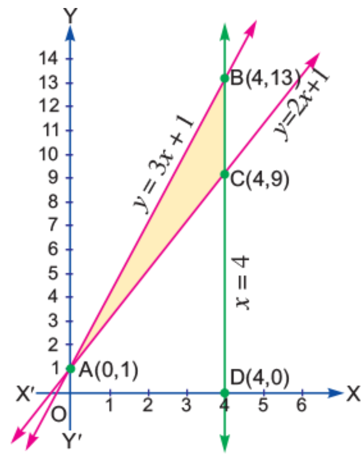
Ans.

The given lines are

$y = 2x + 1 \quad \ldots(i)$

$y = 3x + 1 \quad \ldots(ii)$

$x = 4 \quad \ldots(iii)$



For intersection point of (i) and (iii)

$y = 2 \times 4 + 1 = 9$

Coordinates of intersecting point of (i) and (iii) is $(4, 9)$

For intersection point of (ii) and (iii)

$y = 3 \times 4 + 1 = 13$

i.e., Coordinates of intersection point of (ii) and (iii) is $(4, 13)$

For intersection point of (i) and (ii)

$2x + 1 = 3x + 1 \quad \Rightarrow x = 0$

$\therefore \quad y = 1$

i.e., Coordinates of intersection point of (i) and (ii) is $(0, 1)$.

Shaded region is required triangular region.

$\therefore$ Required Area = Area of trapezium $OABD$ – Area of trapezium $OACD$

$$= \int_0^4 (3x+1)\,dx - \int_0^4 (2x+1)\,dx = \left[3\frac{x^2}{2} + x\right]_0^4 - \left[\frac{2x^2}{2} + x\right]_0^4$$

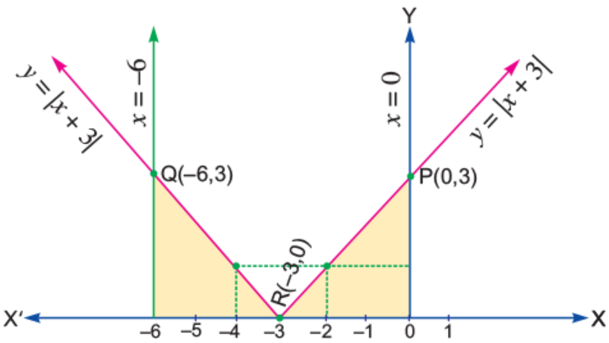
$= [(24 + 4) - 0] - [(16 + 4) - 0] = 28 - 20 = 8$ sq units.

Q.23. Sketch the graph of *y* = |x+3| and evaluate the area under the curve *y* = |x+3| above *x*-axis and between *x* = – 6 to *x* = 0.

Ans.

For graph of $y = |x+3|$

x	0	–3	– 6	– 2	– 4
y	3	0	3	1	1



Shaded region is the required region.

Hence, Required area $= \int_{-6}^{0} |x+3|\, dx$

$= \int_{-6}^{-3} \left|x+3\right| dx + \int_{-3}^{0} |x+3| dx$ [By Property of definite integral]

$= \int_{-6}^{-3} \left|x+3\right| dx + \int_{-3}^{0} |x+3| dx$

$= \int_{-6}^{-3} -(x+3)\, dx + \int_{-3}^{0} (x+3)\, dx$ $\qquad \begin{bmatrix} x+3 \geq 0 \text{ if } -3 \leq x \leq 0 \\ x+3 \leq 0 \text{ if } -6 \leq x \leq -3 \end{bmatrix}$

$= -\left[\frac{x^2}{2} + 3x\right]_{-6}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{0}$

$= -\left[\left(\frac{9}{2} - 9\right) - \left(\frac{36}{2} - 18\right)\right] + \left[0 - \left(\frac{9}{2} - 9\right)\right]$

$= \frac{9}{2} + \frac{9}{2} = 9$ sq units.

Q.24. Using the method of integration, find the area of the region bounded by the lines:

$2x + y = 4$

$3x - 2y = 6$

$x - 3y + 5 = 0$

Ans.

Given lines are $2x + y = 4$ ...(*i*)

$$3x - 2y = 6 \quad ...(ii)$$

$$x - 3y + 5 = 0 \quad ...(iii)$$

For intersection point of (*i*) and (*ii*)

Multiplying (*i*) by 2 and adding with (*ii*), we get

$$\begin{array}{l} 4x + 2y = 8 \\ \underline{3x - 2y = 6} \\ \quad 7x = 14 \end{array} \Rightarrow \quad x = 2 \text{ and } y = 0$$

Hence, intersection point of (*i*) and (*ii*) is $(2, 0)$.

For intersection point of (*i*) and (*iii*)

Multiplying (*i*) by 3 and adding with (*iii*), we get

$$\begin{array}{l} 6x + 3y = 12 \\ \underline{x - 3y = -5} \\ \quad 7x = 7 \end{array} \Rightarrow \quad x = 1 \text{ and } y = 2$$

Hence, intersection point of (*i*) and (*iii*) is $(1, 2)$.

For intersection point of (*ii*) and (*iii*)

Multiplying (iii) by 3 and subtracting from (ii), we get

$$\begin{array}{r} 3x - 2y = 6 \\ \underline{-3x \mp 9y = \mp 15} \\ 7y = 21 \end{array} \quad \Rightarrow \quad y = 3 \text{ and } x = 4$$

Hence, intersection point of (ii) and (iii) is $(4, 3)$.

With the help of intersecting points, required region ΔABC in plotted.

Shaded region is required region.

Required area= area of ΔABC

= Area of trap $ABED$ – Area of ΔADC Area of ΔCBE

$$= \int_1^4 \frac{x+5}{3} dx - \int_1^2 (4-2x)\, dx - \int_2^4 \frac{3x-6}{2} dx$$

$$= \frac{1}{3}\left[\frac{x^2}{2} + 5x\right]_1^4 - [4x - x^2]_1^2 - \frac{1}{2}\left[\frac{3x^2}{2} - 6x\right]_2^4$$

$$= \frac{1}{3}\{(\frac{16}{2} + 20) - (\frac{1}{2} + 5)\} - \left\{(8-4) - (4-1)\right\} - \frac{1}{2}\left\{(\frac{3\times16}{2} - 24) - \left(\frac{3\times4}{2} - 12\right)\right\}$$

$$= \frac{1}{3}\{28 - \frac{11}{2}\} - \left\{4 - 3\right\} - \frac{1}{2}\left\{0 + 6\right\} = \frac{1}{3} \times \frac{45}{2} - 1 - 3 = \frac{7}{2} \text{ sq units}$$

Q.25. Find the area of the region bounded by the parabola $y^2 = 2x$ and the line $x – y = 4$.

Ans.

Given curves are $y^2 = 2x$ ....(i)

and $x - y = 4$ ....(ii)

Obviously, curve (i) is right handed parabola having vertex at $(0, 0)$ and axis along +ve direction of x-axis while curve (ii) is a straight line.

For intersection point of curve (i) and (ii)

$$(x-4)^2 = 2x$$

$$\Rightarrow \quad x^2 - 8x + 16 = 2x$$

$$\Rightarrow \quad x^2 - 10x + 16 = 0$$

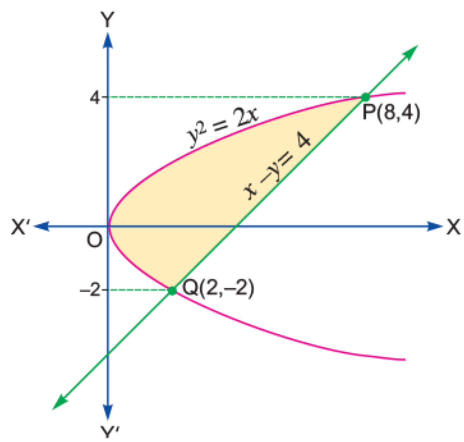
$$\Rightarrow \quad x^2 - 8x - 2x + 16 = 0$$

$$\Rightarrow \quad x(x-8) - 2(x-8) = 0$$

$$\Rightarrow \quad (x-8)(x-2) = 0$$

$$\Rightarrow \quad x = 2, 8$$

$$\Rightarrow \quad y = -2, 4$$



Intersection points are $(2, -2)$, $(8, 4)$

Therefore, required Area = Area of shaded region

$$= \int_{-2}^{4} (y+4)\ \mathrm{dy} - \int_{-2}^{4} \frac{y^2}{2}\mathrm{dy} = \left[\frac{(y+4)^2}{2}\right]_{-2}^{4} - \frac{1}{2}\left[\frac{y^3}{3}\right]_{-2}^{4}$$

$$= \frac{1}{2}.[64 - 4] - \frac{1}{6}[64 + 8] = 30 - \frac{72}{6} = 18 \quad \text{sq units}.$$

Q.26. Using integration find the area of the region $\{(x, y) : x^2 + y^2 \leq 2ax, y^2 \geq ax, x, y \geq 0\}$.

Ans.

Given region R is

$$R = \{(x, y) : x^2 + y^2 \leq 2ax, y^2 \geq ax, x, y \geq 0\}.$$

$\Rightarrow$ $R = R_1 \cap R_2 \cap R_3$, where

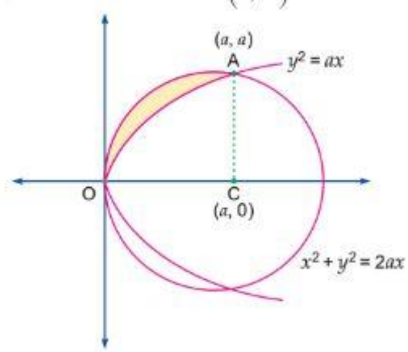
$\Rightarrow$ $R_1 = \{(x, y) : x^2 + y^2 \leq 2ax\}$

$R_2 = \{(x, y) : y^2 \geq ax\}$

and $R_3 = \{(x, y) : x \geq 0, y \geq 0\}$

Obviously, $x^2 + y^2 = 2ax \Rightarrow (x - a)^2 + (y - 0)^2 = a^2$ is a circle having centre at $(a, 0)$ and radius $r = a$.

Therefore the region $R_1 \equiv \{(x, y) : x^2 + y^2 \leq 2ax\}$ is the region inside the circle with centre $(a, 0)$ and radius a.



Also $y^2 = ax$ is right handed parabola with vertex at origin.

So, region $R_2 \equiv \{(x, y) : y^2 \geq ax\}$ is the region out side parabola.

Also, $R_3 \equiv \{(x, y) : x \geq 0, y \geq 0\}$ is region in first quadrant.

Hence, $R = R_1 \cap R_2 \cap R_3$ is the shaded region shown above in figure.

Now for co-ordinate of A, we solve $y^2 = ax$ and

$x^2 + y^2 = 2ax$ as follows

$x^2 + 2ax = 2ax$ [Putting $y^2 = ax$]

$x^2 - ax = 0$

$\Rightarrow x(x - a) = 0$

$\Rightarrow x = 0, a$

For $x = a$, $y = a$

For $x = a,\ y = a$

Hence co-ordinate of A is (a, a)

$\therefore$ Required area $= \int_0^a \sqrt{2ax - x^2}\, dx - \int_0^a \sqrt{ax}\, dx$

$= \int_0^a \sqrt{-(x^2 - 2ax + a^2 - a^2)}\, dx - \sqrt{a} \int_0^a x^{1/2}\, dx$

$= \int_0^a \sqrt{-(x - a)^2 + a^2}\, dx - \sqrt{a} \left[\frac{x^{3/2}}{3/2}\right]_0^a$

$= \int_0^a \sqrt{a^2 - (x - a)^2}\, dx - \frac{2\sqrt{a}}{3} [a^{3/2} - 0]$

$= \left[\frac{1}{2}(x - a)\sqrt{a^2 - (x - a)^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x - a}{a}\right)\right]_0^a - \frac{2a^2}{3}$

$= \left[0 - \left\{0 + \frac{1}{2}a^2\left(-\frac{\pi}{2}\right)\right\}\right] - \frac{2a^2}{3}$

$= \frac{\pi}{4}a^2 - \frac{2a^2}{3} = \left(\frac{\pi}{4} - \frac{2}{3}\right)a^2$ sq unit .

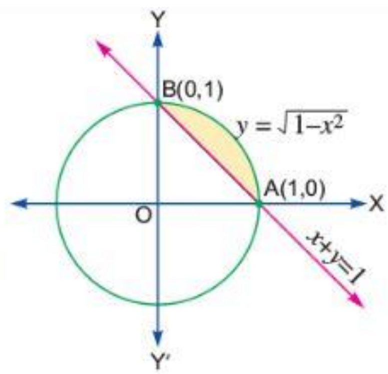
Q.28. Find the area of the region $\{(x, y): x^2 + y^2 \le 1 \le x + y\}$.

Ans.

The required area is the area between the circle

$x^2 + y^2 = 1.$ ...(*i*)

and line $x + y = 1$ ...(*ii*)



Circle (*i*) has centre $(0, 0)$ and radius 1. Line (*ii*) meets *x*-axis at $A(1, 0)$ and *y*-axis at $B(0, 1)$. The circle (*i*) also passes through *A* and *B*. Hence, points of intersection of (*i*) and (*ii*) are $A(1, 0)$ and $B(0, 1)$

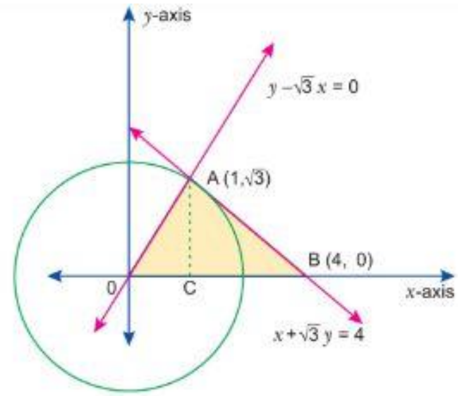
Required area $= \int_0^1 \sqrt{1 - x^2}\, dx - \int_0^1 (1 - x)\, dx$

$= \left[\frac{x.\sqrt{1 - x^2}}{2} + \frac{1}{2}\sin^{-1} x\right]_0^1 - \left[x - \frac{x^2}{2}\right]_0^1$

$= \frac{1}{2}\sin^{-1}(1) - \frac{1}{2} = \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2} = \left(\frac{\pi}{4} - \frac{1}{2}\right)$sq unit.

Q.28. Using integration find the area of the triangle formed by positive *x*-axis and tangent and normal to the circle $x^2 + y^2 = 4\ at\ (1, \sqrt{3})$.

Ans.



Given circle is $x^2 + y^2 = 4$

$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0 \qquad$ [By differentiating]

$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$

Now, slope of tangent at $(1, \sqrt{3}) = \frac{dy}{dx}\Big]_{(1, \sqrt{3})} = -\frac{1}{\sqrt{3}}$.

$\therefore$ Slope of normal at $(1,\sqrt{3}) = \sqrt{3}$

Therefore, equation of tangent is

$$\frac{y - \sqrt{3}}{x - 1} = -\frac{1}{\sqrt{3}}$$

$\Rightarrow \quad x + \sqrt{3}y = 4 \ldots (i)$

Again, equation of normal is

$$\frac{y - \sqrt{3}}{x - 1} = \sqrt{3}$$

$\Rightarrow \quad y - \sqrt{3}x = 0 \quad \ldots (ii)$

To draw the graph of the triangle formed by the lines x-axis, (i) and (ii), we find the intersecting points of these three lines which give vertices of required triangle. Let O, A, B be the intersecting points of these lines.

Obviously, the coordinate of O, A, B are $(0, 0)$, $(1,)\ \sqrt{3}$ and $(4, 0)$ respectively.

Required area= area of triangle OAB = area of region OAC + area of region CAB

$= \int_0^1 y\,dx + \int_1^4 y\ dx$ [Where in 1st integrand $y = \sqrt{3}x$ and in 2nd $y = \frac{4-x}{\sqrt{3}}$]

$= \int_0^1 \sqrt{3}x dx + \int_1^4 \frac{4-x}{\sqrt{3}} dx = \sqrt{3}\left[\frac{x^2}{2}\right]_0^1 - \frac{1}{\sqrt{3}}\left[\frac{(4-x)^2}{2}\right]_1^4$

$= \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}}\left[0 - \frac{9}{2}\right]$

$= \frac{\sqrt{3}}{2} + \frac{9}{2\sqrt{3}} = \frac{12}{2\sqrt{3}} = 2\sqrt{3}$ sq units.

Q.29. Find the area of the region {(x, y) : $x \le y \le |x|$}.

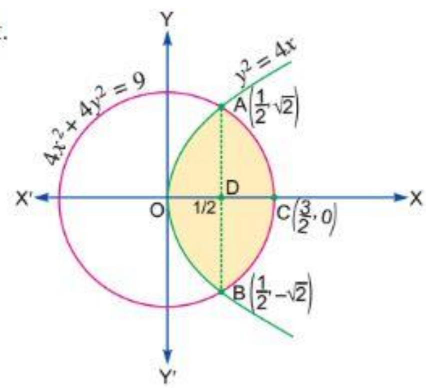
Ans.

Given curves are

$x^2 + y^2 = 16 \qquad \ldots(i)$

$y^2 = 6x \qquad \ldots(ii)$

Obviously curve (i) is a circle having centre at $(0, 0)$ and radius 4 unit. While curve (ii) is right handed parabola having vertex at $(0, 0)$ and axis along +ve direction of x-axis.



Required part in the shaded region.

Now, for intersection point of curve (i) & (ii)

$x^2 + 6x = 16$

$\Rightarrow \qquad x^2 + 6x - 16 = 0$

$\Rightarrow \quad x^2 + 8x - 2x - 16 = 0$

$\Rightarrow \quad x(x+8) - 2(x+8) = 0$

$\Rightarrow \quad (x+8)(x-2) = 0$

$\Rightarrow \quad x = -8$ or 2 $[\because x = -8$ is not possible as y^2 is +ve$]$

$\therefore \quad x = 2$

$\therefore \quad y = \pm\sqrt{12} = \pm 2\sqrt{3}$

Hence, coordinate of B is $(2, 2\sqrt{3})$.

Since, shaded region *i.e.*, required part is symmetrical about x-axis.

Therefore, area of required part

$= 2$ [Area $OBEDO$]

$= 2$ [Area $DEBFD$ – Area $OBFO$]

$$= 2\left[\int_{-4}^{2} \sqrt{16 - x^2}\ \mathrm{dx} - \int_0^2 \sqrt{6x}\ \mathrm{dx}\right]$$

$$= 2\left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_{-4}^{2} - 2\sqrt{6}.\ \frac{2}{3}/x^{3/2}/_0^2$$

$$= \left[\left(2 \times \sqrt{12} + 16\sin^{-1}\frac{1}{2}\right) - (-4) \times 0 + 16\sin^{-1}(-1)\right] - \frac{4\sqrt{6}}{3}[2\sqrt{2} - 0]$$

$$= \left(\frac{8}{3} + 8\right)\pi + \left(4 - \frac{16}{3}\right)\sqrt{3}$$

$$= \left(\frac{32}{3}\pi - \frac{4}{3}\sqrt{3}\right) \text{ sq units.}$$

Long Answer Question-II (OIQ)

[6 marks]

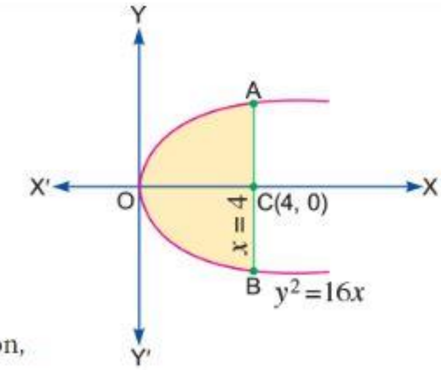
Q.1. Using integration, find the area of the region bounded between the line $x = 4$ and the parabola $y^2 = 16x$.

Ans.

Given parabola and line are

$$y^2 = 16x \quad \ldots(i)$$

$$x = 4 \quad \ldots(ii)$$



Obviously, given parabola (i) is right handed parabola with vertex at $(0, 0)$ and axis along +ve direction of x-axis while line (ii) is a straight line parallel to y-axis meeting x-axis at $x = 4$. Shaded region is required region, which is symmetric along x-axis.

∴ Required area A is given by

$$A = 2(\text{area } OCAO)$$

$$\Rightarrow A = 2\int_0^4 y\,\text{dx} = 2\int_0^4 \sqrt{16x}\text{dx} = 8\int_0^4 \sqrt{x}\,\text{dx} \qquad [\because\ y^2 = 16x\ \ \because\ y = \sqrt{16x}]$$

$$\therefore\ A = 8\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^4 = \frac{16}{3}(4^{3/2} - 0^{3/2}) = \frac{16}{3} \times 8 = \frac{128}{3} \text{ sq units.}$$

Q.2. Find the area of the region bounded by the curve $y^2 = 2y - x$ and the y-axis.

Ans.

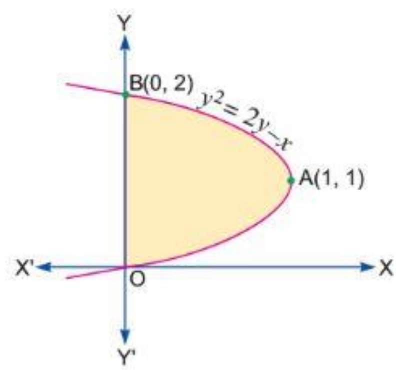
Given curve is

$$y^2 = 2y - x \quad \ldots(i)$$

$$y^2 - 2y = -x$$

$$\Rightarrow \quad y^2 - 2y + 1 = -x + 1$$

$$\Rightarrow \quad (y-1)^2 = -(x-1)$$



Obviously, given curve is left handed parabola having vertex at $(1, 1)$.

Putting $x = 0$ in (i), we get

$$y^2 = 2y \Rightarrow y(y-2) = 0 \Rightarrow y = 0, 2$$

i.e., given parabola meets y axis at $y = 0$ and $y = 2$, so shaded region is required region.

$\therefore$ Required area= Area of region $OABO$

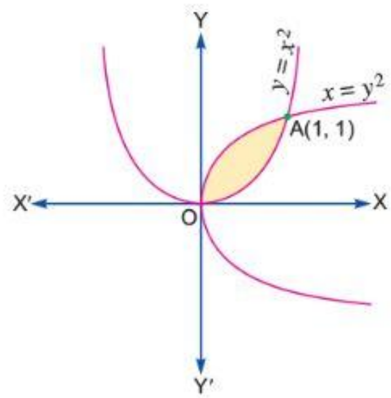
$$= \int_0^2 (2y - y^2)dy = \left[y^2 - \frac{y^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3} \text{ sq units.}$$

Q.3. Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$.

Ans.

The point of intersection of these two parabolas are $O\ (0, 0)$ and $A\ (1, 1)$ as shown in the fig.

Here, we can set $y^2 = x$ or $y = \sqrt{x} = f(x)$ (because region lies above x-axis) and $y = x^2 = g(x)$ where, $f(x) \geq g(x)$ in $[0, 1]$.



Therefore, the required area of the shaded region

$$= \int_0^1 [f(x) - g(x)]\,dx$$

$$= \int_0^1 [\sqrt{x} - x^2]dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq unit}$$

Q.4. Using integration, find the area bounded by the tangent to the curve $4y = x^2$ at the point (2, 1) and the lines whose equations are $x = 2y$ and $x = 3y – 3$.

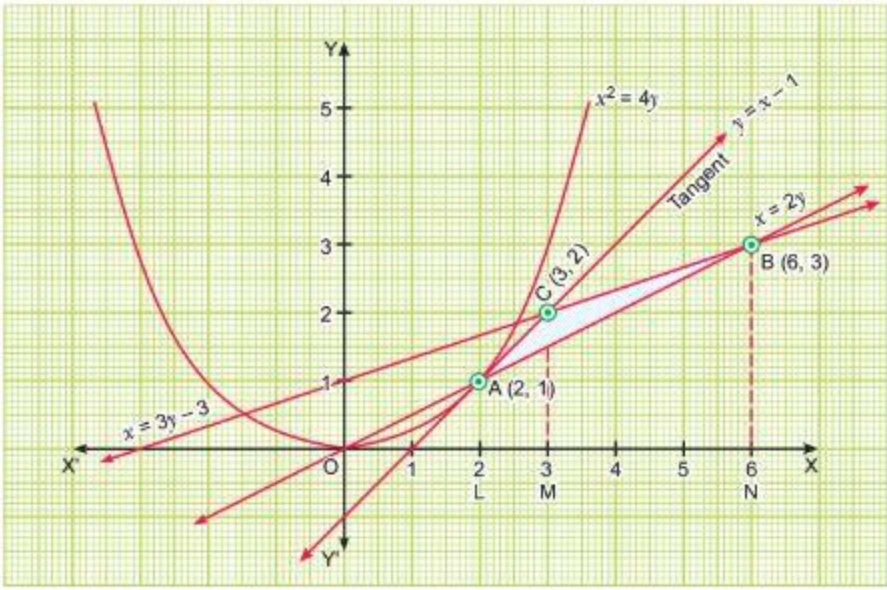
Ans.

Obviously $4y = x^2$ is upward parabola having vertex at origin.

Now $4y = x^2$

$$\Rightarrow 4\frac{dy}{dx} = 2x \quad \Rightarrow \frac{dy}{dx} = \frac{1}{2}x \quad \Rightarrow \left.\frac{dy}{dx}\right]_{(2,1)} = \frac{1}{2} \times 2 = 1$$

$\Rightarrow$ Slope of tangent at $(2, 1)$ to given curve $4y = x^2$ is 1.



Equation of tangent $= \frac{y-1}{x-2} = 1$

$\Rightarrow y - 1 = x - 2 \quad \Rightarrow \quad y = x - 1$

Now, for graph of $x = 2y$

x	0	2
y	0	1

Also for graph of $x = 3y - 3$

x	0	3
y	1	2

After plotting the graph, we get shaded region *ABC* as required region, area of which is to be calculated.

After solving the respective equation, we get

Coordinate of $A \equiv (2, 1)$; $B \equiv (6, 3)$; $C \equiv (3, 2)$

Now, the required area = area of shaded region *ABC*

= ar(region *ALMC*) + ar (region *CMNB*) – ar (region *ALNB*)

$$= \text{ar}(\text{region } ALMC) + \text{ar}(\text{region } CMNB) - \text{ar}(\text{region } ALNB)$$

$$= \int_2^3 (x-1)\text{dx} + \int_3^6 \frac{x+3}{3}\text{dx} - \int_2^6 \frac{x}{2}\text{dx} = \left[\frac{x^2}{2} - x\right]_2^3 + \frac{1}{3}\left[\frac{x^2}{2} + 3x\right]_3^6 - \frac{1}{2}\left[\frac{x^2}{2}\right]_2^6$$

$$= \left[\left(\frac{9}{2} - 3\right) - \left(\frac{4}{2} - 2\right)\right] + \frac{1}{3}\left[\left(\frac{36}{2} + 18\right) - \left(\frac{9}{2} + 9\right)\right] - \frac{1}{4}(36 - 4)$$

$$= \frac{3}{2} + \frac{1}{3}\left(36 - \frac{27}{2}\right) - 8 = \frac{3}{2} + \frac{45}{6} - 8 = 1 \text{ square unit .}$$

Q.5. Sketch the graph *y* =|*x* + 1|. Evaluate $\int_{-3}^{1}|\text{x}+1|\text{dx}.$ What does this value represent on the graph?

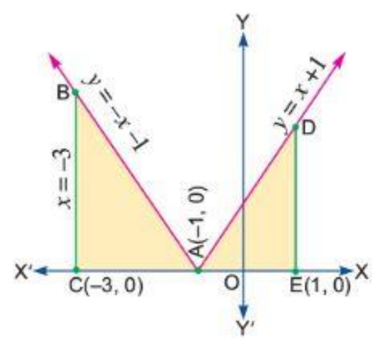
We have, $y = |x+1| = \begin{cases} x+1, \text{ if } x+1 \geq 0 \;\; i.e., \;\; x \geq -1 \\ -(x+1), \text{ if } x+1 < 0 \;\; i.e., \;\; x < -1 \end{cases}$

So, we have $y = x + 1$ for $x \geq -1$ and $y = -x - 1$ for $x < -1$. Clearly, $y = x + 1$ is a straight line cutting x and y-axes at $(-1, 0)$ and $(0, 1)$ respectively. So, $y = x + 1$, $x \geq -1$ represents that portion of the line which lies on the right side of $x = -1$. Similarly, $y = -x - 1$, $x < -1$ represents that part of the line $y = -x - 1$ which is on the left side of $x = -1$. A rough sketch of $y = |x + 1|$ is shown in fig.

Now, $\int_{-3}^{1}\left|x+1\right|dx = \int_{-3}^{-1} -(x+1)\,dx + \int_{-1}^{1}(x+1)\,dx$

$= -\left[\frac{(x+1)^2}{2}\right]_{-3}^{-1} + \left[\frac{(x+1)^2}{2}\right]_{-1}^{1} = -\left[0 - \frac{4}{2}\right] + \left[\frac{4}{2} - 0\right] = 4$ sq units

This value represents the area of the shaded portion shown in figure.

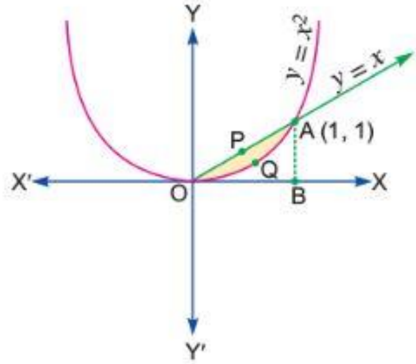


Q.6. Find the area of the region $\{(\mathbf{x}, \mathbf{y}) : \mathbf{x}^2 \leq \mathbf{y} \leq \mathbf{x}\}$

Ans.

We have, $y = x^2 \quad ...(i)$

and $y = x \quad ...(ii)$



We know that $y = x^2$ is an upward parabola and the line $y = x$ is passing through origin.

Now, on solving (i) and (ii), we get

$$x^2 = x \quad \Rightarrow x(x-1)$$

$$x = 0 \text{ or } 1$$

From (ii), $x = 0 \Rightarrow y = 0$ and $x = 1 \Rightarrow y = 1$

So, the point of intersection of (i) and (ii) are $O\ (0, 0)$ and $A\ (1, 1)$.

Draw $AB \perp OX$

Required area= Shaded area shown in figure

= area $OPABO$ – area $OQABO$

$$= \int_0^1 x \ \mathrm{dx} - \int_0^1 x^2 \, \mathrm{dx}$$

$$= \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \ squnit.$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \ squnit.$$

Hence, the required area is $\frac{1}{6}$ sq unit.